

THE ENFORCEMENT OF COLLUSION IN A
FRICTIONLESS OLIGOPOLY II: STABILITY*

by

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0. Introduction

Part I of this paper introduced a model of endogenous collusion between oligopolists in a frictionless environment. It showed that in a generic sense steady states of the dynamical system describing firm behavior are the same as outcomes satisfying the first order conditions for pareto efficiency. This paper continues that analysis by relating stability and the second order conditions for pareto efficiency. No entirely satisfactory relationship is uncovered, except in the symmetric case. Here we can show (section 2) that locally pareto efficient outcomes and steady states are one and the same. Section 3 discusses the implications of locally non-unique steady states in the general case while section 4 applies section 3 to symmetric two player games. Section 1 recalls the key equations of part I and section 5 summarizes our conclusions.

1. Review

Recall the key equations of motion from part I. Movement of output is

$$\dot{x}^j = \sum_{k=1}^N R_k^j y^k \quad (1-1)$$

The autonomous output is (approximately) determined by

$$y^j = \sum_{\ell=1}^N \pi_{\ell}^j R_{\ell}^j \quad (1-2)$$

Finally the reaction coefficients move according to

$$\dot{R}_k^j = b \eta_k^j \left[\pi_j^k \sum_{m=1}^N R_k^m \pi_m^j + \pi_j^j y^k \right] \quad (1-3)$$

2. Symmetric Games

In a symmetric game all firms are identical including in the initial conditions. Such games are easy to study since the game is characterized by two variables: the common output and the common reaction. First the steady state conditions are restated in terms of these variables. An examination of the stability conditions then shows that (local) pareto efficient points and stable steady states coincide.

Let $x^j = z$ and $R_k^j = r$ $j \neq k$ for each firm. In a symmetric game all firms are identical and these variables complete describe the game. The motion of z is given by the equations describing the motion of the x^j (1-1) and (1-2) as

$$\begin{aligned} \dot{z} = \dot{x}^j &= \sum_{m=1}^N R_m^j - \sum_{p=1}^N \pi_p^j R_j^p \\ &= \left(\sum_{m \neq j} R_m^j + 1 \right) \left(\sum_{p \neq j} \pi_p^j R_j^p + \pi_j^j \right) \\ &= ((N - 1) r + 1) ((N - 1) r \pi_k^j + \pi_j^j) \end{aligned} \quad (2-1)$$

The motion of r is given by the equation of motion for R_k^j $j \neq k$ (1-3) as

$$\begin{aligned}
 \dot{r} = \dot{R}_k^j &= b\eta_k^j \left[\pi_j^k \left(\sum_{p=1}^N \pi_p^j R_k^p \right) + \pi_j^j y^k \right] \\
 &= b\eta \left[\pi_j^k \left(\sum_{p=1}^N \pi_p^j R_k^p \right) + \pi_j^j \left(\sum_{p=1}^N \pi_p^j R_j^p \right) \right] \\
 &= b\eta \left[\pi_k^j (\pi_j^j r + \pi_k^j + \pi_k^j (N - 2)r) \right. \\
 &\quad \left. + \pi_j^j (\pi_j^j + (N - 1)r \pi_k^j) \right] \tag{2-2}
 \end{aligned}$$

where the motion y^k is from (1-2) and η is the common value of the η_k^j .

Equating (2-1) and (2-2) to zero and solving for z and r shows that there are three types of steady states

$$r = 1 \quad \pi_j^j + (N - 1) \pi_k^j = 0 \tag{2-3}$$

$$r = -1/(N - 1) \quad \pi_j^j - \pi_k^j = 0 \tag{2-4}$$

$$r = -1/(N - 1) \quad (N - 1) \pi_j^j - \pi_k^j = 0 \tag{2-5}$$

The steady state in (2-3) is autonomous and an extreme point of the weighted sum $\sum_{j=1}^N \pi^j$. The steady state in (2-4) is also autonomous and is an extreme point of the weighted sum $\pi^j - \pi^k$ $j \neq k$. It cannot be pareto efficient, since the weights do not all have the same sign. The steady state in (2-5) is not autonomous.

