

THE FISHER EQUATION UNDER UNCERTAINTY:  
REAL RISK VERSUS MONEY RISK\*

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1. THE FISHER EQUATION

The famous Fisher equation relates the monetary interest rate  $\underline{i}$ , the real interest rate  $\underline{r}$ , and the anticipated rate of inflation  $\underline{a}$ :

$$(1+i) = (1+r)(1+a) \quad (1)$$

This equation is appropriate for a world of certainty, in the sense that all economic agents share fully confident and agreed beliefs about the anticipated rate of inflation (proportionate change in the price level)  $\underline{a}$ .

But what if the future price level is uncertain? Perhaps it will be the case that we need only use, as an instance of a "martingale" relationship, the mathematical expectation:

$$(1+i) = (1+r)E(1+a) \quad (2)$$

Of course, such a mathematical expectation could be defined interpersonally only under some special restrictions about agreement of beliefs. Or perhaps, as has been suggested by other authors,<sup>1</sup> an inverse expression might be valid in the harmonic form:<sup>2</sup>

$$\frac{1}{1+i} = \frac{1}{1+r} E\left(\frac{1}{1+a}\right) \quad (3)$$

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<sup>1</sup>Kochin (1980), Eden and Blejer (1979).

<sup>2</sup>The problem here recurs regularly in "martingale" propositions in prices. We are dealing always with price ratios, and martingale propositions cannot in general be valid both for a ratio and its reciprocal (i.e., the ratio taken the other way). For a discussion, see Woodward (1980) and McCulloch (1973).

The question considered in this paper is whether, or in which circumstances, (2) or (3) may be the correct generalization of the Fisher equation to a world of uncertainty. The correct answer to such a question must come in a general equilibrium context taking account of the fact that uncertainty regarding the future price-level arises from two possible sources: (1) uncertainty about future-date money supplies, and (2) uncertainty about the real consumption availability at future dates. This is an oversimplification, of course, in that individuals might also be uncertain about their own or other people's future preferences, about the future availability of markets, and a host of other considerations. In our analysis, however, we will deal with a simple world in which only future nominal money quantities and future real consumption availabilities are uncertain.

We will also be distinguishing between the Fisher equation in terms of anticipations and in terms of realizations (ex-ante versus ex-post). Our first step is to show that, correctly interpreted -- ex ante -- the Fisher equation holds in its initial form (1) even in a world of uncertainty.

In a simple two-date world, the real interest rate  $\underline{r}$  is defined in terms of the exchange rate between present-date (time-0) and future-date (time-1) claims to real consumption ("corn"):

$$\frac{P_{C1}}{P_{C0}} = \frac{1}{1+r} \quad (4)$$

And the nominal interest rate  $\underline{i}$  can be analogously written:

$$\frac{P_{M1}}{P_{M0}} = \frac{1}{1+i} \quad (5)$$

At the current date-0 when these trades are agreed upon, the present-dated money and corn endowments  $M_0$  and  $C_0$  are known magnitudes while the future-dated  $M_1$  and/or  $C_1$  may be unknown. Nevertheless, the equations relate to unconditional trades: the contracts obligate the parties to make future deliveries regardless of what the social totals may be at those future dates.

Another trading ratio determines the current price-level  $\phi_0$ :

$$\frac{P_{CO}}{P_{MO}} = \phi_0 \quad (6)$$

Now there is of course a fourth corner to this set of trades -- future corn for future money. The fourth trading ratio determines what we will call the forward price-level  ${}_0\phi_1$ :

$$\frac{{}_0^P C_1}{{}_0^P M_1} = {}_0\phi_1 \quad (7)$$

This forward price-level is exactly analogous in interpretation to forward prices as we speak of them in commodity trading. Here the pre-subscript 0 indicates that the prices in equation are determined at date-0 even though the commodity in question may be deliverable at a future date. (However, we will generally suppress the pre-subscript wherever matters are clear enough from the context.)

In this four-cornered trading, arbitrage will assure that:

$$\frac{P_{M0}}{P_{M1}} = \frac{P_{C0}}{P_{C1}} \frac{{}_0\phi_1}{\phi_0} \quad (8a)$$

Or, equivalently:

$$1+i = (1+r) \frac{{}_0\phi_1}{\phi_0} \quad (8b)$$

Thus, implicit in trading at the current date will be a forward price-level  ${}_0\phi_1$  such that, if we make the substitution  $(1+a) = {}_0\phi_1/\phi_0$ , the Fisher equation (1) holds in its original form.

