THE ENFORCEMENT OF COLLUSION IN A FRICTIONLESS OLIGOPOLY I: EQUILIBRIUM*

by

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ABSTRACT

Do oligopolistic firms successfully collude? With adjustment costs and plausible assumptions on the information available to firms the relevant solution concept is that of local almost perfect equilibrium. This paper applies that concept to an environment in which firms respond instantly to each other's output movements. Subject to technical qualifications I show that in the long run complete collusion occurs.

0. Introduction

How and to what extent do self-interested oligopolistic firms successfully collude? The story of oligopolistic interaction ought to be a simple one. With the same firms in a static market over time uncooperative rivals can be punished and helpful rivals rewarded. Only to the extent that retaliatory strategies have associated frictional costs should less than fully collusive behavior be expected.

To date economic theory has not successfully established this point. Supergame models such as Friedman [1], Green [4], Marschak and Selten [12,13] or Radner [16] allow collusive equilibria but many others as well. Other models such as that of Smale [17] employ *ad hoc* behavioral assumptions to get unique equilibria.

In a recent paper [10] I argued that with very large adjustment costs only one (approximately) perfect equilibrium adjustment path is consistent with the bounded rationality of firms. I argued there that firms cannot know how the industry will perform in every contingency, but must extrapolate from local data concerning industry behavior near the status quo. I showed that essentially only one adjustment path is consistent with firms extrapolating income linearly into the future.

In the two parts of this paper I apply the concept of local almost perfect equilibrium developed in [10] to the problem of oligopoly. I study an industry without frictional costs in which firms can respond to one another instantly. I ask: how are long run steady states of the adjustment procedure related to pareto efficient outcomes of the static game? the answer has two parts.

- In "almost all" games steady states and outcomes satisfying the first order conditions for static pareto efficiency "almost" coincide.

- The general stability analysis is intractible. However, it is possible to show that with identical firms and symmetric initial conditions a steady state is stable if and only if it is (locally) efficient.

The paper has two parts. Part I studies equilibrium. Section one introduces the model and derives the adjustment process. Section two studies steady states. Section three considers the very long run. Part II examines stability.

1. The Model

This section describes a simple model of oligopoly without entry in which firms communicate threats,[1] but cannot enter into legally binding contracts. There are four parts. Part one describes the actions available to firms and the information structure by which threats are communicated. Part two describes firm income. Part three models firm behavior. Part four computes the equations of motion of the state variables. The remaining sections of the paper explore the qualitative features of the model.

The Environment: There are N firms, entry is prohibited and each firm j controls its own output x^j. The output vector x is presumed to lie in $\mathcal{X}$, an open subset of $\mathbb{R}^N$. By assuming $\mathcal{X}$ is open, behavior on the boundary is ignored.

Information is exchanged costlessly by a fixed information structure. At time t all firms j announce that they will respond to autonomous output changes by other firms k at a rate R^j_k. At time $t+\Delta t$ all firms k announce their automonous output changes of y^k. At time $t+2\Delta t$ each firm j computes its total output change as

$$\Delta x^j = y^j + \sum_{k \neq j} R^j_K y^k = \sum_{k=1}^{N} R^j_K y^K \qquad (1\text{-}1)$$

where $R^j_j = 1$. Firms are assumed to observe one another's actual output. When Δt is infinitesimal relative to the discount rate j's opponents observe immediately whether or not he fulfilled his announced commitment given in (1-1). Thus lying is detected instantaneously; and (1-1) may be taken to determine the amount by which j actually changes his output. In the continuous time limit as $\Delta t \to 0$ firm j chooses a vector of response rates (also called reaction coefficients) R^j and an autonomous rate of change of output y^j. Firm j's actual output follows the equation

$$\dot{x}^j = \sum_{k=1}^{N} R_k^j y^k \tag{1-2}$$

It is important that in this environment firms react only to an opponent's autonomous output change. They do not counter-react to an opponent's retaliation. The ability of firms to communicate is crucial: it is only the information generated by communication which enables firms to distinguish between voluntary and reactive movements by rival firms.

In addition to changing output firm j may gradually alter its commitment R^j over time. This is given as

$$\dot{R}_k^j = S_k^j \tag{1-3}$$

where S_k^j is the rate at which firm j alters R_k^j. Thus firm j chooses paths for the control variables y^j and S^j in an effort to control the state variables x and R.

<u>Firm Income:</u> The profits of firm j are given by a smooth function $\pi^j : \mathcal{X} \to \mathbb{R}$. Let $\pi_k^j = \partial\pi^j/\partial x^k$. To insure that firms are able to affect

opponent's profits it is assumed that for $x \in \mathcal{X}$ and $j \neq k$ $\pi_k^j(x) < 0$. Let $\pi^\cdot$ be the vector of profit functions, π_k be the column vector $(\pi_k^j)_{j=1}^N$, π^j the row vector $(\pi_k^j)_{k=1}^N$ and π be the matrix with rows π^j. Since $\mathcal{X}$ is open the static game with profit functions π^j may not have any efficient points. To rule out degeneracy at least some point $x \in \mathcal{X}$ should satisfy the first order conditions for pareto efficiency. As a second regularity condition it is assumed that for some $x \in \mathcal{X}$ $\det(\pi(x))=0$. In section two it is shown that this is in fact the first order condition for efficiency.

