REGULARITY IN OVERLAPPING GENERATIONS EXCHANGE ECONOMIES*

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1. INTRODUCTION

Debreu's [1970] theory of the regularity of a smooth finite horizon exchange economy has played an important role in understanding the comparative statics of general equilibrium models. In this paper we develop a regularity theory for stationary overlapping generation exchange economies.

The theory for steady states is developed in section three. We show that generically there are an odd number of steady states on which old people have a non-zero initial endowment of nominal debt (fiat money) and an odd number in which they have no endowment of nominal debt. Generically, these latter steady states have price levels which explode either to zero or to infinity. However, we are also interested in non-steady state perfect foresight paths. As a first step in this direction we analyze the behavior of paths near a steady state. We show that generically they are given by a second order difference equation which satisfies strong regularity properties. We also show that economic theory alone imposes little restriction on these paths: with n-goods, for example, the only restriction on the set of paths converging to the steady state is that they be a manifold of dimension no less than one, no more than $2n$.

The regularity theory we develop here can be applied to analyze the response of an overlapping generations economy to unanticipated shocks. In another paper [1982] we consider the impact of shocks under alternative assumptions about the types of contractual arrangements existing before the shock and the process by which perfect foresight forecasts are formed.

2. THE MODEL

We analyze a stationary overlapping generations model similar to that introduced by Samuelson [1958]. In each period there are n goods. Each generation $-\infty < t < \infty$ is identical and consumes in periods t and $t+1$. The consumption and savings decisions of the (possibly many different types) of consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ in period t and $z(p_t, p_{t+1})$ in period $t+1$. The vector $p_t = (p_t^1, \ldots, p_t^n)$ denotes the prices prevailing in period t. Let $\bar{Q} \quad R_+^{2n}$ be a closed convex cone with non-empty interior and a boundary that is smooth (always taken to mean C^1) except at the origin. Also assume that $q \in \bar{Q}_o \equiv \bar{Q} \{0\}$ has $q \gg 0$. Excess demand is assumed to satisfy

(A.1) $y, z\colon \bar{Q}_o \to R^n$ are smooth functions.

(A.2) (Walras's Law) $p_t' y(p_t, p_{t+1}) + p_{t+1}' z(p_t, p_{t+1}) = 0$.

(A.3) (Homogeneity) y, z are degree zero homogeneous.

Assumption (A.1) has been shown by Debreu [1972] and Mas-Colell [1974] to entail little loss of generality. Assumption (A.2) implies that each consumer faces an ordinary budget constraint in the two periods of his life. This allows the possibility of trade between generations. As we show later this is equivalent to assuming a fixed (possibly negative) stock of fiat money. Assumption (A.3) is standard.

In addition to (A.1)-(A.3), which are familiar from the finite model, we make two boundary assumptions

(B.1) If q is on the boundary of $\bar{Q}_o$ then (y,z) (viewed as a tangent vector to $\bar{Q}_o$ at q) points into the interior of $\bar{Q}_o$.

(B.2) If (for fixed $p \in R_{++}^n$) $\beta \in R_{++}$ cannot be decreased (increased) without violating $(p, \beta p) \in \bar{Q}_o$ then $p' y(p, \beta p) < 0$ (>0).

Assumption (B.1) insures that if the price of a good falls low enough then there is excess demand for that good. Assumption (B.2) is a strengthening of (B.2): it says that if prices in the second period of life fall low enough nominal savings become positive (and vice versa). As we shall see, (B.1) and (B.2) are used to guarantee the existence of interior steady states. Although the theory can be extended to analyze free goods we do not attempt to do so here.

Note that we consider only pure exchange economies and two period lived consumers. We do, however, allow many goods and types of consumers, and the multi-period consumption case can easily be reduced to the case we consider: If consumers live m periods, we simply redefine generations so that consumers born in periods $1,2,\ldots,m-1$ are generation 1, consumers born in periods $-m+2$, $-m+3,\ldots,0$ and $m,m-1,\ldots,2m-2$ are generations 0 and 2 respectively, and so forth. In this reformulation each generation overlaps only with the next generation.