The question about whether to use the expectation operator in its original form (2) or inverted form (3) must then relate to the ex-post or realized version of the Fisher equation. That is, we are asking whether the forward  $(1+a)$  must in general equal the expectation of the realized  $(1+\tilde{a})$  -- where the tilde indicates a random variable. We will usually find it more convenient to express this as the question whether the forward price-level ratio  ${}_0\phi_1/\phi_0$  equals the expectation of the realized ratio  ${}_1\tilde{\phi}_1/\phi_0$ . (Of course, since the denominators are the same, this reduces to simply asking whether the forward  ${}_0\phi_1$  equals the expectation of the realized  ${}_1\tilde{\phi}_1$ .) Or, for all these questions, is it perhaps the inverted form  $\frac{1}{1+a} = E\left(\frac{1}{1+\tilde{a}}\right)$  that represents the valid equality?

## 2. CORN AND MONEY IN INFINITE TIME -- CERTAINTY

The problem just posed turns out to be of non-trivial difficulty when one attempts to model the balance between real consumption and money-holding

from first principles. We would get seriously misleading results from a two-date model: the advantage of money-holding in the terminal period is necessarily understated since money balances that would otherwise survive into the next period have no continuing value. (I.e., the store-of-value aspect of money-holding is lost.) While there are a number of ways of coping with this difficulty, for our purposes it is most convenient to embed our two-date problem in an infinite-time model which, through several devices, can be quite manageable.<sup>1</sup>

The economy is composed of identical individuals with given endowments distributed over time. I.e., there is no production -- not even storage (corn is not a durable commodity in the model). The individual has a preference function involving corn at each date and real balances at each date<sup>2</sup>, which we will write in the special form  $U = \sum_t B^t V(C_t, M_t/\phi_t)$ . Here  $B \leq 1$  is a time-preference parameter, and note that the function is additive over time. We will also assume additivity (absence of positive or negative complementarity) as between corn and real balances.  $V$  is strictly concave in both corn and real balances at each date. Marginal utilities of either commodity at any date will be symbolized by  $v(\cdot)$ , where the additivity assumptions permit omission of all other variables from the parenthesis.

First consider a world of certainty, and in particular "forward" optimization and equilibrium as of date-0 when the economy starts up. The

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<sup>1</sup>The model employed here is a generalization of that in Hirshleifer (1970), see especially pp. 145-146.

<sup>2</sup>As in Patinkin (1965).

standard optimization conditions can be expressed in terms of proportionality between prices and marginal utilities:

$$\frac{P_{Ct}}{B^t v(C_t)} = \frac{P_{C,t+1}}{B^{t+1} v(C_{t+1})} = \dots = \frac{P_{Mt}}{B^t \frac{1}{\phi_t} v\left(\frac{M_t}{\phi_t}\right)} = \frac{P_{M,t+1}}{B^{t+1} \frac{1}{\phi_{t+1}} v\left(\frac{M_{t+1}}{\phi_{t+1}}\right)} = \dots \quad (9)$$

The price ratios between dated corn and money claims, effective in date-0 trading, can then be classified into three groups.<sup>1</sup>

The first type of relation expresses the equality between marginal rates of substitution of successively-dated corn claims and the market trading ratios of those claims:

$$\frac{P_{Ct}}{P_{C,t+1}} = \frac{v(C_t)}{Bv(C_{t+1})} = \frac{\phi_t}{\phi_{t+1}} (1 + i_{t+1}) \quad (10)$$

Note that the trading ratio  $P_{Ct}/P_{C,t+1}$  appears, on the right-hand-side of (10), as a relation involving the money interest rate  $i_{t+1}$  and the successive price-levels  $\phi_t$  and  $\phi_{t+1}$ .<sup>2</sup>

<sup>1</sup>See Hirshleifer, *ibid.*, p. 146, fn 15.

<sup>2</sup>Note also that this right-hand-side is simply equivalent to  $1 + r_{t+1}$ , as follows of course from the Fisher equation (which necessarily holds under certainty for date-0 trading).