Firm j's intertemporal preferences are described by a discount rate ρ^j. The corresponding discount factor is $\delta^j = 1/\rho^j$.

Firm Behavior: Firm behavior will be described by the local almost perfect adjustment path analyzed in Levine [10]. This attempts to model the bounded rationality of firms by assuming they compute present values of income streams by extrapolating existing rates of income growth linearly into the future.

A strategy (or closed-loop strategy) for firm j is a function

$$(y^j, S^j) = F^j(x, R) = (f^j(x, R), g^j(x, R)) \tag{1-4}$$

which assigns a vector of control variables to the vector of state variables. This already embodies a degree of bounded rationality since firm j's choice depends only on the current state variables and not on the entire past history of the market. Suppose that all firms k play $\tilde{F}^k$. Then firm j receives

$$\tilde{A}^j(x, R) = \int_0^\infty \pi^j(x(t)) \exp(-\rho^j t)dt \qquad (1\text{-}5)$$

where x(t) and R(t) satisfy the system of differential equations

$$\dot{x}^j = \sum_{k=1}^{N} R_k^j \tilde{f}^k(x,R) \qquad j = 1,\ldots,N$$

$$\dot{R}^j = \tilde{g}^j(x,R) \qquad j = 1,\ldots,N$$

$$x(0) = x \quad R(0) = R \qquad (1\text{-}6)$$

which are derived by substituting (1-4) into the equations of motion for the state variables (1-2) and (1-3).

Perfect rationality of all firms requires that (for any starting point x_0,R_0) each firm instantly adjust its control variables to maximize (1-5) subject to (1-6). This is an unreasonably strong assumption. I shall instead study a model of bounded rationality in which

(1) firms optimize only approximately: they do not maximize the true present value $\tilde{A}^j$ (which isn't observable) but rather $\hat{A}^j$ (which is observable) an approximation to the true present value.

(2) firms do not adjust instantaneously: they move in the direction that yields the most rapid increase in the maximand $\hat{A}^j$.

Thus, by assumptions (1) and (2)

$$\tilde{f}^j = b\,\kappa^j_j\left\{\sum_k \frac{\partial\hat{A}^j}{\partial x^k}\,\frac{\partial\dot{x}^k}{\partial y^j} + \sum_k\sum_\ell \frac{\partial\hat{A}^j}{\partial R^k_\ell}\,\frac{\partial\dot{R}^k_\ell}{\partial y^j}\right\}$$

$$\dot{\tilde{g}}^j_k = b\,\kappa^j_k\left\{\sum_k \frac{\partial\hat{A}^j}{\partial x^k}\,\frac{\partial\dot{x}^k}{\partial s^j_k} + \sum_k\sum_\ell \frac{\partial\hat{A}^j}{\partial R^k_\ell}\,\frac{\partial\dot{R}^k_\ell}{\partial s^j_k}\right\} \qquad (1\text{-}7)$$

where $\kappa^j_k > 0$ are constant adjustment coefficients, and $b > 0$ is a small scalar constant, as discussed below.

What (1-7) describes is a partial adjustment model with $b\kappa^j_k$ exogenous adjustment rates for the controls. In this model the controls are set to increase the (approximate) level of present value over time. The factor b shows that the controls are small--equivalently that the state variables are adjusted gradually.

Implicitly it is expensive to choose large values of the controls. Indeed, in another paper [10], I show that if there are quadratic costs of choosing the controls the adjustment process (1-7) is almost optimal provided that $D\hat{A}^j$ is uniformly close to $D\tilde{A}^j$ the true present value derivative. Naturally adjustment costs and partial adjustment are dual.

One important reason for a partial adjustment model is bounded rationality. If firm j wishes to globally and instantly set the optimal output level (for example) he must know his demand curve everywhere. If he is content to restrict himself to change output slowly, he need only learn a small segment of his demand curve each day to make an optimal choice subject to his self-imposed constraint. The point is that the faster a firm moves the more quickly it must learn. Bounded rationality and large costs of gathering information imply it will move rather slowly.

How can firms use local information to approximate the present value of income: how do they choose $\hat{A}^j$ close to $\tilde{A}^j$? If the future income stream is not too badly nonlinear, simply extrapolating income linearly should be a good approximation--indeed, casual empiricism indicates this is a widely used technique. Of course in the distant future the extrapolated income isn't too accurate, but with discounting this doesn't matter much. In the present context the coefficient b is a measure of the linearity of future income relative to the discount rate: the smaller b the better linear extrapolation does. Intuitively, if b is small the adjustment rate in (1-7) is small relative to the fixed discount rate. The slower the rate of adjustment the more slowly the (nonlinear) system in (1-6) departs from its linear approximation. These points are discussed rigorously in Levine [10].

There are many kinds of extrapolation a firm might use and linear extrapolation has no particular claim to priority. Fortunately it is shown in [10] that if we require that all firms do no worse (in an order of magnitude sense) than they would using linear extrapolation the equilibrium adjustment path is approximately independent of the extrapolation rule they choose. Thus we may as well examine the most easily computed adjustment path - any other being approximately the same. One easily computed path derived in [10] is the path along which all firms extrapolate linearly and assume that in the future all firms will behave myopically. Let us see what this means in the present context.