Necessary conditions for stability are

$$\dot{a}z/\dot{a}z + \dot{a}r/\dot{a}r \leq 0 \quad (2-6)$$

$$(\dot{a}z/\dot{a}z)(\dot{a}r/\dot{a}r) - (\dot{a}z/\dot{a}r)(\dot{a}r/\dot{a}z) \geq 0 \quad (2-7)$$

sufficient conditions are that (2-6) and (2-7) hold with strict inequality. Do the inefficient steady states in (2-4) and (2-5) satisfy the necessary conditions? Differentiating the expressions for \dot{z} and \dot{r} in (2-1) and (2-2) shows that

$$\dot{a}z/\dot{a}z = ((N - 1)r + 1) .$$

$$((N - 1)r (\pi_{jk}^j + \pi_{kk}^j + (N - 2)\pi_{km}^j) + (\pi_{jj}^j + (N - 1)\pi_{jk}^j)) \quad (2-8)$$

$$\dot{a}r/\dot{a}r = b_n (\pi_k^j \pi_j^j + (\pi_k^j)^2 (N - 2) + \pi_j^j \pi_k^j (N - 1))$$

$$= b_n \pi_k^j (N \pi_j^j + (N - 2) \pi_k^j) \quad (2-9)$$

When $r = -1/(N - 1)$ as in (2-4) or (2-5) we see from (2-8) that $\dot{a}z/\dot{a}z = 0$, and for stability from (2-6) it must be that $\dot{a}r/\dot{a}r \leq 0$. In (2-7)

$$\pi_j^j = \pi_k^j \text{ and from (2-9)}$$

$$\dot{a}r/\dot{a}r = b_{\eta}(\pi_k^j)^2 2(N - 1) > 0 \quad (2-10)$$

implying instability. In (3-5) $\pi_j^j = \pi_k^j/(N - 1)$ and from (3-9)

$$\dot{a}r/\dot{a}r = b_{\eta}(\pi_k^j)^2 (N/(N - 1) + N - 2) > 0 \quad (2-11)$$

implying instability as well.

The steady state in (2-3) must be broken into two cases. Define

$$\begin{aligned} S &\equiv \partial^2 \pi^j / \partial z^2 \\ &= \pi_{jj}^j + (N - 1)\pi_{jk}^j + (N - 1)(\pi_{jk}^j + \pi_{kk}^j + (N - 2)\pi_{km}^j) \end{aligned} \quad (2-12)$$

A necessary condition for pareto efficiency is that each firm's profit be maximized subject to the symmetry constraint. The first order condition for this maximum is the condition in (2-3) $\pi_j^j + (N - 1)\pi_k^j = 0$. The second order necessary condition is $S \leq 0$. In Appendix (A) it is shown that in regular games $S < 0$ together with the first order condition are also sufficient for a symmetric outcome to be locally pareto efficient.

Examination of (2-3) shows $r = 1$ and combining this with (3-8) shows

$$\dot{a}z/\dot{a}z = NS \quad (2-13)$$

while $\pi_j^j = - (N - 1)\pi_k^j$ combined with (2-9) shows

$$\begin{aligned} \dot{a}r/\dot{a}r &= b_{\eta}(\pi_k^j)^2 [(N - 2) - N(N - 1)] \\ &= - b_{\eta}(\pi_k^j)^2 [(N - 1)^2 + 1] < 0 \end{aligned} \quad (2-14)$$

So if $S < 0$ (2-6) holds. To determine when the other half of the sufficient condition (2-7) holds compute from (2-1), (2-2) and (2-3)

$$\begin{aligned} \dot{a}z/\dot{a}r &= (N - 1)((N - 1)\pi_k^j + \pi_j^j + N(N - 1)\pi_k^j) \\ &= N(N - 1)\pi_k^j \end{aligned} \tag{2-15}$$

$$\begin{aligned} \dot{a}r/\dot{a}z &= b_n[(\pi_j^j + \pi_k^j)S] \\ &= -b_n(N - 2)\pi_k^j S \end{aligned} \tag{2-16}$$

Using (2-13), (2-14), (2-15) and (2-16) to compute the expression in (2-7) yields

$$\begin{aligned} (\dot{a}z/\dot{a}z)(\dot{a}r/\dot{a}r) - (\dot{a}z/\dot{a}r)(\dot{a}r/\dot{a}z) \\ &= -b_n(\pi_k^j)^2 S[N[(N - 1)^2 + 1] - N(N - 1)(N - 2)] \\ &= -b_n N^2 (\pi_k^j)^2 S \end{aligned} \tag{2-17}$$

If $S < 0$ the sufficient condition is satisfied and if $S > 0$ the necessary condition fails. Except for the unimportant case $S = 0$ a steady state is stable if and only if it is locally pareto efficient.