The second type of relation equates the date-0 trading ratio between claims to successively dated nominal money units and marginal utilities of correspondingly dated real balances, adjusted appropriately for the price levels so as to represent a preference ratio in nominal-money units:<sup>1,2</sup>

$$(ii) \frac{P_{Mt}}{P_{M,t+1}} = \frac{\frac{1}{\phi_t} v(M_t/\phi_t)}{B \frac{1}{\phi_{t+1}} v(M_{t+1}/\phi_{t+1})} = \frac{i_{t+1}}{i_{t+2}} (1+i_{t+2}) \quad (11)$$

The third type of relation equates the trading ratio between same-dated claims to corn and nominal money and the marginal utility ratio between same-dated corn and real money balances, the latter adjusted appropriately as before for conversion into a preference expression in nominal-money units:

$$\frac{P_{Ct}}{P_{Mt}} = \frac{v(C_t)}{\frac{1}{\phi_t} v(M_t/\phi_t)} = \phi_t \frac{1+i_{t+1}}{i_{t+1}} \quad (12)$$

<sup>1</sup> Relation (ii) states a proportionality condition between the prices and marginal utilities of dated claims to nominal money-holdings. The marginal utility of a nominal money-holding at date-t is of course equal to the marginal utility of a real money-holding at that date divided by the price-level at that date. Thus,  $P_{Mt}$  is proportional to  $(1/\phi_t)v(M_t/\phi_t)$ .

<sup>2</sup> There is one seemingly puzzling feature here. Equation (5) in our simple two-date version was equivalent to  $P_{M0}/P_{M1} = 1+i$ . But on the RHS of equation (11) the corresponding factor  $(1+i_{t+2})$  is multiplied by the ratio  $(i_{t+1}/i_{t+2})$ . This is one of the respects in which a general-date model corrects a misleading aspect of two-date models. The  $P_{Mt}$  terms in our infinite-date equations are associated with marginal utilities as partial derivatives. Thus,  $P_{Mt}$  is proportional to the value of a small increment of money to be held at date-t only -- which is quite different from the value of a small increment of money to be received at date-t and remain in the holder's possession thereafter.



All these marginal utilities and marginal rates of substitution are to be evaluated at the endowment positions, which are of course the equilibrium positions under our representative-individual, pure-exchange assumption. Then we have here three equations in the four variables  $\phi_t$ ,  $\phi_{t+1}$ ,  $i_{t+1}$ ,  $i_{t+2}$ . The system can be closed by imposing a "steady-state" (in the sense of proportionate-growth) assumption assuring that  $i_{t+1} = i_{t+2} = i$ . Then equations (10) and (11) together will determine the nominal interest rate and the growth rate of the price level. (And, with these known, the real interest rate  $r_{t+1} = r_{t+2} = r$  can be solved for via the Fisher equation.) Finally, equation (12) will determine the absolute price level  $\phi_t$ .

It is evident by inspection that the simple quantity theory will hold in this model. Given any proportionate expansion of money balances  $M_t$ , common to all dates (this proviso is necessary in order to maintain the "steady-state" conditions), the price levels  $\phi_t$  will all simply rise in the same proportion. The nominal interest rate  $i$  and the real interest rate  $r$  will be unaffected by such expansions.<sup>1</sup>

#### Numerical Example 1

Let  $B = .8$ , and  $V(C_t, M_t/\phi_t) = \log_e [C_t^4 (M_t/\phi_t)^{-1}]$ . Assume that corn quantities are doubling each period ( $C_{t+1}/C_t = 2$ ), while nominal money balances are constant over time ( $M_{t+1}/M_t = 1$ ).

Equation (10) then takes the specific form:

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<sup>1</sup>This conclusion is not strictly robust, however, to departures from either the representative-individual or the additivity assumptions.

$$\frac{v(C_t)}{Bv(C_{t+1})} = \frac{.4C_{t+1}}{(.8).4C_t} = \frac{.4}{.32} = 2.5 = \frac{\phi_t}{\phi_{t+1}}(1+i)$$

Equation (11) becomes:

$$\frac{\frac{1}{\phi_t} v(M_t/\phi_t)}{\frac{1}{\phi_{t+1}} Bv(M_{t+1}/\phi_{t+1})} = \frac{.1\phi_{t+1}(M_{t+1}/\phi_{t+1})}{(.8).1\phi_t(M_t/\phi_t)} = (1+i)$$

Or, after cancellations:  $1.25 = 1+i$ .