Extrapolating linearly we see that

$$\hat{A}^j = \int_0^{\infty} [\pi^j + \dot{\pi}^j t]\exp(-\rho^j t)dt$$

$$= \delta^j \pi^j + (\delta^j)^2 \dot{\pi}^j \qquad (1\text{-}8)$$

Via some algebraic manipulations and dropping all terms of order b^2 (which is the order of magnitude error in (2-5)) we find

$$\hat{A}^j \approx \delta^j \pi^j + (\delta^j)^2 \sum_{\ell=1}^{N} \pi^j_\ell \sum_{k=1}^{N} R^\ell_k \; k^k_k \sum_{m=1}^{N} R^m_k \left(b \frac{\partial \hat{A}^k}{\partial x^m}\right) \qquad (1\text{-}9)$$

where $\pi^j_\ell \equiv \partial\pi^j/\partial x^\ell$ and

$$\operatorname{sgn}(z) = \begin{cases} +1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases} \qquad (1\text{-}10)$$

The assumption that in the future all firms behave myopically means that in computing $\partial\hat{A}^k/\partial x^m$ on the right hand side of (1-9) firms assume a flat future income profile

$$b\hat{A}^j \approx b\delta^j \pi^k_m \qquad (1\text{-}11)$$

In other words pure myopia means drop terms of order b^2 and higher in computing $b\partial\hat{A}^k/\partial x^m$. Thus our approximation $\hat{A}^j$ is derived by substituting (1-11) in (1-9) to yield

$$\hat{A}^j = \delta^j \pi^j + (\delta^j)^2 b \sum_{\ell=1}^{N} \pi^j_\ell \sum_{k=1}^{N} R^\ell_k \; k^k_k \delta^k \sum_{m=1}^{N} R^m_k \pi^k_m \qquad (1\text{-}12)$$

<u>Nature of the Solution</u>: From (1-7) the partial adjustment equation and (1-12) giving the approximate present value

$$\bar{f}^j = b k_j^j \sum_{\ell=1}^{N} \frac{\partial \hat{A}^j}{\partial x^\ell} R_j^\ell = b\delta^j k_j^j \sum_{\ell=1}^{N} \pi_\ell^j R_j^\ell + O(b^2) \tag{1-13}$$

The analysis is greatly simplified[2] by replacing (1-15) with

$$y^j = \bar{f}^j \approx \sum_{\ell=1}^{N} \tau_\ell^j R_j^\ell \tag{1-14}$$

This incorporates two innovations. The constant $b\delta^j k_j^j$ has been normalized to one. This, however, we are free to do by changing the units in which j's profits π^j are measured. A more significant feature is that the term of order b^2 in (1-12) is dropped. Since the first term is of order b (1-14) should be a good approximation to (1-12). I am <u>not</u> arguing that firms will wish to ignore the second term--to achieve the same degree of accuracy as provided by linear extrapolation, they cannot. I am pointing out instead that important qualitative features of the dynamical system describing the evolution of x and R over time can be understood without reference to the second term in (1-12). In economic terms, I don't care whether a firm's market share is 10 or 12 percent even though the firm itself may care tremendously. The mathematical implications of this approximation are discussed in section three.

From (1-12) and the equations of motion (1-6) the motion of x is given (approximately) by

$$\dot{x}^j = \sum_{k=1}^{N} R_k^j y^k = \sum_{k=1}^{N} R_k^j \sum_{\ell=1}^{N} \pi_\ell^k R_k^\ell \qquad (1\text{-}15)$$

Turning to $\dot{R}_k^j = S_k^j = \bar{g}_k^j$ from (1-7), (1-12) and (1-15)

$$\dot{R}_k^j = b\eta_k^j [\pi_j^k \sum_{m=1}^{N} R_k^m \pi_m^j + \pi_j^j y^k] \qquad (1\text{-}16)$$

where the constant $\eta_k^j = (\delta^j)^2 k_k^j$ and use is made of the normalization rule $b\delta^j k_j^j = 1$. Note that since all terms in (1-16) are of the same order, none can be omitted, even as an approximation.

2. Equilibrium

The remainder of the paper addresses the following question: what is the relationship between stable steady states of the dynamical system described in the first section and pareto efficient outcomes of the game? Mathematically this divides into two issues: how are steady states related to outcomes which satisfy the first order conditions for pareto efficiency (called FOPE), and how are the stability conditions related to the second order conditions for pareto efficiency? This section takes up the first issue and yields the following conclusion: in "almost all" games which satisfy the required regularity conditions steady states and FOPE coincoide "almost exactly." Part II of this paper examines the relationship between stability and local efficiency.

A point x is supportable as a steady state (or is simply called a steady state) if there is some reaction matrix R such that $\dot{x}(x,R) = 0$. In this case the dynamical system is motionless for all time regardless of whether $y = 0$. There are two possible types of steady states. If $y = 0$ by (1-15) $\dot{x} = 0$. This is called an autonomous steady state to reflect the fact that the autonomous action control variables y are stationary. Steady states at which $y \neq 0$ are called non-autonomous. The first half of the section examines autonomous steady states and shows that they are all FOPE. As a partial converse in "almost all" games the FOPE form an N-1 dimensional stratified submanifold of which at worst an N-2 dimensional stratified submanifold fail to be autonomous steady states. The second half of the section examines the possibility that some non-autonomous steady states might fail to be FOPE and demonstrates that in "almost all" games such exceptional steady states are a set of isolated points.