The space of feasible economies $\bar{E}$ are the pairs (y,z) which satisfy (A.1)-(A.3) and (B.1)-(B.2). This is a topological space in the weak C^1 topology described by Hirsch [1976]. Roughly, two economies (y,z) and (y',z') are close if the functions and their first derivatives are close.

It follows from the work of Debreu [1974] and Mantel [1974] that every economy arises as the aggregate excess demand of some group of consumers with convex preferences and budget constraints of the form (A.2).

3. STEADY STATES

A <u>steady state</u> for an economy $(y,z) \in \bar{E}$ is a relative price vector $p \in R^n_+$ and price level growth rate $\beta > 0$ such that $(p,\beta p) \in \bar{Q}_o$ and $z(p,\beta p) + y(\beta p,\beta^2 p) = 0$. By homogeneity this is equivalent to

$z(p,\beta p) + y(p,\beta p) = 0$. In other words, if each period relative prices are given by p and the price level grows at β the market is always in equilibrium. Since claims to good i now cost p^i and claims to good i next period cost βp^i, $\beta-1$ is the steady-state rate of interest.

The nominal steady state savings for the entire economy is $\mu \equiv -p'y(p,\beta p)$. There are two kinds of steady states: real steady states in which $\mu = 0$ and monetized steady states in which $\mu \neq 0$. Gale [1973] refers to real steady states as "balanced." By Walras' law $p'(y+\beta z) = 0$ so that $\beta p'z = -p'y = \mu$. By the equilibrium condition $p'(z+y) = 0$ so $p'z = \mu$. Thus $(\beta-1)\mu = 0$ and in a monetized steady state the interest rate must be zero. We shall see that a real steady state has $\beta = 1$ purely by coincidence. Thus we shall refer to a steady state with $\beta = 1$ as a <u>nominal</u> steady state. Gale refers to these as "golden rule" steady states since they maximize a weighted sum of utilities subject to the steady state constraint.

Only the relative prices in p matter to the steady state and it is natural to normalize these to relative prices by means of a price index. Let $\bar{P}_o \equiv \{p \in R^n_+,\ \beta \in R_+ | (p,\beta p) \in \bar{Q}_o\}$. A <u>price index</u> $h\colon \bar{Q}_o \to R_+$ is a smooth strictly increasing positive function homogeneous of degree one. For each price index h there is a (compact) space of normalized prices $\bar{P}^h_o \equiv \{(p,\beta) \in \bar{P}_o | h(p,\beta p) = 1\}$. For example if $h(q) = q^1$ then good one in period one is numeraire. Note, incidentally, the assumption that $\bar{Q}$ is closed with $\bar{Q}_o$ having strictly positive prices implies that all relative prices (β in particular) are bounded: this is why $\bar{P}^h_o$ is always compact.

Because $\bar{Q}$ is a cone all the normalized price spaces $\bar{P}^h_o$ are naturally diffeomorphic: they constitute different embeddings of a single n-manifold $\bar{P}^*_o$ in $\bar{P}_o$. We refer to $\bar{P}^*_o$ as <u>the</u> space of <u>normalized</u> prices. Since $\bar{P}_o$ is convex we may view $\bar{P}^*_o$ as convex also. The advantage of this point of view is that we need not precommit ourselves to a particular price index, but can choose the index (embedding) most convenient for a particular proof. Because of the homogeneity assumption excess demand are well-defined functions of $(p,\beta) \in \bar{P}_o^*$.

We now examine the number of steady states. Let us first separate the nominal and real cases. If both $\beta = 1$ and $\mu = -p'y = 0$ at a steady state this is characterized by the equations

$$(3\text{-}1) \qquad \begin{aligned} z(p,p) + y(p,p) &= 0 \\ p'y(p,p) &= 0. \end{aligned}$$

By virtue of Walras's law the first n equations $z + y$ may be viewed as a system of $n-1$ equations, while by homogeneity p constitute $n-1$ independent variables. Thus (3-1) may be regarded as n equations in $n-1$ unknowns. Let us therefore assume

(R.1) The system (3-1) has no solution.