So we have  $i = .25$  and  $\frac{\phi_t}{\phi_{t+1}} = 2$ . The nominal interest rate is 25%, and the price level is halved each period. And, from the Fisher equation:

$$1+r = \frac{1+i}{1+a} = \frac{1.25}{.5}$$

so that the real interest rate is  $r = 150\%$ .

It remains only to determine the absolute price level from equation (12). For this we need to specify absolute magnitudes of  $C_t$  and  $M_t$  at some date  $t$ . Let us assume  $C_t = 20$  and  $M_t = 100$ . Then:

$$\frac{v(C_t)}{\frac{1}{\phi_t} v(M_t/\phi_t)} = \frac{.4\phi_t(M_t/\phi_t)}{.1C_t} = \frac{.4(100)}{.1(20)} = 20 = \phi_t \frac{1+i}{i}$$

Since  $i = .25$ , the absolute price level for date- $t$  is  $\phi_t = 4$ .

### 3. CORN AND MONEY IN INFINITE TIME -- UNCERTAINTY

We now want to introduce uncertainty into the model. But the solution will still represent "forward" optimization and equilibrium, viewed as of date-0 when the economy starts up. The preference-scaling function remains the same as before but, under the expected-utility hypothesis,

the overall utility viewed as of date-0 can be written:

$$U = \sum_t B^t E[V(C_t, M_t / \phi_t)] \quad (13)$$

The assumption of separability (additive utility) over time and between the corn and real-balance commodities at each date will be maintained.

In a market regime of complete contingent claims, the economy would have for sale entitlements at any date to either commodity contingent upon the state of the world at that date. The state of the world would be defined by the per-capita magnitudes of the corn crop and the nominal money balances at that date, regarded as exogenously determined by Nature and by the monetary authorities respectively. However, we shall not be assuming here a regime of complete contingent claims, but instead that only unconditional claims to corn or to real balances at any date are purchasable in the market. Then the optimization condition under uncertainty, the analog of equation (9) above, becomes:<sup>1/</sup>

$$\frac{P_{Ct}}{B^t \sum_s \pi_s v(C_{t,s})} = \frac{P_{Mt}}{B^t \sum_s \pi_s \frac{1}{\phi_{t,s}} v(M_{t,s} / \phi_{t,s})} \quad (14)$$

$$= \frac{P_{C,t+1}}{B^{t+1} \sum_\sigma \pi_\sigma v(C_{t+1,\sigma})} = \frac{P_{M,t+1}}{B^{t+1} \sum_\sigma \pi_\sigma \frac{1}{\phi_{t+1,\sigma}} v(M_{t+1,\sigma} / \phi_{t+1,\sigma})}$$

<sup>1</sup>The only slightly tricky feature here is the proportionality relation between the prices  $P_{Mt}$  and the associated dated marginal utilities of nominal money-holdings. In contrast with the similar relation in equation (11), under uncertainty the individual relates his expected marginal utility, of an entitlement to such a money-holding, to the currently-quoted price  $P_{Mt}$  of such a nominal claim. The marginal utility of an extra nominal dollar at date-t, if state-s obtains at that date, will be  $(1/\phi_{t,s})v(M_{t,s}/\phi_{t,s})$ . Viewed as of the current trading date (time-0), the expected marginal utility is  $\sum_s \pi_s (1/\phi_{t,s})v(M_{t,s}/\phi_{t,s})$  which, adjusted by the time-discount factor  $B^t$ , becomes the denominator of the proportionality ratio associated with  $P_{Mt}$  in equation (14).

