

THEORY OF PRICE BUBBLES\*

David Levine

UCLA

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UCLA Department of Economics  
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Despite the long fascination price bubbles have had for economists, their existence and nature remains elusive. A modern literature has explored whether bubbles are consistent with the rationality of traders and it has recently been shown by Tirole (1980) that they are not. A more traditional approach, which has recently found expression in the book of Kindleberger (1978), argues that bubbles do occur, but as a result of irrational behavior.

This paper uses modern analytic tools to study the traditional theory of price bubbles. The model is similar in spirit to that of Zeeman (1974).

Attention focuses on the interaction of two groups of traders:

fundamentalists, whose demand to hold the asset is based on a careful evaluation of its value, and a group of speculators or chartists who believe that changes in asset price are a signal of future price changes. This permits the possibility of bubbles during which speculators bid up prices, convinced that prices in the future will rise even higher.

The goal of this paper is to derive empirically testable propositions from the bubble hypothesis. Three propositions are derived, and data on stock prices from 1926-38 shows that actual price movements exhibit the qualities suggested by the theoretical analysis. Roughly there are two types of fluctuations that should occur if bubbles are present. Prices rise gradually, but fall suddenly when the bubble bursts. In addition significantly more runs of price increases should occur than predicted by random chance, and these should be related in a subtle way to the price history. These statements are given precise meaning in the analysis here.

The paper has five sections. The second section describes the assumptions of the bubble model. Section three describes the qualitative features of price movements in the model. Section four derives quantitative predictions of the model and verifies that the New York price series exhibits

these patterns. The final section sums up the findings of the analysis.

## 2. THE MODEL

This section develops a model of an asset market with two types of speculators. The objective is to give a description of the resulting price dynamics.

It is supposed that there are two types of assets, a riskless asset known to pay a rate of return  $\rho > 0$  in each time period, and a risky asset paying an uncertain return. The price of the riskless asset is normalized to one, while in period  $t$  the risky asset costs  $p^t$ . The "market fundamental" or anticipated present value of the risky asset in period  $t$  is  $V^t$ . There is no anticipated drift in the risky return, so that the return on the risky asset in period  $t$  is  $\rho V^t$ . The market fundamental moves according to the equation

$$(2.1) \quad v^{t+1} = v^t + \epsilon^{t+1}$$

where  $\epsilon^{t+1}$  is the unanticipated innovation in the asset's value due to new information about its future return. The random variables  $\epsilon^t$  are iid and their common distribution is symmetric around zero, and places positive probability on any sub-interval of the finite support  $[-\bar{\epsilon}, \bar{\epsilon}]$ .

All traders are risk neutral and know the market fundamental. There is a fixed finite bound on short sales. Traders belong to one of two homogeneous groups: fundamentalists (f) or speculators (s). Each group is presumed to have sufficient wealth to purchase the entire stock of the asset, including the short sales of the other group. For  $i = f, s$  the  $i$ th group anticipates in period  $t$  that  $p^{t+1}$  will be  $t_p^{t+1}$ . Define

$$(2.2) \quad \pi_i^t = p_i^{t+1} - p^t + \rho(V^t - p^t)$$

to be the time  $t$  anticipated profit (measured in time  $t+1$  dollars) to a trader in group  $i$  who holds a unit of the risky asset for one period. If  $\pi_i^t > 0$  for either group there is market excess demand since short sales by the other group are bounded. If  $\pi_i^t < 0$  for both groups there is market excess supply. If  $\pi_i^t < 0$  for one group and  $\pi_i^t = 0$  for the other group the market is in equilibrium.

Fundamentalists and speculators differ solely by their price expectations. Fundamentalists anticipate  $p_f^{t+1} = V^t$ : that next period price will equal the anticipated value of the asset. Define  $U^t = p^t - V^t$  to be the excess of price over the market fundamental. By a bubble is meant  $U^t > 0$ . Using (2-2) the definition of  $\pi_f^t$

$$(2.3) \quad \pi_f^t = - (1+\rho)U^t$$

It is immediately apparent that  $U^t < 0$  cannot occur - price never falls below the fundamental. If it did there would be excess demand by fundamentalists and the market would not clear. If  $U^t = 0$  the market clears provided  $\pi_s^t < 0$ , that is, there must not be excess demand by speculators. If  $U^t > 0$  the market clears if and only if  $\pi_s^t = 0$ , that is, excess demand by speculators must vanish.

We turn now to the expectations of speculators. This is given by

$$(2.4) \quad p_s^{t+1} = p^t + h(p^t - p^{t-1} | U^t)$$

Speculators anticipate a price increase  $h$  depending on last period's increase

$\Delta^t \equiv p^t - p^{t-1}$  and the excess price of the stock  $U^t \equiv p^t - V^t$ . Observe that speculative profit is given by

$$(2.5) \quad \pi_s^t = h(\Delta^t | U^t) - \rho U^t.$$

The following assumptions concerning  $h$  are made.

(A1) Speculators don't expect to make profits off of rising prices if either the stock is highly overvalued or if there has just been an exorbitant price increase. Thus there are finite positive constants  $U^m$  and  $\Delta^m$  such that  $h(\Delta | U) - \rho U < 0$  if either  $U > U^m$  or  $\Delta > \Delta^m$

(A2) Speculators don't expect prices to rise unless they have previously gone up. Thus  $h(\Delta | U) < 0$  for  $\Delta < 0$ .

If the market isn't in a bubble then  $U^{t-1} = 0$ . If speculators anticipate a loss when  $U^t = 0$ , then  $U^t = 0$  will be a market equilibrium and a bubble will not form at time  $t$ . By induction there can never be a bubble in this case. Thus we should assume that when  $U^{t-1} = 0$  and  $U^t = 0$  there is a positive probability of an anticipated speculative gain. Noticing that

$$(2.6) \quad \Delta^t = U^t - U^{t-1} + \varepsilon^t$$

we see that if a bubble is to occur at time  $t$  then we must have  $h(\varepsilon^t | 0) > 0$ .

