AUCTIONS WITH

ASYMMETRIC BELIEFS*

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UCLA Department of Economics
Working Paper #254
July, 1982
Revised June, 1983

*Discussions with Bob Hansen and Barry Nalebuff are gratefully acknowledged.

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At the root of the rapidly growing theoretical literature on auctions is the revenue equivalence theorem,[1] which states that, under the assumptions of (i) risk neutrality, (ii) independence of preferences and (iii) symmetry of beliefs, any two selling procedures generate the same expected revenue if, in equilibrium, the probability of winning is the same function of the valuation, v. Thus, for example, the open (ascending bid) auction and sealed high bid auction, with announced reserve price r, generate the same expected revenue since, in each, a buyer has an incentive to bid if and only if his valuation exceeds r and the equilibrium bid function is strictly increasing in v.

From this central result research has proceeded along two main branches. First there is the study of auctions which are optimal for the seller, that is, which maximize his expected revenue. Under the three basic assumptions noted above, Myerson [1981] provides a complete characterization of optimal selling procedures as direct revelation mechanisms. In Maskin and Riley [1980, 1982b] it is shown that there exist modified high bid and open auctions which are equivalent to the optimal direct revelation mechanisms. Moreover, with more than one object for sale but each buyer only interested in purchasing one unit, these results are readily generalized.

When the assumption of risk neutrality is relaxed the characterization of expected revenue maximizing selling procedures is much more complicated. In general the seller can extract more revenue by introducing a payment by (or subsidy to) unsuccessful bidders (Matthews [1983], Maskin and Riley [1982a]). Subsidies to unsuccessful high bidders reduce their exposure to risk and hence, ceteris paribus, make them willing to bid more. However there is an offsetting screening role of any auction mechanism. While requiring a payment by those submitting small losing bids lowers the revenue obtainable from such bidders, this is more than offset by the higher bidding of risk averse buyers with high valuations, owing to their greater fear of being unsuccessful.

The second major branch of the new literature contrasts the two common auction procedures (sealed high bid and open bid) when one or more of the three basic assumptions is relaxed. Under increasingly general assumptions, Holt [1980], Riley and Samuelson [1981] and Maskin and Riley [1982a] all show that when buyers are risk averse, the sealed high bid auction generates greater expected revenue than the open auction. Moreover, this ranking continues to hold if the seller also exhibits risk aversion. This is intuitively reasonable. In the open auction each buyer stays in until the asking price equals his own valuation. Thus his bidding strategy is unaffected by risk aversion. On the other hand a risk averse buyer in the sealed high bid auction will shade his bid less than he would if he were risk neutral. Given the revenue equivalence theorem for risk neutral buyers, the result thus follows.

Assumption (ii), independence of preferences, is relaxed by Milgrom and Weber [1982]. If the values that different buyers place on the object for sale are "affiliated" (and hence positively correlated), they show that the open auction generates higher expected revenue than the sealed second bid auction and both of these auctions generate higher expected revenue than the sealed high bid auction.[2]

In this paper assumption (iii), symmetry of beliefs, is relaxed. First a pair of examples are presented for which it is possible to solve analytically for the (asymmetric) equilibrium bidding strategies in both the sealed high bid and open auctions. It is shown that for all parameter values the seller's expected revenue is lower in the open auction. Moreover, from the second of these examples, it is readily established that there exist asymmetric beliefs for which the percentage gain to adopting the sealed high bid auction rather than the open auction is arbitrarily great.

Asymmetry of beliefs is often important in contract bidding. Each potential contractor has essentially the same information about the nature of the project but a different opportunity cost of completing the project, proximity to the job, etc. With some of the latter information common knowledge, beliefs are thus independent and asymmetric.

With such strong ranking results emerging from the two examples it is tempting to conclude that we have provided a sound theoretical explanation for the common use of sealed bid auctions in contract bidding. However, examples are at most suggestive so, to explore matters further, we consider a rather different discrete example in Section II. Under assumptions most closely paralleling those implicit in the earlier examples, the same ranking of the two auctions emerges. However, under different assumptions about the nature of the asymmetry in beliefs, the ranking is reversed.

This discrete example thus rules out any completely general ranking theorem. The theoretical challenge, then, is to obtain necessary and sufficient conditions for a strict ranking of one auction over the other.

Any such analysis would be incomplete without a parallel consideration of existence and uniqueness in high bid auctions. The first effort in this direction is a remarkable and almost unknown paper on sealed high bid auctions with asymmetric beliefs by Griesner, Levitan and Shubik [1967]. For the case of 2 buyers this paper establishes existence and uniqueness under weak restrictions. Unfortunately the result does not immediately generalize to the n buyer case. Elsewhere, in Maskin and Riley [1982c], it is shown that, with n buyers, existence and uniqueness hold under weak assumptions about preferences and <u>symmetric</u> beliefs. However, the issue remains to be resolved with asymmetric beliefs. Some preliminary results to this end are presented in Section III.

I. Equilibrium Bidding With Asymmetric Beliefs — Two Continuous Examples

To focus on essentials we examine the case of two buyers $i = 1,2$. Buyer 1's beliefs about the value buyer 2 places on the object for sale are summarized by the c.d.f. $F_2(v)$. Similarly buyer 2's beliefs about the buyer 1's valuation are summarized by the c.d.f. $F_1(v)$. Suppose, initially, that the seller announces that the object will be sold to the highest bidder, at his bid price, in a sealed bid auction. Then a pair of bid functions

(1) $$b = \psi_i(v), \quad \psi_i'(.) > 0 \quad i = 1,2.$$

constitutes an equilibrium if, when buyer j $(j \neq i)$ adopts $b = \psi_j(v)$ as his bidding behavior, buyer i's best response is to bid $b_i = \psi_i(v)$, for all feasible v.

