LOCAL ALMOST PERFECT EQUILIBRIUM WITH LARGE ADJUSTMENT COSTS*

by

David Levine

University of California, Los Angeles

UCLA Department of Economics
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0. INTRODUCTION

This paper studies continuous time games with closed-loop strategy spaces. A strategy selection such that every agent's strategy is optimal for all initial conditions is called a perfect equilibrium. This is an equilibrium of expectations - regardless of what happens now each agent expects that in the future everyone will behave optimally.[1]

There are two main difficulties with the perfectness concept. First, these equilibria are typically non-unique - in many cases entire continua of equilibria are known to exist.[2] Second, the informational requirements are absurd. Agents are supposed to know what will happen in all contingencies, including contingencies that never have and never will occur and which are far distant from the actual path of the system. This is particularly upsetting since opponents' behavior away from the realized path may not be uniquely determined by what they do along that path.

The objective of this paper is to point out that approximate perfect equilibria that satisfy reasonable informational assumptions exist and that these ε-perfect equilibria[3] are unique in the sense that they are approximately equal to one another.

A reasonable assumption about agents' information is that they know all relevant information locally near states which have actually occured. They could gain this information by making small perturbations in their controls near the status quo.[4] The model of local information I propose is to assume that agents know low order derivatives of relevant functions at the status quo. Naturally this presupposes a relatively smooth environment.

For local data to be used in making intertemporal decisions it must first be extrapolated to global information. I model this by assuming that agents extrapolate income streams linearly into the future. Casual empiricism indicates that the use of such techniques is widespread. My view is this: agents entering an environment for the first time do so with the working hypothesis that linear extrapolation works relatively well. As long as their experience does not contradict this belief they will continue to use naive extrapolation to compute almost optimal paths. Hence ε-perfect equilibria sufficiently smooth that linear extrapolation works, if they exist, are the only economic equilibria likely to be observed.

Let me briefly indicate how the local adjustment procedure works. At each moment of time agents extrapolate income streams linearly and based on this make an (almost) optimal choice at that moment. By continually reoptimizing in this manner at all future times it follows from Bellman's principle that the path chosen this way will be (almost) optimal. Notice that agents need not actually determine what they will do in the future, indeed if at some future time their decision were to be based on linear extrapolations from some far past date they would not do well at all.

One important smooth ε-perfect equilibrium, which I call the fundamental solution, is derived by basing linear extrapolation on the presumption that all agents behave myopically. The loss from doing this is of the same order of magnitude as that from complete linear extrapolation. As a result local almost perfect equilibria avoids the game theoretic infinite regress. It is true that each agent's behavior depends on what he thinks his opponents think he thinks, and so forth. In a model passing through time, however, the higher levels in this chain have consequences in the more distant future. With discounting these higher level conjectures may as well be abandoned. This leads to a finite recursion. At the final stage a solution is uniquely determined because everyone is presumed myopic. This accords well with reality, for although game theoretic paradoxes pose a curious and interesting intellectual challenge, game theoretic regressions don't seem to be an important part of observed economic behavior.

An important fact that follows from the recursive nature of the fundamental solution is that <u>all</u> ε-perfect equilibria smooth enough to permit accurate linear extrapolation and with ε small enough are close to the fundamental solution: the adjustment path is essentially unique.

The organization of the paper by sections is: (1) describes a simple dynamic model with adjustment costs; (2) describes myopic and linear extrapolation, the myopic and fundamental solutions and characterizes them in the main theorem; (3) contrasts the full optimum with the fundamental solution in a control problem with quadratic objective; (4) examines the fundamental solution in a simple duopoly problem; (5) states an extension of the earlier results; and (6) summarizes the salient conclusions.

1. THE MODEL

This section describes a simple dynamic economic environment with adjustment costs. Extensions are considered in section six.

There are N agents $j=1,...,N$. Agent j's vector of control variables is denoted $w^j \in \mathbb{R}^{m_j}$. The vector of state variables is $v \in \mathbb{R}^m$. The control and state spaces are assumed unbounded. The equation of motion is

$$\dot{v} = gw = \sum_{k=1}^{N} g_k w^k \tag{1-1}$$

where the g_k are fixed m x m_j matrices. You may wish to think of the scalar case with v^j firm j's capital stock, w^j its level of investment and $g^j \equiv [0,...,1,0...]$ with 1 in the jth position.

Agent j's objective is the present value of income. His discount rate is $0<\rho^j<\infty$ with corresponding discount factor $\delta^j \equiv 1/\rho^j$; his time horizon is infinite. The income of agent j is given by

$$a^j(v,w^j) \equiv \alpha^j(v) - \beta^j(w^j) \tag{1-2}$$

where a^j is net income, α^j is gross income and β^j are adjustment costs. It is assumed

$$\beta^j(w^j) = (1/b)(1/2)(w^j)^T w^j \tag{1-3}$$

where T denoted the transpose of a vector,

$0 < b << 1$ is a "small" scalar constant and adjustment costs are correspondingly "large". Since α^j may be non-linear, if agents move rapidly relative to the discount factor the income stream will be highly non-linear and linear extrapolation won't work well. To prevent this adjustment costs have been assumed large.

There are several reasons why it may be costly to adjust quickly. It may be physically difficult to change variables such as the capital stock or the physical location of the agent. From the perspective of this paper there is a more fundamental source of adjustment costs which needs to be made explicit: the cost of determining an optimal policy. Suppose I am selling apples and need to know how many I should pick today to sell tomorrow. I know I sold twenty-five apples today at one dollar each and can be fairly confident I can do the same tomorrow. If I contemplate changing output by five or less apples I must know my demand curve in the range fifteen to twenty-five apples in order to make an optimal decision subject to the self-imposed restriction that I'm not going to change output by more than five apples. This I can do fairly easily by inspecting prices and sales over the last several weeks, by recalling conversations I've had with customers, and the like. If, however, I wish to contemplate changing output by up to fifteen apples

I must know my demand curve in the range five to thirty-five apples. To form a reliable determination of demand over so broad a range in so little time as a day will require an appreciable expenditure of time and effort on my part: telephone marketing surveys, poring over sales/revenue records from many years, and the like. The point is that adjustment costs may stem from bounded rationality - the faster I move the more resources I must expend to determine how best to move.

Throughout this paper the following assumption is maintained

Assumption (A): α^j is five times continuously differentiable and for some $1 \leq K_0 < \infty$ $p=0,1,2,3,4,5$ $|D^p \alpha^j| \leq K_0$; $|g| \leq K_0$; $|\rho^j| \leq K_0$; $|\delta^j| \leq K_0$

Here D^p denotes pth order differentiation. The fact that all the bounds in assumption (A) are the same is a convention. The restrictive assumption is that α^j be smooth and that the derivatives remain bounded as $|v|$ grows large.

Definition (1-1): A (closed-loop) strategy for agent j is a bounded continuously differentiable function $f^j : \mathbb{R}^m \to \mathbb{R}^{m_j}$. This means that at v when playing f^j agent j chooses $w^j = f^j(v)$.

This strategy space is limited in the sense that agents don't choose their controls as a function of the entire past, merely as a function of the state variables. While this is restrictive with the present formulation of the equation of motion, in the more general state equation of section five the state space can be extended to include various informational variables (including time). The quite reasonable interpretation of definition (1-1) in this more general context is that agents' decisions are based on a finite set of variables which have a Markovian structure - i.e. evolve according to an ordinary differential equation.

Suppose f is a strategy selection. Since f is bounded and continuously differentiable there is a unique flow $\phi_t(f,v)$ defined for $-\infty<t<\infty$ and for all $v \in \mathbb{R}^m$ which satisfies

$$\phi_0(f,v) = v$$

$$D_t\phi_t(f,v) = gf(\phi_t(f,v)) \qquad (1\text{-}4)$$

and is the path of the state variables when all agents k play f^k and the system starts at v at time zero. By standard arguments it can be shown that

$$D_v\phi_t = \exp\left[g\int_0^t Df(\phi_s)ds\right] \qquad (1\text{-}5)$$

or equivalently that $D_v\phi_t$ satisfies the variational equation

$$D_t(D_v\phi_t) = (gDf(\phi_t))(D_v\phi_t) \tag{1-6}$$

with initial condition $D_v\phi_0 = I$ the identity matrix.