Autonomous steady states are characterized by the conditions $y = 0$ $\dot{R} = 0$. To analyze the set of points x which are supportable as autonomous steady states R must be eliminated from these equations. The resulting condition will then be contrasted with the first order efficiency conditions to show that autonomous steady states are FOPE.

Equating the expressions for y and $\dot{R}$ from (1-14) and (1-16) to zero gives the condition for an autonomous steady state

$$\sum_{p=1}^{N} \pi_p^j R_j^p = 0 \qquad j = 1,\ldots, N \qquad (2\text{-}1)$$

$$\pi_k^j \sum_{p=1}^{N} \pi_p^j R_k^p = 0 \qquad j = k = 1,\ldots, N \quad j \neq k \qquad (2\text{-}2)$$

Since the $\pi_k^j \neq 0$ (by assumption) they can be eliminated from (2-2) to yield the equivalent condition

$$\sum_{p=1}^{N} \pi_p^j R_k^p = 0 \qquad j, k = 1,\ldots, N \quad j \neq k \qquad (2\text{-}3)$$

It is instructive to combine the N^2 equations in (2-1) and (2-3) into the N vector equations

$$\pi R_k = 0 \qquad k = 1,\ldots,N \qquad (2\text{-}4)$$

where $\pi=\{\pi_k^j\}$ with j subscripting rows and k subscripting columns and $R_k=(R_k^j)_{j=1}^N$

If it is to be possible to solve (2-4) for N vectors satisfying $R_k^k = 1$ then π must be singular and admit in its null space a vector $\gamma = (\gamma^j)_{j=1}^N$ with $\gamma^j \neq 0$ for any j. I will call such a matrix <u>regular</u> <u>singular</u>. If π is regular singular setting $R_k^j = \gamma^j/\gamma^k$ solves (2-4) and satisfies the restriction $R_k^k = 1$. So it is necessary and sufficient for x to be an autonomous steady state that $\pi(x)$ be regular singular.

How does this compare with the first order condition for pareto efficiency? At a pareto efficient point for a non-zero vector of weights $\mu = (\mu^j)_{j=1}^N$ the weighted sum $\sum_{j=1}^N \mu^j \pi^j$ must be maximized. The first order conditions (which must be satisfied since $\mathcal{X}$ is an open set) are

$$\sum_{j=1}^N \mu^j \pi_k^j = 0 \qquad k = 1,\ldots, N \qquad (2\text{-}5)$$

which in matrix notation is

$$\mu'\pi = 0 \qquad (2\text{-}6)$$

or the condition that π be singular. This condition will be taken as the definition of a FOPE, the restriction that the weights μ^j have the same sign being viewed in this terminology as part of the second order conditions for efficiency. Note incidentally the difference between the weights γ corresponding to reactions and the weights μ corresponding to utility weights: the former satisfies $\pi\gamma = 0$, the latter $\mu'\pi = 0$.

All autonomous steady states are FOPE. The converse is false since not all singular matrices are regular singular. To proceed further we must recall some basic ideas from singularity theory. A good reference here is Yu [19] pp. 30-31.

An L dimensional stratified submanifold S (in a suitably high dimensional Euclidean space M) is a finite disjoint union of submanifolds S_0, $S_1, \ldots,$ S_k with maximal dimension L and such that $S_0 \cup S_1 \ldots \cup S_i$ is closed in M. A map meets S transversally iff it meets each component submanifold transversally. Since the number of strata S_i is finite all the usual genericity theorems concerning transversality apply to stratified submanifolds as well.

As is well known singular N×N matrices are a N^2-1 dimensional stratified submanifold. Not surprisingly appendix (A) shows that non-regular singular matrices (which satisfy the additional restriction that every vector in its null space has at least one component vanish) are contained in an N^2-2 dimensional stratified submanifold. Thus we are led to conclude that almost all singular matrices are regular singular.

To infer genericity in the space of regular games we utilize the Whitney C^2 topology[3] on the space G of all C^2 mappings $\mathscr{X} \to IR^N$. Appendix (B) shows that the set of regular games G^R is an open set plus a portion of its boundary in G so that standard genericity theorems work in G^R.

Thus the jet transversality theorem implies

Proposition (2-1): If S is an L dimensional stratified submanifold then for almost all games (the intersection of a residual set in G with G^R) the set of $x \varepsilon \mathscr{X}$ which satisfy the restriction $\pi(x) \varepsilon S$ is a N-(N^2-L) dimensional stratified submanifold.

An immediate corollary is

Proposition (2-2): All autonomous steady states are FOPE. In almost all games FOPE are N - 1 dimensional and are all autonomous steady states except possibly in an N - 2 dimensional stratified submanifold.

Exceptional Steady States: At an autonomous steady state y = 0, and the first order conditions for pareto efficiency are satisfied. It is possible, however, that $\dot{x} = 0$ and $y \neq 0$ in which case the preceding analysis does not apply. Here it is shown that in almost all games there are at most an isolated set of points which are steady states but not FOPE. There are two steps in this undertaking: first R is eliminated from the equations $\dot{x} = 0$ $\dot{R} = 0$; then an application of Proposition (2-1) gives the desired genericity result.