Rather than attempting to solve directly for the equilibrium bid functions, it is convenient to define a pair of inverse bid functions

(2) $$v = \phi_i(b) \equiv \psi_i^{-1}(b).$$

Then if buyer 1 submits a bid of b he wins if and only if the other buyer bids less,[3] that is if $v_2 < \phi_2(b)$.

Buyer 1's expected gain is therefore

(3) $$\Pi_1(b;v) = (v-b)\ \text{Prob}\{v_2 < \phi_2(b)\} = F_2(\phi_2(b))(v-b).$$

Clearly if buyer 2's maximum bid is $\bar{b}_2$, it never pays buyer 1 to bid more than $\bar{b}_2$. Similarly if buyer 1's maximum bid is $\bar{b}_1$ it never pays buyer 2 to bid more than $\bar{b}_1$. Then $\bar{b}_1 = \bar{b}_2$, that is, there exists some common maximum bid $\bar{b}$. Furthermore a buyer with a zero valuation will not make a positive bid. Thus the equilibrium inverse bid functions must satisfy

(4) $$\phi_i(0) = 0, \quad F_i(\phi_i(\bar{b})) = 1 \quad i = 1,2$$

We now define $\phi_1(b)$ and $\phi_2(b)$ to be the solution to the pair of differential equations,

$$(5)\quad \begin{aligned} &\text{(a)}\quad \phi_1(b)\, F_2'(\phi_2)\phi_2' - \frac{d}{db}\,(b\, F_2(\phi_2)) = 0 \\ &\text{(b)}\quad \phi_2(b)\, F_1'(\phi_1)\phi_1' - \frac{d}{db}\,(b\, F_1(\phi_1)) = 0. \end{aligned}$$

satisfying the boundary condition (4).

Differentiating (3) with respect to b we obtain

$$\frac{\partial \Pi_1}{\partial b}(b;v) = vF_2'(\phi_2)\phi_2' - \frac{d}{db}(bF_2(\phi_2)).$$

With ϕ_2 satisfying (5) we can substitute from (5a) to obtain

$$\frac{\partial \Pi_1}{\partial b}(b;v) = (v - \phi_1(b))\, F_2'(\phi_2)\phi_2'$$

Thus, if the equilibrium bid functions are strictly increasing, so that ϕ_1' and ϕ_2' are positive, $\partial \Pi_1/\partial b$ is positive for $b < \phi_1^{-1}(v) \equiv \psi_1(v)$ and negative for $b > \psi_1(v)$. That is, $\psi_1(v)$ is buyer 1's optimal response when buyer 2 is adopting the bidding strategy $b = \psi_2(v)$. Of course a symmetrical argument applies for buyer 2. Thus if ϕ_1 and ϕ_2, the solution to (4) and (5), are both strictly increasing they are indeed equilibrium bid functions.[4]

We now consider the special case in which $F_i(v) = a_i v$, $v \in [0,1/a_i]$, $a_1 < a_2$. Without loss of generality[5] we may redefine units so that

$$(6)\quad \begin{aligned} F_1(v) &= (1-\alpha^2)v, \quad v \in (0,1/(1-\alpha^2)) \\ F_2(v) &= (1+\alpha^2)v, \quad v \in (0,1/(1+\alpha^2)). \end{aligned}$$

Then (5) becomes

$$(7)\qquad \begin{cases} \text{(a)} \quad \phi_1\phi_2' - \frac{d}{db}(b\phi_2) = 0 \\ \\ \text{(b)} \quad \phi_1'\phi_2 - \frac{d}{db}(b\phi_1) = 0. \end{cases}$$

Adding (7a) and (7b) and integrating we obtain

$$(8)\qquad \phi_1\phi_2 = b(\phi_1+\phi_2) + K.$$

But, from (4) $\phi_i(0) = 0$ so $K = 0$. Also, from (4), $F_i(\phi_i(\bar{b})) = 1$. Then

$$(9)\qquad \begin{cases} \phi_1(\bar{b}) = \dfrac{1}{1-\alpha^2} \\ \\ \phi_2(\bar{b}) = \dfrac{1}{1+\alpha^2} \end{cases}$$

Substituting (9) into (8) we obtain

$$(10)\qquad \bar{b} = 1/2.$$

Thus the range of equilibrium bids, $[0,1/2]$, is independent of the parameter α.

Next using (8) to substitute for ϕ_1 in (7a), and rearranging, we obtain the following first order differential equation for ϕ_2.

$$(11)\qquad \frac{b\phi_2' + \phi_2}{b^2\phi_2^2} = \frac{1}{b^3}.$$

Expressed in this form, the differential equation is easily solved to obtain

$$\phi_2(b) = \frac{2b}{1+Lb^2}.$$

For this to satisfy (9) and (10), we have $L = 4\alpha^2$. Then substituting into

(8) we obtain, finally,

(12) $$\begin{cases} \phi_1(b) = \dfrac{2b}{1-(2\alpha b)^2} \\ \\ \phi_2(b) = \dfrac{2b}{1+(2\alpha b)^2}. \end{cases}$$

Note that $\phi_1(b) > \phi_2(b)$ for all $b > 0$, so that $\psi_1(v) \equiv \phi_1^{-1}(v) < \psi_2(v)$. Thus buyer 1, whose valuation of the object is perceived by buyer 2 to be drawn from a more favorable distribution, bids more conservatively,[6] for all v.