3. Strong Quasi Stability

When steady states are not isolated it is possible to make arbitrarily small movements away from a steady state to another steady state. Stability is impossible. A weaker condition which I call strong quasi-stability requires only that a small movement away from a steady state leads to a nearby steady state. For autonomous steady states which are N-1 dimensional this is the relevant stability concept.

Define the stability matrix

$$A = \begin{bmatrix} \dot{\partial x / \partial x} & \dot{\partial x / \partial R} \\ \dot{\partial R / \partial x} & \dot{\partial R / \partial R} \end{bmatrix} \quad (3-1)$$

A necessary condition for strong quasi-stability is that the real parts of the eigenvalues of A be non-positive. If steady states form an N - 1 dimensional manifold N - 1 of the N² eigenvalues of A must vanish, the corresponding eigenvectors indicating directions on the steady state manifold. In Appendix (B) it is shown that if (x*, R*) is a steady state at which N² - N + 1 of the eigenvalues of A have strictly negative real parts and if there is an open set surrounding (x*, R*) in which the set of steady states are an N - 1 dimensional manifold then (x*, R*) and all nearby steady states are strong quasi-stable.

The previous section showed that the points x^* which are supportable as steady states are usually $N - 1$ dimensional. This does not tell, however, the dimensionality of steady states in (x, R) space. Appendix (C) shows that if the matrix $\lambda(x^*)$ defined by

$$\lambda_{kj}^j = \begin{cases} 1 & j = k \\ -\pi_j^j / \pi_k^j & j \neq k \end{cases} \quad (3-2)$$

does not admit a non-zero vector with non-negative components in its null space, and in addition for all k $[\pi \mid \lambda_k]$ has full rank, then there is a unique R^* such that (x^*, R^*) is a steady state and a neighborhood of this steady state in which steady states form an $N - 1$ dimensional manifold.

This discussion yields the following conclusion concerning quasi-stability.

Proposition (3-1): if (x^*, R^*) is an autonomous steady state and

- o $[\pi(x^*) \mid \lambda_k]$ has full rank for all k
- o $\lambda(x^*)_{\mu} = 0 \Rightarrow \mu \neq 0$
- o $A(x^*, R^*)$ has $N^2 - N + 1$ eigenvalues with strictly negative real parts

then there is an open set $U \ni (x^*, R^*)$ in which all steady states

- o form an $N - 1$ dimensional manifold
- o are strongly quasi-stable.

4. Two Firm Symmetric Games

Symmetric pareto efficient points in symmetric games are stable with respect to symmetric shocks. As an application of Proposition (3-1) it is demonstrated here that in two firm games symmetric pareto efficient points are strong quasi-stable; that is with respect to asymmetric as well as symmetric shocks.

The first order condition for symmetric pareto efficiency was given in (2-3) as

$$\pi_j^j + (N - 1) \pi_k^j = 0 \quad (4-1)$$

$$\pi_j^j + \pi_k^j = 0 \quad (4-2)$$

where (4-2) follows from $N = 2$. The matrix π is

$$\pi = \begin{bmatrix} -\pi_k^j & \pi_k^j \\ \pi_k^j & -\pi_k^j \end{bmatrix} \quad (4-3)$$

which has rank one since $\pi_k^j \neq 0$ and is regular singular since it admits $e = (1,1)'$ in its null space.

The matrix λ is computed from (3-2)

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j / \pi_k^j & j \neq k \end{cases} \quad (4-4)$$

$$\lambda = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (4-5)$$

From inspection of (4-3) and (4-5) these matrices satisfy the hypotheses of proposition (3-1). To apply the proposition requires in addition that three eigenvalues of A have strictly negative real parts.

In the two firm cases the dynamic equations for agent one from (1-1), (1-2) and (1-3) are

$$\dot{x}^1 = [\pi_1^1 + R_1^2 \pi_2^1 + R_2^1 (\pi_2^2 + R_2^1 \pi_1^2)] \quad (4-6)$$

$$\dot{R}_2^1 = b\eta [\pi_1^2 (\pi_1^1 R_2^1 + \pi_2^1) + \pi_1^1 (\pi_2^2 + \pi_1^2 R_2^1)] \quad (4-7)$$