The conditions for a (not necessarily autonomous) steady state are given by equating the expressions for $\dot{x}$ and $\dot{R}$ from (1-15) and (1-16) to zero

$$\sum_{k=1}^{N} R_k^j y^k = 0 \qquad j = 1,\ldots, N \qquad (2\text{-}7)$$

$$[\pi_k^j(\sum_{p=1}^{N} \pi_p^j R_k^p) + \pi_j^j y^k] = 0 \qquad j \neq k \qquad (2\text{-}8)$$

To eliminate R from these equations use $\pi_k^j \neq 0$ $j \neq k$ to define

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j/\pi_k^j & j \neq k \end{cases} \qquad (2\text{-}9)$$

Then (2-8) is equivalent to

$$\sum_{p=1}^{N} \pi_p^j R_k^p = y^k \lambda_k^j \qquad j \neq k \qquad (2\text{-}10)$$

The equation (1-14) defining y^k is

$$y^k = \sum_{p=1}^{N} \pi_p^k R_k^p \qquad (2\text{-}11)$$

which can be expressed as

$$\sum_{p=1}^{N} \pi_p^k R_k^p = y^k \lambda_k^k \qquad (2\text{-}12)$$

since $\lambda_k^k = 1$ by definition. The N^2 equations (2-10) and (2-12) can be combined into the N vector equations

$$\pi R_k = y^k \lambda_k \qquad (2\text{-}13)$$

Throughout the remainder of this discussion it is assumed that the steady state is not a FOPE. In this case π is non-singular and (2-13) can be solved for the reaction coefficients.

$$R_k = y^k \pi^{-1} \lambda_k \qquad (2\text{-}14)$$

The equation for R_k^k (which is one by definition) is

$$1 = R_k^k = y^k (\pi^{-1})^k \lambda_k \qquad (2\text{-}15)$$

from which the autonomous controls are

$$y^k = 1/(\pi^{-1})^k \lambda_k \qquad (2\text{-}16)$$

Observe that this implies $y^k \neq 0$ for any k, which is consistent with the earlier finding that the steady state must be non-autonomous since it is not a FOPE.

The N vector equations (2-14) can be rewritten using (2-16) as the N^2 scalar equations

$$R_k^j = (\pi^{-1})^j \lambda_k / [(\pi^{-1})^k \lambda_k] \qquad (2\text{-}17)$$

So far the first steady state condition $\dot{x} = 0$ given in (2-7) has not been used. To eliminate R from the steady state conditions (2-7) and (2-8) substitute (2-17) into (2-7) to get the necessary conditions

$$\sum_{k=1}^{N} (\pi^{-1})^j \lambda_k/[(\pi^{-1})^k \lambda_k]^2 = 0 \tag{2-18}$$

which since π^{-1} is non-singular is equivalent to

$$\sum_{k=1}^{N} \lambda_k^j/[(\pi^{-1})^k \lambda_k]^2 = 0 \tag{2-19}$$

A useful implication of (2-19) is that $\lambda_k^j \neq 0$ and $\pi_j^j \neq 0$. If not since for $j \neq k$ $\lambda_k^j = -\pi_j^j/\pi_k^j$ it must be that $\pi_j^j = 0$ and so $\lambda_k^j = 0$ for every $k \neq j$. In this case since $\lambda_k^k = 1$ by definition

$$\sum_{k=1}^{N} \lambda_k^j/[(\pi^{-1})^k \lambda_k]^2 =$$

$$\lambda_k^k/[(\pi^{-1})^k \lambda_k]^2 > 0 \tag{2-20}$$

contradicting the steady state condition (2-19).

Let H_E^N be the set of matrices π which satisfy (2-19). If H_E^N is N^2 dimensional then by Proposition (2-1) almost all games will have only an isolated set of points x at which $\pi(x)\epsilon H_E^N$. Define the surjective mapping $U:H_E^N \rightarrow H_U^N$ by

$$U_k^j = \text{sgn}(\pi_k^k) \cdot \lambda_k^j/[(\pi^{-1})^k \lambda_k]^2 \tag{2-21}$$

Since $\pi_k^k \neq 0$ this map is smooth. To show that H_E^N is $N^2 - N$ dimensional it suffices to show that H_U^N is $N^2 - N$ dimensional and diffeomorphic to H_E^N. Since U is smooth H_U^N and H_E^N are diffeomorphic provided U has a smooth inverse.

Let e be a vector of ones. By (2-19) if $U \epsilon H_U^N$ $Ue = 0$. This shows that H_U^N is an $N^2 - N$ dimensional linear subspace.

To solve for π given U it is necessary only to solve for λ and π_j^j $j = 1,\ldots,N$. The π_k^j $j \neq k$ are then given by reversing the definition of λ in (2-9) as

$$\pi_k^j = -\pi_j^j/\lambda_k^j \qquad j \neq k \tag{2-22}$$

which is smooth since $\lambda_k^j \neq 0$. Solving for λ_k^j $j \neq k$ is straightforward since by the definition of U in (2-21)

$$U_k^j/U_k^k = \lambda_k^j \tag{2-23}$$

where $U_k^k \neq 0$ since π_k^k and λ_k^k are non-zero. To solve for π_j^j using λ and U observe from (2-22) and $\lambda_j^j = 1$ that π factors as

$$\pi = \beta d \qquad \beta_k^j = \begin{cases} 1 & j=k \\ -1/\lambda_k^j & j \neq k \end{cases} \qquad d_k^j = \begin{cases} \pi_j^j & j=k \\ 0 & j \neq k \end{cases} \tag{2-24}$$