Using this flow the present value of agent j's income starting at v until the horizon t when f is played is

$$A_t^j(f,v) \equiv \int_0^t a^j(\phi_s(f,v), f^j(\phi_s(f,v)))\exp(-\rho^j s)ds \tag{1-7}$$

Since f is bounded a^j is bounded and

$$A^j(f,v) \equiv \lim_{t\to\infty} A_t^j(f,v) \tag{1-8}$$

exists and is finite.

We shall need a smoothness criterion for strategies.

Definition (1-2): A strategy selection $\tilde{f}$ is called K-smooth iff for $K \geq K_0$, $p=0,1,2,3$ $|D^p\tilde{f}| \leq bK$

Thus f must be thrice continuously differentiable with bounded derivatives. The dependence of the bound on b makes good sense - if adjustment costs are large of order (1/b) it is to be expected that the adjustment rate will be small of order b. In section three it is shown that f K-smooth with $b \ll 1/K$ is precisely the condition required for linear extrapolation to provide good results.

Let f^{-j} be the strategy selection for all agents but agent j.

Definition (1-3): A strategy $\tilde{f}^j$ is called ε-optimal for j given $\tilde{f}^{-j}$ iff for all v

$$\tilde{A}^j(v) \equiv A^j(\tilde{f},v) \geq A^j(f,v) - \varepsilon$$

if, in other words; j is within ε of the optimum.

Definition (1-4): A strategy selection $\tilde{f}$ is called an ε-perfect equilibrium iff for all j $\tilde{f}^j$ is ε-optimal for $\tilde{f}^{-j}$.

This says that each agent ε-optimizes having rational expectations about everyone's behavior in all contingencies. While bounded rationality is consistent with ε-perfect equilibrium the converse isn't necessarily true: why if everyone has rational expectations do they sub-optimize? An alternative interpretation, and the one relevant to this paper, is that agents make small expectational errors and this causes them to undertake sub-optimal decisions relative to the full optimum.

A key technical fact is that in a smooth environment approximate optimization occurs if and only if Bellman's equation is approximately satisfied. To state this precisely some notation is required. If $M(x)$ is a function the notation $M(x) = O^p(K)$ means that there is some positive real function $\bar{M}(\cdot)$ which <u>does not depend on the economic environment in any way</u> such that for $n = 0,1,\ldots,p$ $|D^n M(x)| \leq \bar{M}(K)$. The superscript T denotes the transpose of a matrix or vector, while $D\tilde{A}^j$ is the row vector of derivatives of $\tilde{A}^j$ with respect to the state variables. Then

<u>Proposition (1-1)</u>: $\tilde{f}$ K-smooth and $b<1/K$ imply

(A) $\tilde{A}^j = O^2(K)$ and therefore

$$\bar{f}^j(v) \equiv b g_j^T D\tilde{A}^{jT}(v) = bO^1(K)$$

(B) <u>Sufficiency</u>

$\tilde{f}^j = \bar{f}^j + (b\varepsilon)^{\frac{1}{2}} O(K)$ implies $\tilde{f}^j$ ε-optimal for j given $\tilde{f}^{-j}$

(C) <u>Necessity</u> if $b\varepsilon \leq 1$

$\tilde{f}^j$ ε-optimal for j given $\tilde{f}^{-j}$ implies

$\tilde{f}^j = \bar{f}^j + (b\varepsilon)^{\frac{1}{2}} O(K)$

Notice that there is no reason that the symbols $O(K)$ in (B) and (C) should stand for the same function: the sole reason for introducing this notation is to avoid having to continually recompute bounds.

To interpret proposition (1-1) it should be realized that $\bar{f}^j$ maximizes the approximate Hamiltonian that arises when $D\tilde{A}^j$ are used as costate variables in place of the present value derivatives along the optimal path. Note that the assumption of linear response and quadratic adjustment costs guarantees that $\bar{f}^j$ is unique. An implication of proposition (1-1) from the case $\varepsilon = 0$ is that $\tilde{f}^j = \bar{f}^j$ is necessary and sufficient for an optimum: this is simply Bellman's principle.

An ε-equilibrium requires that each agent ε-optimize over all closed-loop strategies. Although this rules out the use of strategies which depend on the entire past history of the game it is important to realize that this constraint is not binding: if a closed-loop strategy is ε-optimal (relative to other closed-loop strategies and for given initial conditions) then no strategy of any kind (provided the differential equation (1-1) has a unique solution) can yield a gain of more than ε over the closed-loop strategy.

The reason for this is fairly straightforward. The proof of proposition (1-1) actually shows that a smooth closed-loop strategy is ε-optimal relative to other closed-loop strategies if and only if it is ε-optimal relative to all smooth open-loop control paths originating at the same initial conditions - this requires only notational changes in the proofs. Since any path for which (1-1) has a unique solution can be approximated arbitrarily closely by smooth paths it follows that a path ε-optimal with respect to smooth paths is ε-optimal with respect to all paths.

The significance of this is that while the restriction of agents to closed-loop strategies requires bounded rationality it does <u>not</u> entail any loss to agents - they do as well with closed-loop strategies as they would if all strategies were available.

A proof of proposition (1-1) can be found in the appendix.

2. LOCAL PERFECT EQUILIBRIA

This section investigates environments in which linear extrapolation works well. This means first that assumption (A) must be satisfied. If this fails - if the non-linearity of the exogenously given functions as measured by K_0 is large, and the adjustment costs as measured by (1/b) is small - agents will move rapidly over a non-linear income surface and the resulting income streams will be highly non-linear. As we will show, more than this is required. The agents' strategies must also be K-smooth with b small relative to K. In other words, even if the exogenous environment is relatively smooth, agents might choose to behave in such highly non-linear ways that linear extrapolation does poorly. In this section it is shown that in relatively smooth environments the loss from linear extrapolation is of order b^5. Thus I call K-smooth Kb^5-perfect equilibria K-linearly perfect equilibria to reflect the fact that they are smooth enough to permit relatively accuratc linear extrapolation, and agents do no worse in order of magnitude loss than if they extrapolated linearly.

The goal of this section is to characterize linearly perfect equilibria. An adjustment path called the fundamental solution is explicitly computed and shown to be linearly perfect, thus establishing existence. The fundamental solution is shown also to have an important dominance property that implies approximate uniqueness : every linearly perfect equilibrium is close to the fundamental solution.

The proofs of these facts are based on Taylor series representations simple in principle but complicated in detail. The first subsection is devoted to analyzing various bounds and approximations needed in the proof of the main theorem. Subsection two describes linear extrapolation and the fundamental solution, and states the main theorem. The final two subsections prove the main theorem.

To deal efficiently with high order derivatives the perspective that they are symmetric multi-linear maps is adopted. Thus if $M(x)$ is a function $D^pM(x)[y^1,\ldots,y^p]$ is the p-linear map $D^pM(x)$ applied to $y^1,\ldots,y^p$.

Expansions and Bounds

Let $\tilde{f}$ be K-smooth with $b<1/K$. The pth time derivative of income to j at v is denoted by $\tilde{a}_p^j(v)$ and can be computed recursively from

$$\tilde{a}_0^j(v) \equiv a^j(v,\tilde{f}^j(v))$$

$$\tilde{a}_p^j(v) = D\tilde{a}_{p-1}^j\left[g\tilde{f}(v)\right] \qquad (2\text{-}1)$$

The derivatives $D^p\tilde{a}_0^j$ needed to compute (2-1) recursively are

$$D\tilde{a}_0^j[y] = D\alpha^j[y] + (1/b)\tilde{f}^{jT}D\tilde{f}^j[y]$$

$$\begin{aligned} D^2\tilde{a}_0^j[y^1,y^2] &= D^2\tilde{a}_0^j[y] \\ &= D^2\alpha^j[y] + (1/b)\{D\tilde{f}^{jT}[y^1]D\tilde{f}^j[y^2] \\ &\quad + \tilde{f}^{jT}D^2\tilde{f}^j[y]\} \end{aligned}$$

$$\begin{aligned} D^3\tilde{a}_0^j[y^1,y^2,y^3] &= D^3\tilde{a}_0^j[y] \\ &= D^3\tilde{a}_0^j[y] + (1/b)\{\textstyle\sum D\tilde{f}^{jT}[y^{i_1}]D^2\tilde{f}^j[y^{i_2},y^{i_3}]\} \\ &\qquad\qquad (i_1,i_2,i_3)=\text{all permutations of } (1,2,3) \\ &\quad + (1/b)\tilde{f}^{jT}D^3\tilde{f}^j[y] \end{aligned} \qquad (2\text{-}2)$$

Using assumption (A) and K-smoothness with (2-1) and (2-2) it can be established that

$$\tilde{a}_p^j = b^p O^2(K) \quad p=0,1 \quad \tilde{a}_2^j = b^2 O^1(K) \qquad (2\text{-}3)$$

and inspection shows that all parts of K-smoothness are required to bound $D\tilde{a}_2^j$.