Having solved for the equilibrium inverse bid functions it is a straightforward matter to obtain an expression for the c.d.f. $G_H(b;\alpha)$, of the winning bid in the high bid auction.

$$\begin{aligned} G_H(b;\alpha) &= \text{Prob}\{b_1 \text{ and } b_2 < b\} \\ &= \text{Prob}\ \{v_1 < \phi_1(b) \text{ and } v_2 < \phi_2(b)\} \\ &= F_1(\phi_1(b))F_2(\phi_2(b)). \end{aligned}$$

Substituting from (6) and (12) we obtain

(13) $$G_H(b;\alpha) = 4b^2\left(\frac{1 - \alpha^4}{1 - (2\alpha b)^4}\right).$$

Then expected revenue from the high bid auction is

(14) $$\begin{aligned} R_H(\alpha) &= \int_0^{1/2} b dG_H(b;\alpha) \\ &= \frac{1}{2} - \int_0^{1/2} G_H(b,\alpha)db. \end{aligned}$$

Note that at $\alpha = 0$ the distribution of the winning bid reduces to $G_H(b;0) = 4b^2$. This, of course is Vickrey's example of a symmetric auction with valuations uniformly distributed on [0,1]. Substituting into (14) we obtain

$$R_H(0) = 1/3.$$

Furthermore we may rewrite (13) as

$$G_H(b;\alpha) = \frac{1}{4b^2}\left(1 - \frac{1-(2b)^4}{1-(2\alpha b)^4}\right). \tag{13'}$$

Expressed in this way it is easy to see that $G_H(b;\alpha)$ is a strictly decreasing concave function of α for all $b \in (0,1/2)$. Then from (14), expected revenue, $R_H(\alpha)$, is a strictly increasing convex function of α.

We now examine equilibrium in an open auction, or, equivalently, in sealed bid auctions with the high bidder paying the second highest bid for the object. Just as in the open auction with symmetric beliefs, each buyer's dominant strategy is to remain in the auction until the asking price equals his true valuation. Then the seller receives a payment b equal to the second highest valuation. Writing the c.d.f. of this payment as $G_0(b;\alpha)$ we have

$$\begin{aligned} G_0(b;\alpha) &= \text{Prob}\{\text{second highest valuation is less than } b\} \\ &= 1 - \text{Prob}\{\text{both valuations exceed } b\} \\ &= 1 - (1 - F_1(b))(1 - F_2(b)). \end{aligned}$$

Substituting from (6) we obtain

$$G_0(b;\alpha) = 2b - (1-\alpha^4)b^2, \quad b \in [0,\frac{1}{1+\alpha^2}]. \tag{15}$$

Then expected revenue from the open auction is

$$\begin{aligned} R_0(\alpha) &= \int_0^{1/(1+\alpha^2)} b\,dG_0(b;\alpha) \\ &= \int_0^{1/(1+\alpha^2)} [2b - 2(1-\alpha^4)b^2]db \end{aligned} \tag{16}$$

$$= [b^2 - \frac{2}{3}(1-\alpha^4)b^3]_0^{1/(1+\alpha^2)}$$

$$= \frac{1}{3}\frac{(1 + 3\alpha^2 + 2\alpha^4)}{(1 + \alpha^2)^3} = \frac{1}{3}(1 - (\frac{\alpha^2}{1+\alpha^2})^2).$$

As Vickrey observed, for the limiting case in which $\alpha = 0$, so that beliefs are symmetric, we have

$$R_0(0) = R_H(0) = 1/3.$$

The two auctions therefore generate the same expected revenue. However, as is readily confirmed, $R_0(\alpha)$ is a strictly decreasing function of α which approaches 1/4 as α approaches 1. Since we have already shown that $R_H(\alpha)$ is a strictly increasing convex function the main claim is established. That is, <u>with beliefs asymmetric and uniform, the sealed high bid auction generates strictly greater expected revenue than the open auction</u>.

Given the convexity of $R_H(\alpha)$ and the extent of the reductions in $R_0(a)$ as α rises towards unity, it is tempting to conclude that the net gain to adopting the high bid auction will increase relatively rapidly with α.

To see that this is the case we note that for the sealed high bid auction, the distribution function for the winning bid can be rewritten as

$$G_H(b;\alpha) = \frac{1}{8\alpha^3}[\frac{2\alpha}{1-2\alpha b} + \frac{2\alpha}{1+2\alpha b} - \frac{4\alpha}{1+(2\alpha b)^2}].$$

Each term in the bracketed expression is easily integrated. Then substituting into (14) we obtain

(17) $$R_H(\alpha) = \frac{1}{2} - \frac{(1-\alpha^4)}{8\alpha^3}(\log(\frac{1+\alpha}{1-\alpha}) - 2\tan^{-1}(\alpha)).$$

From (16) and (17) the results presented in Table 1 are readily computed.

α^2	R_0	R_H	Percentage Gain
.0	.3333	.3333	0
.25	.3290	.3392	6
.33	.3125	.3443	10
.4	.3061	.3510	14
.45	.3012	.3533	17
.5	.2962	.3590	21
.9	.2585	.4411	71
1.0	.25	.5	100

Table 1: Comparison of the Sealed High Bid and Open Auction

Note that even with $\alpha^2 = .33$, so that v_1 is distributed uniformly on $[0,3/2]$ and v_2 is uniformly distributed on $[0,3/4]$, the percentage gain is far from being insignificant.

