Extrapolative Investment Equilibrium*

by

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1. INTRODUCTION

This paper studies the strategic aspects of investment by firms in a single industry. An important part of this problem is the way in which firms forecast the future return on investment. In an open-loop equilibrium firms have perfect foresight but do not believe that rivals will respond to deviations from the equilibrium path even though it would be in their best interest to do so. This type of equilibrium has been studied by Flaherty [1]. In a closed-loop equilibrium firms have perfect foresight and realize that rivals will reoptimize in response to deviations. This is similar to Selten [14]'s notion of the "perfect" equilibrium of an extensive form game. It was first studied in a continuous time framework by Starr/Ho [17] and in the investment problem by Spence [16] and by Fudenberg/Tirole [5].

In this paper I am interested in closed-loop equilibrium without perfect foresight. I investigate equilibria in which firms use simple rules-of-thumb and in which it is not costly for them to do so. I note that other authors--Friedman [4], Laitner [10] and Smale [15]--have studied rules-of-thumb in similar problems but in a direction orthogonal to this paper.

The type of environment in which simple rules can be expected to work well for forecasting is one of gradual change--a world in which tomorrow is only slightly different than today. In section four I give a precise definition of gradual change. This imposes two restrictions on the environment. First, the exogenous data--the conditions of cost and demand--must permit gradual change. For example, if the marginal cost of investment is low, the return high and if the return changes erratically as the capital stock varies, then the future return can't be accurately forecast by simple extrapolation. Second, attention must be restricted to those equilibria which change gradually. Despite the data it may be desirable for each firm to choose an erratic investment plan, <u>given that its rivals are doing so</u>.

The idea that extrapolation rules should work well is captured by the notion of an ε-extrapolative equilibrium, which I define in section three. This is similar to Radner [12]'s notion of an ε-equilibrium: the gain to replacing the rule-of-thumb with perfect foresight can't exceed ε. In section five I use dynamic programming to characterize ε-extrapolative equilibria in an environment of gradual change.

To analyze choice among alternative rules I introduce costs of using them and a minimax criterion for choosing among them (section six). This enables me to characterize the circumstances favoring a particular rule. An important aspect of the gradual change environment is that simple rules have approximate uniqueness and dominance properties (section seven). This means that equilibrium paths aren't too sensitive to the types of rules firms use, and is important since there is a great deal of arbitrariness in singling out particular rules.

I also study extrapolative equilibrium paths when firms are identical. In section eight I explore conditions which cause asymmetries in initial conditions to lead to long run differences between firms. In section nine I study how steady state capital varies with the type of extrapolation used. Section ten concludes the paper.

2. THE MODEL

I study the strategic investment of a fixed set of firms $i=1,\ldots,N$ in a single industry. The capital stock of firm i is measured in logarithmic units and is denoted by $x^i \varepsilon 1R$. The vector of capital stocks of all firms is denoted by $x \equiv (x^i)_{i=1}^n \varepsilon 1R^n$. The technical artifice of measuring capital in logarithmic units eliminates the possiblity of entry and exit by forcing all firms to have positive levels of capital. This restricts attention to the behavior of existing firms in an industry free from threat of entry. The net investment of firm i is denoted by $y^i \varepsilon 1R$ so that $\dot{x}^i = y^i$.

The instantaneous profit of firm i is a function $\Pi^i(x,y^i)$ which depends only on the capital vector and firm i's investment rate. Thus investment externalities are ruled out. This isn't entirely realistic--if one firm is liquidating its capital stock by scrapping and selling it, rival firms are probably able to snap up the used capital at bargain-basement prices. While allowing "mild" externalities won't change my results, they would require an unreasonable increase in notation.

The instantaneous profit functions are assumed to smooth. Implicitly this presumes a unique product market equilibrium which depends smoothly on the capital stock. It is possible to impose restrictions on cost and demand which insure this. For a detailed analysis see Flaherty[3]. Partial derivatives are denoted by subscripts so that $\Pi^i_j \equiv \partial\Pi^i/\partial x^j$, $\Pi^i_j \equiv \partial\Pi^i/\partial y^i$, and so forth.

The profit function is assumed to be concave in y^i. In fact I make the stronger assumptions that for scalar constants $a > 0$ and $\lambda \geq 1$

$$\lambda a^{-1} \geq -\Pi^i_{yy} \geq a^{-1} . \tag{2.1}$$

Thus a^{-1} measures the curvature of Π^i with respect to y^i and λ measures the extent to which Π^i fails to be quadratic in y^i. Assumption (2.1) insures that there is a unique investment rate $\gamma^i(x,z)$ which equates the marginal cost of investing with the rate of return z; which solves the equation $\Pi^i_y(x,\gamma^i) + z = 0$.

I assume that all firms face a common interest rate. For notational convenience time will be measured in units such that the interest rate is one.

The assumptions of a common interest rate, that each firm has one kind of capital and that profits don't depend on time are all made for pedagogical reasons. Extension to more general cases is straightforward.

3. EXTRAPOLATION

The objects of choice by firms are contingency plans or closed-loop strategies $y^i = f^i(x)$ giving the investment rate as a function of the capital vector. I wish to examine the behavior of firm i holding the strategies of other firms fixed. An extrapolation rule for firm i is a real valued (smooth) function $P^{iG}(x)$, where G is used to distinguish between rules. It is firm i's estimate of the maximum present value it can get starting at x.

Firm i's estimated rate of return on investment is P_i^{iG} the rate of increase in present value in response to a change in the capital stock. It chooses its strategy to equate the estimated rate of return with the marginal cost of investment: it chooses f^{iG} so that $\Pi_y^i(x, f^{iG}) + P_i^{iG}(x) = 0$. By the concavity assumption (2.1) this has a unique solution

$$y^i = f^{iG}(x) = \gamma^i(x, P_i^{iG}(x)) \ . \tag{3.1}$$

A consequence of Bellman's principle and the concavity assumption is that if the estimate P^{iG} is correct then the extrapolative strategy f^{iG} is optimal.

Hereafter I fix a vector of rules P^{iG}, one for each firm. Corresponding to these are the strategies f^{iG} given in (3.1). Of equal importance is $\phi^{tG}(x)$ which is the capital vector of time t when the initial vector is x and firms play the strategies f^{iG}. It is called the capital flow. The capital flow, in other words, is the solution of the initial value problem

$$\frac{\partial \phi^{tG}}{\partial t} = f^G(\phi^{tG}(x)) \qquad \phi^{oG}(x) = x \ . \tag{3.2}$$

There is also the profit flow $\Pi^{iG}(x) \equiv \Pi^i(x, f^{iG}(x))$, and the present value function

(3.3) $$\hat{P}^{iG}(x) \equiv \int_0^{\infty} \Pi^{iG}(\phi^{tG}(x))e^{-t}dt$$

I can now define the notions of ε-optimality and ε-equilibrium. An ε-optimal extrapolation rule guarantees that the gain from switching to another extrapolation rule doesn't exceed ε regardless of initial conditions.

Definition (3.1): P^{iG} is ε-optimal against P^{jG} $j\neq i$ iff for every x and alternative rule P^{iH} ($P^{jH}=P^{jG}$ $j\neq i$) $\hat{P}^{iH}(x)\leq\hat{P}^{iG}(x)+\varepsilon$.

If all firms are using ε-optimal rules this is an ε-equilibrium

Definition (3.2): P^{iG} $i=1,\ldots,N$ is an ε-extrapolative equilibrium iff for all firms i P^{iG} is ε-optimal against P^{jG} $j\neq i$.

A 0-optimal rule and 0-equilibrium are called optimal and an equilibrium respectively.