(A3) Bubbles can form with positive probability. Thus for some  $\varepsilon$  in the support  $[-\bar{\varepsilon}, \bar{\varepsilon}]$   $h(\varepsilon | 0) > 0$ .

(A4) Speculative expectations are "well-behaved" in the sense that  $h(\Delta|U)$  is smooth and strictly concave for

$$0 < \Delta < \Delta^m \quad \text{and} \quad 0 < U < U^m$$

(A5) Increases in the excess price of the stock lower anticipated speculative profit, so that  $\partial h/\partial U - \rho < 0$ .

In Figure 2-1  $h(\Delta|U)$  satisfying (A1)-(A5) is graphed for a typical value of  $U$ .

It is useful to define  $Z^t \equiv U^{t-1} - \epsilon^t$  and use (2.5) and (2.6) to find speculative profits as

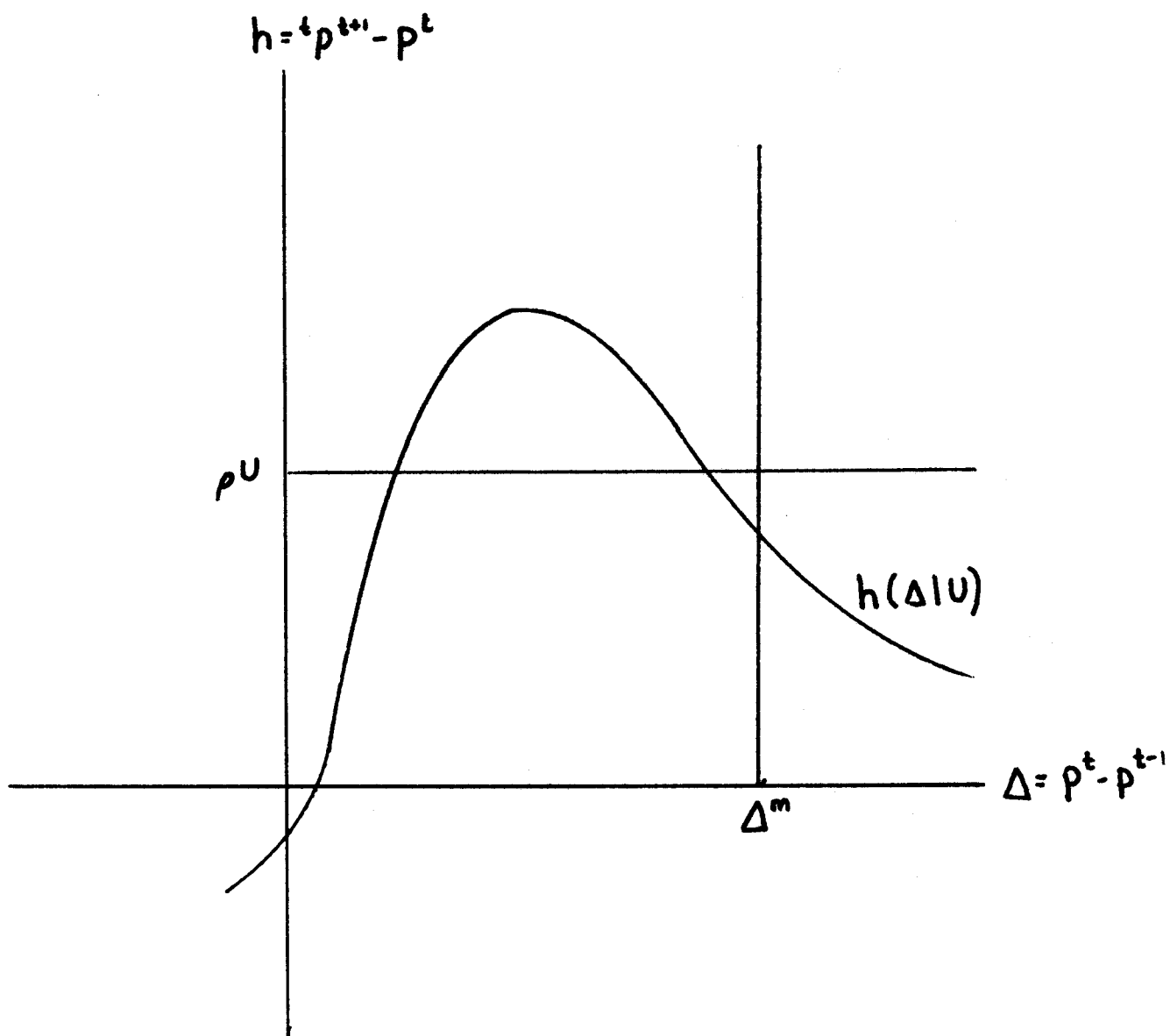
$$(2.7) \quad \pi_s^t = h(U^t - Z^t|U^t) - \rho U^t.$$

Since market equilibrium can occur only where either  $U^t = 0$  and  $\pi_s^t < 0$  or  $U^t > 0$  and  $\pi_s^t = 0$  we are led to examine the region  $R$  in  $Z^t$ - $U^t$ -space where  $U^t > 0$  and  $\pi_s^t > 0$ : this is the region of speculative excess demand and fundamentalist excess supply. This is described by the following proposition, and illustrated in Figure 2-2. Proof of the proposition is straightforward and omitted for brevity.