We now consider an example with n buyers. Once again the form of the asymmetry is chosen so that it is possible to solve explicitly for the equilibrium bid functions. We assume that each of the n buyers' beliefs about the other $n-1$ buyers valuations are given by the following cumulative distribution functions.

$$(18) \qquad F_i(v_i) = \left(\frac{(s-\theta_i)v_i}{1+s-\theta_i}\right)^{\theta_i}, \quad v_i \in \left[0, \frac{1+s-\theta_i}{s-\theta_i}\right] \quad i=1,\ldots,n$$

where

$$(19) \qquad \theta_1 < \theta_2 < \ldots < \theta_n,$$

and

$$(20) \qquad s \equiv \sum_{i=1}^{n} \theta_i.$$

We shall confirm that the bidding strategies

$$(21) \qquad b_i(v_i) = \frac{(s-\theta_i)v_i}{1+s-\theta_i} \quad i = 1,\ldots,n$$

are equilibrium bidding strategies for the sealed high bid auction.

Suppose that buyers $2,\ldots,n$ adopt the bidding strategies given by (21). From (18) and (21) the c.d.f. of $b \equiv \max\{b_2,\ldots,b_n\}$ is

$$(22) \qquad G_1(b) = \prod_{j=2}^{n} b^{\theta_j} = b^{s-\theta_1}$$

If buyer 1, with valuation v_1, bids b_1 he wins $v_1 - b_1$ if all other buyers bid less. Thus his expected gain is

$$\Pi_1(b_1;v_1) = (v_1-b_1)G_1(b_1),$$
$$= (v_1-b_1)b_1^{s-\theta_1}.$$

Differentiating by b_1 we obtain,

$$\frac{\partial \Pi_1}{\partial b_1}(b_1;v_1) = (\frac{s-\theta_1}{1+s-\theta_1} v_1 - b_1)(1+s-\theta_1)b_1^{s-\theta_1-1}.$$

Then $\Pi_1(b_1;v_1)$ has its global maximum at

$$b_1 = (\frac{s-\theta_1}{1+s-\theta_1})v_1.$$

Thus with buyers $2,\ldots,n$ adopting the bidding strategies given by (21), buyer 1's optimal response is to adopt (21) also. Given the symmetry of the example the same must hold for buyers $2,\ldots,n$. Then (21) indeed describes equilibrium bidding strategies of the n buyers in the sealed high bid auction.

Again from (18) and (21) the c.d.f. of

$$b_o = \max\{b_1,\ldots,b_n\}$$

is

$$G_o(b_o) = \prod_{j=1}^{n} b_o^{\theta_j} = b_o^s$$

Then the expected value of the successful bid is

$$\int_0^1 b_o dG_o(b_o) = \frac{s}{1+s}$$

Suppose, for concreteness, that $\theta_n = 1$. Then

$$\frac{s}{1+s} \equiv \frac{1 + \sum_{j=1}^{n-1} \theta_j}{2 + \sum_{j=1}^{n-1} \theta_j} > \frac{1}{2}$$

Thus, in the high bid auction, the expected revenue of the seller is bounded from below by one half.

We now consider the open auction, or equivalently, the sealed "second bid" auction. As with symmetric beliefs each buyer has an incentive to bid his own reservation value. Then the expected revenue of the seller is just the expected value of the second highest reservation value.

Define

$$\hat{v} = \max\{v_1,\ldots,v_{n-1}\}$$

From (18) the c.d.f. of $\hat{v}$ is

$$H(\hat{v}) = \prod_{j=1}^{n-1} \left(\frac{(s-\theta_j)\hat{v}}{1+s-\theta_j}\right)^{\theta_j}$$

$$= A(\theta)\hat{v}^{s-\theta_n}$$

where

$$A(\theta) = \prod_{j=1}^{n-1} \left(\frac{s-\theta_j}{1+s-\theta_j}\right)^{\theta_j}$$

Given (19), it follows that as $\theta_{n-1} \downarrow 0$ $A(\theta) \uparrow 1$ and $s-\theta_n \downarrow 0$. Hence, for all $\hat{v} > 0$, $H(\hat{v}) \uparrow 1$. Thus as $\theta_{n-1} \downarrow 0$ the probability that the largest of the valuations of the first $n-1$ buyers is less than $\hat{v}$ approaches 1. Since this holds for all $\hat{v} > 0$, it follows that the expected value of the second highest valuation must approach zero as $\theta_{n-1} \downarrow 0$.

We have therefore established that, as $\theta_{n-1} \downarrow 0$, the expected seller revenue from the second bid auction approaches zero. Since we have already established that the expected value from the high bid auction is bounded from below by 1/2, it follows that the proportional gain to adopting the high bid auction has no upper bound.

From this and the previous example it seems reasonable to conjecture that, for a much broader family of distributions than those considered here, the high bid auction will generate a significantly larger expected revenue than the open auction, whenever divergence from symmetry of beliefs is non-negligible. However, we now show that the reverse ranking of the two auctions is also possible.

II. A Discrete Example

Suppose there are just 2 buyers. Buyer i has valuation v_i with probability π_i, otherwise places no value on the object for sale. The seller announces that he will sell at the bid price to the buyer submitting the highest sealed bid in excess of the "reserve" price r, where $r < v_i$, $i = 1,2$. Obviously buyer i will only submit a bid if his valuation is v_i. In this case suppose that his bid distribution is $F_i(b)$.