Next consider

$$D\tilde{A}^j(v) = \int_0^\infty D\tilde{a}_0^j(\tilde{\phi}_s)\exp(-\rho^j s)\,ds \qquad (2\text{-}4)$$

Expand (2-4) under the integral sign in Taylor series

$$D\tilde{A}^j(v) = \int_0^\infty \{D\tilde{a}_0^j(v) + D\tilde{a}_1^j(\phi_s^*)\}\exp(-\rho^j s)ds$$
$$= \delta^j D\tilde{a}_0^j(v) + bO(K) \quad (2\text{-}5)$$

where (2-3) is used to bound the remainder. Similarly

$$D^2\tilde{A}^j(v) = \delta^j D^2\tilde{a}_0^j(v) + bO(K) \quad (2\text{-}6)$$

Finally, expand (2-4) into a first order Taylor series with remainder

$$D\tilde{A}^j(v) = \int_0^\infty \{D\tilde{a}_0^j(v)+D\tilde{a}_1^j(v)+(1/2)D\tilde{a}_2^j(\phi_s^*)\}\exp(-\rho^j s)ds$$
$$= \delta^j D\tilde{a}_0^j(v) + (\delta^j)^2 D\tilde{a}_1^j(v) + b^2 O(K) \quad (2\text{-}7)$$

This final equation forms the basis of linear extrapolation techniques.

Linear Extrapolation

Suppose all agents play $\tilde{f}$ K-smooth with $b<1/K$. Agent j computes approximately the present value of income at v to be $\tilde{\alpha}^j$ and chooses $\tilde{f}^j$ according to

$$\tilde{f}^j(v) = bg_j^T D\,\tilde{\alpha}^{jT}(v) \quad (2\text{-}8)$$

If $\tilde{\alpha}^j = \tilde{A}^j$ so that no approximation is involved $\tilde{f}^j$ would be optimal by proposition (1-1). Also by proposition (1-1) necessary and sufficient conditions for $\tilde{f}^j$ to be ε-optimal are

$$|\tilde{f}^j(v) - \bar{f}^j(v)| \leq (b\varepsilon)^{½}O(K) \qquad (2\text{-}9)$$

(where however different functions O(K) may be involved depending on whether necessity or sufficiency are involved.) What does this mean in terms of $\tilde{\mathcal{a}}^j$?

$$|\tilde{f}^j(v) - \bar{f}^j(v)| = |bg_j^T[D\tilde{\mathcal{a}}^{jT}(v) - D\tilde{A}^{jT}(v)]|$$
$$\leq bK_0|D\tilde{\mathcal{a}}^j(v) - D\tilde{A}^j(v)| \qquad (2\text{-}10)$$

Thus necessary and sufficient conditions for $\tilde{f}^j$ to be ε-optimal are

$$|D\tilde{\mathcal{a}}^j(v) - D\tilde{A}^j(v)| \leq (\varepsilon/b)^{½}O(K) \qquad (2\text{-}11)$$

The objective is to consider methods of extrapolating future income from local information. A naive form of extrapolation is the purely myopic method

$$\tilde{\mathcal{a}}^j(v) = \delta^j\tilde{a}_0^j(v) \qquad (2\text{-}12)$$

which assumes v won't change in the future. From the Taylor expansion of $\tilde{A}^j$ in (2-5) we see that $\tilde{\mathcal{a}}^j = \tilde{A}^j + bO^1(K)$, where remember O^1 implies the first derivatives are close. Thus from (2-11) we are led to conclude (2-12) is $b^3O(K)$-optimal for j. This leads us to define

Definition (2-1): $\tilde{f}^j$ is K-myopically optimal (for j given $\tilde{f}^{-j}$) iff $\tilde{f}^j$ is b^3K-optimal.

In other words, strategies that do as well in order of magnitude term as myopic extrapolation are called myopically optimal.

A more sophisticated extrapolation technique is to project the income stream linearly into the future, setting

$$\tilde{a}^j = \delta^j \tilde{a}_0^j + (\delta^j)^2 \tilde{a}_1^j \qquad (2\text{-}13)$$

Using the expansion (2-7) and (2-11) we see that linear extrapolation is $b^5 O(K)$-optimal. This leads us to define

Definition (2-2): $\tilde{f}^j$ is K-linearly optimal (for j given $\tilde{f}^{-j}$) iff $\tilde{f}^j$ is b^5K-optimal.

Definition (2-3): $\tilde{f}$ is K-linearly perfect iff $\tilde{f}$ is K-smooth and $\tilde{f}^j$ is K-linearly optimal for j given $\tilde{f}^{-j}$.

Linear perfect equilibria are exactly those equilibria smooth enough to permit linear extrapolation, and in which agents do no worse than they would if they used linear extrapolation (in order of magnitude). As pointed out above, in an environment of the type under consideration these are the only economically interesting equilibria.

The fundamental myopic (or myopic) solution for j is to set

$$\tilde{\alpha}^j_m \equiv \delta^j \alpha^j(v) \tag{2-14}$$

which is the same as (2-12) except that adjustment costs are ignored. Since these are of order b anyway, we shouldn't expect any greater order of magnitude loss from (2-14) than from (2-12).

The fundamental linear (or fundamental) solution for j is to set

$$\tilde{\alpha}^j_\ell \equiv \delta^j a^j(v, \tilde{f}^j_m) + (\delta^j)^2 D\alpha^j(v) g\tilde{f}_m(v) \tag{2-15}$$

which is derived by extrapolating linearly under the assumption that everyone is myopic (and also ignores an adjustment cost term of order b^2).

<u>Theorem (2-1)</u> [Main Theorem]:

(A) <u>Smoothness</u>: $\tilde{f}^j_m$ and $\tilde{f}^j_\ell$ are $O(K_0)$-smooth

(B) <u>Existence</u>: If $b<O(K_0)$ $\tilde{f}_\ell$ is $O(K_0)$-linearly perfect

(C) <u>Dominance</u>: If $\tilde{f}$ is K-smooth $K \geq O(K_0)$ $b<1/K$

(1) $\tilde{f}^j_m$ and $\tilde{f}^j_\ell$ are $O(K)$-myopically optimal

(2) if $\tilde{f}^{-j}$ is K-myopically optimal for $-j$

$\tilde{f}^j_\ell$ is $b^4 O(K)$-optimal for j

(3) if $\tilde{f}$ is K-linearly perfect for any $0 \leq \eta < 1$

$\tilde{f}^j_\ell$ is $b^{4+\eta} O(K)$-optimal for j

(D) Uniqueness: if $\tilde{f}$ is K-linearly perfect $K \geq O(K_0)$

for any $0 \leq \eta < 1$ $|\tilde{f}-\tilde{f}_\ell| = (b^{5+\eta})^{1/2} O(K)$

Note that the functions O(K) need not all be the same, and in particular in (C3) and (D) they may depend on the choice of η.

The main theorem uses the fundamental solution to categorize linearly perfect equilibria. For b small enough and K large enough part (B) shows linearly perfect equilibria exist and part (D) shows that they are essentially unique in the sense that they are all close to the joint fundamental solution. Part (C) is also of interest: it shows that as long as opponents are reasonably well-behaved the fundamental solution does reasonably well.

The next two subsections prove the main theorem.

Outline of the Proof

This subsection states a number of lemmas and shows how they imply the main theorem. The next subsection demonstrates the truth of the lemmas.