Proposition (R): If (A1) - (A5) hold then

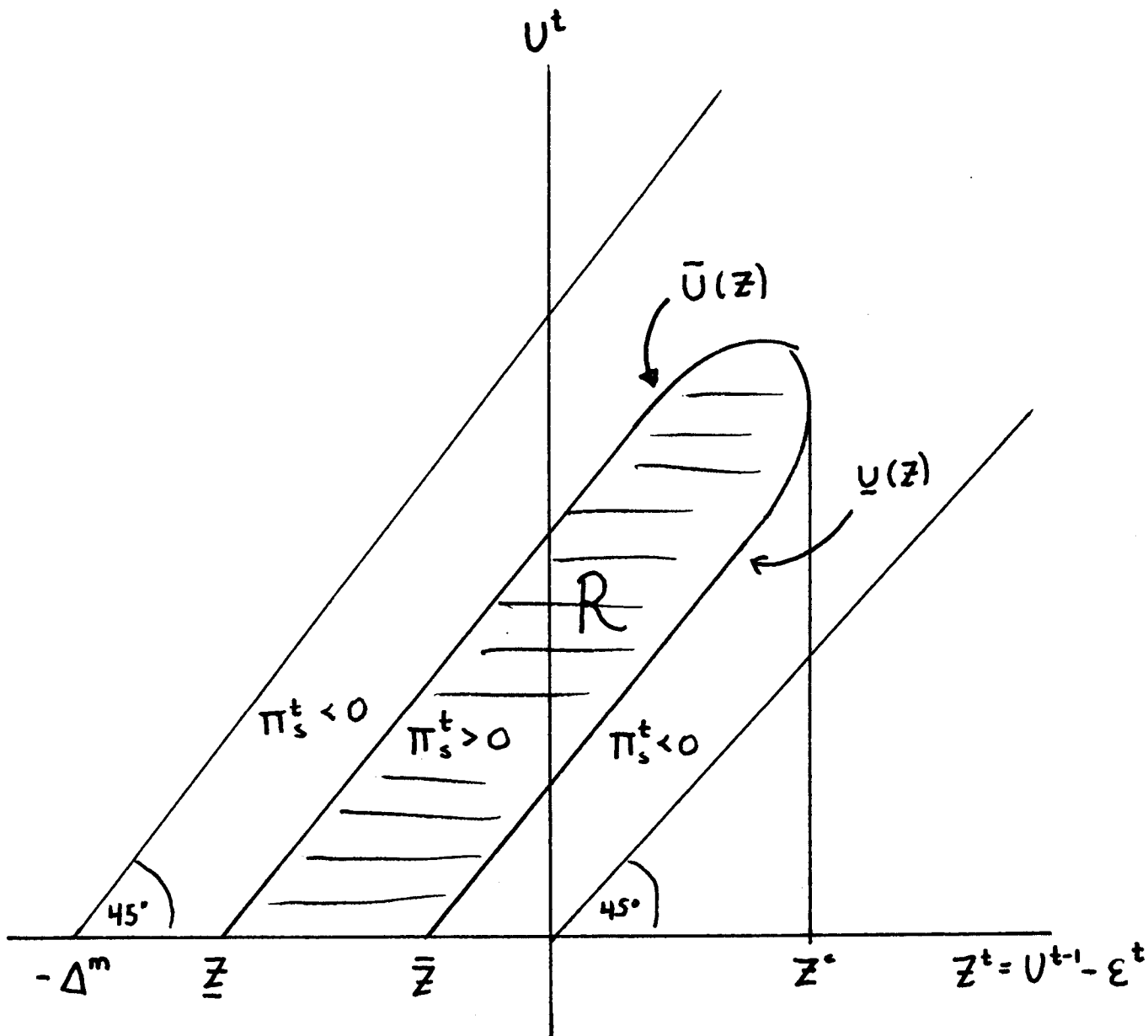
(1)  $R$  is a compact convex set bounded above by a smooth concave function  $\bar{U}(Z^t)$  and below by a smooth convex function  $\underline{U}(Z^t)$  and the  $Z^t$ -axis. The function  $\bar{U}$  is defined on  $[\underline{Z}, Z^c]$  and  $\underline{U}$  on  $[\bar{Z}, Z^c]$ .

FIGURE 2-1



Expectations of Speculators

FIGURE 2-2



Speculative Excess Demand  
(shaded area - Region R)



(2) Speculative excess demand vanishes in R if and only if either  $U^t = \bar{U}(Z^t)$  or  $U^t = \underline{U}(Z^t)$ .

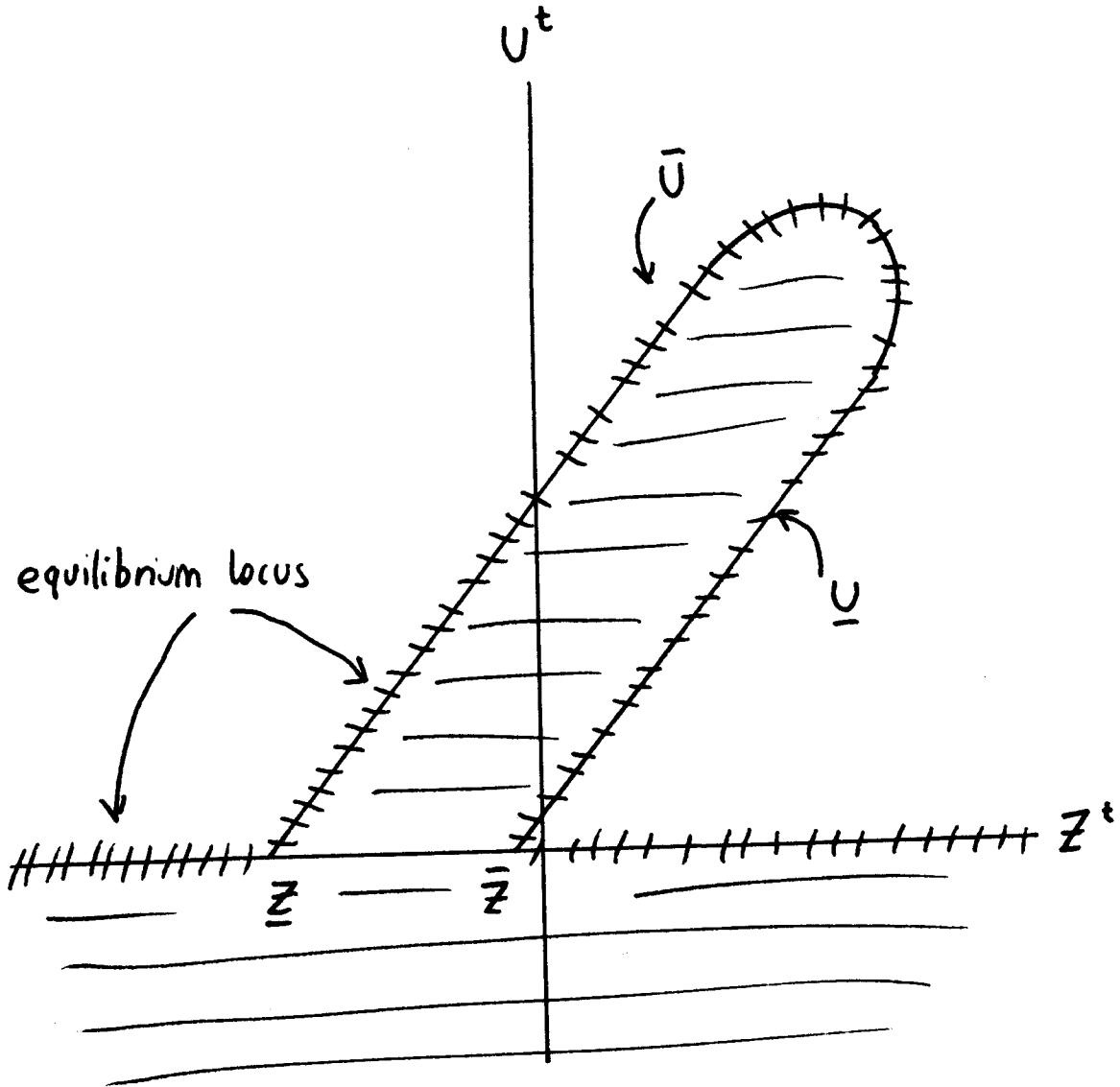
(3) In R  $\Delta^m + Z^t > U^t > Z^t$ .