Differentiating these equations, using symmetry and the equilibrium conditions in (4-2) shows that

$$\partial \dot{x}^1 / \partial R_2^1 = [\pi_2^2 + R_2^1 \pi_1^2 + R_2^1 \pi_1^2] = b\pi_k^j$$

$$\partial \dot{x}^1 / \partial R_1^2 = \pi_k^j$$

$$\partial \dot{R}_2^1 / \partial R_2^1 = -b\eta (\pi_k^j)^2$$

$$\partial \dot{R}_2^1 / \partial R_1^2 = 0 \quad (4-8)$$

By symmetry the stability matrix A defined in (3-1) is

$$A = \begin{bmatrix} \partial \dot{x}^1 / \partial x^1 & \partial \dot{x}^1 / \partial x^2 & \pi_k^j & \pi_k^j \\ \partial \dot{x}^1 / \partial x^2 & \partial \dot{x}^1 / \partial x^1 & \pi_k^j & \pi_k^j \\ \partial \dot{R}_2^1 / \partial x^1 & \partial \dot{R}_2^1 / \partial x^2 & -b\eta (\pi_k^j)^2 & 0 \\ \partial \dot{R}_2^1 / \partial x^2 & \partial \dot{R}_2^1 / \partial x^1 & 0 & -b\eta (\pi_k^j)^2 \end{bmatrix} \quad (4-9)$$

Let $e_0 = (0, 0, 1, -1)'$; then $Ae_0 = -b\eta(\pi_k^j)^2 e_0$ so e_0 is an eigenvector corresponding to a negative eigenvalue. Furthermore there are two other eigenvalues corresponding to symmetric departures from equilibrium and by the analysis of symmetric shocks these are strictly negative as well.

Proposition (3-1) now applies and the efficient steady state and nearby steady states are strong quasi-stable.

5. Conclusion

When firms can engage in costless retaliatory policies we anticipate that they will reward opponents who make pareto improving output adjustments and punish those who selfishly try to increase output. This paper has shown that, subject to technical qualifications, this is true. The important qualifications:

- o In asymmetric games pareto efficient outcomes may be unstable steady states. A small subset of the pareto efficient outcomes may not be steady states at all.

- o In asymmetric games there may be stable steady states which satisfy the first order conditions for pareto efficiency, but not the second order conditions. A small number of steady states may not even satisfy the first order conditions.

APPENDIX (A) --Symmetric Efficiency

The objective is to show that in a symmetric game with $\pi_k^j < 0$ $j \neq k$ $\pi_j^j + (N - 1) \pi_k^j = 0$ and $S < 0$ imply local efficiency. It was already shown that this implies no small symmetric perturbation makes any firm better off. Suppose π has rows π^j as always. I will show that if z is an asymmetric N -vector then $\pi_k z < 0$ for some k thus implying any non-symmetric perturbation makes at least one firm worse off via the mean value theorem.

Lemma (A): If z is non-symmetric for some k $\pi_k z < 0$.

Proof: Suppose conversely z is asymmetric and $\pi z \geq 0$. Since $\pi_k^j \neq 0$ $j \neq k$ a check shows that π has rank $N - 1$. Since $\pi e = 0$ where e is symmetric (by assumption) it cannot be that $\pi z = 0$. Thus we may assume $\pi z \geq C$ where $C_1 = 1$ $C_k = 0$ $k > 1$. By a theorem on linear inequalities found for example as theorem 2.7 in Gale [3] the system $\pi z \geq C$ has a solution if and only if the system $y' \pi = 0$ $y' C = 1$ has no non-negative solution. Letting $e = (1, \dots, 1)'$ we see that $e' \pi = 0$ and $e' C = 1$. Thus $\pi z \geq C$ has no solution, a contradiction.

Q.E.D.

Appendix (B) -- Strong Quasi-Stability

Definitions: We consider a dynamical system $\dot{x} = f(x)$ defined on an open subset of \mathbb{R}^{m+n} . The vector field f is assumed to be C^1 and by the existence and uniqueness theorem there is a unique maximal flow $\phi_t(x)$ satisfying $\partial\phi_t/\partial t = f(\phi_t)$ and $\phi_0(x) = x$. If x^* is an equilibrium so $f(x^*) = 0$ then the stability matrix $A \equiv Df_{x^*}$. An equilibrium x^* is strongly quasi-stable if for every $\epsilon > 0$ there is an open neighborhood $U_\epsilon \ni x^*$ such that $x \in U$ implies $\lim_{t \rightarrow \infty} \phi_t(x)$ exists and lies within ϵ of x^* . Sufficiently small perturbations from x^* lead to an equilibrium close to x^* .