Since $\pi_j^j \neq 0$ d is non-singular, $\pi^{-1} = d^{-1}\beta^{-1}$ and

$$(\pi^{-1})^k = (1/\pi_j^j)(\beta^{-1})^k \tag{2-25}$$

Substitute this expression into (2-21) the definition of U_j^j to find

$$U_j^j = \text{sgn}\,(\pi_j^j)\lambda_j^j/[(1/\pi_j^j)(\beta^{-1})^j\lambda_j]^2 \tag{2-26}$$

and since $\lambda_j^j = 1$ (2-26) solves as (2-27)

$$\pi_j^j = \mathrm{sgn}\,(U_j^j)\sqrt{U_j^j}\;\left|(\beta^{-1})\;\lambda_j\right| \tag{2-27}$$

which is smooth since it can never vanish.

The preceding discussion can be summarized as

Proposition (2-3): In almost all games the set of steady states which are not FOPE are a set of isolated points.

3. The Very Long Run

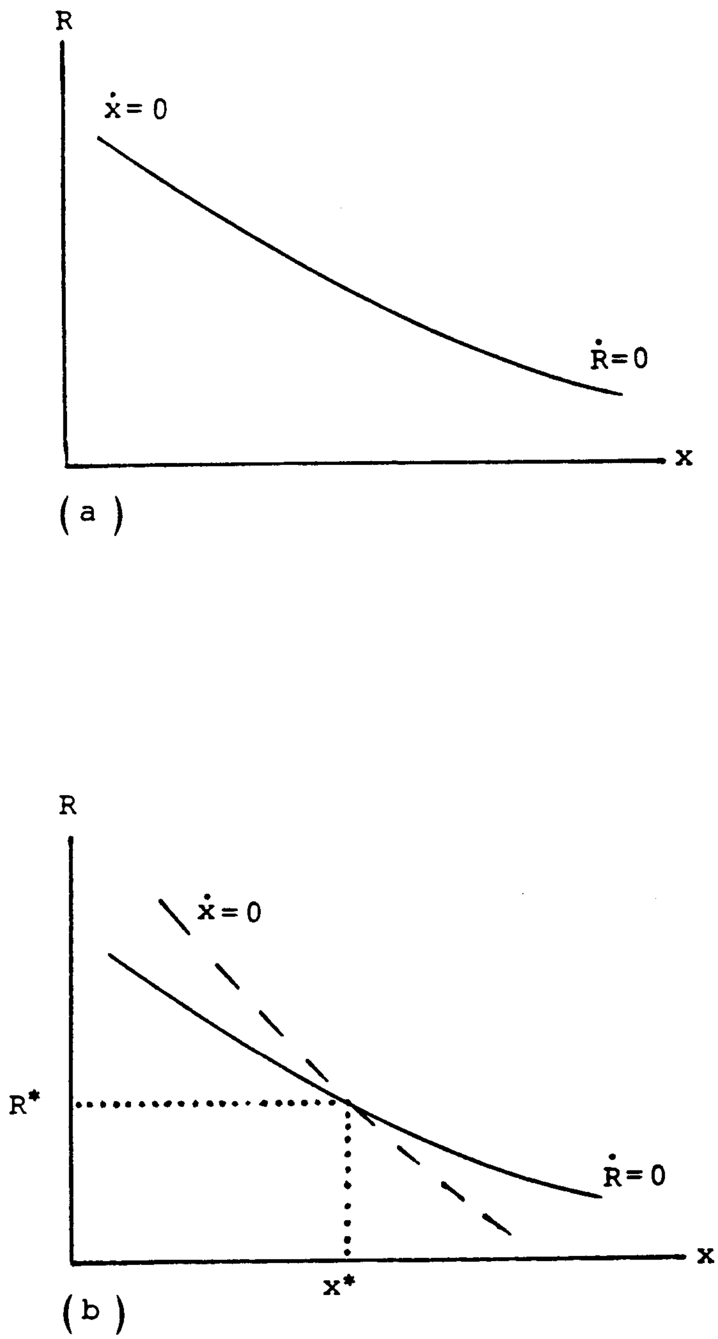
The system we have been studying was derived by approximating the true equation for y (1-13) by (1-14). Thus the true system is a (small) perturbation of the system we have studied. How does this perturbation affect our results?

The first point is that a small perturbation can't have steady states far distant from the steady states of the unperturbed system. This means that the perturbed steady states are (approximately) a subset of the steady states we have studied. In effect, a perturbation can destroy steady states and it can introduce new steady states but the new ones must be close to the old ones.