### 3. THE MARKET EQUILIBRIUM

How is market equilibrium determined in each period? As previously indicated, if  $U^t < 0$  there is market excess demand, due to excess demand by fundamentalists. If  $U^t = 0$  and  $Z^t < \underline{Z}$  or  $Z^t > \bar{Z}$  the market is in equilibrium. If  $U^t = 0$  and  $\underline{Z} < Z^t < \bar{Z}$  there is excess demand. When  $U^t > 0$  excess demand is determined entirely by speculators as shown in Figure 2-2 -- it vanishes along the curves  $\bar{U}$  and  $\underline{U}$ , is positive between them, and negative otherwise. Figure 3-1 plots the possible market equilibria with regions of excess demand shaded and excess supply unshaded.

An important feature of equilibria is that for  $\bar{Z} < U^{t-1} - \epsilon^t < Z^c$  there is no unique market clearing value of  $U^t = p^t - v^t$ : there are three possible equilibria at  $\bar{U}$ ,  $\underline{U}$  and at 0. To determine the motion of the system as a function solely of the past, that is, to determine how  $U^t$  (or equivalently  $p^t$ ) depends on  $U^{t-1}$  and  $\epsilon^t$  (equivalently  $p^{t-1}$ ) we must specify how a particular equilibrium is chosen. At each moment of time, we may imagine what the asset is sold by auction at an exchange. Some initial price  $p_0^t = U_0^t + v^t$  is quoted. If there is excess demand price is raised and if excess supply it is lowered, and this tatonemont continues until the market is cleared. In this case the particular equilibrium is determined by the initial price quote  $U_0^t$ . If  $U_0^t > \underline{U}$  from Figure 3-1 the equilibrium price lies on  $\bar{U}$ . If  $U_0^t = \underline{U}$  it lies on  $\underline{U}$ , while if  $U_0^t < \underline{U}$  it falls to the fundamental  $U = 0$ . The most obvious initial pricing rule is to take  $U_0^t = U^{t-1}$ ; however since  $\underline{U}(Z) > Z$  this means

FIGURE 3-1



Market Clearing Equilibria  
 (shaded area is market excess demand)

that bubbles will collapse unless supported by random shocks. In other words if  $\varepsilon^t = 0$   $\underline{U}(Z^t) > Z^t = U^{t-1} + \varepsilon^t = U^{t-1} = U_0^t$  implying that  $U^t = 0$  if this initial pricing rule is followed. To eliminate the possibility that bubbles burst solely because of the auctioning procedure, and also to prevent them from occurring solely for that reason the following initial pricing rule is presumed