Lemma (2-1): $\tilde{f}^j_m = bO^3(K_0)$ $\tilde{f}^j_\ell = \tilde{f}^j_m + b^2 O^3(K_0)$

Lemma (2-2): if $\tilde{f}$ K-smooth $b<1/K$ then

(A) $|\tilde{f}^j - \tilde{f}^j_m| \leq b^2 K$ implies $\tilde{f}^j$ O(K)-myopically optimal

(B) $\tilde{f}^j$ K-myopically optimal implies $|\tilde{f}^j - \tilde{f}^j_m| \leq b^2 O(K)$

Lemma (2-3): if $\tilde{f}$ K-smooth $K \geq O(K_0)$ $b<1/K$ and $|D^p\tilde{f} - D^p\tilde{f}_m| \leq (b^{3+\eta})^{1/2}K$ where $0 \leq \eta \leq 1$ $p=0,1$ then

(A) $\tilde{f}^j_\ell$ is $b^{4+\eta}O(K)$-optimal

(B) if in addition $\tilde{f}$ is K-linearly perfect

$$|\tilde{f}^j - \tilde{f}^j_\ell| \leq (b^{5+\eta})^{1/2}O(K)$$

Lemma (2-4): $M(x)$ twice continuously differentiable $|M| \leq Kb^\mu$ $|D^2M| \leq Lb$ imply $|DM| \leq (b^{\mu+1})^{1/2}\sqrt{8LK}$

Let's see how the main theorem follows from these lemmas. Part (A) follows from lemma (2-1) and the definition of K-smoothness. To prove part (B) we need only show that $\tilde{f}_\ell$ satisfies the hypothesis of lemma (2-3A) with $\eta = 1$. This is a direct consequence of lemma (2-1).

Part (C-1) is just lemma (2-2A). To get (C-2) observe by lemma (2-2B) $|\tilde{f}^{-j}-\tilde{f}^{-j}_m| \leq b^2O(K)$. Since $\tilde{f}^{-j}_m$ and $\tilde{f}^{-j}$ are K-smooth lemma (2-4) enables us to conclude that $|D\tilde{f}^{-j}-D\tilde{f}^{-j}_m| \leq b^{3/2}O(K)$. Then part (C-2) follows from lemma (2-3A).

Parts (C-3) and (D) can be proven simultaneously by induction on η. Take first $\eta=0$. Since $\tilde{f}$ linearly perfect implies $\tilde{f}^k$ myopically optimal (all k) we see as in part (C-2) $|D\tilde{f} - D\tilde{f}_m| \leq b^{3/2}O(K)$. By lemma (2-3A) this proves (C-3) for $\eta=0$, while by lemma (2-3B) it proves also (D) for $\eta=0$.

We now examine the implication of (D) for $\eta=0$. By lemma (2-4) it enables us to conclude $|D\tilde{f} - D\tilde{f}_\ell| \le b^{7/4}O(K)$. By lemma (2-1) this in turn shows $|D\tilde{f} - D\tilde{f}_m| \le b^{7/4}O(K)$. Using lemma (2-3) as before shows (C-3) and (D) hold for $\eta = 1/2$ (and therefore $0 \le \eta \le 1/2$). Continuing on in this way we get η arbitrarily close to one.

Proof of the Lemmas

Lemma (2-1): $\tilde{f}_m^j = bO^3(K_0)$ $\quad \tilde{f}_\ell^j = \tilde{f}_m^j + b^2O^3(K_0)$

proof: From (3-14) defining $\tilde{\mathcal{A}}_m^j$ and (3-8) giving $\tilde{f}_m^j$

$$\tilde{f}_m^j = b\delta^j g_j^T D\alpha^{jT} \tag{2-16}$$

By assumption (A) $|D^p\alpha^j| \le K_0 \quad 0 \le p \le 5$ which with (2-16) shows $\tilde{f}_m^j = bO^3(K_0)$. From (2-15) defining $\tilde{\mathcal{A}}_\ell^j$ and (2-16)

$$\tilde{\mathcal{A}}_\ell^j = \tilde{\mathcal{A}}_m^j - (1/2)b(\delta^j)^2 D\alpha^j g_j g_j^T D\alpha^{jT}$$
$$+ b(\delta^j)^2 D\alpha^j \textstyle\sum_k \delta^k g_k g_k^T D\alpha^{kT} \tag{2-17}$$

Observing from (2-8) that $\tilde{f}_\ell^j = bg_j^T D\tilde{\mathcal{A}}_\ell^{jT}$ and using assumption (A) once again shows $\tilde{f}_\ell^j = \tilde{f}_m^j + b^2O^3(K_0)$.

Q.E.D.

Lemma (2-2): if $\tilde{f}$ is K-smooth $b<1/K$

(A) $|\tilde{f}^j-\tilde{f}^j_m| \le b^2K$ implies $\tilde{f}^j$ O(K)-myopically optimal

(B) $\tilde{f}^j$ K-myopically optimal implies $|\tilde{f}^j-\tilde{f}^j_m|\le b^2O(K)$

proof:

For any K-smooth strategy selection we have from the Taylor expansion (2-5)

$$D\tilde{A}^j = \delta^j D\tilde{a}^j_0 + bO(K)$$

$$= D\tilde{\alpha}^j_m - (1/b)\delta^j(1/2)\tilde{f}^{jT}\tilde{f}^j + bO(K)$$

$$= D\tilde{\alpha}^j_m + bO(K) \qquad (2\text{-}18)$$

From Proposition (1-1) giving $\bar{f}^j$ and (2-8) giving $\tilde{f}^j_m$ (2-18) reads

$$\bar{f}^j = \tilde{f}^j_m + b^2O(K) \qquad (2\text{-}19)$$

If $|\tilde{f}^j - \tilde{f}^j_m| \le b^2K$ (2-19) implies $|\tilde{f}^j-\bar{f}^j| \le b^2O(K)$ and by proposition (1-1B) this implies $\tilde{f}^j$ $b^3O(K)$-optimal against -j. Conversely by proposition (1-1C) $\tilde{f}^j$ b^3K-optimal implies $|\tilde{f}^j - \bar{f}^j| \le b^2O(K)$. Using (2-19) and the triangle inequality then shows $|\tilde{f}^j_m-\bar{f}^j| \le b^2O(K)$.

Q.E.D.

Lemma (2-3): if $\tilde{f}$ K-smooth $K\ge O(K_0)$ $b<1/K$ and $|D^p\tilde{f} - D^p\tilde{f}_m| \le (b^{3+\eta})^{1/2}K$ $0\le\eta\le 1$ $p=0,1$

(A) $\tilde{f}^j_\ell$ is $b^{4+\eta}O(K)$-optimal against -j

(B) if in addition $\tilde{f}$ is K-linearly perfect

$$|\tilde{f}^j - \tilde{f}^j_\ell| \le (b^{5+\eta})^{\frac{1}{2}} O(K)$$

<u>proof</u>:

The idea of the proof is similar to that of lemma (2-2), using however the Taylor expansion (2-7) in place of (2-5). Throughout the proof K is assumed large enough that $\tilde{f}^j_\ell$ is K-smooth, where lemma (2-1) shows this is possible.

For any K-smooth $\tilde{f}$ from (2-7)

$$\begin{aligned} D\tilde{A}^j &= \delta^j D\tilde{a}^j_0 + (\delta^j)^2 D\tilde{a}^j_1 + b^2 O(K) \\ &= \delta^j [D\alpha^j - (1/b)\tilde{f}^{jT} D\tilde{f}^j] \\ &\quad + (\delta^j)^2 [D^2\alpha^j g\tilde{f} + D\alpha^j g D\tilde{f}] + b^2 O(K) \end{aligned} \tag{2-20}$$

so that from (2-8) and (2-15)

$$\begin{aligned} \bar{f}^j &= \tilde{f}^j_\ell - \delta^j (1/b)[\tilde{f}^{jT} D\tilde{f}^j - \tilde{f}^{jT}_m D\tilde{f}^j_m] \\ &\quad + (\delta^j)^2 [D^2\alpha^j g[\tilde{f} - \tilde{f}_m] + D\alpha^j g[D\tilde{f} - D\tilde{f}_m]] \\ &\quad + b^2 O(K) \\ &= \tilde{f}^j_\ell + |\tilde{f} - \tilde{f}_m| O(K) + |D\tilde{f} - D\tilde{f}_m| O(K) + b^2 O(K) \end{aligned} \tag{2-21}$$

If in addition $|D^p\tilde{f} - D^p\tilde{f}_m| \le (b^{3+\eta})^{\frac{1}{2}} K \quad p=0,1$ then

$$\bar{f}^j = \tilde{f}^j_\ell + (b^{3+\eta})^{\frac{1}{2}} O(K) \tag{2-22}$$

Taking $\tilde{f}^j = \tilde{f}^j_\ell$ and observing that $\tilde{f}^j_\ell$ satisfies all the required hypothesis shows by proposition (1-1B) $\tilde{f}^j_\ell$ is $b^{4+\eta}O(K)$-optimal against $\tilde{f}^{-j}$. If in addition some other $\tilde{f}^j$ is K-linearly optimal against -j it follows from (2-22), proposition (1-1C) and the triangle inequality that $|\tilde{f}^j - \tilde{f}^j_\ell| \leq (b^{5+\eta})^{1/2}O(K)$.