The Theorem: Our objective in this section is to prove the following sufficient condition for quasi-stability:

Theorem (B-1): Suppose there is an open set $U \ni x^*$ in which the set of equilibria $M \equiv \{x \in U \mid f(x) = 0\}$ are an m dimensional submanifold of U and that the stability matrix $A(x^*)$ has n eigenvalues with strictly negative real parts. Then there is an open set $W \ni x^*$ such that

- (a) $x \in M \cap W$ implies x is strongly quasi-stable
- (b) the rate of convergence in the definition of quasi-stability is exponentially fast

Proposition (B-2): Suppose $x = (x^1, x^2)'$; $f = (f^1, f^2)'$ where $x^1 \in \mathbb{R}^m$, $x^2 \in \mathbb{R}^n$, that there is $U \ni 0$ with

$$f(x^1, 0) = 0 \quad (x^1, 0) \in U$$

and that Df_0 has n eigenvalues with negative real parts. Then there are $g, G > 0$, $W \ni 0$ and map $p: W \rightarrow \mathbb{R}^m \times 0$ continuous at 0 such that $x \in W$ implies for $t \geq 0$

$$|\phi_t(x) - p(x)| \leq Ge^{-gt}$$

Theorem (B-1) follows from (B-2) by the use of local coordinates and the continuity of $A(x)$.

Proof of the Proposition: The idea behind the proof is much like that in the usual case where all eigenvalues of A have negative real parts: we show that the linearized system is stable and that trajectories remain so close to the initial equilibrium that the linearized system is a good approximation to the actual system. We do this by examining the motion of the system in a small cylinder around the equilibrium manifold. The important extension to the ordinary stability proof is to show that the fact that the equilibrium manifold has the right dimension guarantees the system cannot drift out the end of the cylinder.

Our procedure is to define $z^1 = |x^1|$ and $z^2 = (x^2, Bx^2)^{1/2}$. The variable z^1 measures how far down the cylinder around the equilibrium manifold $x^2 = 0$ the system has drifted. The variable z^2 measures how far the system is from the equilibrium manifold in the metric induced by the positive definite matrix B . Our proof proceeds via a series of lemmas.

Lemma (B-1): Suppose $z^1(t), z^2(t) \geq 0$ and for some $A, \gamma >$ and

$$A \geq z^1, z^2 \geq 0 \quad \dot{z}^2/|z^1| < -\gamma \text{ then } z^2(0) \leq \min(1, \gamma) [A - z^1(0)]$$

implies $0 \leq z^1(t), z^2(t) \leq A$ for $t \geq 0$.

Proof: Obvious.

Lemma (B-2): For $z^1 = |x^1|$ $z^2 = (x^{2'} B x^2)^{1/2}$ and

B positive definite

$$\dot{z}^2 / |\dot{z}^1| \leq \delta \left[\frac{x^{2'} B \dot{x}^2}{|x^2|} \frac{|x^1|}{|x^{1'} \dot{x}^1|} \right]$$

where $\delta > 0$ depends only on B.

Proof:

$$\dot{z}^1 = x^{1'} \dot{x}^1 / |x^1|$$

$$\dot{z}^2 = (x^{2'} B \dot{x}^2) / (x^{2'} B x^2)^{1/2}$$

$$\frac{|z^2|}{|z^1|} = \frac{x^{2'} B x^2}{|x^2|} \frac{|x^1|}{|x^1, \dot{x}^1|} \cdot \frac{|x^2|}{(x^{2'} B x^2)^{1/2}} \quad (B-1)$$

and the lemma follows since $|x^2| / (x^{2'} B x^2)^{1/2} < \delta$
by equivalence of norms. QED

Lemma (B-3): Set $A = Df_0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};$

then the eigenvalues of A_{22} are the non-zero eigenvalues of A .