We begin by showing that the support of F_i (supp F_i) must contain r. To see this suppose that

$$\min\{\min \operatorname{supp} F_1, \min \operatorname{supp} F_2\} = \min \operatorname{supp} F_1 = b^* > r.$$

Suppose b^* were an atom for F_2. If $b^* > v_1$ a v_1 bidder could bid less than v_1 and obtain a positive expected payoff. Then b^* could not be an atom for F_1. If $b^* < v_1$ a v_1-buyer could slightly lower his bid and discontinuously raise his payoff. Then again b^* could not be an atom for F_1. But with b^* not an atom for F_1, a v_2-buyer could bid $\frac{1}{2}(r+b^*)$ and have the same probability of winning as bidding b^*. Hence b^* cannot be an atom for F_2. But then a v_1-buyer could bid $\frac{1}{2}(r+b^*)$ and raise his payoff above that from b^*, a contradiction. Hence if

$$\min\{\min \operatorname{supp} F_1, \min \operatorname{supp} F_2\} = \min \operatorname{supp} F_1,$$

then $\min \operatorname{supp} F_1 = r$. Now suppose that $\min \operatorname{supp} F_2 > \min \operatorname{supp} F_1 = r$. If there exist $\tilde{b} \in \text{supp } F_1 \quad (r, \min \text{supp } F_2)$, then a v_1-buyer could bid $\frac{1}{2}(r+\tilde{b})$ and have the same probability of winning as from bidding $\tilde{b}$. Hence $\text{supp } F_1 \quad (r, \min \text{supp } F_2) = \phi$. But then a v_2-buyer is better off bidding $\frac{1}{2}(\min \text{supp } F_2 + r)$ than $\min \text{supp } F_2$. We conclude that $\min \text{supp } F_1 = \min \text{supp } F_2 = r$.

Suppose next that $F_2(r) > 0$. Then buyer 1 is strictly better off bidding $r + \varepsilon$ than r, for ε sufficiently small. Thus $F_2(r) > 0 \rightarrow F_1(r) = 0$.

To summarize, we have established that

$$(23) \qquad \begin{aligned} &(v_1-b)[(1-\pi_2) + \pi_2 F_2(b)] = (v_1-r)(1-\pi_2(1-F_2(r))), \ b \in \text{supp } F_1 \\ &(v_2-b)[(1-\pi_1) + \pi_1 F_1(b)] = (v_2-r)(1-\pi_1(1-F_1(r))), \ b \in \text{supp } F_2 \end{aligned}$$

where

$$(24) \qquad \min\{F_1(r), F_2(r)\} = 0$$

Let $\bar{b}_i = \max \text{supp } F_i$ so that $F_i(\bar{b}_i) = 1$. If, say, $\bar{b}_1 < \bar{b}_2$, then for $\varepsilon > 0$ and sufficiently small, the bid $\bar{b}_2 - \varepsilon$ has a probability 1 chance of winning, so no one would ever bid $\bar{b}_2$. Hence $\bar{b}_2 = \bar{b}_1 = \bar{b}$. From (23)

$$(25a) \qquad v_1 - \bar{b} = (v_1-r)(1-\pi_2+\pi_2 F_2(r))$$

$$(25b) \qquad v_2 - \bar{b} = (v_2-r)(1-\pi_1+\pi_1 F_1(r)).$$

We have already argued that $F_1(r)$ and $F_2(r)$ cannot both be strictly positive. Suppose $F_2(r) = 0$ and $F_1(r) > 0$. Then, from (25)

$$\begin{cases} v_1 - \bar{b} = (v_1-r)\ (1-\pi_2) \\ v_2 - \bar{b} > (v_2-r)\ (1-\pi_1) \end{cases}$$

Subtracting the second of these conditions from the first and rearranging we obtain

$$\pi_2(v_1-r) < \pi_1(v_2-r).$$

An identical argument establishes that this inequality is reversed if $F_1(r) = 0$. Therefore,

$$\text{sign}\{F_2(r) - F_1(r)\} = \text{sign}\{\pi_2(v_1-r) - \pi_1(v_2-r)\}. \tag{26}$$

If follows immediately from (25) and (26) that

$$\bar{b} = r + \min[\pi_1(v_2-r),\pi_2(v_1-r)] \tag{27}$$

Conditions (25) and (27) implicitly define $\max\{F_1(r),F_2(r)\}$. Also, from (23) we have

$$\begin{aligned} 1 - \pi_2 + \pi_2F_2(b) &= \frac{v_1-\bar{b}}{v_1-b} \\ 1 - \pi_1 + \pi_1F_1(b) &= \frac{v_2-\bar{b}}{v_2-b} \end{aligned} \tag{28}$$

Conditions (25), (27) and (28) then completely characterize an equilibrium (in fact the <u>unique</u> equilibrium).

We now consider expected seller revenue. From buyer 1 the seller expects to receive

$$H_1 = \pi_1 \int_r^{\bar{b}} b(1-\pi_2+\pi_2F_2(b))F_1'(b)db.$$

Substituting from (28) we can rewrite H_1 as

$$H_1 = \int_r^{\bar{b}} \frac{(v_1-\bar{b})(v_2-\bar{b})bdb}{(v_1-b)(v_2-b)^2} \quad . \tag{29}$$

But

$$\frac{b}{(v_1-b)(v_2-b)^2} = \frac{v_1}{(v_2-v_1)^2}\left[\frac{1}{v_1-b} - \frac{1}{v_2-b}\right] - \frac{v_2}{(v_2-v_1)(v_2-b)^2} \quad .$$

Substituting into (29) and integrating we obtain the following expression for the first buyer's expected payments.

$$(30) \qquad H_1 = \frac{v_1(v_1-\bar{b})(v_2-\bar{b})}{(v_2-v_1)^2} \log \left[\frac{(v_1-r)(v_2-\bar{b})}{(v_2-r)(v_1-\bar{b})}\right] - \frac{v_2(v_1-\bar{b})}{(v_2-v_1)} \frac{(\bar{b}-r)}{(v_2-r)}$$