The importance of this regularity assumption is that it is generic.

Proposition (3-1): The set of economies satisfying (R.1) is open dense.

<u>Proof:</u> Openness is obvious. To prove density let $v_1 \in R^n$, $v_2 \in R$ and construct the perturbation

$$y^i(v) = y^i + \frac{\Sigma_{j=1}^n p_1^j v_1^j}{\Sigma_{j=1}^n p_1^j} - v_1^i + \frac{p_2^i}{p_1^i} v_2$$

$$z^i(v) = z^i - v_2.$$

A check shows for v small enough that $(y(v),z(v)) \in \bar{E}$; that is (A.1)-(B.2) are satisfied. To show that (R.1) is dense, it suffices by parametric transversality to show that the derivative of (3-1) with respect to v has rank n whenever it vanishes. Writing out this derivative we have

$$\begin{vmatrix} ep'/(e'p - I) & 0 \\ 0 & e'p \end{vmatrix}$$

where e is a vector of ones. This matrix has rank n as required.

Q.E.D.

Now we examine nominal steady states. Such steady states are characterized by $z(p,p) + y(p,p) = 0$. Since $z(p,p) + y(p,p)$ has the formal properties of the excess demand function of an n good pure exchange economy the theory of nominal steady states can be derived directly from the finite theory. Existence of a nominal steady state is immediate and from (B.1) it lies in the interior. From Dierker (1972) the relevant regularity assumption is

(R.2) $D_1z(p,p) + D_2z(p,p) + D_1y(p,p) + D_2y(p,p)$ has rank $n-1$ at nominal steady states.

This implies an odd number of normalized nominal steady states. Furthermore since the map from $\bar{E}$ to n good exchange economies is a continuous open map (R.2) is generic in $\bar{E}$.

We turn now to the real steady states. These may be characterized by the equations

$$z(p,\beta p) + y(p,\beta p) = 0 \tag{3-2}$$
$$p'y(p,\beta p) = 0.$$

By Walras's law we also have $p'z(p,\beta p) = 0$ at the steady state and thus (3-2) is equivalent to

$$(I-pp')\ (z(p,\beta p)+y(p,\beta p)) = 0 \tag{3-3}$$
$$p'y(p,\beta p) = 0$$

By choosing the price index $h(q) = \Sigma_i\ q_i^2$ we see that (3-3) may be viewed as a vectorfield on the tangent spaces of $\bar{P}^*_o$. By (B.1) and (B.2) this vectorfield points inwards on the boundary. Thus the vectorfield vanishes and a real steady state exists. The relevant regularity condition is

$$\begin{vmatrix} (I-pp')\ (D_1z+\beta D_2z+D_1y+\beta D_2y) & (I-pp')\ (D_2z+D_2y)p \\ y' + p'(D_1y+\beta D_2y) & p'D_2yp \end{vmatrix} \tag{R.3}$$

has rank n.

From index theory these economies have a finite odd number of normalized steady states. We also need

<u>Proposition (3-2)</u>: Under (R.1) (R.3) is also generic.

<u>Proof</u>: The openness of R.3 is immediate from the stability of transverse intersection and the continuity of the derivatives of (y,z). To prove

density define for $v \in R^n$

$$y^i(v) \equiv y^i + v^i$$
$$z^i(v) \equiv z^i - (p_1^i/p_2^i)v^i.$$

A check shows $(y(v),z(v)) \in E$. Also $z^i(v) + y^i(v) = z^i + y^i + (1-\beta^{-1})v$. Differentiating (3-3) with respect to v we get

$$\begin{vmatrix} (1-\beta^{-1})\ (I-pp') \\ (1-\beta^{-1})p' \end{vmatrix}$$

By (R.1) $\beta \neq 1$ and this matrix has rank n so the lemma follows from parametric tranversality. Q.E.D.