An ε-equilibrium is a continuous time version of a concept due to Radner [12]. One interpretation is as a closed-loop equilibrium: each firm has rational expectations about its rivals' responses to deviations from the equilibrium path, but makes errors (no larger than ε) in computing its own optimum. Actually, this doesn't make too much sense: why if everyone has rational expectations do they suboptimize? An alternative interpretation, and one relevant to this paper, is that firms optimize fully, but make small expectational errors. This latter interpretation blurs the distinction between open and closed-loop equilibrium to some extent--an open-loop equilibrium can be an extrapolative equilibrium in which the expectational errors arise from the (incorrect) assumption that rivals won't respond to deviations.

I will be particularly interested in four specific rules. The null extrapolation rule is $P^{iN}\equiv 0$. This is useful for comparative purposes, for if a rule is any good at all it should at least outperform the null rule.

The <u>myopic extrapolation rule</u> is to forecast future profit equal to today's profit under the assumption that all firms' strategies are derived from following the null extrapolative rule; to set

$$P^{iM}(x) \equiv \int_o^\infty \Pi^{iN}(x)e^{-t}dt = \Pi^{iN}(x) \equiv \Pi^i(x,f^{iN}) \tag{3.4}$$

where recall from (3.1) that $f^{iN} \equiv \gamma^i(x,P_i^{iN})$. Note that the null and myopic rules are really open-loop concepts--firms do not consider each other's responses.

The <u>linear extrapolation rule</u> is to forecast future profits by fitting a linear trend under the assumptions that all firms follow the myopic rule. Denote the time rate of change of Π^{iG} by $\Pi_t^{iG} \equiv \sum_j \Pi_j^{iG} f^{jG}$. The linear rule is

$$P^{iL}(x) \equiv \int_o^\infty [\Pi^{iM}(x) + \Pi_t^{iM}(x)t]e^{-t}dt = \Pi^{iM}(x) + \Pi_t^{iM}(x) \ . \tag{3.5}$$

This is not an open-loop concept: firms explicitly consider that they have some effect on rivals' capital.

Both the myopic and linear rules are based soley on the use of <u>local</u> information--low order taylor expansions of the exogenous data. An important part of what follows will be the introductions of assumptions which guarantee that local information is indeed relevant in the proximate future.

Finally, the <u>perfect foresight</u> rule serves as an upper bound on how accurate extrapolation can be. It is denoted by P^{iF} and characterized by the condition that $P^{iF} = \hat{P}^{iF}$.

4. DEFINITION OF GRADUAL CHANGE

Simple extrapolation rules do not work well in all environments. They ought to work in an environment which changes only gradually, however. An example clarifies what I mean by this. Suppose that profits are linear in x; that

$$\Pi^i(x,y^i) = \sum_j \Pi^i_j x^j + \Pi^i(0,y^i) \tag{4.1}$$

where the Π^i_j are fixed constants. This linear world has an important property: the decision problem of firms do not depend on the initial capital vector. This is because $\Pi^i(x_0+\Delta x,y^i)$ and $\Pi^i(x_1+\Delta x,y^i)$ differ only by the fixed constant $\sum_j \Pi^i_j(x^j_0-x^j_1)$, and adding a constant to the objective function doesn't change the decision problem.

In an environment like this it is natural to look for <u>static</u> equilibria in which investment plans don't depend on the capital vector, in which $f^i(x)=f^i$ a constant. It is easy to prove that there is a unique static equilibrium given by

$$f^i = \gamma^i(0,\Pi^i_i) \; . \tag{4.2}$$

I call the static equilibrium in a linear model an environment of <u>no change</u>, for given the strategies of rival firms tomorrow looks exactly like today. An environment of no change requires two restrictions: the profit functions have the linear form (4.1) <u>and</u> the equilibrium is static.

Although I don't know if non-static equilibria can exist in this model, the restriction to static equilibria is not innocuous. The linear game is similar in certain respects to a repeated matrix game with a one-period dominant strategy equilibrium (such as the prisoners' dilemma). In both cases the decision problem of firms is independent of history and the current decisions of rival firms. John Bryant [2] has pointed out that firms repeatedly playing their dominant strategies no matter what is the unique static equilibrium in the matrix case. However, repeated games also have non-static perfect equilibria as demonstrated by Rubinstein [13], for example. In the current context only static equilibria have the no change property. They involve considerably less guesswork and computation than other equilibria. Myopic and linear extrapolation work perfectly as do any other extrapolation rules based on fitting a reasonable class of functions to the data. This cannot be true in other than an environment of no change.

Any function looks locally like the linear form (4.1). The curvature of Π^i measures how rapidly this local picture changes as x changes. If the net investment rate is small relative to this curvature then the local picture is valid in the proximate future, while discounting implies that firms don't much care about the distant future. This is what I mean by an environment of gradual change: intuitively, tomorrow looks only a little different than today.

To make this precise requires that the net investment rate and curvature of the present value function be bounded. First, a bound on the return to net investment (measured myopically) is given in

Definition (4.1): The maximum absolute return, or return, is

$$R \equiv \sup|\Pi^i_j(x,y^i)| + \sup|\Pi^i_y(x,0)|$$

The second term is the marginal cost of maintaining the capital stock at its current level.

Next I give a bound on how the return changes in response to the capital vector

<u>Definition</u> <u>(4.2)</u>: The <u>curvature</u> of the Π^i is

$$\kappa \equiv \sup \{ |\Pi^i_{jk}|, |\Pi^i_{yk}| \}$$

The net investment rate is roughly aR where $a \equiv \sup|\Pi^i_{yy}|^{-1}$ was defined in (2.1). Roughly, gradual change requires that $aR\kappa$ is small. To give an exact definition let $0 < \beta < 1$ and $B \equiv 1/(1-\beta)$ be scalar constants. These numbers will serve as fixed bounds through the rest of the paper, β being smaller than the interest rate (which is one) and B being less than infinity. Also define the constants

(4.3) $$C^n_{43} \equiv B^2[(n+1)!]^{n+5}$$

<u>Definition</u> <u>(4.3)</u>: The vector of pairs (Π^i, P^{iG}) $i = 1,\ldots,N$ is an <u>environment</u> <u>of</u> <u>gradual</u> <u>change</u> (of order m) provided

(A) $\lambda \leq B$

(B) $\sup\limits_{a_i \in \{1,\ldots,N,y\}} |\Pi^i_{a_1,\ldots,a_n}| \leq (B-1)\kappa \quad 3 \leq n \leq m$

(C) $$\sup |P^i_{j_1,\ldots,j_n}| \leq (B-1) \begin{cases} R & n = 1 \\ \kappa & n = 2 \\ \sup\limits_{a_i \in \{1,\ldots,N,y\}} |\Pi^i_{a_1,\ldots,a_n}| & 3 \leq n \leq m \end{cases}$$

(D) $aR \leq B-1$

(E) $C^m_{43}\, a\kappa \leq \beta$.

Part (A) is merely a convenience - it requires that λ defined in (2.1) be smaller than B. Part (B) requires that the Π^i have a high degree of differentiability and that higher order derivatives are not greatly in excess of the second derivatives. This is essential if extrapolation based on local information is to work. Actual markets are not especially smooth and have all kinds of random fluctuations. As a practical matter extrapolation must be based on averaging out these fluctuations. I regard the Π^i as the approximation that results after fluctuations have been smoothed away.

The requirement that third (or higher) derivatives can't be much larger than the second derivatives means that the second derivatives can't fluctuate rapidly between their upper and lower bounds. For example, the function $\alpha \sin (x_1 \alpha^{-.4})$ has small second and large third derivatives when α is small. Functions like this with ripples in their second and higher order derivatives are ruled out. It seems reasonable that when firms smooth out random fluctuations to obtain the approximations Π^i they smooth out ripples at the same time.

Part (C) requires that the derivatives of the P^{iG} don't greatly exceed those of the Π^i. This is a restriction on the equilibrium which insures that only equilibria of gradual change are considered. Since it isn't true that optimal strategies are necessarily smooth it remains to be seen whether ε - equilibria of gradual change exist for small ε. However, if rivals play strategies which aren't smooth, extrapolation isn't going to work.