Q.E.D.

<u>Lemma (3-4)</u>: Let $M(x)$ be twice continuously differentiable with $|M| \leq b^\mu K$ and $|D^2M| \leq bL$. Then $|DM| \leq (b^{\mu+1})^{1/2}\sqrt{8LK}$.

<u>proof</u>: Since $|DM(x)|$ is the largest directional derivative on the unit circle it suffices to consider the case where x is a scalar. Assume without loss of generality $DM(x) \geq 0$. The idea of the proof is similar to that in lemma (A-5). Since $|D^2M| \leq Lb$ the earliest DM reaches zero is at $x+DM(x)/Lb$ so that by the fundamental theorem of calculus

$$\begin{aligned} 2b^\mu K &\geq M(x + DM(x)/bL) - M(x) \\ &= \int_x^{x+DM(x)/bL} DM(z)dz \\ &\geq \int_x^{x+DM(x)/2bL} [DM(x)/2]dz \\ &= [DM(x)]^2/4bL \end{aligned} \qquad (2\text{-}23)$$

from which the lemma follows directly.

Q.E.D.

3. AN APPLICATION TO OPTIMAL CONTROL THEORY

The theory of the previous section can be applied to solve one-agent control problems. Because only one agent is involved the subscript j is dropped in this section. Let

$$\alpha(v) = (1/2)v^T Mv \quad M \text{ negative definite}$$

$$g = I \quad \text{the identity matrix} \tag{3-1}$$

In other words the control problem is

$$\underset{w(\cdot)}{\text{maximize}} \int_0^\infty (1/2)[v^T Mv-(1/b)w^T w]\exp[-\rho s]ds$$

$$\text{subject to } \dot{v} = w \tag{3-2}$$

First this is solved exactly, then compared to the fundamental solution.

The Hamiltonian is

$$H(w) = (1/2)[v^T Mv - (1/b)w^T w] + \lambda w \tag{3-3}$$

where $\lambda = DA$ along the optimal path are the costate variables. Maximizing the Hamiltonian in w gives

$$w = b\lambda \tag{3-4}$$

while from control theory

$$\dot{\lambda} = \rho\lambda - D_v H = \rho\lambda - Mv \tag{3-5}$$

Thus the first order condition is

$$\begin{bmatrix} \dot{w} \\ \dot{v} \end{bmatrix} = G \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} \rho I & -bM \\ I & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{3-6}$$

First the eigenvectors and eigenvalues of G must be computed. Let γ_i^1, γ_i^2 be the eigenvalues of G with corresponding eigenvectors x_i^1, x_i^2. Let μ_i be the eigenvalues of M and y_i the corresponding eigenvectors. A simple computation shows

$$\gamma_i^1 = \frac{\rho - \sqrt{\rho^2 - 4\mu_i b}}{2}$$

$$\gamma_i^2 = \frac{\rho + \sqrt{\rho^2 - 4\mu_i b}}{2}$$

$$x_i^j = \begin{bmatrix} x_{i1}^j \\ x_{i2}^j \end{bmatrix} = \begin{bmatrix} \gamma_i^j \, y_i \\ y_i \end{bmatrix} \tag{3-7}$$

Fix v, then w must be chosen so that the system (3-6) is stable; this means $[w^T, v^T]^T$ must be a linear combination of eigenvectors of G whose corresponding eigenvalues have negative real parts. From (3-7) these are the eigenvectors x_i^1.

Let $C = [y_1, \ldots, y_m]$. Then

$$v = \sum_{i=1}^{m} \eta_i y_i = C\eta \qquad (3\text{-}8)$$

where η is an m-tuple of coefficients. From (4-8)

$$\eta = C^{-1}v \qquad (3\text{-}9)$$

Finally

$$\begin{aligned} w_e &= \sum_{i=1}^{m} \eta_i x_{i1}^{1} \\ &= \sum_{i=1}^{m} \eta_i \gamma_i^1 y_i \\ &= C \operatorname{diag}(\gamma_i^1)\eta \\ &= C \operatorname{diag}(\gamma_i^1)\, C^{-1}v \qquad (3\text{-}10) \end{aligned}$$

which gives the exact solution in closed-loop form.

Consider now the fundamental solution. The myopic present value is $\tilde{\mathcal{a}}_m = (1/2)(1/\rho)v^T Mv$, and the myopic solution is

$$\tilde{f}_m = w_m = bg^T D\tilde{\mathcal{a}}_m^T = (b/\rho)Mv \qquad (3\text{-}11)$$

The fundamental present value is

$$\begin{aligned} \tilde{\mathcal{a}}_\ell &= (1/\rho)[\alpha(v)-\beta(w_m)] + (1/\rho^2)[D\alpha(v)w_m] \\ &= (1/\rho)(1/2)[v^T Mv-(b/\rho^2)v^T M^2 v]+(1/\rho^2)(b/\rho)[v^T M^2 v] \\ &= (1/2)(1/\rho)\; v^T Mv + (1/2)(b/\rho^3)v^T M^2 v \qquad (3\text{-}12) \end{aligned}$$

Thus the fundamental solution is

$$\tilde{f}_\ell = w_\ell = bg^T D \mathcal{a}_m^T$$
$$= (b/\rho)[M + (b/\rho^2)M^2]v \qquad (3\text{-}13)$$

How are (3-10) the exact solution and (3-13) the fundamental solution related? From the main theorem w_ℓ and w_e differ by at most order b^3. Since (3-13) contains no terms of order b^3 it must be the second order Taylor's series approximation to (3-10) viewed as a function of b. Let's verify that this is so. The only expression in (3-10) that depends on b are the γ_i^1 which from (3-7) are

$$\gamma_i^1 = \frac{\rho}{2}\left[1 - \sqrt{1 - \frac{4\mu^i b}{\rho^2}}\right] \qquad (3\text{-}14)$$

Expanding (3-14) into a second order Taylor series around b = 0

$$\gamma_i^1 \simeq \frac{\mu_i b}{\rho} + \frac{\mu_i^2 b^2}{\rho^3} \qquad (3\text{-}15)$$

Consequently to second order

$$w_e \simeq (b/\rho)C\,[\mathrm{diag}(\mu_i)+(b/\rho^2)\mathrm{diag}(\mu_i^2)]\,C^{-1}v$$
$$= (b/\rho)[M+(b/\rho^2)C\,\mathrm{diag}(\mu_i)\,C^{-1}\,C\,\mathrm{diag}(\mu_i)C^{-1}]v$$
$$= (b/\rho)[M + (b/\rho^2)M^2]v \qquad (3\text{-}16)$$

which is exactly (3-13).

Two points should be raised from a purely analytic standpoint. First, the fundamental solution was easy to derive and could have been derived equally easily if α wasn't quadratic, while deriving the exact solution was tedious and was possible <u>only</u> because α was quadratic. Second, the expression for the fundamental solution (2-12) is relatively easy to interpret, while the exact solution is a rather mysterious function of eigenvalues and eigenvectors.

A final note is that the problem given here doesn't actually satisfy assumption (A) since α and $D\alpha$ are unbounded. This reflects merely the fact that the main theorem holds under quite general conditions - assumption (A) is intended largely to simplify the proofs.

4. APPLICATION TO A DUOPOLY PROBLEM

There are two firms, and v^j is firm j's output. Since each firm j chooses w^j, the rate of change of its own output, the matrix of responses of the state to control variables b = I the identity matrix. Each firm's discount rate $\delta^j \equiv 1$ and both firms have identical gross profit functions

$$\alpha^j = [\ 1 - v^1 - v^2\ ]v^j \qquad (4\text{-}1)$$

which corresponds to constant marginal cost, a linear demand curve with unitary slope and price minus marginal cost at zero industry output equal to one.