Here, to cut down on notational complexity,  $s$  is used to index states at date- $t$  while  $\sigma$  is used to index states at date- $t+1$ . The analogs of equations (10) through (12) then become:

$$(i) \quad \frac{P_{Ct}}{P_{C,t+1}} = \frac{\sum_s \pi_s v(C_{t,s})}{B \sum_{\sigma} \pi_{\sigma} v(C_{t+1,\sigma})} = \frac{\phi_t}{\phi_{t+1}} (1+i_{t+1}) \quad (15)$$

$$(ii) \quad \frac{P_{Mt}}{P_{M,t+1}} = \frac{\sum_s \pi_s \frac{1}{\phi_{t,s}} v(M_{t,s}/\phi_{t,s})}{B \sum_{\sigma} \pi_{\sigma} \frac{1}{\phi_{t+1,\sigma}} v(M_{t+1,\sigma}/\phi_{t+1,\sigma})} = \frac{i_{t+1}}{i_{t+2}} (1+i_{t+2}) \quad (16)$$

$$(iii) \quad \frac{P_{Ct}}{P_{Mt}} = \frac{\sum_s \pi_s v(C_{t,s})}{\sum_s \pi_s \frac{1}{\phi_{t,s}} v(M_{t,s}/\phi_{t,s})} = \phi_t \frac{1+i_{t+1}}{i_{t+1}} \quad (17)$$

At first sight it appears that we have a seriously underdetermined system here. Let us adopt the same "steady-state" (proportionate growth) assumption as before to assure that  $i_{t+1} = i_{t+2} = \bar{i}$ , cutting down the number of variables by one. But we still have the entire sets of contingent realized price levels  $\phi_{t,s}$  at date- $t$  and  $\phi_{t+1,\sigma}$  at date- $t+1$  as additional variables to solve for, in comparison with the certainty model of equations (10) through (12). However, the situation is rescued once we specify that our "steady-state" assumption will embody also the property which we call "forward-looking-constancy" (FLC). This means that, regardless of whatever state obtains at any date- $t$ , the economy looking forward must (in terms of prospective proportionate change) be the same. To illustrate, we might have the FLC property under the following conditions: Regardless of whatever state occurs at date- $t$ , the next date's corn crop will (say) with equal probability either stay constant or grow 10% -- while the nominal-money magnitude will (say) with equal probability either stay constant or grow 100%. It follows from FLC that the realized nominal money magnitude at any date- $t$  cannot affect utility. (Note that this implies

the strict quantity theory as between alternative money-risk-determined states at date-t.) That is, at any date the ratio  $M_{t,s}/\phi_{t,s}$  will be a constant over  $s$  for all those states where  $M_t$  varies for given  $C_t$ . For such states we may write:

$$m_t = \frac{M_{t,1}}{\phi_{t,1}} = \dots \quad \text{and} \quad m_{t+1} = \frac{M_{t+1,1}}{\phi_{t+1,1}} = \dots \quad (18)$$

On the other hand, for states determined by real (consumption) risk,  $C_t$  and therefore  $\phi_t$  may vary even for given  $M_t$ . This suggests, as will indeed turn out to be the case, that it may be simpler to analyze cases of "money risk only" than cases of "real risk only" -- not to mention the still more difficult combined-risk cases.

#### 4. SOLUTION WITH LOGARITHMIC UTILITY

To actually obtain a solution, in general the best we can do is to arrive at a numerical result, by solving recursively starting with given (realized) corn and nominal-money endowments at some specified date -- say, date-0. For a logarithmic utility function as in (19) below however, the equations reduce to simpler form permitting an analytic solution. The analogs of equations (15) through (17) when we have also allowed for stationarity in the FLC sense are:

$$\frac{\sum_s \pi_s / C_{t,s}}{B \sum_\sigma \pi_\sigma / C_{t+1,\sigma}} = \frac{\phi_t}{\phi_{t+1}} (1+i) \quad (15')$$

$$\frac{\sum_s \pi_s / M_{t,s}}{B \sum_\sigma \pi_\sigma / M_{t+1,\sigma}} = 1+i \quad (16')$$

$$\frac{\alpha \sum_s \pi_s / C_{t,s}}{\beta \sum_s \pi_s / M_{t,s}} = \phi_t \frac{1+i}{1} \quad (17')$$

Notice that all the contingent realized price variables  $\phi_{t,s}$  and  $\phi_{t+1,\sigma}$  have cancelled out. Then, as in the certainty case the first two equations can be solved for the interest rate and the ratio of price-level change. Given initial realized values for corn and nominal-money endowments at some date-0, the third equation then permits solving for the absolute price levels.