Let $\bar{E}^R$ be the subset of $\bar{E}$ which satisfies (R.1)-(R.3). We can summarize this discussion by

Proposition (3-3): $\bar{E}^R$ is open dense in $\bar{E}$. Each economy in $\bar{E}^R$ has an odd number of real and of nominal normalized steady states and no real steady state has $\beta = 1$. Furthermore the number of steady states of each type are constant on connected components of $\bar{E}^R$ and vary continuously with the economy.

In the sequel we will wish to show that for a generic economy certain properties are satisfied at all steady states. Mathematically, it is more convenient to prove that for a generic economy these properties are satisfied at a particular steady state. A useful fact about regular economies is that the latter property implies the former. To formalize this let $\bar{F}^R$ $\bar{E}^R \times \bar{P}^*_o$ be the set of (y,z,p,β) for which (p,β) is a steady state of (y,z). Let $\bar{F}^G$ be open dense in $\bar{F}^R$. Define $\bar{E}^G$ to be the subset of $\bar{E}^R$ such that if $(y,z) \in \bar{E}^R$ and $(y,z,p,\beta) \in \bar{F}^R$ then $(y,z,p,\beta) \in \bar{F}^G$, that is, such that all steady states have the property $\bar{F}^G$. It follows directly from Proposition (3-3) and the fact that finite intersections of open dense sets are open dense that $\bar{E}^G$ is open dense in $\bar{E}^R$. Thus, in the sequel, we prove all theorems about genericity in $\bar{F}^R$, with the understanding that this carries over also into $\bar{E}$.

4. RESTRICTIONS ON DEMAND DERIVATIVES

We are interested in discovering the properties of the demand derivatives $y_1 \equiv D_1y(p,\beta p)$, $y_2 \equiv D_2y(p,\beta p)$, $z_1 \equiv D_1z(p,\beta p)$ and $z_2 \equiv D_2z(p,\beta p)$ at steady states (p,β). The most convenient way to do this is to introduce the jet mapping $\bar{d}\colon \bar{F}^R \to \bar{D}$ where $\bar{D}$ is a subset of the space of 6-tuples $(y_1,y_2,z_1,z_2,p,\beta)$ and the mapping $\bar{d}$ applied to an economy/steady state (y,z,p,β) yields the excess demand derivatives evaluated there.

What restrictions should we place on the 6-tuples in D? Differentiating Walras's law (A.2) we see

$$\begin{aligned} &y' + p'y_1 + \beta p'z_1 = 0 \\ &z' + p'y_2 + \beta p'z_2 = 0 \end{aligned} \tag{4-1}$$

while from the steady state condition $z' + y' = 0$. Thus elements of D should satisfy

(D.2) $$p'(y_1+y_2+\beta z_1+\beta z_2) = 0.$$

Differentiating the homogeneity condition (A.3)

(D.3) $$(y_1+\beta y_2)p = 0$$
$$(z_1+\beta z_2)p = 0.$$

Thus we define $\bar{D}$ to be the 6-tuples satisfying (D.2)-(D.3) and for which $(p,\beta) \in \bar{P}_o^*$. The following theorem implies that the space $\bar{D}$ captures all the important restrictions on demand derivatives.

Proposition (4-1): The jet mapping $\bar{d}$ is a continuous open mapping of $\bar{F}^R$ onto a dense subset of $\bar{D}$.

Thus any generic set in $\bar{D}$ is a generic property in $\bar{E}$. Furthermore any open set in $\bar{D}$ corresponds to a non-void open set in $\bar{E}$. Proposition (4-1) enables us to restrict our study entirely to the space $\bar{D}$.