Part (D) like the other parts requires that derivatives be bounded at infinity. For a model of a stationary market of finite extent I think this is acceptable. Otherwise, like part (A), part (D) is merely a convenience.

Part (E) requires $a\kappa$ be small and rules out many cases of possible interest. It is needed in proving that the present value function is differentiable, and it seems sensible to assume at least enough gradual change to make the mathematics work out.

How gradual is change in an environment of gradual change? In computing this it is useful to let C's denote constants which may depend on B, but on nothing else. The subscript indicates the equation in which the constant first appears so that C^n_{43} appears first in equation 4.3, for example. Each constant is a complicated polynomial in B (with positive integer coefficients) and I won't write down its exact value unless it has some special significance.

Lemma (4.1): In an environment of gradual change

$$(4.4) \qquad |f^{iG}| \le C^1_{44}\, aR$$

$$|f^{iG}_{j_1,\ldots,j_n}| \le C^{n+1}_{44}\, a\kappa \le \beta/2N \qquad 1 \le n \le m-1$$

$$(4.5) \qquad |\Pi^{iG}_j| \le C^1_{45}\, R$$

$$|\Pi^{iG}_{j_1,\ldots,j_n}| \le C^n_{45}\, \kappa \qquad 2 \le n \le m-1$$

$$(4.6) \qquad |\Pi^{iG}_{tj_1,\ldots,j_n}| \le C^n_{46}\, aR\kappa \qquad 1 \le n \le m-2$$

Equation (4.4) shows how gradual change restricts the extrapolative strategies. It implies that the capital flow ϕ^{tG} exists for all $-\infty < t < \infty$ and grows no faster than linearly.

Equation (4.5) shows that the profit flow $\Pi^{iG} = \Pi^i(x,f^{iG})$ has derivatives with bounds similar to those on the Π^i.

Finally equation (4.6) shows that $aR\kappa$ is indeed a measure of how gradual change is. For example, when n=1, it says that the time rate of change of the (myopic) rate of return is $C^1_{47}\, aR\kappa$.

Since lemma (4.1) is as plausible as it is tedious to compute the various bounds, I omit the proof.

An important consequence of (4.4) (which follows from part (E) of gradual change) is that the present value function is twice continuously differentiable.

Corollary (4.1.1): In an environment of gradual change

(4.7) $|P_j^{iG}| \leq C_{47}^1 R$ for $m \geq 2$

$|P_{jK}^{iG}| \leq C_{47}^2 \kappa$ for $m \geq 3$.

Proof: I bound $\hat{P}_j^{iG}$; $\hat{P}_{jK}^{iG}$ is similar. From (3.3) the present value derivative is $\hat{P}_j^{iG} = \int_o^\infty [\sum_k \Pi_k^{iG}(\phi^{tG})\phi^{ktG}]e^{-t}dt$ provided that the integrand is absolutely integrable. By lemma (4.1) $|\Pi_k^{iG}| \leq C_{45}^1 R$ giving the bound $|\hat{P}_j^{iG}| \leq NC_{45}^1 R \int_o^\infty \sup_x |\phi_j^{ktG}|e^{-t}dt$. Let ϕ_x^{tG} be the matrix $\{\phi_j^{ktG}\}$. This is given by the variational equation $\phi_x^{tG} = \exp[\int_o^t f_x^G(\phi^{sG})ds]$, from which it can be computed that $|\phi_j^{ktG}| \leq \exp[t\ N \sup|f_j^i|]$. In other words $|f_j^i| \leq \beta/N$ is a sufficient condition that the derivative of the flow grows more slowly than the interest rate, and that $\int_o^\infty \sup_x|\phi_j^{ktG}|e^{-t}dt \leq B$. I note that the stronger condition in (4.4) that $|f_j^i| \leq \beta/2N$ is required to bound P_{jK}^{iG} since the integrand has a term of the form $|\phi_j^{ktG}|^2$. Q.E.D.

In the previous section I defined three simple rules: the null, myopic (in (3.4)) and linear (in (3.5)) rules. When are these rules consistent with gradual change? A direct computation shows that

Corollary (4.1.2): If $D^m \equiv B^{1/5}[(m+2)!]^{-2m-12}$ and

(A) $\lambda \leq D^m$

(B) $\sup_{a_i \in \{1,\ldots,N,y\}} |\Pi^i_{a_1,\ldots,a_n}| \leq (D^m-1)K$ $3 \leq n \leq m+1$

(D) $aR \leq D^m-1$

(E) $C_{43}^m\ a\kappa \leq \beta$

then for $G = N, M, L$ the vector (Π^i, P^{iG}) satisfies gradual change (of order m).

If the conditions of this corollary are satisfied I will say that Π^i is <u>almost linear</u> (to order m).

It is essential to know that there are interesting models that are almost linear. Obviously the linear form (4.1) satisfies this since $\kappa = 0$. There is an instructive technique for constructing more general Π^i which are almost linear. Suppose that $\Pi^i(x,y)$ has bounded derivatives (except Π^i_y) and satisfies the concavity assumption (2.1). Many such functions exist. Then the function

$$(4.8) \qquad \Pi^i(x,y^i) - \alpha^{-1}(y^i)^2 \qquad \alpha > 0$$

satisfies part (A)-(D) of gradual change for B sufficiently large, and has R, κ bounded above independent of α. However, by taking α small enough, a can be made as small as desired so that part (E) is also satisfied. Thus for arbitrary Π^i and α sufficiently small (4.8) is amost linear. This has an economic interpretation: if the cost of <u>net</u> (not gross) investment is large enough the exogenous data is consistent with gradual change. This construction shows that profits can depend on the capital stock in virtually any manner desired. For example, (4.8), unlike the linear form (4.1), can have Π^i bounded above, which is an obvious restriction from an economic viewpoint.

5. CRITERIA FOR NEAR OPTIMALITY

To compare the merits of different extrapolation rules I give a bound on how badly they perform. Since Bellman's principle says that the perfect foresight rule $P^{iF} = \hat{P}^{iF}$ is optimal I develop the obvious corollary: the degree of optimality of a rule can be measured by how close P^{iG} and $\hat{P}^{iG}$ are.

Corresponding to the rule P^{iG} is the Hamiltonian

$$(5.1) \qquad \hat{H}^{iG}(x,y^i) \equiv \Pi^i(x,y^i) + \hat{P}_i^{iG}(x)y^i + \sum_{j\neq i} \hat{P}_j^{iG}(x)f^{jG}(x) .$$

giving total momentary profits: dividends plus capital gains. If the environment is one of order two gradual change $\hat{P}^i$ is continuously differentiable by corollary (4.1.1) and the Weierstrass formula asserts that the gain from using P^{iH} in place of P^{iG} starting at x is given by the time integral of the difference in momentary profits

$$(5.2) \qquad \varepsilon^{iG}(x,P^{iH}) = \int_0^\infty [\hat{H}^{iG}(\phi^{tH},f^{iH}) - \hat{H}^{iG}(\phi^{tH},f^{iG})]e^{-t}dt .$$

This is discussed, for example, in Young [18]. Let $\hat{f}^{iG}$ maximize $\hat{H}^{iG}$ and let $\hat{\phi}^{tG}$ be the corresponding capital flow. It follows from (5.2) that

$$(5.3) \qquad \sup\,[\hat{H}^{iG}(x,\hat{f}^{iG}) - \hat{H}^{iG}(x,f^{iG})] \geq \sup\,\varepsilon^{iG} \geq \int_0^\infty[\hat{H}^{iG}(\hat{\phi}^{tG},\hat{f}^{iG}) - \hat{H}^{iG}(\hat{\phi}^{tG},f^{iG})]e^{-t}dt$$

while repeated application of Taylor's remainder formula shows that for some y_x^i, $\bar{y}_x^i$

$$(5.4) \qquad \begin{aligned} &\hat{H}^{iG}(x,\hat{f}^{iG}) - \hat{H}^{iG}(x,f^{iG}) \\ &= -(1/2)\Pi_{yy}^i(x,y_x^i)\{[\Pi_{yy}^i(x,\bar{y}_x^i)]^{-1}(\hat{P}_i^{iG} - P_i^{iG})\}^2 . \end{aligned}$$