As a first thought-experiment suppose the corn crop is not random but grows at a fixed proportionate rate  $g$  over all time. This of course determines a stationary real interest rate  $r_t = r_{t+1} = r$  over all time (from equation (10), and making use of the Fisher equation). Suppose in addition that the risk on the money side takes a very special form. To wit, at any date the monetary authorities will choose one of two possible proportionate expansions of nominal money balances, with conditional probabilities unchanging over time. Evidently, this satisfies the "forward-looking-constancy" (FLC) condition. That is, regardless of whatever has actually been realized up to and at any date- $t$ , looking forward from that date the entire economy still looks the same in terms of prospective proportionate changes.

#### Numerical Example 2 (Risk on the money side only)

We will continue to use here the logarithmic preference-scaling function of N.E.1. As indicated in the text, this permits a simplification of equations (15) through (17) beyond that produced by the steady-state condition. Specifically, we will use the steady-state condition

and of course the FLC assumption, with logarithmic utility in the form:

$$v_t = \log_e [C_t^\alpha (M_t / \phi_t)^\beta] \quad (19)$$

where  $U = \sum_t B^t v_t$ .

Concretely, the logarithmic utility function of N.E.1 corresponds to  $\alpha = .4$  and  $\beta = .1$ . We let  $B = .8$  as before. Let us assume that the (non-stochastic) growth rate  $g$  for corn endowments is 100% -- that is, corn supplies double each period, so that  $C_{t+1}/C_t = 1 + g = 2$ . As for nominal-money, suppose that with equal probability at any date it will either remain unchanged or double. Let us express these as alternative growth rates  $\mu' = 0$  in the first case and  $\mu'' = 1$  in the second case.

Then for the (non-stochastic) equation (15') we have:

$$\frac{\alpha/C_t}{B\alpha/C_{t+1}} = \frac{1}{.8} \frac{C_{t+1}}{C_t} = \frac{2}{.8} = 2.5 = \frac{\phi_t}{\phi_{t+1}} (1+i)$$

We can get a direct solution of (16') under the special assumptions here by noticing that to every single state  $s$  at date- $t$  there will be exactly two associated (assumed equally probable) states at date- $t+1$ . Call these  $s'$  (the state associated with  $\mu'$ ) and  $s''$  (the state associated with  $\mu''$ ). We now no longer need  $\sigma$  to index states at date- $t+1$ , but can use the  $s$ -index at both dates. Then:

$$\frac{\frac{\beta \Sigma \pi_s / M_{t,s}}{s}}{\frac{\pi_s / 2}{(1+\mu') M_{t,s}} + \frac{\pi_s / 2}{(1+\mu'') M_{t,s}}} = \frac{\frac{\Sigma \pi_s / M_{t,s}}{s}}{\frac{B}{2} \left[ \frac{1}{1+\mu'} + \frac{1}{1+\mu''} \right] \Sigma \frac{\pi_s}{M_{t,s}}}$$

$$= \frac{1}{\frac{B}{2} \left[ \frac{1}{1+\mu'} + \frac{1}{1+\mu''} \right]} = 1+i \quad (16'')$$

Or, numerically:

$$\frac{1}{\frac{.8}{2} \left[ \frac{1}{1} + \frac{1}{2} \right]} = \frac{1}{.4(1.5)} = \frac{1}{.6} = 1+i$$

Thus (16''), the version of (16') applicable to the special assumption here, permits solution all of itself for the nominal interest rate  $i = 2/3$ . Substituting numerically back in the previous equation, we see that  $\phi_t / \phi_{t+1} = 3/2$  or  $1+i = 2/3$ . (That is, the price level is shrinking by  $1/3$  each period -- whereas in the previous Numerical Example 1, with no money-supply growth at all, it was halving each period.)

Finally, we want to obtain the absolute price-level from equation (17'), and we would also like to know the contingent realized prices  $\phi_{t,s}$ . These will depend upon the entire history of the system, even in the logarithmic-utility case. Let us deal only with the first two periods, i.e., let  $t = 0$  and  $t+1 = 1$ . We will also need to specify a starting-point at date-0 (i.e. date-0 is non-stochastic). Specifically, assume that  $C_0 = 2$  and  $M_0 = 10$ . Then, by successively applying equation (17') we obtain:

$$(a) \quad \frac{P_{C0}}{P_{M0}} = \frac{.4}{\frac{.1}{10}} = 20 = \phi_0 \frac{1+i}{1} = \phi_0 \left( \frac{5}{2} \right)$$

Thus,  $\phi_0 = 8$ .



$$(b) \frac{P_{C1}}{P_{M1}} = \frac{.4}{.1} \frac{\frac{1}{4}}{\frac{1}{2} + \frac{1}{2}} = \frac{40}{3} = \phi_1 \frac{1+i}{1} = \phi_1 \left(\frac{5}{2}\right)$$

Thus,  $\phi_1 = \frac{16}{3}$ .