Similarly, expected revenue from buyer 2 is

$$(31) \qquad H_2 = \pi_2(1-\pi_1)rF_2(r) + \pi_2 \int_r^{\bar{b}} b(1-\pi_1+\pi_1F_1(b))F_2'(b)db$$

$$= \pi_2(1-\pi_1)rF_2(r) + \int_r^{\bar{b}} \frac{(v_1-\bar{b})(v_2-\bar{b})bdb}{(v_1-b)^2(v_2-b)}$$

$$= r(1-\pi_1)\left[\frac{v_1-\bar{b}}{v_1-r} - (1-\pi_2)\right]$$

$$+ \frac{v_2(v_1-\bar{b})(v_2-\bar{b})}{(v_2-v_1)^2} \log \left[\frac{(v_2-r)(v_1-\bar{b})}{(v_1-r)(v_2-\bar{b})}\right] - \frac{v_1(v_2-\bar{b})(\bar{b}-r)}{(v_1-v_2)(v_1-r)} \quad .$$

Adding (30) and (31), and rearranging, expected seller revenue, H, can be expressed as

$$(32) \qquad H = \frac{1}{v_2-v_1}\left[(v_1-\bar{b})(v_2-\bar{b})\log \left[\frac{(v_2-r)(v_1-\bar{b})}{(v_1-r)(v_2-\bar{b})}\right]\right] + \frac{(\bar{b}-r)(v_1v_2-\bar{b}r)}{(v_1-r)(v_2-r)}$$

$$+ r(1-\pi_1)\left[\frac{v_1-\bar{b}}{v_1-r} - (1-\pi_2)\right]$$

In the open or "second bid" auction the object sells for r if one buyer has a positive valuation and for $\min\{v_1,v_2\}$ if both have positive valuations. Expected seller revenue is then

$$(33) \qquad S = [\pi_1(1-\pi_2) + \pi_2(1-\pi_1)]r + \pi_1\pi_2\min\{v_1,v_2\}$$

We are now in a position to compare expected revenue in the two auctions. To simplify matters we begin by considering the limiting case as the reserve price r is made very small.[7]

Suppose first that

$$(34) \qquad \frac{\pi_2-\pi_1}{v_2-v_1} > \frac{\pi_1}{v_1}$$

In terms of Figure 1, (π_2,v_2) lies on or above the line AOB. Rearranging (34) we obtain

$$\pi_1 v_2 < \pi_2 v_1$$

Then, from (27), since $r = 0$

$$(35) \qquad \bar{b} = \pi_1 v_2$$

It is also convenient to define

$$(36) \qquad k \equiv v_2/v_1$$

Then substituting for $\bar{b}$ and v_2 in (33), the difference in expected revenue can be written as

$$(37) \qquad H - S = \frac{v_1 k(1-\pi_1)(1-k\pi_1)}{k-1} \log\left(\frac{1-\pi_1 k}{1-\pi_1}\right) + \pi_1 v_2 - \pi_1\pi_2 \min\{v_1,v_2\}$$

$$= v_1 k(1-\pi_1)\pi_1 D\left(\frac{1-\pi_1 k}{1-\pi_1}\right) + \pi_1^2 k v_1 - \pi_1\pi_2 \min\{v_1,v_2\}$$