Since by (2.1) $\lambda a^{-1} \geq -\Pi_{yy}^i \geq a^{-1}$

$$(5.5) \qquad \varepsilon^{iG} \leq C_{55}\, a \sup\,(\hat{P}_i^{iG} - P_i^{iG})^2 .$$

To find a lower bound on the degree of optimality use (5.3) and (5.4) to compute

$$\text{(5.6)} \qquad \sup \varepsilon^{iG} \geq (1/2)\lambda^{-2} a \int_o^\infty |\hat{P}_i^{iG} - P_i^{iG}|^2 e^{-t} dt \ .$$

The possibility that $\hat{P}_i^{iG}$ and P_i^{iG} are far apart while their integral is close is ruled out by assuming gradual change of order three. In this case the greatest rate of decrease of $|\hat{P}_i^{iG} - P_i^{iG}|^2$ is

$$\text{(5.7)} \qquad 2 \sup |\hat{P}_i^{iG} - P_i^{iG}| \sum_j |\hat{P}_{ij}^{iG} - P_{ij}^{iG}||\hat{f}^{jG}| \leq C_{57}\ aR\kappa \sup |\hat{P}_i^{iG} - P_i^{iG}|,$$

which follows from lemma (4.1) and corollary (4.1.1). Thus at worst $|\hat{P}_i^{iG} - P_i^{iG}|^2$ declines linearly to zero at time given by its initial value divided by (5.7). This makes it possible to find as a lower bound on (5.6)

$$\text{(5.8)} \qquad \sup \varepsilon^{iG} \geq (C_{58} R\kappa)^{-1} |\hat{P}_i^{iG} - P_i^{iG}|^3 \ .$$

6. CHOICE AMONG EXTRAPOLATIVE RULES

The drawback of extrapolation is that it doesn't do as well as complete optimization. The advantage is that it requires less information and computation. I will now examine this tradeoff in more detail.

I will evaluate the choice by firm i among the rules G = N, M, L and F assuming that the rules of firms $j \neq i$ are held <u>fixed</u> at some P^{jG}. By S^{iG} I mean the present value of savings on information and computation when i uses the rule G in place of the perfect foresight rule F. Thus $S^{iF} \equiv 0$ and I assume $0 < S^{iL} < S^{iM} < S^{iN}$ so that less complicated rules are cheaper. The null rule N which requires no computation saves the most, the myopic rule which requires the firm know only its own profit function saves less, and the linear rule requiring knowledge of all profit functions saves the least. Note that the units in which the profits Π^i are measured are now fixed.

I also assume that firm i's knowledge about the loss it incurs when it uses G in place of perfect foresight takes the form of an upper bound $\bar{\varepsilon}^{iG}$. Naturally $\bar{\varepsilon}^{iF} = 0$. Given this information the firm uses a minimax criterion to evaluate rules - it chooses the rule for which the greatest possible loss is least. It chooses, in other words, the rule for which $\bar{\varepsilon}^{iG} - S^{iG}$ is smallest.

An upper bound on the loss to using a rule can also be computed from the results of previous sections. To complete the model I assume rational expectations - the firm uses the same bounds generated by my computations. To do so the firm requires <u>some</u> global information: it must know a, R and κ. However, it seems reasonable to suppose that it is easier to estimate upper bounds on profit derivatives from past experience than to figure out the entire profit functions.

There are legitimate criticisms of this model of choice among extrapolation rules. The firm may try to pin down the loss to using a rule more precisely than an upper bound: it may try to find a lower bound, or probability distribution over the loss. It may also be able to obtain more refined bounds than my crude computations. However, the model is pedagogically useful: it brings out the environmental considerations which will tend to favor one rule over another.

First I will compare the rules N, M and F, leaving the linear rule L for later. About other firms $j \neq i$ I assume only that their rules satisfy gradual change of order m. The environment is assumed to be almost linear of order m, so when firm i plays either N or M (or L) by corollary (4.1.2) the total environment including i is one of gradual change of order m.

How badly does the null rule $P^{iN} \equiv 0$ do? For $m \geq 2$ by corollary (4.1.1)

$$(6.1) \qquad |\hat{P}_i^{iN} - P_i^{iN}| = |\hat{P}_i^{iN}| \leq C_{48}^1 R \, .$$

By (5.5) the corresponding loss is

$$(6.2) \qquad \varepsilon^{iN} \leq C_{55} a \sup |\hat{P}_i^{iN} - P_i^{iN}|^2 \leq C^N a R^2 \equiv \bar{\varepsilon}^{iN}.$$

If aR^2 is small enough that $S^{iN} - \bar{\varepsilon}^{iN} > 0$ the completely naive rule N dominates perfect foresight! This does make sense. When aR^2 is small profitability isn't very sensitive to what i does and he might as well minimize his informational and computational costs by using N.

The myopic rule M is given in (3.4) as $P^{iM} = \Pi^{iN}$. Using a first order taylor expansion it follows that for some $0 < s(t) < t$

$$(6.3) \qquad \hat{P}_i^{iM} = \int_0^\infty \sum_k \Pi_k^{iM} \phi_i^{kMt} e^{-t} dt = \Pi_i^{iM} + \int_0^\infty \sum_k \Pi_{kt}^{iG}(\phi^{Gs}) \phi_i^{kGt}(\phi^{Gs}) t \, e^{-t} dt,$$

that is the present value of profits equals the current level plus the integral of the difference between future and current profits (evaluated as a rate of change times time by the mean value theorem). The first term Π_i^{iM} is the direct effect of firm i's capital on its profits. This is (roughly) what is captured by the myopic rule. The second (integral) term is the indirect effect i's investment has on future profits by changing the investment plans of rivals. The key fact about gradual change with $m \geq 3$ is that (by lemma (4.1) and the proof of corollary (4.1.1)) the second term has a bound of order $aR\kappa$. When change ($aR\kappa$) is gradual enough the direct effect of i's investment is more important than the indirect effect on rivals' plans - the interactions between firms are small (they vanish in the linear/static case). This is the source of all substantive theorems in this paper. As a formal matter, it is easy to show

(6.4) $$|\hat{P}_i^{iM} - P_i^{iM}| \leq C_{46} aR\kappa$$

which combines with (5.5) to yield the loss

(6.5) $$\varepsilon^{iM} \leq C^M a^3 R^2 \kappa^2 \equiv \bar{\varepsilon}^{iM} \quad .$$

Thus, the myopic rule is in error only insofar as the rate of change of the system as measured by $a^3R^2\kappa^2$ is large. If this is zero, the future is static and myopic (static) expectations work perfectly.

Now let's compare N, M and F. The null rule is best iff

(6.6) $$S^{iN} - \bar{\varepsilon}^{iG} = S^{iN} - C^n aR^2 > \max \{ S^{iF} - \bar{\varepsilon}^{iF}, S^{iM} - \bar{\varepsilon}^{iM} \}$$
$$= \max \{ 0, S^{iM} - C^M a^3 R^2 \kappa^2 \} \ .$$

A sufficient condition for (6.6) is

(6.7) $$aR^2 < (S^{iN} - S^{iM})/C^N \ .$$

In other words, if your profit doesn't depend much on what you do, don't put effort into figuring out what to do.