As a check, we can verify the Fisher equation:

$$(1+r)(1+a) = (1+\frac{3}{2})(\phi_1/\phi_0) = (\frac{5}{2})(\frac{2}{3}) = \frac{5}{3} = 1+i.$$

We also want the contingent realized prices at date-1. Because of the FLC assumption, the forward-looking economy still looks the same whatever state is realized at that date, and hence (17') may still be used -- simply substituting the appropriate realized numerical values for  $C_{1,s}$  and  $M_{1,s}$ .

(c) If  $s = s'$  at date-1:

$$\frac{P_{C1,s'}}{P_{M1,s'}} = \frac{.4(\frac{1}{4})}{.1(\frac{1}{10})} = 10 = \phi_{1,s'} \frac{1+i}{1} = \phi_{1,s'} \left(\frac{5}{2}\right)$$

Thus,  $\phi_{1,s'} = 4$ .

(d) If  $s = s''$  at date-1:

$$\frac{P_{C1,s''}}{P_{M1,s''}} = \frac{.4(\frac{1}{4})}{.1(\frac{1}{20})} = 20 = \phi_{1,s''} \left(\frac{5}{2}\right)$$

Thus,  $\phi_{1,s''} = 8$ .

Now we can verify numerically that:

$$\frac{1}{\phi_1} = \frac{3}{16} = E\left(\frac{1}{\phi_{1s}}\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{8}\right)$$

Thus, in this Numerical Example, the "harmonic" version (3) would be correct

for expressing the Fisher equation in terms of realizations of contingent price levels.

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We need to examine, of course, whether this is only an accidental result of our special numerical assumptions, or a general property of models with logarithmic utility functions as in (18).

More generally, with both money and real risk, from equation (17') we can write the following equations for the forward price-level  ${}_0\phi_1$  and the contingent realized price-levels  ${}_1\phi_{1s}$ :

$${}_0\phi_1 = \frac{i}{1+i} \frac{\alpha E(1/C_{1s})}{\beta E(1/M_{1s})} \quad (20)$$

$${}_1\phi_{1s} = \frac{i}{1+i} \frac{\alpha(1/C_{1s})}{\beta(1/M_{1s})} \quad (21)$$

It is then elementary to verify the following "Expectational Rules":

(A) With logarithmic utility<sup>1</sup> and <sup>FLC</sup> money risk only, the numerators in both (20) and (21) involve a constant  $C_1$  instead of stochastically varying  $C_{1s}$ , and so:

$$\frac{1}{{}_0\phi_1} = E\left(\frac{1}{{}_1\phi_{1s}}\right) \quad (22)$$

(B) With logarithmic utility and real risk only, the denominators in both

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<sup>1</sup>Of course, we are maintaining our other special assumptions: that all individuals are identical, that utility is additive over time, and "forward-looking constancy" FLC.

(20) and (21) involve a constant  $M_1$  instead of stochastically varying  $M_{1s}$ , and so:

$$\phi_1 = E(\phi_{1s}) \quad (23)$$

So, quite generally for logarithmic utility functions, in the realization version of the Fisher equation we need to use the expectational formula of equation (2) if the risk is on the real side only but the inverse-expectational or harmonic formula of equation (3) if the risk is on the money side only. It will be quite evident that, in all mixed-risk cases, neither formula will be exactly correct.

#### 5. EXPECTATIONAL RULES WITH GENERAL UTILITY FUNCTION

Let us now see whether we can obtain similarly satisfying results with a general utility function.