Proof of Porposition (4-1): Continuity of $\bar{d}$ is obvious. To prove the remainder of the proposition we need to know how to convert elements of $\bar{D}$ into elements of $\bar{E}$. Suppose $d \in \bar{D}$. Let h be the price index for which good one when young is numeraire. Let $\hat{X}^d$ be the matrix of demand derivatives with first row and column deleted. From Walras's law in (4-1) we see we should define $y' = -p'(y_1 + \beta z_1)$ and $z' = -p'(y_2 + \beta z_2)$. Let $\hat{q}$ be the vector $(p,\beta p)$ with the first component deleted and let $\hat{x}^d(\hat{q})$ be the vector (y,z) with the first component deleted. Let $\hat{q}_t$ be an arbitrary $2n-1$ vector. We define the linear affine function $\hat{x}^d$: $R^{2n-1} \to R^{2n-1}$ by

$\hat{x}^d(\hat{q}_t) = \hat{x}^d(\hat{q}) + \hat{X}^d(\hat{q}_t - \hat{q})$. Suppose $x \in \bar{E}$ and $\hat{x}$ is the last $n-1$ components of x viewed as a function on R^{2n-1} by setting $q_1 = 1$. We now define $\hat{x}^\lambda$ to be the weighted average

$$\hat{x}^\lambda(\hat{q}_t) = \lambda(\hat{q}_t)\hat{x}^d(\hat{q}_t) + (1 - \lambda(\hat{q}_t))\hat{x}(\hat{q}_t).$$

By a construction of Hirsch if $\bar{B}$ is a ball around $\hat{q}$ of radius $\varepsilon > 0$ we may assume λ to be C^1 with $0 < \lambda < 1$, $|D\lambda| < 3/\varepsilon$, $\lambda(\hat{q}) = 1$ and for $\hat{q}_t \notin \bar{B}$ $\lambda(\hat{q}_t) = 0$. Thus $\hat{x}^\lambda$ coincides with $\hat{x}$ outside of B; but $\hat{x}^\lambda(\hat{q}) = \hat{x}^d(\hat{q})$ and $D\hat{x}^\lambda(\hat{q}) = \hat{X}^d$. Furthermore there is a unique extension of $\hat{x}^\lambda$ to x^λ: $\bar{Q}_o \to R^{2n}$ which satisfies Walras's law and homogeneity. For ε small enough x^λ and x coincide on the boundary of $\bar{Q}_o$ and the boundary assumptions are satisfied. Thus we may assume $x^\lambda \in \bar{E}$. Finally, a direct computation shows that $\bar{d}(x^\lambda,p,\beta) = d$.

Let us first use this construct to show that $\bar{d}$ is open. Let $\bar{d}(x,p,\beta) = d$ and let $d^k \to d$. Choosing $\varepsilon_k = \max\{|q^k - q|^{1/2}, |\hat{x}^{dk}(\hat{q}) - \hat{x}(\hat{q})|^{1/2}, |\hat{X}^{dk} - Dx(q)|\}$ then $\varepsilon_k \to 0$ and a computation using the mean value theorem shows $x^{\lambda k} \to x$. Since $\bar{E}^R$ is open in $\bar{E}$ $x^{\lambda k}$ is eventually in $\bar{E}^R$ and this proves $\bar{d}$ is open.

Next we show $\bar{d}(\bar{F}^R)$ is dense in $\bar{D}$. Indeed let $d \in \bar{D}$ $\bar{d}(\bar{F}^R)$. Since $x^\lambda \in \bar{E}$ there is $x^* \to x^\lambda$ with $x^k \in \bar{E}^R$. However the steady state (p,β) is itself regular in the ball $\bar{B}$ of fixed radius ε. Thus the x^k must have a steady state $(p^k,\beta^k) \to (p,\beta)$. Thus $(x^k,p^k,\beta^k) \in \bar{F}^R$ and $\bar{d}(x^k,p^k,\beta^k) \to d$.

<u>QED</u>.

It is of interest to see what (R.1) to (R.3) mean in $\bar{D}$. From (4-1) we see that $p'y = 0$ if and only if $p'(y_1 + \beta z_1)p = 0$. Thus (R.1) holds if and only if

(DR.1) $p'(y_1 + \beta z_1)p = 0$ implies $\beta \neq 1$.

Let us define $J \equiv z_1 + \beta z_2 + y_1 + \beta y_2$. From homogeneity (D.3) $Jp = 0$. The restriction (R.2) is

(DR.2) J has rank $n-1$.