Sufficient conditions for the myopic rule to be best are

$$(6.8) \qquad aR^2 > S^{iN}/C^n$$
$$a^2\kappa^2 < C^N S^{iM}/C^M S^{iN} \; .$$

The first condition says that it must pay to put some effort into figuring out what to do, the second that if the rate of return doesn't depend too heavily on what you do (κ small) there is no point in expending resources to figure out exactly how the rate of return varies.

An instructive way to compare N, M and F is to fix a and plot the values of R^2 and κ^2 for which each rule is best. As figure (6-1) shows small values of R favor N, small values of κ favor M and large values of both favor F.

Let us now examine the linear rule L given in (3.5) as $P^{iL} \equiv \Pi^{iM} + \Pi_t^{iM}$. Reasoning as in (6.3), but using a second order taylor expansion, shows that for $m \geq 4$

$$(6.9) \qquad |\hat{P}_i^{iL} - P_i^{iL}| \leq C_{69}[aR\kappa(aR + a\kappa) + a\kappa \sup |P_j^{jG} - P_j^{jM}| + aR \sup |P_{jk}^{jG} - P_{jk}^{jM}|]$$
$$\leq C^L C_{64}\, aR\kappa .$$

From (5.5)

$$(6.10) \qquad \varepsilon^{iL} \leq C^L C^M a^3 R^2 \kappa^2 = C^L \bar{\varepsilon}^{iM}$$

so that (compare (6.5)) in general the linear rule is worse than the simpler myopic rule. Remember - the linear rule anticipates opponents responses by assuming they are myopic. If opponents don't play myopically, or nearly so, this assumption is misleading and degrades the performance of the rule. However, if M works well so that $\bar{\varepsilon}^{iM}$ is small $C^L \bar{\varepsilon}^{iM}$ will also be small. In other words, even if opponents aren't myopic the firm stands to make no great loss if it assumes they are.

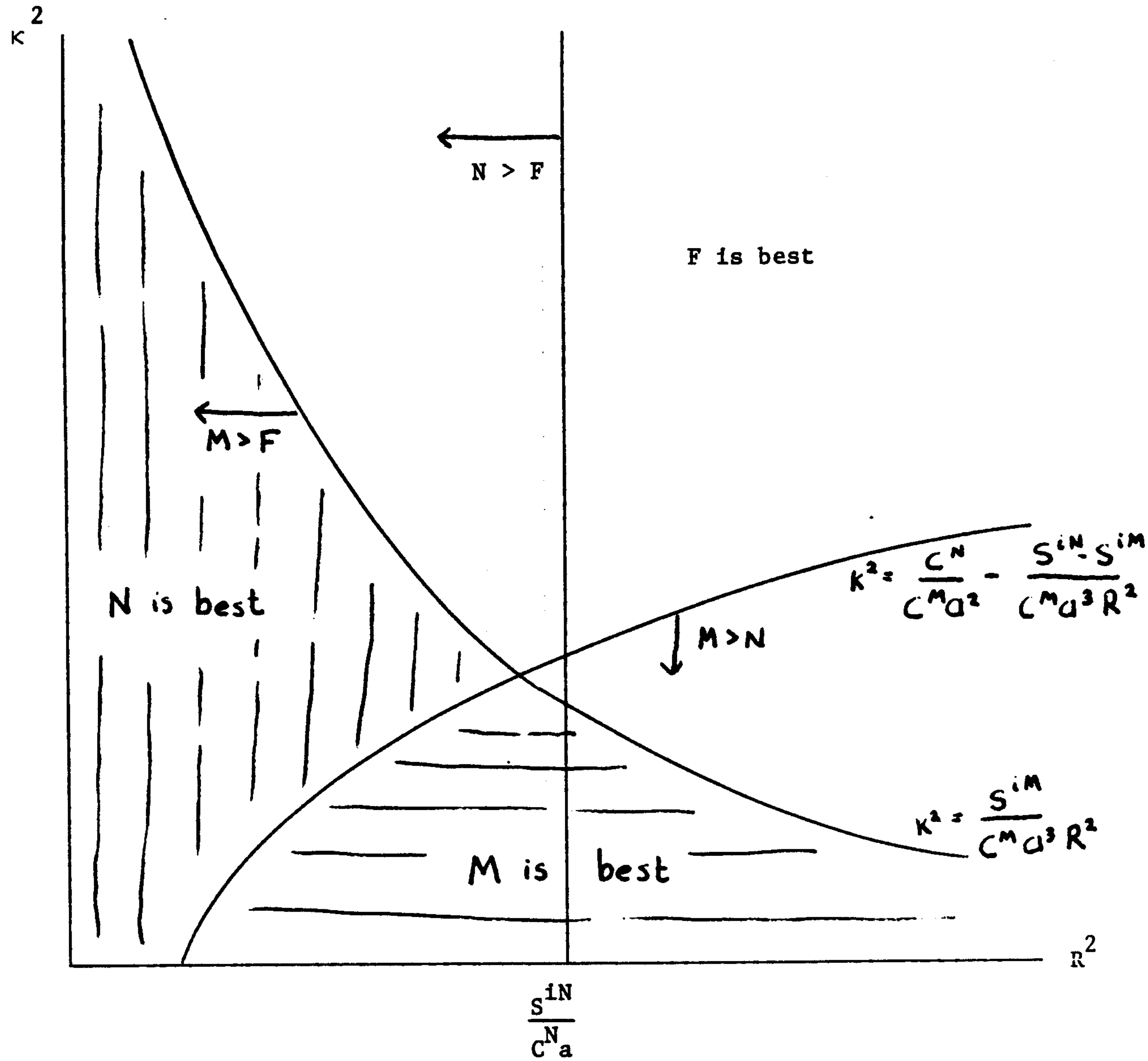

Figure (6-1): Choice Among the Rules N, M and F

Suppose on the other hand that opponents are using <u>either</u> the myopic rule <u>or</u> the linear rule. Lemma (4.1) implies

(6.11) $$|P_j^{jL} - P_j^{jM}| \leq C_{611}^1 aR\kappa$$

$$|P_{jk}^{jL} - P_{jk}^{jM}| \leq C_{611}^2 \kappa(aR + a\kappa)$$

and it follows from (6.9) and (5.5) that

(6.12) $$\varepsilon^{iL} \leq C^L a^3 R^2 \kappa^2 (aR + a\kappa)^2 \equiv \bar{\varepsilon}^{iL}$$

Thus, <u>if i is willing to assume</u> rivals are using either M or L (which I show in the next section isn't a bad assumption), he won't lose more than $\bar{\varepsilon}^{iL}$. Under this assumption when is L best? This is most easily answered by example: let $R = \kappa = a^{-3/4}$. Then $\bar{\varepsilon}^{iL} = 4C^L a^{1/2}$, $\bar{\varepsilon}^{iM} = C^M$ and $\bar{\varepsilon}^{iN} = C^N a^{-1/2}$. Thus for small enough a, L will be best. The relevant case, in other words, is R and κ large, but a^{-1} larger still.

7. EXISTENCE, DOMINANCE AND UNIQUENESS

I propose that ε-extrapolative equilibrium of gradual change is an appropriate solution concept in an almost linear environment. Some basic theoretical questions about these equilibria remain unanswered, however. One is the question of existence. I don't know whether almost linearity guarantees the existence of perfect foresight equilibria, let alone ones of gradual change (although this is true in the special case where the Π^i are quadratic). Are there at least ε-eqùilibria of gradual change for small values of ε? An immediate implication of (6.5) and (6.12) is that when a, R and κ are small the answer is affirmative.

Theorem (7.1)[Existence]: If the Π^i are almost linear of order m

(A) if $m \geq 3$ the P^{iM} are an ε^M-extrapolative equilibrium of gradual change of order m with $\varepsilon^M = C^M a^3 R^2 \kappa^2$.