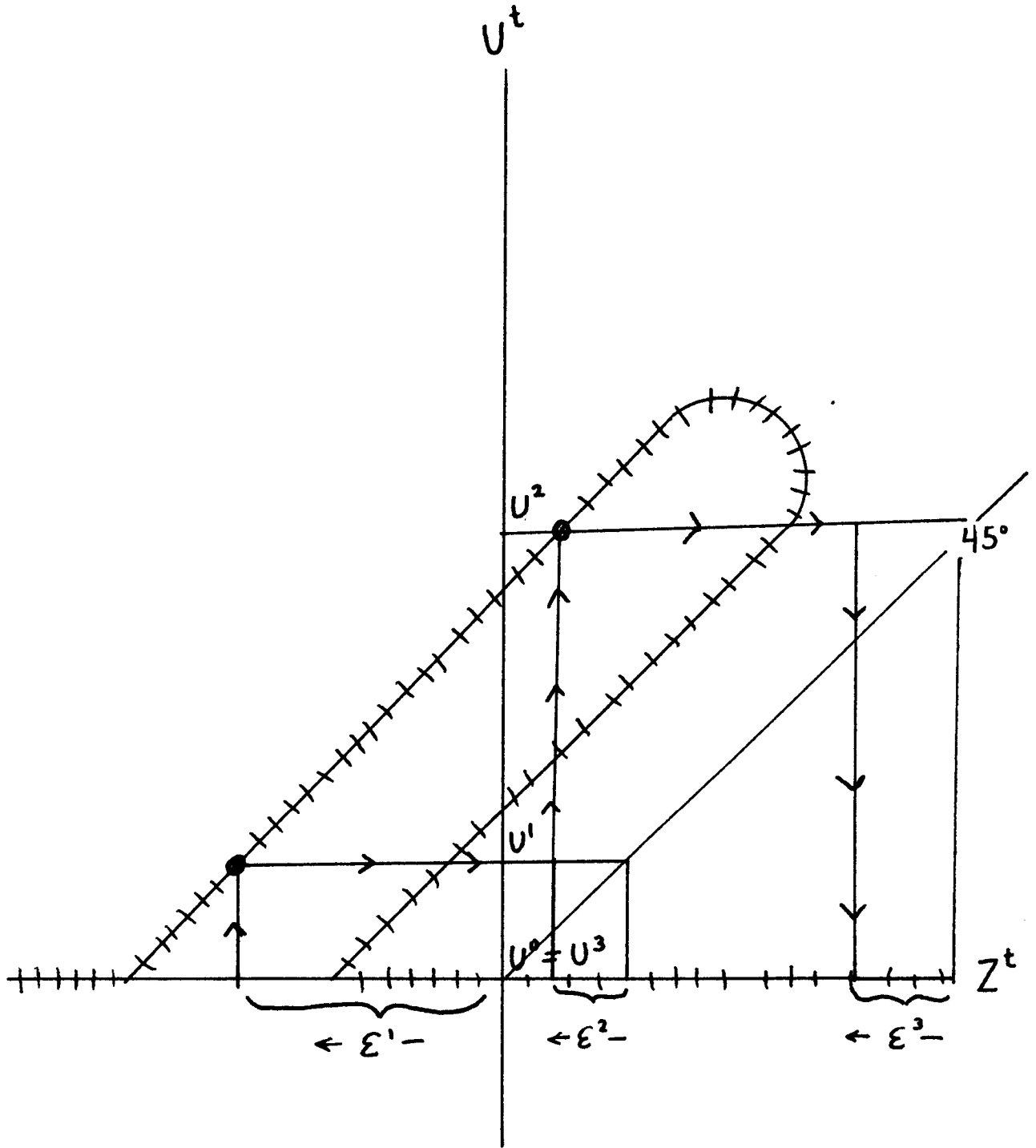
$$(B) \quad U_0^t = \begin{cases} +\infty & U^{t-1} > 0 \\ -\infty & U^{t-1} < 0 \end{cases}$$

This rule implies that if  $U^{t-1} > 0$  the equilibrium in period  $t$  lies along  $\bar{U}$  while if  $U^{t-1} < 0$  the equilibrium in period  $t$  is fundamental with  $U^t = 0$ . Remember that this is only when  $\bar{Z} < Z^t < Z^c$ ; otherwise the unique equilibrium corresponding to  $Z^t$  is chosen. We assume, in other words, a bubble remains a bubble if at all possible, but will not form unless it must. This seems the most reasonable hypothesis.

Market equilibrium is now completely determined. As a simple review Figure 3-2 illustrates a possible sequence of events. At time 0 the market is fundamental and  $U^0 = 0$ . Then at time 1 a shock  $\underline{Z} < -\varepsilon^1 < \bar{Z}$  occurs causing a bubble to form with  $U^1 > 0$ . The following period  $\varepsilon^2 > 0$  and the bubble continues with  $U^2 > 0$ . However in period 3 despite  $\varepsilon^3 > 0$   $Z^3 > Z^c$  and the bubble bursts with  $U^3$  crashing to the fundamental  $U^3 = 0$ .

Some terminology helps understand the qualitative features of the market equilibrium over time. Time periods can be categorized into two types: fundamental periods during which  $U^t = 0$ ; and bubble periods in which  $U^t > 0$ . There are two possible types of transitions in a fundamental period  $t$ . If the subsequent period is a bubble period a trigger shock is said to occur and  $\varepsilon^{t+1}$  must lie between  $\underline{Z}$  and  $\bar{Z}$  which occurs with probability  $0 < \delta < 1/2$  since

FIGURE 3-2



Price Dynamics

$\bar{Z} < 0$  and  $\epsilon^t$  is symmetric. In this case  $\Delta^{t+1} \equiv p^{t+1} - p^t = U^{t+1} + \epsilon^{t+1} > 0$  and is also bounded above by  $\bar{U}(Z)$ . If  $\epsilon^{t+1} < \underline{Z}$  or  $\epsilon^{t+1} > \bar{Z}$  then the subsequent period is fundamental period, and a fundamental transition occurs. These possibilities are summarized in Figure 3-3 and Table 3-1.

Bubbles may end in one of two ways. If  $U^t - \epsilon^{t+1} < \underline{Z}$  then  $U^{t+1} = 0$ . This is called a terminal shock --  $\epsilon^{t+1}$  is so large that the market fundamental catches up to the bubble. In this case  $\Delta^{t+1} = \epsilon^{t+1} - U^t > -\underline{Z} > 0$ , so prices rise even though the bubble ends. Alternatively, if the bubble rises so high that  $U^t - \epsilon^{t+1} > Z^c$  a crash occurs and  $U^{t+1} = 0$ . In this case  $\Delta^{t+1} = \epsilon^{t+1} - U^t < -Z^c$ ; in a crash prices fall.

Finally, a bubble transition in which  $U^t$  and  $U^{t+1}$  are both positive may occur. Then by Proposition (R) since  $\bar{U}(Z) > Z$

$$(3.1) \quad \begin{aligned} \Delta^{t+1} &= U^{t+1} - U^t + \epsilon^{t+1} \\ &= \bar{U}(U^t - \epsilon^{t+1}) - (U^t - \epsilon^{t+1}) > 0 \end{aligned}$$