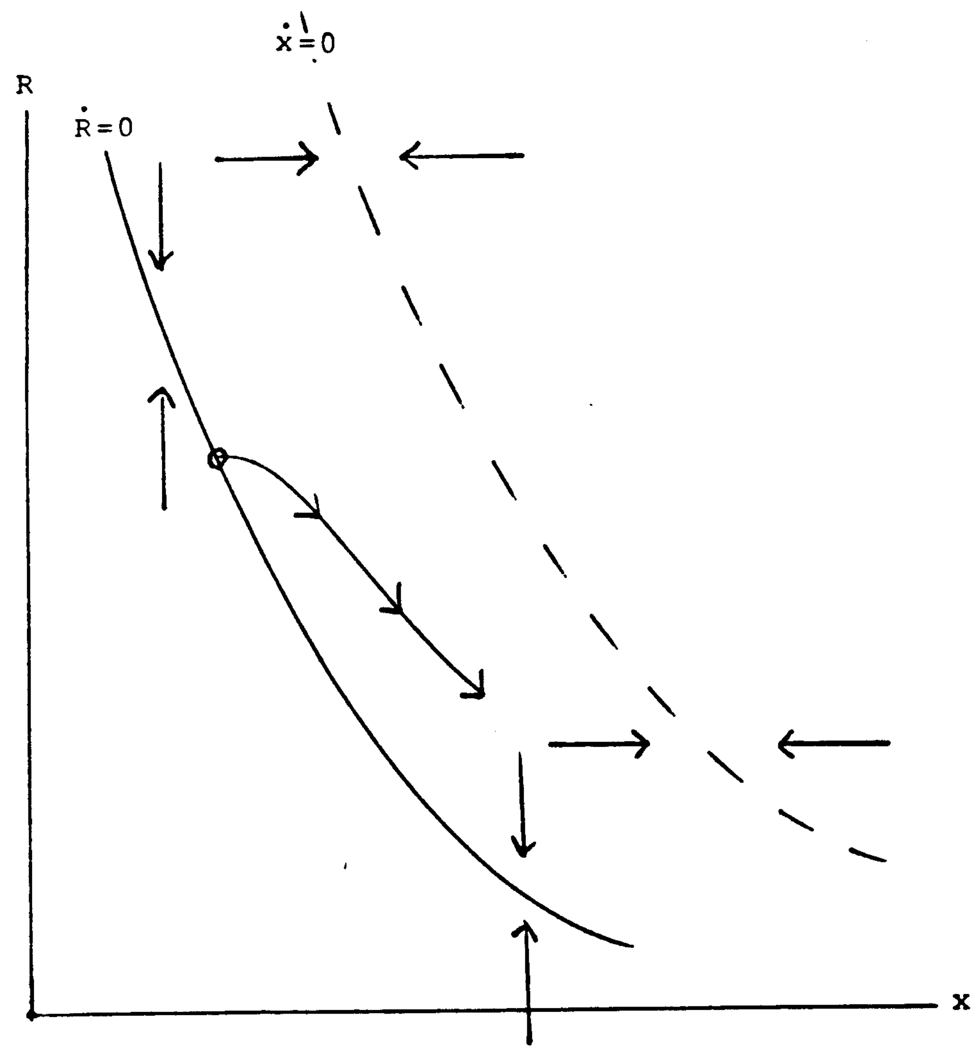
The exceptional steady states are generically hyperbolic and thus won't be affected much by small perturbations. The situation near the manifold of autonomous steady states is very different. Generically, perturbations of a dynamical system have locally isolated steady states. It is to be expected that when the system is perturbed the steady state manifold will be replaced by a set of isolated steady states lying near the manifold.

This deserves a bit of explanation. Figure (3-1) illustrates what the system might look like before and after a perturbation. Initially the $\dot{x} = 0$ and $\dot{R} = 0$ curves coincide constituting a manifold of steady states. When (1-13) is replaced by (1-14) the $\dot{x} = 0$ curve shifts slightly by an amount proportional to b. Only the steady state (x^*, R^*) remains.

What happens along the old steady state manifold? Suppose both $\dot{x} = 0$ and $\dot{R} = 0$ are attractors. Initially R doesn't move, while x moves towards the new $\dot{x} = 0$ curve. As x approaches $\dot{x} = 0$ R is no longer in equilibrium and begins to move along the old steady state manifold as il-



Figure(3-1): Perturbation of the Steady State Manifold



Figure(3-2): Drift Along the Unperturbed Steady State Manifold lustrated in Figure (3-2).

If the old steady state manifold is a repellor the situation is quite different: a small perturbation of the system will typically cause the system to move away from the old manifold entirely.

In the attractive case, how rapid is the drift along the steady state manifold? From (1-14) and (1-16) $\dot{x} = bF(x, R)$ and $\dot{R} = b^2G(x, R)$ where remember b is small. As the system moves along the old manifold x is approximately in equilibrium at $\dot{x} = 0$ since it equilibrates faster than $\dot{R}$. This means, since $\dot{x} = 0$ and $\dot{R} = 0$ lie apart by order b, that the distance of the system from $\dot{R} = 0$ is $\Delta x \cong bH(R)$. Thus, $\dot{R} = b^2G_x\ \Delta x = b^3G_xH(R)$ where G_x are the derivatives of G with respect to x.

In the short run x equilibrates most rapidly at rate O(b) so R is determined by initial conditions and x by $\dot{x} = 0$. In the long run R equilibrates at rate $O(b^2)$ causing the system to move towards the unperturbed manifold of steady states. In the very long run, however, the perturbation causes the system to drift along the unperturbed steady state manifold at rate $O(b^3)$. This is similar to the notion of fast and slow manifolds introduced by Zeeman [20] chapter 3.

The exact nature of very long run steady states where output shares are determinate is an interesting question for future research.

APPENDIX (A)--Singularity and Regular Singularity

The objective is to prove two lemmas:

Lemma (A-1): m x n matrices of rank r are a set of codimension (m - r) x (n - r).