(B) if $m \geq 4$ the P^{iL} are an ε^L-extrapolative equilibrium of gradual change of order m with $\varepsilon^L = C^L a^3 R^2 \kappa^2 (aR + a\kappa)^2$.

While theorem (7.1) asserts the existence of the type of equilibrium we are interested in it is not entirely adequate. Suppose opponents use rules which aren't M or L. Does this degrade the performance of M and L, or do they have dominance properties? Are there extrapolative equilibrium strategies with different qualitative features, or are they approximately unique? I consider the myopic case first.

Theorem (7-2)[Myopic Dominance and Uniqueness]: If the Π^i are almost linear of order $m \geq 3$ and the P^{iG} satisfy gradual change of order m then

(A) P^{iM} is ε^M-optimal against the P^{iG}

(B) if the P^{iG} are an ε^M-equilibrium $|f^{iG} - f^{iM}| < C^{MU} a^2 R\kappa$.

Part (A) is a direct consequence of (6.5) which required only gradual change by opponents. Myopic works equally well no matter how rational or irrational opponents, provided they move gradually. Note the nature of this result. The rule M loses no more than ε^M when compared with <u>any</u> alternative strategy and when <u>opponents are restricted</u> to rules of gradual change.

<u>Proof of (B)</u>: The lower bound on ε (5.8) implies

(7.1) $$\varepsilon^M = C^M a^3 R^2 \kappa^2 \geq (C_{58} R\kappa)^{-1} |\hat{P}_i^{iG} - P_i^{iG}|^3 .$$

As in (6.3) a first order taylor expansion shows

(7.2) $$\hat{P}_i^{iG} - P_i^{iG} = P_i^{iM} - P_i^{iG} + \Pi_i^{iG} - \Pi_i^{iN} + \int_o^\infty \sum_k \Pi_{kt}^{iG} \phi_i^{iGt} t\, e^{-t} dt.$$

Reasoning as in (6.4) and making use of (7.1) shows

(7.3) $$|P_i^{iG} - P_i^{iM}| \leq C_{73} aR\kappa .$$

Since $f^{iG} - f^{iM} = \gamma^i(x, P_i^{iG}) - \gamma^i(x, P_i^{iM})$ and $\gamma_Z^i = -(\Pi_{yy}^i)^{-1}$

the mean value theorem and the concavity assumption (2.1) imply (B). Q.E.D.

By lemma (4.1) gradual change implies only that $|f^{iG} - f^{iM}| \leq C_{44}^1 aR$. Thus if $a\kappa$ is small (B) means that ε^M-equilibria are closer to M, and thus each other, than they are to arbitrary rules of gradual change. In this sense ε^M-equilibria are approximately unique when $a\kappa$ is small.

I now consider the linear case.

Theorem (7.3)[Linear Dominance and Uniqueness]: If the Π^i are almost linear of order $m \geq 4$, the P^{iG} satisfy gradual change of order m and $\varepsilon^L \leq \varepsilon^M$

(A) P^{iL} is $C^L \varepsilon^M$-optimal against the P^{iG} .

(B) if the P^{iG} are an ε^M-equilibrium P^{iL} is ε^{LD}-optimal against the P^{jG} with $\varepsilon^{LD} = C^{LD} a^3 R^2 \kappa^2 (aR + a^2 \kappa^2)$.

(C) if the P^{iG} are an ε^L-equilibrium $|f^{iG} - f^{iL}| \leq C^{LU} a^2 R\kappa (aR + a\kappa)^{2/3}$.

Part (A) restates (6.10) and says that the linear rule does pretty well against anything gradual by opponents.

<u>Proof of part (B)</u>: Part (B) will follow from (5.5) and (6.9) provided that there are constants C_{74}, C_{75} with

(7.4) $$|P_j^{jG} - P_j^{jM}| \leq C_{74} aR\kappa$$

(7.5) $$|P_{jk}^{jG} - P_{jk}^{jM}| \leq C_{75} \kappa \sqrt{aR} \, .$$

In other words I must show that all ε^M-equilibria are close to M in both first and second derivative. Equation (7.3) implies (7.4) with $C_{74} = C_{73}$. Lemma (4.1) implies

(7.6) $$|P_{jk\ell}^{jG} - P_{jk\ell}^{jM}| \leq C_{76} \kappa \, .$$

I claim that (7.4) and (7.6) together imply (7.5): that the second derivative of a function can be bounded by its first and third derivatives (compare (5.7) and (5.8)). Consideration of directional derivatives shows that (7.5) follows from

Lemma (7.4): If $h: \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable with

$$|h| \leq H \text{ and } |h''| \leq H_2 \text{ then } |h'| \leq \sqrt{8HH_2} \, .$$

Lemma (7.4) can be verified from the fundamental theorem of calculus.

Q.E.D.

From this we see that when aR and $a\kappa$ are small and rivals are rational to the extent that they do at least as well as myopically firm i does better playing L than playing M. In this sense L is good even if opponents use good extrapolation rules other than M and L.

Proof of part (C): Equation (7.3) implies $|P_i^{iG} - P_i^{iL}| \leq 2C_{73}aR\kappa$ while as in (7.6) $|P_{jk\ell}^{jG} - P_{jk\ell}^{jL}| \leq C_{76}\kappa$. Thus by lemma (7.4)

(7.7) $$|P_{jk}^{jG} - P_{jk}^{jL}| \leq C_{77}\kappa\sqrt{aR} \, .$$

These inequalities, the lower bound (5.8), the analog of (7.2) using a second order taylor expansion, lemma (4.1) and the proof of corollary (4.1.2) imply the analog of (7.3)

(7.8) $$|P_i^{iG} - P_i^{iL}| \leq C_{78}\, aR\kappa(aR + a\kappa)^{2/3}$$

Q.E.D.

Thus when aR and aκ are small and $\varepsilon^L \leq \varepsilon^M$, ε^L-equilibria (and perfect foresight equilibria if they exist) are closer to each other than they are to M.

8. SYMMETRY AND ASYMMETRY

Extrapolative equilibria of gradual change exist and are approximately unique in almost linear environments. What do they look like? One way of examining this issue is to ask when firms, beginning with different capital stocks but otherwise identical, will differ in the long run. This is a question examined by previous authors studying strategic investment: Flaherty [3], Fudenberg/Tirole [5] and Spence [16].

Hereafter I assume all firms have an identical technology, so that $\Pi^i(x^1,\ldots,x^i,\ldots) = \Pi^1(x^i,\ldots,x^1,\ldots)$, for example. The Π^i are fixed smooth profit functions almost linear to order $m \geq 3$. I shall study the space of rules P^G with corresponding strategies f^G which satisfy

F1. The P^G are a smooth, symmetric ε^M-equilibrium of gradual change of order m.

Thus I limit attention to rules that do as well as M and in addition I assume that the errors of firms are symmetric, so that $P^i(x^1,\ldots,x^i,\ldots) = P^1(x^i,\ldots,x^1,\ldots)$ and so forth. This assumption is satisfied by M itself and by L if $\varepsilon^L \leq \varepsilon^M$.

F2. [boundedness] For some $\delta^G, Z^G > 0$

$$x^i \geq Z^G \quad \text{implies} \quad f^{iG} \leq -\delta^G$$
$$x^i \leq Z^G \quad \text{implies} \quad f^{iG} \geq \delta^G$$

This means that there is a capital stock so small that it will always be profitable to invest and so large that it will be profitable to disinvest. I shall examine later the economic and strategic issues in requiring (F2).

A <u>steady state</u> $\bar{x}^G$ is such that no firm following G will choose to move, such that $f^G(\bar{x}^G) = 0$. The variational matrix is

$$(8.1) \qquad A^G(x) \equiv \begin{bmatrix} f_1^{1G}(x) & \ldots & f_N^{1G}(x) \\ \vdots & & \\ f_1^{NG}(x) & \ldots & \end{bmatrix}.$$

F3. [regularity] At steady states x^G the real parts of eigenvalues of $A^G(x^G)$ don't vanish.