The myopic present value for firm j is $\tilde{\alpha}^j_m = \alpha^j$ and the myopic strategy is $\tilde{f}^j_m = bD_j\alpha^j = 1-v^k-2v^j \quad k\neq j$. It is easy to see that if both firms behave myopically the symmetric steady state $\tilde{f}^1_m = \tilde{f}^2_m = 0 \quad v^1 = v^2 = \bar{v}_m$ occurs at the static Cournot-Nash output $\bar{v}_m = (1/3)$.

The fundamental solution is easily computed to be

$$\tilde{f}^j_\ell = b\{D_j\alpha^j+b[D_j\alpha^jD^2_{jj}\alpha^j+D^2_{jk}\alpha^jD_k\alpha^k+D_k\alpha^jD^2_{jk}\alpha^k]\}$$
$$j \neq k \qquad (4\text{-}2)$$

which can be solved for the symmetric steady state output

$$\bar{v}_f = (1/3)(1 + b/3) \qquad (4\text{-}3)$$

Thus when both firms play their fundamental strategies the industry is more competitive and both firms worse off than when they act myopically. This is reminiscent of the Stackelberg warfare discussed by Bishop [1]. Both firms realize that when they increase output their opponent's myopic response is to cut output, thus at the static Cournot-Nash output both firms will increase output slightly until the steady state in (4-3) is reached. This is an outstanding example of how, because of a public goods problem, increased rationality of agents can make all agents worse off (except consumers who aren't represented in the game).

Notice that fundamental steady state output in (4-3) is close to the static Cournot-Nash output. It is easy to see why: the condition for a myopic steady state is the first order condition for a static Nash equilibrium of the game with payoff functions α^j. Since the fundamental solution is a small perturbation of the myopic solution its steady states will typically lie nearby.

5. EXTENSIONS

The model of the previous sections had a very strong structure. While this greatly simplified the algebra in the proofs this structure is by no means necessary for our results. In this section several easy extensions are given.

What is important in the types of results here? Two assumptions are key: the exogenous environment must be such as to permit smooth adjustment by agents, and adjustment costs must be large enough that agents will choose to adjust slowly relative to the discount rate and the intrinsic non-linearity of the payoff structure. For example, bang-bang control problems will never fit into this framework - linear extrapolation won't work if large discontinuous changes will take place in the future. A sample of the type of problem that does fit the framework here is

$$\dot{v} = bh(v,w) + g(v)w \qquad (5\text{-}1)$$

$$a^j(v,w) = \alpha^j(v,w) - \beta^j(v,w) \qquad (5\text{-}2)$$

$$(1/2)(1/b)w^T\underline{B}^j w \leq \beta^j(v,w) \leq (1/2)(1/b)w^T\bar{B}^j w \qquad (5\text{-}3)$$

where $\underline{B}^j$ and $\bar{B}^j$ are positive definite and β^j is strictly concave. The quadratic bounds on the adjustment costs illustrated in figure (5-1) insure that agents will choose small values of the

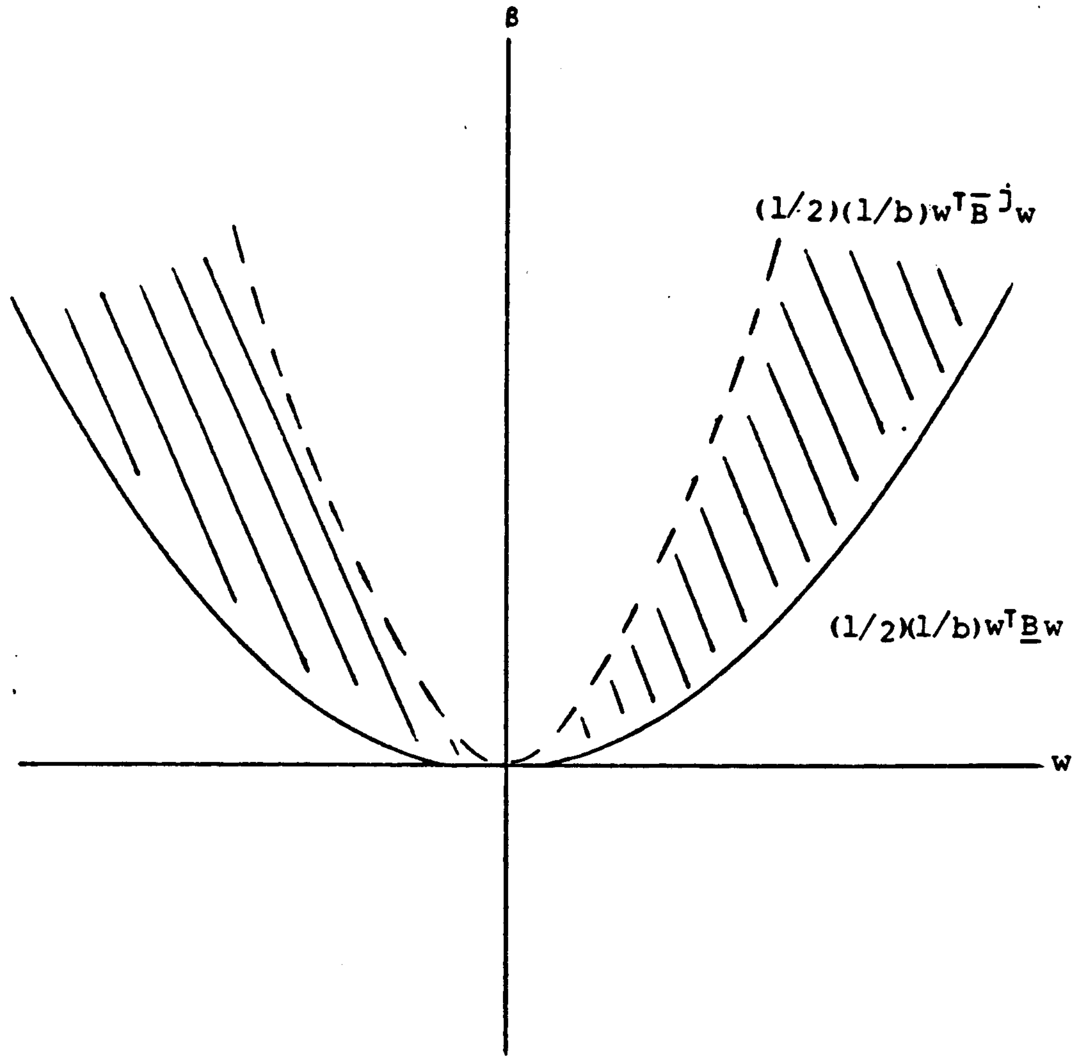

Figure (5-1): Bounded Adjustment Costs

controls; the adjustment equation (5-1) insures that when w is of order b $\dot{v}$ will be as well, so that extrapolated paths will be good approximations. Provided that the functions h,g,α and β are sufficiently differentiable with uniformly bounded derivatives all of the theorems of the previous sections go through unchanged. Note that in (5-3) adjustment costs can depend on all agents' controls. This complicates the expressions for the Hamiltonian and for $\bar{f}$ but otherwise has no significant impact on our results.

One important technical assumption was that the state space be unbounded. As long as (almost) optimal paths don't actually hit the boundary this makes no difference. If the paths do hit the boundary this creates a minor problem: payoff paths will be discontinuous when the boundary is reached. However, as long as the system is sufficiently far from the boundary the discontinuity will be far enough in the future that the discounted error from extrapolation will be small.

Behavior near the boundary is problematic. A far more serious problem, however, is that in economic models there are frequently important non-convexities on the boundary - start-up costs and the like - which are unrelated to the smooth structure in the interior. About these types of problems this framework has nothing to say.

A second assumption that was primarily technical in nature was the assumption that various functions and their derivatives were bounded at infinity. It is not hard to see that what is really needed here is that income streams and their derivatives not grow faster than the discount rate. Although the algebra may be messy, there is no reason in principle why the results of this paper can't be extended to growth models.

The results can also be extended to discrete time models. This requires replacing assumptions on derivatives with assumptions on finite differences and using first differences as the basis of linear extrapolation. While algebraically messy this is conceptually straightforward.

6. CONCLUSION

What does it mean for b to be "small"? Apparently as b goes to zero adjustment costs become infinite and nobody moves - not a very interesting limit system. This, however, is an artifact of the units in which things are measured as the limit is approached. Two systems of measurement yield non-degenerate limits and give insight into the type of environment to which the results of this paper are applicable.