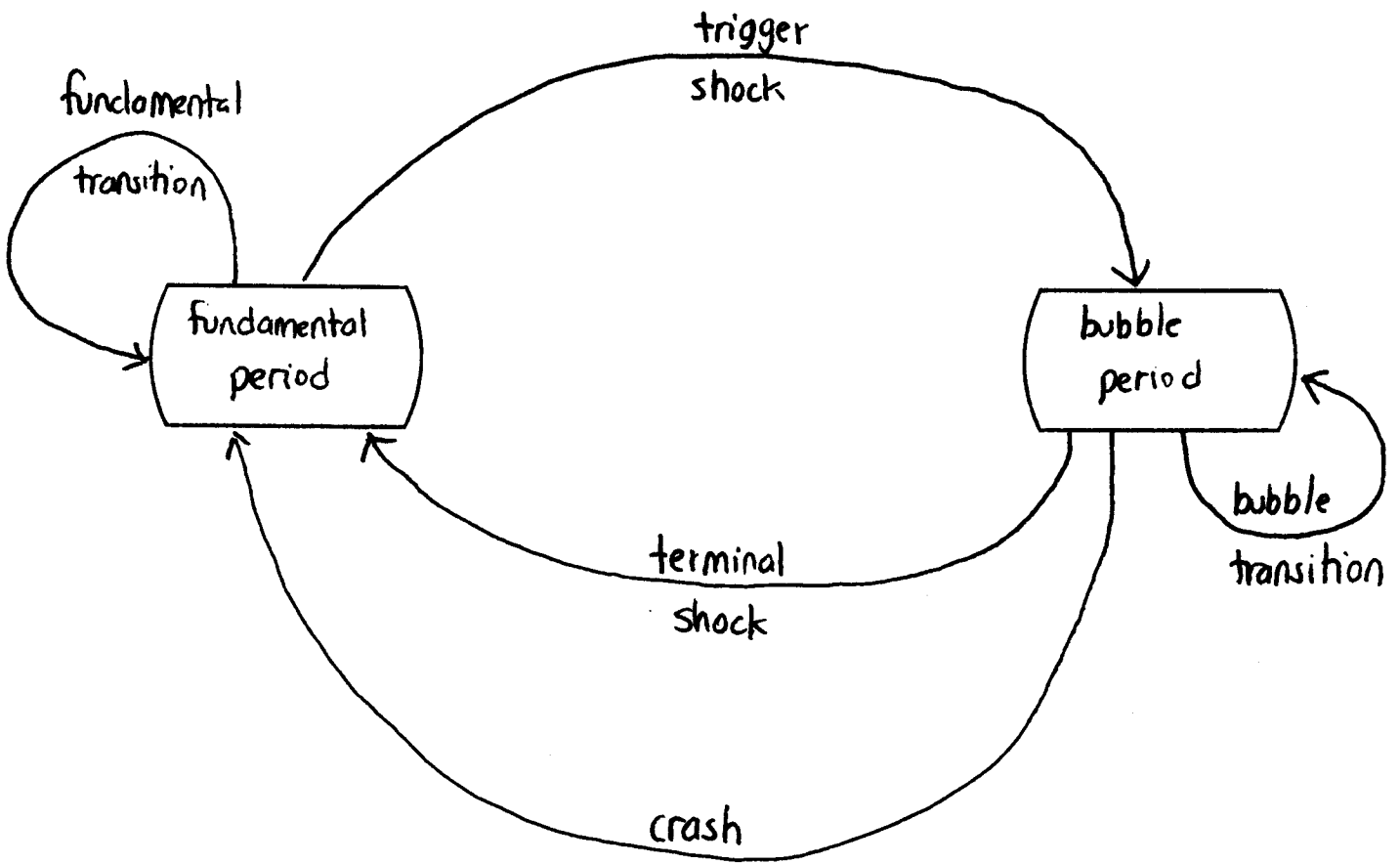
Thus, during a bubble prices always go up.

This discussion, summarized in Table 3-1 and Figure 3-3 gives a general description of price movements.

<u>Type of Transition</u>	<u><math>u^t</math></u>	<u><math>u^{t+1}</math></u>	<u><math>\Delta^{t+1}</math></u>
Fundamental	0	0	?
Trigger	0	+	+
Bubble	+	+	+
Terminal	+	0	+
Crash	+	0	-

TABLE 3-1: TRANSITIONAL EVENTS

FIGURE 3-3



#### 4. PRICE FLUCTUATIONS

The goal of this section is to derive the distinctive quantitative features of price fluctuations and test for their presence in actual stock market data. To perform the tests monthly data on price movements in the New York Exchange from 1926-38 was gathered and is reproduced in Table 4-1. Notice that the net change in the index over the period is only two points, considerably less than the monthly fluctuations.

The choice of monthly data is somewhat arbitrary. However, the results should not be too sensitive to the observational period provided that it is not so long as to preclude the possibility of observing bubbles, nor so short that movements reflect only the fact that even asset markets do not truly clear instantaneously. One justification for the use of monthly data is that (ex post) it exhibits the type of fluctuations predicted by the theory. While the non-parametric statistical analysis based on this data cannot replace a more detailed econometric study, it does indicate that the simple model developed in this paper does have strong predictive power.

Random Walk: Before examining implications particular to the bubble hypothesis, we should verify the important assumption that the market fundamental follows a random walk -- this might not be true if there are systematic trends in retained earnings, for example. From Table 3-1 we see that if price did not rise the market is in a fundamental period. As long as the market remains fundamental price rises if and only if  $\epsilon^{t+1} > 0$  -- if the fundamental goes up. If the market fundamental follows a random walk then prices should be equally likely to rise or fall provided that prices did not rise in the preceding period. Let  $m$  denote the probability that prices go up in a period following one in which prices did not rise. From Table 4-1 we find that prices either increased or decreased following a period in which

From Kindleberger, Charles, The World in Depression (UC Press; 1973), p. 111.