Proof: Since $f(x^1, 0) = 0$ $A_{11} = A_{21} = 0$ and

$\det(A - \lambda I) = (-\lambda)^m \det(A_{22} - \lambda I)$. Since A has at most m zero eigenvalues the lemma follows directly. QED

Lemma (B-4): Suppose A_{22} is a real matrix and that the largest real part of any eigenvalue of A_{22} is dominated by $-\beta'$. Then there is a non-singular matrix C such that for any x^2

$$x^{2'} C' C A_{22} x^2 \leq -\beta' x^{2'} C' C x^2$$

Proof: According to ch. 7 §1 lemma of Hirsch and Smale [8] there is a real basis such that in the corresponding inner product, $\langle \cdot, \cdot \rangle$ $\langle A_{22} x^2, x^2 \rangle \leq -\beta \langle x^2, x^2 \rangle$. Letting C

transform the basis in which the coordinates of A_{22} are given to this new basis shows that this inner product can be written in the desired form. QED

Lemma (B-5): There is $\alpha', \beta' > 0$ and a positive definite matrix B such that

$$x^{2'} B A_{22} x^2 \leq -\alpha' |x^2|^2, \quad -\beta' x^{2'} B x^2$$

Proof: By lemmas (B-3) and (B-4) there is non-singular matrix C and $\beta' > 0$ such that

$$x^{2'} C' C A_{22} x^2 \leq -\beta' x^{2'} C' C x^2 \quad (B-2)$$

Taking $B = C' C$ and observing by equivalence of norms that $x^{2'} B x^2 \geq \gamma |x^2|^2$ shows that $\alpha' = \beta' \gamma$. QED

Lemma (B-6): There is a neighborhood W of 0 with its closure $\bar{W} \subset U$ compact and a $\alpha, \beta > 0$ such that for $x \in W$

$$x^{2'} B x^2 \leq -\alpha |x^2|^2, \quad \beta x^{2'} B x^2$$

Proof: Take \bar{W} a closed ball of radius r at 0 with $\bar{W} \subset U$. By Taylor's theorem

$$\begin{aligned} \dot{x}^2 &= f^2(x^1, x^2) = f^2(x^1, 0) + A_{22}(x^1, \bar{x}^2)x^2 \\ &= A_{22}(x^1, \bar{x}^2)x^2 \end{aligned} \tag{B-3}$$

where $(x^1, \bar{x}^2) \in \bar{W}$. Hence, $|(x^1, \bar{x}^2)| < r$. Since by lemma (B-5)

$$x^{2'} B A_{22}(0) x^2 \leq -\alpha' |x^2|^2, \quad -\beta' x^{2'} B x^2 \tag{B-4}$$

it follows that since A_{22} is continuous for any $0 < \alpha < \alpha'$ $0 < \beta < \beta'$ r can be chosen sufficiently small that the lemma holds. QED

Lemma (B-7): There is $\mu > 0$ such that for $x \in W$

$$|\dot{x}^1| \leq \mu |x^2|$$

Proof: By Taylor's theorem

$$\begin{aligned} |\dot{x}^1| &= |f^1(x^1, 0) + A_{12}(x^1, \bar{x}^2)x^2| \\ &\leq \sup_{x \in \bar{W}} |A_{12}(x)| |x^2| \end{aligned} \tag{B-5}$$

and since \bar{W} compact and A_{12} continuous

$$\sup_{x \in \bar{W}} |A_{12}(x)| \leq \mu \tag{B-6}$$

QED

Lemma (B-8): Let $V \supset 0$ be open. Then there is $V' \supset 0$ such that for $x \in V'$ $\phi_t(x) \in V$.

Proof: For $x \in W$ by lemmas (B-2), (B-6) and (B-7)

$$\begin{aligned} \dot{z}^2 / |z^1| &\leq \delta \left[\frac{x^{2'} B \dot{x}^2}{|x^2| |x^1|} \right] \\ &\leq \frac{\delta}{\mu} \left[\frac{x^{2'} B \dot{x}^2}{|x^2|^2} \right] \\ &\leq -\frac{\delta \alpha}{\mu} = -\gamma \end{aligned} \tag{B-7}$$

Now choose A so that the ball around zero of radius A lies in $W \cap V$ and choose V' to have radius $A \min(1, \gamma)$. An application of lemma (B-1) yields the desired conclusion. QED

Lemma (B-9): There is $p: W' \rightarrow \mathbb{R}^m \times 0$ continuous at zero and $g, G > 0$ such that for $x \in W'$ and $t \geq 0$ $|\phi_t(x) - p(x)| \leq Ge^{-gt}$.