One way of rescaling the system is to change the units in which time is measured choosing new time units $\bar{t} = bt$. In these new time units the equation of motion (1-1) becomes

$$\dot{v} = (1/b)gw \tag{6-1}$$

and as b goes to zero both the myopic and fundamental solution approach

$$\begin{aligned}\dot{v} &= (1/b)g\tilde{f}_m \\ &= \textstyle\sum_k \delta^k g_k g_k^T D\alpha^k \end{aligned} \tag{6-2}$$

in which each agent moves the state variables in the direction which improves his own individual income. In the new time scale the discount rate is $\bar{\rho}^j = \rho^j/b$. As $b \to 0$ $\bar{\rho}^j \to \infty$ and with an infinite discount rate agents behave completely myopically. If the discount rate is large (but not infinite) agents will still do reasonably well behaving myopically, but do even better using linear extrapolation or the fundamental solution.

An alternative method of rescaling the system is to change the units in which both income and the state variables are measured setting $\bar{a}^j = ba^j$ and $\bar{v} = v/b$. In these units with initial conditions v_0 the system approaches

$$\dot{v} = \sum_k \delta^k g_k g_k^T D\bar{\alpha}^k \tag{6-3}$$

where the limit income function $\bar{\alpha}^k(\bar{v}) = D\alpha^k(v_0)[\bar{v}]$ is linear. In other words the myopic solution is exact if gross income is a linear function. If gross income is " almost " linear in the sense that $D\alpha^k$ changes slowly along the equilibrium adjustment path the myopic solution does reasonably well, but the fundamental solution does even better. In actual economic environments, of course, the gain to using a linear extrapolation rule may outweigh the informational and computational costs, and the use of these rules will be important.

The discussion of limiting systems is useful in understanding what it means for b to be "small". The non-linearity in gross income should be small over a period of time during which substantial discounting occurs. From an economic perspective, however, this discussion misses the point. The objective of this paper has been to study a particular class of rules of thumb - forecasting techniques which extrapolate income linearly. Casual empiricism suggests that in many economic environments such rules are widely used and work relatively well. For this reason it is interesting to make assumptions which guarantee that linear extrapolation works. The central point of this paper is a simple one - in these environments there is a (nearly) unique adjustment path. It is characterized by the fact that along this path each agent can assume without significant loss that his opponents behave myopically. This is to say that in an economic environment in which linear extrapolation is a good rule of thumb game-theoretic regression is not an important aspect of behavior. This may explain why, although the paradoxes of game theory are an important intellectual question, they don't seem from an empirical standpoint to have much relevance to economics.

APPENDIX: ALMOST OPTIMAL CONTROL

In smooth systems Bellman's equation is necessary and sufficient for a smooth optimal path. This appendix demonstrates proposition (1-1): that in a smooth system a necessary and sufficient condition for a smooth almost optimal path is that Bellman's equation almost hold.

Throughout this section $\tilde{f}$ is assumed to be K-smooth with $b<1/K$. In outline the strategy of the proof is : (1) establish the smoothness of the state valuation functions $\tilde{A}^j$; (2) derive an expression for the loss from switching from $\tilde{f}^j$ to f^j in terms of $\bar{f}^j$ which maximizes the Hamiltonian, and (3) use (2) to find a necessary and a sufficient condition for an ε-optimal path.

To deal efficiently with high order derivatives it is convenient to view them as symmetric multi-linear operators. If $M(x)$ is a function its pth derivative $D^pM(x)[y^1,\dots,y^p]$ is a symmetric p-linear function of the variables y^i which have the same dimension as x. By definition the norm $|D^pM(x)|$ is the smallest number such that for all y

$$|D^pM(x)[y]| \le |D^pM(x)| \; \Pi_{i=1}^{p} |y^i|$$

Smoothness of the State Valuation Function

Lemma (A-1): $\tilde{A}^j = O(K)$

proof:

$$a^j(v,\tilde{f}) = \alpha^j(v) - (1/2)(1/b)\tilde{f}^{jT}\tilde{f}^j$$
$$= O(K_0) + O(bK^2/2)$$
$$= O(K) \qquad \text{(A-1)}$$

using $|\alpha^j(v)| \le K_0$ from assumption (A), $|\tilde{f}^j| \le bK$ from K-smoothness and the maintained hypothesis $b<1/K$. Integrating and using $1/\rho^j = \delta^j \le K_0$ from assumption (A) shows

$$\tilde{A}^j(v) = \int_0^\infty a^j(\tilde{\phi}_s, \tilde{f}^j(\tilde{\phi}_s))\exp[-\rho^j s]ds$$
$$= O(K) \int_0^\infty \exp[-\rho^j s]ds$$
$$= O(K) \quad \delta^j = O(K) \qquad \text{(A-2)}$$

Q.E.D.

To bound $D^p\tilde{A}^j$ the derivatives of the flow $\tilde{\phi}_t$ must be bounded.

Lemma (A-2): $|D_v\tilde{\phi}_t| \le \exp[K^2bt]$

proof: By (1-4)

$$D_v\tilde{\phi}_t = \exp[g \int_0^t D\tilde{f}(\tilde{\phi}_s)ds] \qquad \text{(A-3)}$$

and the lemma follows from $|g| \le K_0$ $|D\tilde{f}| \le K$

Q.E.D.

Lemma (A-3): $|D_v^2 \tilde{\phi}_t| \leq O(K)bt \exp[K^2bt]$

proof: From (A-3)

$$D_v^2\tilde{\phi}_t[y^1,y^2] = \{(g/2)\int_0^t\{D^2\tilde{f}[y^1,D_v\tilde{\phi}_s[y^2]] + D^2\tilde{f}[y^2,D_v\tilde{\phi}_s[y^1]]\}ds\} \exp[g\int_0^t D\tilde{f}(\tilde{\phi}_s)ds] \qquad \text{(A-4)}$$

Using lemma (A-2) and the additional bound $|D^2\tilde{f}|\leq bK$ yields the desired result.

Q.E.D.

Lemma (A-4): $D\tilde{A}^j = O^2(K)$

proof: Since

$$\tilde{A}^j(v) = \int_0^\infty a^j(\tilde{\phi}_s,\tilde{f}^j(\tilde{\phi}_s))\exp[-\rho^j s]ds \qquad \text{(A-5)}$$

$$D\tilde{A}^j = \int_0^\infty\{D\alpha^j - D\beta^j D\tilde{f}^j\}D_v\tilde{\phi}_s \exp[-\rho^j s]ds \qquad \text{(A-6)}$$

$$\begin{aligned} D^2\tilde{A}^j[y^1,y^2] = \\ &\int_0^\infty(\{D^2\alpha^j[y] - D^2\beta^j[D\tilde{f}^j[y^1],D\tilde{f}^j[y^2]] \\ &-(1/2)D\beta^j[y^1]D\tilde{f}^j[y^2] - (1/2)D\beta^j[y^2]D\tilde{f}^j[y^1]\}D_v\phi_s \\ &+\{D\alpha^j - D\beta^j D\tilde{f}^j\}D_v\phi_s[y])\exp[-\rho^j s]ds \end{aligned} \qquad \text{(A-7)}$$

the bounds in lemmas (A-2) and (A-3) with (A) and K-smoothness now yield the desired result.

Q.E.D.

This proves part (A) of proposition (1-1).

The Hamiltonian

For agent j the Hamiltonian corresponding to $\tilde{f}$ is defined as

$$\tilde{H}^j(v,w^j) \equiv a^j(v,w^j) + D\tilde{A}^j(v)g\tilde{f} \tag{A-8}$$

which exists by lemma (A-4) and the maintained hypothesis that $\tilde{f}$ is K-smooth with b<1/K. This differs slightly from the usual definition of the Hamiltonian since $\tilde{A}^j$ is evaluated along $\tilde{\phi}$ which is not necessarily an optimal path for agent j.