New York Stock Price Changes

(Standard Statistics Index)

	<u>1926</u>	<u>1927</u>	<u>1928</u>	<u>1929</u>	<u>1930</u>	<u>1931</u>	<u>1932</u>
J	0	+2	-2	-1	+7	-7	-1
F	-6	+1	+6	+4	+11	+2	+1
M	-3	+1	+9	-3	+4	-12	-12
A	0	+3	+5	0	+9	-19	-4
M	+4	+1	-7	-2	-17	+6	-4
J	+3	+3	0	+12	-3	+3	-2
J	+3	-5	+5	+7	-2	-1	+16
A	+1	+17	+9	+6	0	-13	-4
S	-2	-1	+4	-22	-21	-11	-3
O	+1	+3	+13	-39	-9	+13	-3
N	+2	+5	-1	+2	-7	-14	0
D	+1	0	+15	+2	+1	0	0
	<u>1933</u>	<u>1934</u>	<u>1935</u>	<u>1936</u>	<u>1937</u>	<u>1938</u>	
J	-3	+4	-1	+6	+6	-1	
F	-1	-3	-5	+4	0	-3	
M	+5	+3	+4	0	-10	-10	
A	+16	-8	+7	-6	-6	0	
M	+12	+1	+2	+1	-4	+6	
J	+7	-1	+4	+9	+8	+18	
J	-5	-3	+3	+3	+2	-3	
A	+2	-1	+3	+2	-20	+1	
S	-5	0	+2	+8	-19		
O	+1	-4	+10	+5	-9		
N	+2	0	0	-2	-4		
D	+5	+1	+4	+4	+5		

TABLE 4-1: PRICE MOVEMENTS



they failed to increase on 59 occasions. On  $\hat{m} = 44\%$  of these periods prices went up. This is reasonably close to 50%. Furthermore under the null hypothesis  $m = 50\%$   $\hat{m} - m$  should be approximately normally distributed with a standard error  $\hat{S} = 6\%$ . Dividing we find that  $(\hat{m}-m)/\hat{S} = 1.00$ , while the probability that a standard normal exceeds 1.00 in absolute value is 32% indicating that at conventional levels of significance (10% say) we should accept the null hypothesis. While this does not prove that the fundamental in fact followed a random walk it is certainly strong evidence in favor of this hypothesis.

Bubbles: We would like to examine the empirical implications of "substantial" bubbles. From Figure (3-1) we see that bubbles will last longer and grow higher the larger is  $Z^c$  (which is the largest that  $Z^t$  can grow before the bubble bursts.) In the next subsections we shall examine the implications of the hypothesis " $Z^c$  is large."

Crashes: One implication of the bubble model is that while prices rise rather gradually even during a bubble, they fall quite radically during a crash. A formal statement of this is

Proposition (3-1): Suppose  $Z^c > \Delta^m$ . Then

$$\sup_{\Delta^{t+1} > 0} \Delta^{t+1} < \sup_{\Delta^{t+1} < 0} |\Delta^{t+1}|$$

Proof:

Price rises during a fundamental transition are at most  $\bar{\epsilon}$ . During a bubble transition or trigger shock by Proposition (R)

$$(3.1) \quad \Delta^{t+1} = \bar{U}(U^{t-\epsilon}{}^{t+1}) - (U^{t-\epsilon}{}^{t+1}) < \Delta^m.$$

During a terminal shock

$$(3.2) \quad \Delta^{t+1} = \epsilon^{t+1} - U^t < \epsilon^{t+1} < \bar{\epsilon}$$

The largest value of  $\Delta^{t+1} > 0$  then is

$$(3.3) \quad \sup_{\Delta^{t+1} > 0} \Delta^{t+1} < \max(\Delta^m, \bar{\epsilon})$$

The greatest fall in price occurs during a crash where  $\Delta^{t+1} = \epsilon^{t+1} - u^t$ .

Since a crash can occur with  $|U^t - U^m|, |\epsilon^{t+1} - \bar{\epsilon}| < \psi$  for  $\psi$  arbitrarily small

$$(3.4) \quad \sup_{\Delta^{t+1} < 0} |\Delta^{t+1}| > U^m + \bar{\epsilon} > Z^c + \bar{\epsilon}$$

QED

What Proposition (3-1) says about observed price series is that extreme price drops should be larger in absolute magnitude than extreme price increases. Table 3-2 lists the largest five absolute rises and declines. Since the fifth largest absolute drop is larger than the largest monthly increase the data is strongly consistent with Proposition (3-1).

Trigger Shocks: Unless bubbles collapse immediately, prices should rise for several periods in a row once a bubble forms. Let N denote the least number of bubble periods between a trigger shock and a crash.

## MONTHLY INCREASES AND DECLINES

<u>Largest Price Drops</u>	<u>Largest Price Increases</u>
-39	+18
-22	+17
-21	+16
-20	+16
-19	+13

TABLE 3-2

Proposition (3-2): If  $Z^c > 2\Delta^m + \bar{\epsilon}$  then  $N > 2$

Proof:

Suppose a trigger shock occurs at time  $t$ . Then  $U^{t+1} < \Delta^m$  by Proposition (R) part (3). Similarly  $U^{t+2} < 2\Delta^m + \bar{\epsilon} < U^m$ . Hence a crash does not occur at least until  $U^{t+3}$  implying  $N > 2$ . QED.

The following implication of the theory now suggests itself to mind. Suppose price falls so  $\Delta^t < 0$  then rises so that  $\Delta^{t+1} > 0$ . If  $\Delta^{t+1} > \underline{Z}$  or  $\Delta^{t+1} < \bar{Z} + \bar{U}(\underline{Z})$  a bubble does not form and  $\Delta^{t+2}$  should be positive and negative with equal probability. On the other hand, if  $\underline{Z} > \Delta^{t+1} > \bar{Z}$  a bubble has formed and by Proposition (3-2)  $\Delta^{t+2} > 0$  with probability one, since a crash does not occur. This suggests that a price rise following a decline can be broken into a small, intermediate and large range. In the small and large range 50% of subsequent period price changes should be positive, while in the

intermediate range considerably more than 50% of the subsequent period price changes should be positive. Inspection of Table 3-1 suggests that the three groups should be 1-3, 4-8 and 9 or more points. Table 3-3 presents the relevant statistical information. According to the preceding discussion the small and large initial rises of 1 to 3 and 9 or more points should have a 50% chance of subsequent period price increases, a hypothesis which cannot be rejected at the 10% level. The intermediate range of 4 to 8 should have a greater than 50% chance of a subsequent period price rise, which is supported by the fact that following an initial intermediate price increase price rose in the following period 83% of the time. This fraction is sufficiently large that we can reject the null hypothesis of 50% in favor of the one sided alternative that the probability of increase is greater than 50% at the 1% level of significance.