Lemma (A-2): regular singular n x n matrices are a set of codimension 2.

Proof of (A-1): From McCoy [14] section 15.5 a matrix A of rank r has an r x r non-singular submatrix A_{11}. Assume

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} r & n-r \\ \left[\begin{array}{c} A_{11} \\ A_{21} \end{array}\right. & \left.\begin{array}{c} A_{12} \\ A_{22} \end{array}\right] \end{array} \begin{array}{c} \\ r \\ m-r \end{array} \qquad \text{(A-1)}$$

Any matrix of rank r is obtained from a matrix of the form (A-1) by permuting rows and columns. Since only a finite set of such permutations is possible it suffices to prove the lemma for a matrix of the form (A-1). Following Guilleman and Pollack [6] chapter 1.4 problem 13 define

$$B = \begin{bmatrix} I & -A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix} \qquad \text{(A-2)}$$

Since B is nonsingular rank (AB) = rank (A). Also

$$AB = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1} A_{12} \end{bmatrix} \tag{A-3}$$

so that A has rank r if and only if

$$F(A) = A_{22} - A_{21} A_{11}^{-1} A_{12} = 0 \tag{A-4}$$

Since $\partial F/\partial A_{22} = I$ the transformation F has full rank. Thus since F(A) is $(m - r) \times (n - r)$ the lemma follows. Q.E.D.

Proof of (A-2): Let A be square, singular and non-regular. Any such matrix can be obtained from a matrix that admits a vector of the form $(0, X_2, \ldots, X_n)'$ in its null space by a finite permutation of columns, so assume A has this form. Set

$$A = [a_{11}, A_{12}] \tag{A-5}$$

where a_{11} is an n-vector. Then A has the required form if and only if A_{12} has rank $(n - 2)$ or less. By the same reasoning used in lemma (A-1) this implies a codimension of 2. Q.E.D.

APPENDIX (B)--Genericity in Regular Games

Let G be the set of C^2 mappings $\pi^{\cdot}:\mathcal{X} \dashrightarrow \mathbb{R}^N$ in the Whitney C^2 topology where $\mathcal{X}$ is open in $\mathbb{R}^N$. Let G^R be the subset of $\pi^{\cdot} \varepsilon G$ which also satisfy $\pi^j_k(x) < 0 \quad j \neq k$ and $\det(\pi(x)) \neq 0$. The objective is to prove

Lemma (B-1): $G^r \subset$ closure (interior G^R)).

Define G^1 to be the subset of G which satisfies $\pi^j_k(x) < 0 \quad j \neq k$ and G^2 the subset which satisfies $\det(\pi(x)) \neq 0$. Obviously small perturbations of π satisfying $\pi^j_k(x) < 0$ will satisfy the restriction so G^1 is open. Thus it suffice to show $G^2 \subset$ closure (interior G^2)). Define G^3 to be the subset of G^2 such that for $\pi^{\cdot} \varepsilon G^3$ there is an x_1 and x_2 with $\det(\pi(x_1)) > 0$ and $\det(\pi(x_2)) < 0$. Obviously G^3 is open. To prove lemma (B-1) it is then necessary to only show that

Lemma (B-2): $G^2 \subset$ closure (G^3)

This requires a preliminary lemma.

Lemma (B-3): If A is singular there is a matrix B such that for $\lambda > 0 \quad \det(A + \lambda B) > 0$.

Proof:

By the Jordan decomposition theorem $A = C^{-1}JC$ where J is upper triangular, has zeroes in the first (N - r) diagonal positions and the non-zero eigenvalues of A in the remaining r diagonal positions. Assume without loss of generality that the product of the non-zero eigenvalues of A is positive. Let D be a diagonal matrix with ones in the first r positions, zeroes in the other (N - r). Then for $\lambda > 0$ $\det(J + \lambda D) > 0$. Thus $B = C^{-1}DC$ satisfies the required property.

Q.E.D.

Now suppose $\pi^{\cdot} \in G^2 - G^3$. Ignoring the case $\det(\pi)$ vanishes identically (left as an exercise) suppose without loss of generality for some x_1 $\det(\pi(x_1)) < 0$. By assumption for some x_2 $\det(\pi(x_2)) = 0$. Let B be a ball centered on x_2. Because $\mathcal{X}$ is open in $\mathbb{R}^N$ we may assume $B \subset \mathcal{X}$ and $x_1 \notin$ closure (B). Let π^* be (by lemma B-3)) such that $\det(\pi(x_2) + \lambda\pi^*) > 0$ for $\lambda > 0$. Using techniques similar to those of chapter 2 section 2 of Hirsch [7] there is a function $\bar{\pi}^{\cdot}: \mathcal{X} \to \mathbb{R}^N$ which is C^∞, vanishes outside closure (B) and has $D\bar{\pi}^{\cdot}(x_2) = \pi^*$. Then $(\pi^{\cdot} + \lambda\,\bar{\pi}^{\cdot}) \in G^3$ for $\lambda > 0$ and approximates π^* arbitrarily well. This proves lemma (B-2) and thus (B-1).

Notes

(1) Oligopoly without communication is discussed in Levine [9].

(2) With this modification the model is conceptually similar to that of Guttman [5].

(3) See Hirsch and Smale [8].

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