Proof: Lemmas (B-2) and (B-6) imply for some $g > 0$ and $x \in W$

$$\dot{z}^2 \leq -2gz^2 \tag{B-8}$$

By lemma (B-8) if $x \in W'$ initially (B-11) holds at later times and by the Bellman-Gronwall growth lemma

$$|z^2| \leq \Delta e^{-gt} \quad \text{for some } \Delta > 0 \quad (\text{B-9})$$

which is to say

$$|x^2| \leq \Delta e^{-gt} \quad (\text{B-10})$$

By lemma (B-7)

$$|\dot{x}^1| \leq \mu \Delta e^{-gt} \quad (\text{B-11})$$

implying the existence of $p^1(x)$ such that

$$|x^1 - p^1(x)| \leq (\mu \Delta / g) e^{-gt} \quad (\text{B-12})$$

Setting $p^2(x) = 0$ and $G = \Delta(\mu/g + 1)$ we see

$$\begin{aligned} |x - p(x)| &\leq |x^1 - p^1(x)| + |x^2| \\ &\leq G e^{-gt} \end{aligned} \quad (\text{B-13})$$

The continuity of p at 0 follows directly from lemma (B-8).

QED

APPENDIX (C)--The Manifold of Autonomous Steady States

The objective is to prove that when the hypotheses of proposition (3-1) are satisfied at (x^*, R^*) there is an open set $U \supset (x^*, R^*)$ in which steady states form a manifold of dimension $(N - 1)$. This is done by means of four lemmas.

Lemma (C-1): If $\lambda(x^*)$ does not admit a weak positive null vector then there is an open $U \supset x^*$ such that $x \in U$ and (x, R) a steady state implies $\pi(x)$ is singular.

Proof: Since $\lambda(x)$ is continuous choose U so that $x \in U$ implies $\lambda(x)$ has no weak positive null vector. If $\pi(x)$ is non-singular the steady state is exceptional and from (2-20) satisfies
$$\sum_{k=1}^N \lambda_k^j / [(\pi^{-1})^k \lambda_k]^2 = 0$$
 contradicting the fact λ doesn't admit weak positive null vectors.

Q.E.D.

Lemma (C-2): If $[\pi(x^*) \mid \lambda_k(x^*)]$ has full rank for all k then there is open $U \supset x^*$ such that $x \in U$ and $\pi(x)$ singular imply (x, R) can't be a non-autonomous steady state.

Proof: Since $\pi(x)$ and $\lambda(x)$ are continuous choose U so that $x \in U$ implies $[\pi(x) \mid \lambda_k(x)]$ has full rank for all k . From part I (2-13) a necessary condition for (x, R) to be a steady state is

$$[\pi \mid -\lambda_k] \begin{bmatrix} R_k \\ y^k \end{bmatrix} = 0 \quad \text{all } k \quad (C-1)$$

Since π is singular and $[\pi \mid \lambda_k]$ has full rank the only solution to (C-1) is $\pi R_k = 0$ and $y^k = 0$. Thus all $y^k = 0$ and if (x, R) is a steady state it is an autonomous one. Q.E.D.

Corollary C-3: If (x^*, R^*) are as in proposition (C-1) there is $U \supset (x^*, R^*)$ in which all steady states are autonomous.

Now define $w_k^j = \pi^j R_k$. From (1-1) and (1-3) the equations of motion are

$$\dot{x}^j = \sum_{k=1}^N R_k^j w_k^k$$

$$\dot{R}_k^j = b \pi_k^j [\pi_j^k w_k^j + \pi_j^j w_k^k] \quad (C-2)$$

Let $\pi_{-k}^j = (\pi_1^j, \dots, \pi_{k-1}^j, \pi_{k+1}^j, \dots, \pi_N^j)$. The derivative of w_k^j with respect to all the state variables is the row vector

$$Dw_k^j = [(D\pi^j R_k)' \mid 0 \mid \dots \mid \pi_{-k}^j \mid \dots \mid 0] \quad (C-3)$$

1 k

where $D\pi^j$ is the matrix of second derivatives of π^j . Observe that

$R_k^* = \gamma/\gamma_k$ so that this can be written as

$$Dw_k^j = [(D\pi^j \gamma)' / \gamma_k \mid 0 \mid \dots \mid \pi_{-k}^j \mid \dots \mid 0] \quad (C-4)$$

The matrix Dw is formed by stacking the Dw_k^j .

Lemma (C-4): If U contains only autonomous steady states and at steady states $\text{corank}(Dw) \leq (N-1)$ then autonomous steady states are a manifold in U .

Proof: Since π is continuous by choosing U small enough we may assume whenever π is singular the first row (say) is a linear combination of the other rows. Since all steady state in U are autonomous $w = 0$ is necessary and sufficient for a steady state there. Define \bar{w} to be the sub-matrix of w formed by deleting the $(N-1)$ rows corresponding to $w_2^1, w_3^1, \dots, w_N^1$. I claim that $\bar{w} = 0$ implies $w = 0$ in U and $D\bar{w}$ has full rank, which will prove the lemma.