Arguments from differential topology show that under (F2) and (F3) there are only finitely many steady states. Furthermore, in a topological sense, "almost all" equilibria which satisfy (F1) and (F2) satisfy (F3), so that it should not be regarded as a stong restriction. Details on these types of argument can be found in Guillemin/Pollack [6] or Hirsch [7].

There are three questions about the symmetry and asymmetry of steady states which I wish to pose. First, what mathematical conditions on the f^G give rise to symmetry and asymmetry? Second, from what economic conditions do these derive? Finally, when do <u>all</u> equilibria in F share the <u>same</u> symmetric or asymmetric character?

To mathematically describe f^G requires index theory which is developed in Guillemin/Pollack [6] or Hirsch [7]. An <u>invariant</u> <u>manifold</u> V^G of f^G has $\phi^{tG}(x) \in V^G$ whenever $x \in V^G$ and $t \geq 0$. For example, by (F2), the set $Z^G \equiv \{x \mid |x^i| \leq Z^G\}$ is invariant as is the symmetry axis $S \equiv \{x \mid x^1 = x^2 = \dots = x^N\}$. If V^G is an invariant manifold and $x^G \in V^G$ is a steady state then the <u>index</u> of x^G in V^G $\mathrm{ind}(x^G,V^G)$ is the sign of the products of those eigenvalues of $A^G(x^G)$ which have generalized eigenspaces contained in the (complexification) of the tangent space to V^G at x^G. If $V^G = Z^G$ this reduces to the sign of the determinant of A^G since the tangent space to A^G equals all of $\mathbb{R}^N$. If $V^G = S$ then the tangent space to S is spanned by the symmetric vector $e \equiv (1,1,\dots,1)'$. When $x \in S$ it is easily shown that the eigenvalues of A^G are $f_i^{iG} + (N-1)f_j^{iG}$ with eigenspace spanned by e, and $f_i^{iG} - f_j^{iG}$ with eigenspace equal to the orthogonal complement of e. Thus $\mathrm{ind}\,(x^G,S) = \mathrm{sgn}\,[f_i^{iG}(x^G) + (N-1)f_j^{iG}(x^G)]$. The key theorem in index theory is

<u>Index Theorem</u>: If V^G is compact and convex with steady states x_k then

$$\sum_k \mathrm{ind}(x_k, V^G) = (-1)^{\dim(V^G)} .$$

<u>Corollary (8.1)</u>: There is a symmetric steady state stable with respect to <u>symmetric</u> changes in the initial conditions.

<u>Proof</u>: $V^G = S \cap Z^G$ satisfies the hypothesis of the index theorem and $\dim(V^G) = 1$. So there must be at least one steady state x^G with $\mathrm{ind}(x^G, V^G) = -1$. Above we computed that this implies at x^G $f_i^{iG} + (N-1)f_j^{iG} < 0$ which following Hirsch/Smale [8] is precisely the condition for the stability of the one dimensional dynamical system derived from restricting f^G to V^G.

Q.E.D.

If the symmetric steady state is actually to be stable then the second eigenvalue $f_i^{iG} - f_j^{iG}$ must also be negative: this eigenvalue corresponds to perturbations orthogonal to the symmetry axis. If conversely

(A) $\qquad f_i^{iG} - f_j^{iG} \geq 0 \quad$ on $\quad S \cap Z^G$

long run asymmetric behavior might be anticipated.

<u>Corollary (8.2)</u>: Under (A) all symmetric steady states are unstable and there are at least $2^N - 2$ asymmetric steady states.

<u>Proof</u>: Let q be a non-void proper subset of $\{1,\ldots,N\}$, and define $V^q = \{x \mid x^i = x^j \text{ if } i,j \in q \text{ or } i,j \notin q\} \cap Z^G$, that is, unless $i \in q$ and $j \notin q$ (or vice versa) $x^i = x^j$. These are two dimensional manifolds, invariant by the symmetry of f^G. If q and q' are complements $V^q \equiv V^{q'}$, if not $V^q \cap V^{q'} = S \cap Z^G$. Thus there are $2^{N-1}-1$ such manifolds meeting only on the symmetry axis. Each V^q satisfies the hypothesis of the index theorem. I will complete the proof

by showing each has at least two asymmetric steady states. Indeed, let x_k^S be the symmetric steady states and x_j^q the asymmetric steady states in V^q. By (A) $\text{ind}(x_k^S,S) = \text{ind}(x_k^S,V^q)$ since all the asymmetric eigenvalues of x_k^S are positive. From the index theorem since $\dim(V^q) = 2$

$$(8.2) \qquad \sum_k \text{ind}(x_k^S,S) = -1$$

$$(8.3) \qquad \sum_k \text{ind}(x_k^S,S) + \sum_j \text{ind}(x_j^q,V^q) = \sum_k \text{ind}(x_k^S,V^q) + \sum_j \text{ind}(x_j^q,V^q) = 1 .$$

Combining (8.2) and (8.3) gives $\sum_j \text{ind}(x_j^q,V^q) = 2$ meaning at least two asymmetric steady states in V^q.

Q.E.D.

Corollary (8.2) does not and cannot say anything about the stability of asymmetric steady states. Unfortunately not every equilibrium path is necessarily converging to a steady state: some may cycle or exhibit even more complicated asymptotic behavior (even generically). If, however, every path converges to a steady state and a generic technical condition due to Kupka-Smale and described in Irwin [9] is satisfied then

<u>Corollary (8.3)</u>: Under (A) and the condition above there at at least N stable asymmetric steady states.

<u>Proof</u>: By the Morse inequalities there is at least one stable steady state. By corollary (8.2) it isn't symmetric, and by interchanging the roles of firms there must be at least N.

Q.E.D.

We see then that condition (A) means that initial asymmetries will result in long run asymmetries. Roughly what is required is that f_i^{iG} should be large (positive) and that f_i^{jG} should be small (negative) along S: increases in i's capital should lead him to invest faster and his rivals more slowly.

What economic assumptions will lead to $f_i^{iG} - f_j^{iG} > 0$? Let us assume

Π1. $\Pi_y^i(x,\bar{\gamma}) = 0$ so that $\gamma^i(x,0) = \bar{\gamma}$

that is the optimal depreciation rate doesn't depend on x. Also

Π2. $\Pi_{yj}^i = 0 \qquad j \neq i$.

so that the cost to i of investing isn't affected by j's capital stock. Then compute

(8.4) $$f_i^{iG} = -[\Pi_{yy}^i]^{-1}(\Pi_{yi}^i + P_{ii}^{iG})$$

(8.5) $$f_j^{iG} = -[\Pi_{yy}^i]^{-1} P_{ij}^{iG} \qquad j \neq i .$$

Thus we see that f_i^{iG} will be large if Π_{yi}^i and P_{ii}^{iG} are large: if increases in i's capital lower his cost of investment and raise his rate of return. If $G = M$ then $P_{ii}^{iG} = \Pi_{ii}^i$ and this is just the myopic rate of return. Also f_j^{iG} will be small provided P_{ij}^{iG} is small: if increases in j's capital lower i's return on investment. All in all this makes good economic sense: if increases in i's capital improve his position to the detriment of his rivals then when he begins with a slight edge he will tend to reinforce it over time.

In fact if firms begin with unequal capital stocks the ranking of capital stocks must remain unchanged for all time. If not, then some firm i would have to catch up to some other firm j.creating a tie. Once tied by symmetry both firms capital would remain equal for all time. This would mean that the system would have to hit the invariant manifold where $x^i = x^j$ in finite time. But reversing the direction of time shows that no point on this manifold can be reached from outside it in finite time.