We start with the case of money risk only. Then, using the stationarity (FLC) assumption as before, from (17) the forward price level  $\phi_1$  can be written:

$$\phi_1 = \frac{i}{1+i} = \frac{v(C_1)}{E \frac{1}{\phi_{1s}} v(M_{1s}/\phi_{1s})} = \frac{i}{1+i} \frac{v(C_1)}{v(m_1)E(1/\phi_{1s})} \quad (24)$$

We do not have any more the nice cancelling out of  $\phi_{1s}$ . But note that, in the denominator of the last expression, under the "money-risk-only" assumption the ratio  $M_{1s}/\phi_{1s}$  is a constant  $m_1$  regardless of state, as in (18). This is what enabled us to pass  $v(m_1)$  outside the expectation operator. Then, as it stands, equation (24) is already a relation between  $\phi_1$  and  $E(1/\phi_{1s})$  that can be re-written as:

$$\frac{1}{\phi_1} = \frac{1+i}{i} \frac{v(m_1)}{v(C_1)} E\left(\frac{1}{\phi_{1s}}\right) \quad (24a)$$

Note here that the ratio  $v(m_1)/v(C_1)$  is the Marginal Rate of Substitution between two certainties -- a trade-off between deterministic real consumption at date-1 and deterministic real balances at date-1. Then, since the "price of a unit of real balances at date t" can be written as  $P_{Mt}/\phi_t$ , it follows from (12) that -- given FLC --  $v(m_1)/v(C_1) = i/(1+i)$ .

So Proposition (A) that held for logarithmic utility can be generalized into the much stronger version:

(A\*) For any utility function<sup>1</sup> in a world of money risk only, the inverse-expectational or harmonic formula (given "forward-looking constancy") holds in the form:

$$\frac{1}{\phi_1} = E\left(\frac{1}{\phi_{1s}}\right) \quad (25)$$

There will not, however, be a correspondingly nice (B\*) formula for the case of real risk only. The reason is that, in equation (24), real risk imports a stochastic term into both the numerator and the denominator of the equation corresponding to (24) above:

$$\phi_1 = \frac{i}{1+i} \frac{Ev(C_{1s})}{E \frac{1}{\phi_{1s}} v(M_{1s}/\phi_{1s})} \quad (26)$$

In this model, real risk has more complicated effects than money risk,

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<sup>1</sup>More precisely, any utility function meeting our maintained assumptions: additivity over time, and between corn and real balances. The assumption of identical individuals is also required, in general, to yield this result.

since  $\phi_{1s}$  will respond to the various levels of  $C_{1s}$  and  $M_{1s}/\phi_{1s}$  will not be a constant over states of the world defined by variations in availability of real consumption quantities.

## 6. CONCLUDING REMARKS

The question motivating this paper was how to adapt the Fisher equation --  $(1+i) = (1+r)(1+a)$  -- to a world of uncertainty. And, in particular, whether the equation should be written in either of the forms:

$$\text{Expectational: } 1 + i = (1+r)E(1+\tilde{a})$$

$$\text{Harmonic: } \frac{1}{1+i} = \frac{1}{1+r} E\left(\frac{1}{1+\tilde{a}}\right)$$

We found, first of all, that if the Fisher equation is interpreted in terms of ex-ante choices as of the current date, so that  $1+a = \phi_1/\phi_0$  where  $\phi_1$  here is the forward price-level, arbitrage enforces the original version of the equation. So the issue in question arises only for the ex-post Fisher equation, the equation defined in terms of realizations of  $\tilde{a}$  (or equivalently of  $\tilde{\phi}_1$ ). Then the "expectational" version above can be expressed as: Is the forward price level equal on average to the future price level? And the "harmonic" version can be read: Is the forward deflator equal on average to the realized deflator?

Our analysis showed that the problem cannot really be solved in a 2-date model, because such models mis-specify the "store-of-value" aspect of money. Our analysis employed an infinite-date model, with a number of important simplifying assumptions: all individuals are identical, pure exchange applies, utility at any date is additive over the two goods (corn and real balances) and is also additive over time, and most interestingly

there is "forward-looking constancy" (FLC) -- after the realized state at any date, the proportionate relations between present and future endowments look exactly the same as before.

Under these assumptions, we obtained the important results:

- (1) If risk is on the money side only, and of an FLC nature, then the harmonic version of the Fisher equation in terms of realizations is correct.
- (2) If risk is on the real side only, then no such simple result applies in general. However, in the special case of a logarithmic utility function the expectational version of the Fisher equation in terms of realizations is valid.

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