Suppose $\bar{w} = 0$. Then $w_1^1, w_1^2, \dots, w_1^N = 0$ which reads $\pi R_1 = 0$. Since $R_1^1 = 1$ this implies π is singular. Does $w_k^1 = 0$? This reads $\pi' R_k = 0$. But $\pi^j R_k = 0$ for $j \neq k$ and π^1 is a linear combination of the π^j . Thus $\pi' R_k = 0$. So $w = 0$.

Since $\text{corank}(Dw) \leq (N-1)$ to show $D\bar{w}$ has full rank it suffices to show that Dw_k^1 is a linear combination of the rows of $D\bar{w}$. Observe from (D-4) that

$$\gamma_k Dw_k^m - \gamma_1 Dw_1^m = [-\gamma_1 \pi_{-1}^m \mid 0 \mid \dots \mid \gamma_k \pi_{-k}^m \mid 0] \quad (C-5)$$

Thus

$$\gamma_k Dw_k^1 - \gamma_1 Dw_1^1 = [-\gamma_1 \pi_{-1}^1 \mid 0 \mid \dots \mid \gamma_k \pi_{-k}^1 \mid 0] \quad (C-6)$$

k

Suppose $\pi^1 = \sum_{i=2}^N \mu_i \pi^i$. Then from (C-5) and (C-6)

$$Dw_k^1 = (1/\gamma_k) [\gamma_1 Dw_1^1 + \sum_{i=2}^N (\gamma_k Dw_k^1 - \gamma_1 Dw_1^i) \mu_i] \quad (C-7)$$

is the desired linear combination.

Q.E.D.

Lemma (C-5): At an autonomous steady state if the stability matrix has corank $(A) \leq (N-1)$ then corank $(Dw) \leq (N-1)$.

Proof: From (C-2) the equations of motion can be written as

$$\dot{v} = Lw \quad (C-8)$$

where v is the vector of state variables and L is an $N^2 \times N^2$ matrix. Since $w = 0$ at the autonomous steady state $A = LDw$. Thus corank $(A) \leq (N-1)$ implies corank $(Dw) \leq (N-1)$. Q.E.D.

The proof of proposition (3-1) now follows from the fact that corollary (C-3) and lemma (C-5) imply the hypothesis of lemma (C-4).

References

- [1] James Friedman, "Non-cooperative Equilibria for Supergames," RE Stud, January 1971.
- [2] _____, Oligopoly and the Theory of Games, North-Holland, 1977.
- [3] David Gale, Theory of Linear Economic Models, McGraw-Hill, 1960.
- [4] Edward Green, "Non-cooperative Price Taking in Large Dynamic Markets," JET, April 1980.
- [5] Joel Guttman, "Understanding Collective Action: Matching Behavior," AER, May 1978.
- [6] Guillemin and Pollack, Differential Topology, Prentice-Hall, 1974.
- [7] Morris Hirsch, Differential Topology, Prentice-Hall, 1974.
- [8] Hirsch and Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, 1974.
- [9] David Levine, "The Enforcement of Collusion in a Simple Oligopoly," UCLA Working Paper #211, August 1981.
- [10] _____, "Local Almost Perfect Equilibria," UCLA Working Paper #210, August 1981.
- [11] _____, "Some Result on Dynamical Systems," MIT, 1981.
- [12] Marschak and Selten, General Equilibrium with Price Making Firms, Springer-Verlag, 1974.
- [13] _____, "Restabilizing Responses, Inertia Supergames, and Oligopolistic Equilibria," QJE, February 1978.
- [14] Neal McCoy, Fundamentals of Abstract Algebra, Allyn and Bacon, 1973.
- [15] John Milnor, Topology from the Differential Viewpoint, University Press of Virginia, 1976.
- [16] Roy Radner, "Collusive Behavior in Epsilon Equilibria," JET, April 1980.
- [17] S. Smale, "The Prisoner's Dilemma," Econometrica, November 1980.
- [18] R. Selten, "A Re-examination of the Perfectness Concept," Int. J. Game Theory, 21 (1979), pp. 1-9.
- [19] Yung-Chen Lu, Singularity Theory, Springer Verlag, 1976.
- [20] E.C. Zeeman, Catastrophe Theory, Addison-Wesley, 1974.