Turning to (R.3) Walras's law (4-1) implies the matrix in (R.3) equals

$$\begin{vmatrix} (I - pp')J & (I - pp')(z_2 + y_2)p \\ \beta p'(y_2 - z_1) & p'y_2p \end{vmatrix}$$

and a second application of Walras's law (D.2) shows that this has the same rank as

$$\begin{vmatrix} J & (z_2 + y_2)p \\ \beta p'(y_2 - z_1) & p'y_2p \end{vmatrix} .$$

Also (D.2) implies that if $Jx = 0$ then $p'(y_2 - z_1)x = 0$. Thus a necessary condition for (R.3) is

(DR.3) J has rank $n-1$.

Let $\bar{D}^R$ be the subset of $\bar{D}$ which satisfies (DR.1)-(DR.3). It follows directly from Proposition (4-1) that $\bar{D}^R$ is open dense in $\bar{D}$.

As a final note we observe that if there is a vector $\hat{q}$ with $\hat{q}'J = 0$ and $\hat{q}'(z_2 + y_2)p \neq 0$ and J has rank $n-1$ then (R.3) is satisfied. It is straightforward to show that the former condition is generic given the latter, and thus (DR.3) is "almost" the same as (R.3).

5. PATHS NEAR STEADY STATES

A path (or perfect foresight path) is a finite or infinite sequence of prices $(\ldots,p_{t-1},p_t,p_{t+1},\ldots)$ such that $(p_t,p_{t+1}) \in \bar{Q}_o$ and $z(p_{t-1},p_t) + y(p_t,p_{t+1}) = 0$. Our goal is to find generic conditions under which paths near steady states are well behaved, which means that they should follow a "nice" second order difference equation.

Fix a steady state (p,β). As before we let the excess demand derivatives at the steady state be denoted by $y_1 \equiv D_1y(p,\beta p)$ and so forth. Note that these are homogeneous of degree minus one. The equilibrium condition can be linearized as

(5-1) $$z_1(p_{t-1}-\beta^{t-1}p) + (z_2+\beta^{-1}y_1)(p_t-\beta^t p) + \beta^{-1}y_2(p_{t+1}-\beta^{t+1}p) = 0$$

Suppose that condition

(NS1) y_2 is non-singular

holds. Then the linearized system can be solved to find

(5-2) $$(q_{t+1}-\beta^{t+1}q) = G(q_t-\beta^t q)$$

$$G = \begin{vmatrix} 0 & I \\ G_2 & G_1 \end{vmatrix}, \quad G_1 = \beta y_2^{-1}z_1 \quad \text{and} \quad G_2 = y_2^{-1}(\beta z_2+y_1).$$

where $$q = (p,\beta p), \quad q_t = (p_{t-1},p_t),$$

A direct implication of the implicit function theorem is

Proposition (5-1): Under (NS1) there is an open cone $\bar{U}$ around q and a unique function

$g: \bar{U} \to \bar{Q}_o$ which is smooth (C^1), homogeneous of degree one and such that

A. If $\{p_t\}$ is a path and $q_t, q_{t+1} \in \bar{U}$ then $q_{t+1} = g(q_t)$.

B. If $\{p_t\}$ has $q_t \in \bar{U}$ at all times and $q_{t+1} = g(q_t)$ then it is a path.

Furthermore $Dg(\beta^t q, \beta^{t+1} q) = G$.

Our goal is to establish that there are generic restrictions on the demand derivatives y_1, y_2, z_1, z_2 such that (NS1) holds and such that G is a "nice" matrix, and to prove that under these conditions g is a "nice" dynamical system.

6. RESTRICTIONS ON THE LINEARIZED SYSTEM

We are interested in discovering the properties of the linearized system as represented by the matrix G. It is convenient to work not in $\bar{D}$ or $\bar{D}^R$ but in the subset $\bar{D}^{NS}$ of $\bar{D}$ for which (NS1) holds and also

(NS2) $K \equiv y_1 + y_2 + \beta z_1 + \beta z_2$ has rank $n-1$.