where

$$(38) \qquad D(x) = \frac{x\log x}{1-x} + 1$$

Appealing to l'Hôpital's Rule,

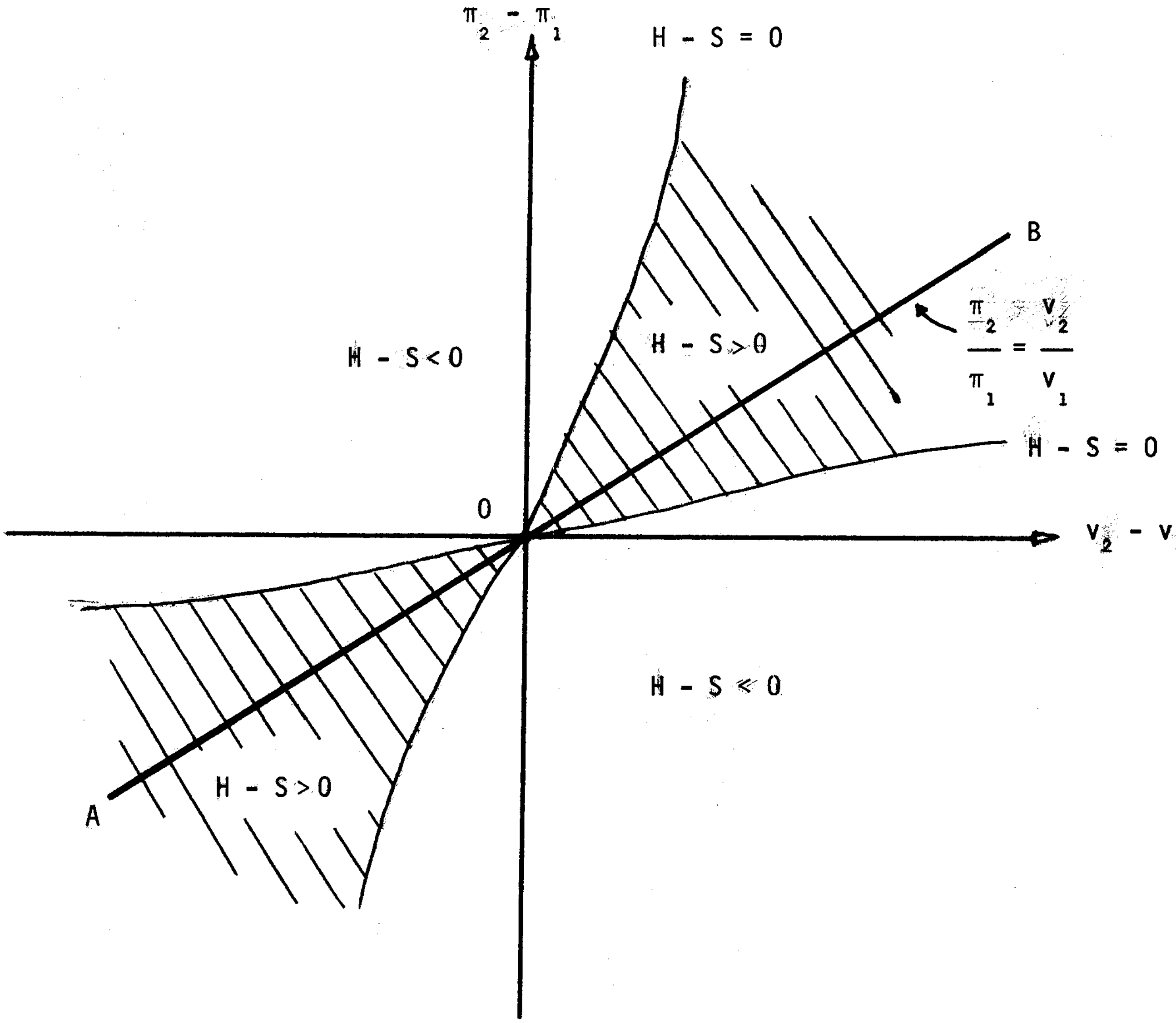

Figure 1: Comparison of expected returns from the high and second bid auctions.

$$D(1) = 1 + \lim_{x\to 1} \frac{\frac{d}{dx}\, x\log x}{\frac{d}{dx}\,(1-x)}$$
$$= 0$$

Also, differentiating (38) we obtain

(39) $$D'(x) = \frac{1-x+\log x}{(1-x)^2}$$

It is readily confirmed that the numerator is negative except at $x = 0$. Then $D(x)$ is a strictly decreasing function changing sign at $x = 1$. Then, setting $x = (1-\pi_1 k)/(1-\pi_1)$ we can conclude that

(40) $D(\frac{1-\pi_1 k}{1-\pi_1})$ is a strictly increasing function of k which changes sign at $k = 1$.

If v_2 is no greater than v_1, so that $k < 1$, (37) reduces to

(41) $$H - S = v_1 k[\pi_1(1-\pi_1)D(\frac{1-\pi_1 k}{1-\pi_1}) + \pi_1(\pi_1-\pi_2)],\ k < 1$$

First of all, along AO (34) holds with equality, that is $k = \pi_2/\pi_1$. Substituting into (41) we obtain

(42) $$H - S = v_1 k\pi_1(1-\pi_1)[D(\frac{1-\pi_1 k}{1-\pi_1}) + \frac{\pi_1}{1-\pi_1}(1-k)]$$

$$= v_1 k\pi_1(1-\pi_1)[D(x) + x - 1],$$

where $x = (1-\pi_1 k)/(1-\pi_1) > 1$, since $k < 1$. Substituting from (38)

$$D(x) + x - 1 = \frac{x\log x}{1-x} + x = \frac{x}{1-x}[\log x + 1 - x]$$

The term in parentheses is negative except at $x = 1$. Then, for $k < 1$ so that $x > 1$, $D(x) + x - 1$ is positive. Then $H - S$ is positive.

Note also that the bracketed expression in (41) is strictly decreasing in π_2. Then, for π_2 sufficiently large $H - S$ is negative. Moreover the bracketed expression is strictly increasing in k and hence v_2. Then the boundary of the region above the line segment AO over which $H - S$ is positive, has an everywhere positive slope. Since the two auctions generate the same revenue in the symmetric case, this curve must also pass through the origin O.

We next suppose v_2 exceeds v_1 so that $k > 1$. Expression (37) then reduces to

$$H - S = v_1[k\pi_1(1-\pi_1)D(\frac{1-\pi_1 k}{1-\pi_1}) + \pi_1(k\pi_1-\pi_2)], \ k > 1 \tag{43}$$

Along OB the second term in the bracketed expression is zero and, from (40), the first term is positive. Then, as depicted, $H - S$ is positive. Once again the bracketed expression is strictly increasing in k and strictly decreasing in π_2. Then, above the line segment OB the boundary of the region over which $H - S$ is positive has an everywhere positive slope.

It remains to consider all those points in the Figure below the line AOB. Since the equilibrium expected revenue is, for both auctions, symmetric in (π_1, v_1), (π_2, v_2) all the above arguments continue to hold, except in opposite quadrants.

The boundary curves below AOB must therefore have the depicted slopes with $H - S$ strictly positive in the interior of the shaded region.

One feature of this discrete example which distinguishes it from the continuous examples of the previous section is that, in general, one buyer bids the minimum price with finite probability. In order to avoid this occurring in the discrete case we must choose parameter values so that $F_1(v) = F_2(r) = 0$. With $r = 0$ it follows from (26) that $\pi_1 v_2 = \pi_2 v_1$ and hence that condition (34) holds so that the auction is on the line AOB. Thus, in this closest counterpart to the earlier continuous examples, the high bid auction continues to dominate.

However, as the Figure makes clear, there are a wide variety of parameter pairs for which the ranking is reversed. Note that even if $v_2 > v_1$ and $\pi_2 > \pi_1$ so that F_2 exhibits first order stochastic dominance over F_1, it remains possible for the second bid auction to generate greater expected revenue.