Suppose now that for some $\delta^M, Z^M > 0$

$$\Pi 3. \qquad x^i \geq Z^M \quad \text{implies} \quad \Pi^i_i \leq -\delta^M$$

$$x^i \leq Z^M \quad \text{implies} \quad \Pi^i_i \geq \delta^M$$

so that the return on capital drops to negative when the capital stock his large enough or positive when small enough. Since by (Π1) $f^{iN} = \bar{Y}$ a constant this implies (F2) for G = M. Suppose also that for some $\delta^{MA} > 0$

$$\Pi 4. \qquad \Pi^i_{yi} + \Pi^i_{ii} - \Pi^i_{ij} > \delta^{MA} \quad \text{for} \quad x \in S \cap Z^M, \delta^S$$

so that by (8.4) and (8.5) (A) is satisfied for G = M. In other words Π1–Π4 are sufficient conditions on the economic data that at least one ε^M-equilibrium of gradual change, namely M, satisfies also (F2) and the asymmetry condition (A). Can we conclude from this that all ε^M-equilibria of gradual change G exhibit asymmetric tendencies? Certainly, provided for some $0 < \mu < 1$ $|P^{iG}_i - P^{iM}_i| \leq \mu\delta^M$ and $|P^{iG}_{ij} - P^{iM}_{ij}| \leq \mu\delta^{MA}/2$. Looking back to (7.3) we see $|P^{iG}_i - P^{iM}_i| \leq C_{73}aR\kappa$ and from (7.5) $|P^{iG}_{ij} - P^{iM}_{ij}| \leq C_{75}\kappa\sqrt{aR}$. Putting this together we see that (Π1)–(Π4) and

$$\Pi 5. \qquad C_{73}\, aR\kappa \leq \mu\delta^M$$

$$C_{75}\,\kappa\sqrt{aR} \leq \mu\delta^{MA}/2$$

mean that all ε^M-equilibria of gradual change tend towards asymmetry. Note incidentally that gradual change alone forces δ^M of order R and δ^{MA} of order κ, but no more.

9. COMMITMENT AND CAPITAL

The preceding section examined the shared properties of all ε^M-equilibria of gradual change. This section shall instead contrast the myopic equilibrium with the linear and related equilibria.

As before all firms are identical - in addition I now assume they are identical in initial conditions. I shall only be interested in steady states on the symmetry axis S. Furthermore I assume strong properties of the profit functions:

C1. $$\Pi^i(x,y^i) = \Pi^i(x,0) + \Pi^i(0,y^i)$$

so that profits are additively separable in capital and investment. Profits equal gross profits $\Pi^i(x,0)$ minus investment cost $-\Pi^i(0,y^i)$. Equivalently the optimal investment rate $\gamma^i(x,z) = \gamma^i(z)$ depends only on the rate of return and not the capital stock.

C2. $$\Pi^i(0,y^i) \text{ is maximal at } y^i = 0$$

so that the optimal depreciation rate $\gamma^i(0) = 0$. Equivalently the null rule has the corresponding strategy $f^{iN} \equiv 0$ of not investing at all.

An important consequence of (C1)-(C2) is the implication it has for the myopic rule. Since $f^{iN} \equiv 0$, $f^{iM} = \gamma^i(\Pi^i_i(x,0))$ and x^M is a steady state if and only if $\Pi^i_i(x^M,0) = 0$. This is exactly the first order for the static Nash equilibrium of the one period simultaneous move game with strategies x^i and payoffs $\Pi^i(x,0)$. The myopic rule is a continuous version of the "behavioral" dynamic for this static game: it says to move in the direction that increases own profits taking opponents capital as given. Unlike the discrete time full optimization version of this rule, which makes little sense, the myopic dynamic emerges as almost optimal behavior when adjustment costs $\Pi^i(0,y^i)$ are large.

The final simplifying assumption is

C3. On the symmetry axis S $\Pi^i_{ii}, \Pi^j_{ji}, \Pi^j_i < 0$ and there exists a symmetric steady state x^M.

Thus increases in i's capital decrease everyone's rate of return and lower the absolute level of rivals' profits. The existence of a symmetric steady state can be guaranteed by a more primitive assumption such as (Π3) of the last section, if desired.

Is steady state capital larger or smaller when everyone plays L than when they play M? Since the change itself is small (when change is sufficiently gradual), by the myopic uniqueness theorem the steady state condition $f^L(x^L) = 0$ can be approximated by

$$(9.1) \qquad A^M(x^M)[x^L - x^M] + [f^L(x^M) - f^M(x^M)] \approx 0 \; .$$

solving for $x^L - x^M$ and using $f^M(x^M) = 0$ shows

$$(9.2) \qquad x^L - x^M \approx \left[\frac{\Pi^i_{yy}}{\Pi^i_{ii} + (N-1)\Pi^i_{ij}} \right] f^L(x^M)$$

where by (2.1) and (C3) the bracketed expression is positive. Furthermore, when $f^M = 0$,

$$(9.3) \quad f^{iL} = \gamma^i(\Pi^{iM}_i + \Pi^{iM}_{ti}) = \gamma^i((N-1)\Pi^i_j(-\Pi^i_{yy})^{-1}\Pi^i_{ij}) > 0$$

proving that the symmetric steady state capital is larger under L than under M. It is equally true that under L capital is larger than in static Nash equilibrium. Furthermore, by the linear uniqueness theorem, when $\varepsilon^L \leq \varepsilon^M$ and a, R and κ are all small all symmetric ε^L-equilibria of gradual change must have symmetric steady state capital greater than the static Nash level. It is easy to show under (C3) that this actually makes firms worse off. This, apparently, is perverse - the extra effort of firms to find the optimum by using ε^{L}-optimal rules in place of the ε^{M}-optimal rule M actually lowers their profits!

This result, however perverse it may seem, merely recognizes the nature of capital as commitment. Because it is costly to undo investment changing capital is credible as a threat. Assumption (C3) means that each firm can force its rivals to reduce their capital by committing itself to increasing its own capital. It is optimal to make this commitment, ceteris paribus, whether or not rivals are known to be making similar commitments. Under rule M corresponding to static Nash behavior firms do not recognize this opportunity to influence rivals' behavior. Under the rule L, or any other rule which does as well, they do, and by means of simultaneous commitment make each other worse off. This is reminiscent of the Stackleberg warfare discussed by Bishop [1]. It is even more closely related to the work on oligopoly with overlapping commitment done by Maskin/Tirole [11].

In general simultaneous commitment need not make all firms worse off - indeed commitment to strategies of punishment and reward will typically lead to pareto improvements. The rule L is important because, unlike M, it captures the incentives created by the possibility of commitment.

10. CONCLUSION

I think that perfect nash equilibrium can be legitimately criticized on three grounds. First, the costs of gathering the information required for optimization and of computing the optimum can be quite large. Second, if opponents make errors it is better to exploit those errors than to assume that they optimize - indeed a strategy based on optimization by opponents can do quite badly if they make mistakes. Third, if there are multiple equilibria how can a firm know which equilibrium strategy its rivals will play? On the other hand simple rules-of-thumb can be legitimately criticized on the grounds that if they do very badly there is a large incentive for firms to try and find better rules.

The conclusion of this paper is that in an environment of gradual change none of these objections have any force. Simple rules-of-thumb don't do badly, they do well. Consequently the cost of finding nearly optimal strategies is small. The dominance properties of near equilibrium strategies insures that there is little to be gained by exploiting opponents errors, while uniqueness means that there is no ambiguity about which equilibrium opponents will choose. Furthermore it doesn't matter whether open- or closed-loop equilibrium occurs. Because gradual change forces the effect of a firm's investment on rivals future capital to be less important than the effect on its own future capital the strategic interactions between firms is small. It is this which gives rise to such pleasant conclusions.

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