	(1)	(2)	(3)	(4)	(5)
	Size of Price Rise After a Decline	Observa- tions	Fraction of Subsequent Period Price Increases	Standard Error*	**
(1)	1 to 3	14	0.36	0.13	-1.08 (28%)
(2)	4 to 8	12	0.83	0.14	2.36 (1%)
(3)	9 plus	4	0.25	0.25	-1.00 (32%)
(4)	1 to 3, 9 plus	18	0.33	0.12	-1.42 (16%)

\*Under the null hypothesis of a 50% probability of increase.

\*\*First number is column (3) minus 0.50 divided by column (4) (under the null hypothesis this is approximately drawn from a standardized normal distribution); parenthesized number is probability that a standard normal exceeds the test statistic in absolute magnitude in rows (1), (3) and (4) (the two-sided test level significance); and that a standard normal exceeds the statistic in row (2) (the one-sided test level significance).

TABLE 3-3: TRIGGER SHOCKS -- SOME EVIDENCE

Runs: Now let  $E_t^k$  be the event that a price decline in period  $t-k$  has been followed by  $k$  price rises in a row, and let  $q^k = \text{pr}(\Delta^{t+1} > 0 | E_t^k)$ . Recall  $N$  is the least number of bubble periods between a trigger shock and craash. If  $k < N$  we might expect that  $q^k$  increases in  $k$ , since each period has an independent chance of entering a bubble if it is not already a bubble.

Proposition (3-3): For  $k < N$   $q^k$  strictly increases in  $k$  and if  $\text{pr}(\varepsilon^t < \underline{Z}) < \text{pr}(\underline{Z} < \varepsilon^t < \bar{Z})$  then  $q^1 > 1/2$ .

Proof: Recall that the probability of a trigger shock is  $0 < \delta < 1/2$ .

$q^1 > 1/2$  follows from Proposition (3-2) and the positive probability of a trigger shock. The key to proving the rest of the proposition is to observe that  $\lambda^k \equiv \text{pr}(\text{terminal shock} | E_t^k) < \delta$  so that the probability that the market is in a bubble goes up. Since a terminal shock requires  $U^t - \varepsilon^{t+1} < \underline{Z}$  and  $U^t > 0$  it requires that  $\varepsilon^{t+1} < \underline{Z}$ . But a trigger shock requires  $\underline{Z} < \varepsilon^t < \bar{Z}$  so by the assumption  $\delta > \lambda^k$ .

Now let  $r^k = \text{pr}(\text{bubble} | E_t^k)$ . Since  $k < N$  a bubble ends only with a terminal shock and

$$(3.5) \quad \begin{aligned} r^k &= r^{k-1}(1-\lambda^k) + \delta(1-r^{k-1}) \\ r^k - r^{k-1} &= (\delta-\lambda^k)r^k + \delta > 0. \end{aligned}$$

This shows that the probability of being in a bubble increases.

To complete the proof, we must use (3-5) to prove that  $q^k$ , the probability of yet another price increase, goes up. The probability of a price increase in a bubble is one, while during a fundamental period it is  $1/2$  so

$$(3.7) \quad q^k = r^k + (1/2) (1-r^k) = (1/2) (1+r^k)$$

Since by (3-5)  $r^k$  increases strictly,  $q^k$  does as well. QED

To test Proposition (3-3) estimates  $\hat{q}^1$ ,  $\hat{q}^2$  and  $\hat{q}^3$  were compiled in Table 3-5. According to Proposition (3-3) if  $N > 3$   $q^3 > q^2 > q^1 > 0.50$  which is true of the estimates  $q^1$ . The second half of Table 3-4 ascertains the extent to which these observed relationships are statistically significant. Of the inequalities tested  $q^2 > q^1$  at the 10% level and  $q^3 > q^1$  at the 8% level; the remaining relationships are not statistically significant. In many ways the estimates  $q^1$ ,  $q^2$  and  $q^3$  are the most convincing evidence in favor of the bubble hypothesis — not only do they satisfy the expected relationship that a small number of price increases increases the probability that price will rise yet higher — but it is hard to see how this could be explained in the absence of price bubbles.

## 5. CONCLUSION

Unlike the modern literature which has studied the relationship between rationality and price bubbles, this paper has modelled price bubbles when speculators are irrational. A number of implications of the model were derived and tested: all were supported by the evidence. One message of this paper is that testing the bubble hypothesis is a subtle matter — the market is not always in a bubble, and tests involve either extreme values of price jumps, or probabilities of price increases conditional on carefully specified events. The second message is that once appropriate tests are performed there is strong evidence that price bubbles do occur.

<u>k</u>	$\hat{q}^k$	<u>Observations</u>	<u>Standard Error of <math>\hat{q}^k</math></u>
1	0.55	29	0.09
2	0.73	15	0.11
3	0.80	10	0.16

(1)	(2)	(3)	(4)
<u>Test</u>	<u>Statistic*</u>	<u>Standard Error*</u>	<u>**</u>
$q^1 - 0.50 > 0$	0.05	0.09	0.56 (29%)
$q^2 - q^1 > 0$	0.18	0.14	1.29 (10%)
$q^3 - q^2 > 0$	0.07	0.19	0.37 (36%)
$q^3 - q^1 > 0$	0.25	0.18	1.39 (8%)

\*The statistic is  $\hat{q}^i - \hat{q}^j$  where  $\hat{q}^0 \equiv 0.50$ ; the standard error is  $\{(\hat{S}^i)^2 + (\hat{S}^j)^2\}^{1/2}$  where  $\hat{S}^i$  is the standard error of  $\hat{q}^i$  and  $S^0 = 0$ .

\*\*First number is Column (2) divided by Column (3) which is approximately drawn from a standard normal distribution, parenthesized number is the probability that a standard normal exceeds the first number — the one-sided significance level of the hypothesis  $q^i - q^j = 0$  against the alternative in (1).

TABLE 3-5: CONSECUTIVE PRICE INCREASES

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