III. Characterization of Equilibrium Bidding

In order to make any general comparison of the two auctions we must first characterize equilibrium bidding in the sealed high bid auction. Let v_i, buyer i's valuation, be a random draw from the distribution $F_i(\cdot)$ where $F_i(0) = 1 - F_i(\bar{v}_i) = 0$ and F_i is a twice continuously differentiable strictly increasing function over $[0,\bar{v}_i]$ $i = 1,\ldots,n$. Elsewhere in Maskin and Riley [1982c] we consider a broad family of auctions in which the payoff to winning the auction, with parameter v and bid b is some increasing function $U(-b;v)$. From Lemmas 1 and 2 of that paper we have the following result.

Proposition 1: Characterization of equilibrium bidding.

Suppose buyer i with parameter value v_i who pays b for the object has an increase in utility of $U(-b,v_i)$, where v_i is a random draw from

$F_i(.)$ and both U and F_i, $i = 1,\ldots,n$ are strictly increasing twice continuously differentiable functions.

Suppose further that

$$U > 0 \rightarrow \frac{\partial}{\partial v}\left(\frac{U_1}{U}\right) < 0$$

Then if $\tilde{b}_i(v_i)$, $i = 1,\ldots,n$ is an equilibrium bidding strategy (possibly a mixed strategy), $\tilde{b}_i(v_i)$ is strictly increasing when it exceeds the reserve price r. Moreover if $n = 2$ $\tilde{b}_i(v_i)$ is a continuous function.

For the simple case considered in this paper

$$U(-b,v_i) = v_i - b$$

and all the conditions of the proposition are satisfied. With only two buyers an argument almost identical to that in the previous section establishes that both cannot make the minimum bid, r, with positive probability. Moreover both must have the same maximum bid $\bar{b}$. Then equilibrium bid functions can be characterized as follows

$$b_i(v_i) \begin{cases} = r & v_i \in [r,\hat{v}_i], \quad \min_{i=1,2}\{\hat{v}_i\} = r \\ > r & \text{and strictly increasing for } v_i > \hat{v}_i \end{cases} \tag{39}$$

Further arguments in Maskin and Riley [1982c], and indeed in Griesmer, Levitan and Shubik [1967] imply that for $n=2$ the inverse bid functions

$$v_i = \phi_i(b), \quad b > r, \quad \phi_i = b_i^{-1}(.)$$

are differentiable. We now ask under what conditions bid functions are strictly increasing for all $v_i > r$ so that there is no mass point at r.

Suppose this to be the case. Then, from (5)

$$(\phi_1-b)F_2'\phi_2' = F_2 \quad \text{and} \quad (\phi_2-b)F_1'\phi_1' = F_1.$$

By hypothesis $\phi_1(b) \downarrow r$ as $b \downarrow r$. Then

$$\lim_{b\downarrow r} \phi_1'(b) = \infty$$

Hence if $\lim_{b\downarrow r} \frac{\phi_2'}{\phi_1'}$ exists

$$\lim_{b\downarrow r} \left(\frac{\phi_2'-1}{\phi_1'-1}\right) = \lim_{b\downarrow r} \frac{\phi_2'}{\phi_1'}$$

But

$$\frac{\phi_2'}{\phi_1'} = \frac{F_2(\phi_2(b))F_1'(\phi_1(b))}{F_1(\phi_1(b))F_2'(\phi_2(b))} \frac{(\phi_2-b)}{(\phi_1-b)} \quad .$$

Applying l'Hopital's Rule

$$\lim_{b\downarrow r} \frac{\phi_2'}{\phi_1'} = \frac{F_2(r)F_1'(r)}{F_1(r)F_2'(r)} \lim_{b\downarrow r} \frac{\phi_2'-1}{\phi_1'-1} \quad .$$

But then it must be the case that

$$(40) \qquad \frac{F_1'(r)}{F_1(r)} = \frac{F_2'(r)}{F_2(r)} \quad \text{or} \quad F_1(r) = F_2(r) = 0.$$

Since both the examples in Section 1 satisfy the second of these two conditions it seems quite possible that the results are somewhat special. Moreover we cannot rule out the counter example in Section 2 simply by characterizing the equilibrium in the continuous case. The obvious, but challenging, next step will be to attempt a general revenue comparison when condition (40) holds.

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FOOTNOTES

[1]This was independently derived by Myerson [1981] and Riley and Samuelson [1981]. Twenty years earlier Vickrey [1961] established the equivalence of the two common auctions (open and sealed high bid) under the assumption that valuations were independent draws from a uniform distribution.

[2]With three or more buyers these inequalities are strict under only weak additional assumptions.

[3]We shall assume that for all v such that $0 < F_i(v) < 1$, F_i is a strictly increasing differentiable function. Then as long as the equilibrium bid is strictly increasing with v we can ignore ties.

[4]We conjecture that the arguments used by Riley and Maskin [1982c] to establish existence and uniqueness with symmetric beliefs will be applicable to the case of asymmetric beliefs as well.

[5]If $F_i(x_i) = a_i x_i$ define $v_i = (a_1+a_2)x_i/2$ and $\alpha^2 = (a_2-a_1)/(a_1+a_2)$. Then v_1 and v_2 are distributed according to (6).

[6]In general, with v_i distributed on $[0,\bar{v}_i]$ and $\bar{v}_1 < \bar{v}_2$ we must have $\psi_1(v) < \psi_2(v)$ for sufficiently large valuations, since $\psi_i(\bar{v}_i) = \bar{b}$. However it is conjectured that this inequality need not always hold for all v.

[7]In the limit itself, with $r=0$, a buyer with a zero valuation is indifferent between not bidding and bidding the reserve price. As long as each buyer chooses the former strategy the equilibrium bidding by buyers with positive valuations is simply that characterized above with $r=0$.