Consider the problem of maximizing $\tilde{H}^j(v,w^j)$ over w^j. From (A-8), (1-1) and (1-2)

$$\tilde{H}^j(v,w^j) = \alpha^j(v) - (1/b)(1/2)[w^j]^T w^j + D\tilde{A}^j(v)\ \{ \sum_{k\neq j} g_k \tilde{f}^k(v) + g_j w^j \ \} \tag{A-9}$$

which is quadratic in w^j. The unique value of w^j denoted $\bar{f}^j(v)$ which satisfies the first order condition $D_2\ \tilde{H}^j(v,w^j) = 0$ is the unique maximizing choice of w^j. The first order condition is

$$-(1/b)\ w^j + g_j^T\ D\tilde{A}^{jT}(v) = 0 \tag{A-10}$$

so that

$$\bar{f}^j(v) = bg_j^T\ D\tilde{A}^{jT}(v) \tag{A-11}$$

Also since $\tilde{H}^j$ is quadratic in w^j if $\bar{H}^j(v) \equiv \tilde{H}^j(v, \bar{f}^j(v))$ is the value of the maximized Hamiltonian

$$\tilde{H}^j(v, w^j) = \bar{H}^j(v) - (1/b)(1/2)(w^j - \bar{f}^j(v))^T \quad (w^j - \bar{f}^j(v)) \tag{A-12}$$

The objective of this section is to use (A-12) to evaluate the gain to j from switching to f^j from $\tilde{f}^j$

$$\tilde{\Delta}^j(f^j, v) \equiv A^j(f^j, \tilde{f}^{-j}, v) - \tilde{A}^j(v) \tag{A-13}$$

To this end define

$$\tilde{Q}_t^j(f^j, v) = A_t^j(f^j, \tilde{f}^{-j}, v) + e^{-\rho^j t}\tilde{A}^j(\phi_t(f^j, \tilde{f}^{-j}, v)) \tag{A-14}$$

to be the present value of playing f^j to time t then switching to $\tilde{f}^j$. Since by lemma (A-1) $\tilde{A}^j$ is bounded

$$\begin{aligned} \tilde{\Delta}^j(f^j, v) &= \tilde{Q}_\infty^j(f^j, v) - \tilde{Q}_0^j(f^j, v) \\ &= \int_0^\infty D_t Q_s^j(f^j, v) ds \end{aligned} \tag{A-15}$$

where the last line follows from the fundamental theorem of calculus. Using (A-14) it is seen that

$$D_t Q_s^j(f^j, v) = [\tilde{H}^j(\phi_s, f^j(\phi_s)) - \rho\tilde{A}^j(\phi_s)]e^{-\rho^j s} \tag{A-16}$$

where $\phi_s = \phi_s(f^j, \tilde{f}^{-j}, v)$. Furthermore, by the usual properties of present values

$$\rho\tilde{A}^j(v) = \tilde{H}^j(v,\tilde{f}^j(v)) \quad \text{(A-17)}$$

Equations (A-15), (A-16) and (A-17) combine to show

$$\tilde{\Delta}^j(f^j,v) = \int_0^\infty [\tilde{H}^j(\phi_s,f^j(\phi_s))-\bar{H}^j(\phi_s,\tilde{f}^j(\phi_s))]e^{-\rho^j s}ds \quad \text{(A-18)}$$

To evaluate (A-18) observe that

$$\tilde{H}^j(v,w^j) - \tilde{H}(v,\tilde{w}^j) = [\tilde{H}^j(v,w^j) - \bar{H}^j(v)] - [\tilde{H}^j(v,\tilde{w}^j) - \bar{H}^j(v)] \quad \text{(A-19)}$$

so that using (A-12) (A-18) becomes

$$\tilde{\Delta}^j(f^j,v) = (1/b)(1/2)\{\int_0^\infty(\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)e^{-\rho^j s}ds - \int_0^\infty(f^j-\bar{f}^j)^T(f^j-\bar{f}^j)e^{-\rho^j s}ds\} \quad \text{(A-20)}$$

evaluated along ϕ_s. Some simple implications of (A-15) are that $\tilde{f}^j$ is optimal if and only if $\tilde{f}^j=\bar{f}^j$ which is Bellman's equation, and that if $\tilde{f}^j\neq\bar{f}^j$, $\bar{f}^j$ yields strictly greater present value than $\tilde{f}^j$.

<u>Characterization of ε-Optima</u>

Suppose as always $\tilde{f}$ is K-smooth with $b<1/K$. Set

$$\tilde{\gamma}^j \equiv \sup_v |\tilde{f}^j(v) - \bar{f}^j(v)| \quad \text{(A-21)}$$

This section completes the proof of proposition (1-1) by showing that bounds on $\tilde{\gamma}^j$ of order $(b\varepsilon)^{1/2}O(K)$ are necessary and sufficient for an ε-optimum for j given $\tilde{f}^{-j}$.

Sufficiency is easy. Suppose $\tilde{\gamma}^j \leq (2/K_0)^{1/2}(b\varepsilon)^{1/2}$. Then the first term in (A-20) is

$$(1/b)(1/2)\int_0^\infty(\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)\exp(-\rho^j s)ds$$
$$\leq (1/b)(1/2)\delta^j(\tilde{\gamma}^j)^2$$
$$\leq (1/b)(1/2)K_0[(2/K_0)b\varepsilon] \leq \varepsilon \qquad \text{(A-22)}$$

and from (A-20) we see this implies that $\tilde{f}^j$ is indeed ε-optimal for j. This proves (B) of proposition (1-1).

Consider now the converse, part (C). Suppose $\tilde{f}^j$ is ε-optimal. Then by definition the gain from playing $\bar{f}^j$ $\tilde{\Delta}^j(\bar{f}^j(v)) \leq \varepsilon$ for all v. From (A-20) we see that this means

$$(1/b)(1/2)\int_0^\infty(\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)\exp(-\rho^j s)ds \leq \varepsilon \qquad \text{(A-23)}$$

when the integral is evaluated along $\bar{\phi}_s$. However, along $\bar{\phi}_s$ $D_t\tilde{f}^j$ and $D_t\bar{f}^j$ are absolutely bounded. Thus if $\tilde{\gamma}^j$ is too large $|\tilde{f}^j-\bar{f}^j|$ can't decrease fast enough for (A-23) to be satisfied. To formally conclude the proof of proposition (1-1) we prove

<u>Lemma (A-5)</u>: (A-23) implies $\tilde{\gamma}^j \leq (b\varepsilon)^{1/2}O(K)$ provided

$b\varepsilon \leq 1$.

<u>proof</u>:

By part (A) $\bar{f}^j = bO^1(K)$ while by K-smoothhess $\tilde{f}^j = bO^1(K)$. Thus

$$
\begin{aligned}
&|D_t[\tilde{f}^j(\bar{\phi}_t) - \bar{f}^j(\bar{\phi}_t)]| \\
&\leq |D\tilde{f}^j(\bar{\phi}_t) - D\bar{f}^j(\bar{\phi}_t)||D_t\bar{\phi}_t| \\
&\leq bO(K)|g_j\bar{f}^j + \Sigma_{k\neq j}\, g_k\tilde{f}^k| \\
&\leq b^2O(K) \qquad \text{(A-24)}
\end{aligned}
$$

Let $d \equiv |\tilde{f}^j(v) - \bar{f}^j(v)|$. By (A-24) $|\tilde{f}^j - \bar{f}^j|$ declines no faster than linearly to zero at time $d/b^2O(K)$, so

$$
\begin{aligned}
&(1/b)(1/2)\int_0^\infty(\tilde{f}^j-\bar{f}^j)^T(\tilde{f}^j-\bar{f}^j)\exp(-\rho^j s)ds \\
&\geq (1/b)(1/2)\int_0^{d/2b^2O(K)}(d/2)^2\exp(-\rho^j s)ds \\
&= (1/b)(1/2)(d^2/4)\{1-\exp[-\rho^j d/2b^2O(K)]\} \\
&\geq (1/b)\;[O(K)d]^2\{1-\exp[-O(K)d]\} \\
&= (1/b)\;[M(O(K)d)]^2 \qquad \text{(A-25)}
\end{aligned}
$$

where $M(x) = x\sqrt{1 - \exp(-x)}$. Using the hypotheses (A-23) and (A-25) shows

$$
d \leq O(K)[M^{-1}(b\varepsilon)]^{\frac{1}{2}} \qquad \text{(A-26)}
$$

and it is easy to verify that for $0 \leq b\varepsilon \leq 1$

$$M^{-1}(b\varepsilon) \leq K_1 b\varepsilon \tag{A-27}$$

where K_1 is a fixed constant. This is the desired conclusion.

Q.E.D.

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NOTES

(1) The concept of perfect equilibrium is due to Selten and discussed in Selten [7].

(2) This is pointed out in Rubinstein [4].

(3) Epsilon equilibria are due to Radner and are discussed in [3].

(4) Learning by small perturbations is examined formally in Levine [2].