Note that by Walras's law $p'K = 0$ so K can't have full rank. It is essential to know that $\bar{D}^{NS}$ is generic.

Proposition (6-1): $\bar{D}^{NS}$ is open dense in $\bar{D}$.

Proof: Openness is obvious. Density of (NS1) follows from the perturbation of d given by $y_1^\lambda = y_1 + \lambda\beta I$, $y_2^\lambda = y_2 - \lambda I$, $z_1^\lambda = z_1 - \lambda\beta I$, $z_2^\lambda = z_2 + \lambda I$,

p and β held fixed. A check of (D.2) and (D.3) shows $d^\lambda \varepsilon \bar{D}$. Further for $\lambda \neq 0$ but small enough y_2^λ is non-singular. Density of (NS2) follows from observing we may add terms of the form $\lambda(I-pp')$ (where $p'p = 1$) to (say) y_1 without violating (D.2) or (D.3). Thus $K^\lambda = K + \lambda(I - pp')$ and if $p_o'K^\lambda = 0$ then $[p_o' - (p_o'p)p'] [K + \lambda I] = 0$ implying for $\lambda \neq 0$ and small enough $p_o = (p_o'p)p$ and thus that the left null space of K^λ has dimension one. QED.

Our next step is to consider the mapping $\bar{g}^+: \bar{D}^{NS} \to \bar{G}^+$ where $\bar{G}^+$ are 6-tuples $(y_1,y_2,G_1,G_2,p,\beta)$ satisfying appropriate conditions. The map $\bar{g}^+$ is the identity on the first two and last two components while G_1 and G_2 are given by (5-2) as $G_1 = \beta y_2^{-1} z_1$ and $G_2 = y_2^{-1}(\beta z_2 + y_1)$. Since y_2 is non-singular on $\bar{D}^{NS}$ $\bar{g}^+$ is obviously continuous. Equally important it has a continuous inverse on $\bar{g}^+(\bar{D}^{NS})$ given by the identity on the first two and last two components and by

$$[z_1,z_2] = -[y_1,y_2]G/\beta$$

where $G = \begin{vmatrix} 0 & I \\ G_1 & G_2 \end{vmatrix}$ as in (5-2).

Thus $\bar{g}^+$ is a homeomorphism onto $\bar{G}^+ \equiv \bar{g}^+(\bar{D}^{NS})$. It remains to identity $\bar{G}^+$. Walras's law (D.2) holds if and only if

$(G^+.2)$ $p'y_2[I - G_1 - G_2] = 0.$

Note that this implies $[p'y_2G_1,p'y_2]G = [p'y_2G_1,p'y_2]$ and thus that G has an eigenvalue equal to one. Homogeneity (D.3) holds if and only if

$(G^+.3)$ $(y_1 + \beta y_2)p = 0$

$Gq = \beta q$

where $q = (p,\beta p)$. Thus G has an eigenvalue equal to β. Next (NS.1) is unchanged, while (NS.2) becomes

$(NS^+.2)$ $I - G_1 - G_2$ has rank $n-1$.

Thus $(G^+.2)$-$(G^+.3)$ and $(NS^+.1)$-$(NS^+.2)$ uniquely characterize $\bar{G}^+$.

Finally we focus in on G itself considering $\bar{g}$: $\bar{G}^+ \to \bar{G}$ where $\bar{G}$ are certain 4-tuples (G_1,G_2,p,β) and $\bar{g}$ is the projection map. Thus $\bar{g}$ is continuous, and we will show that it is an open map onto $\bar{g}(\bar{G}^+)$ while identifying $\bar{G} \equiv \bar{g}(\bar{G}^+)$.

We examine $(G^+.3)$ first. Since y_1 doesn't appear except in this condition $(y_1 + \beta y_2)p = 0$ serves merely to determine y_1 once y_2 is given. Obviously y_1 may be locally chosen as a continuous function of β, y_2 and p. The second condition is

(G.3) $Gq = \beta q$.

Next we impose $(NS^+.2)$

(G.2) $I - G_1 - G_2$ has rank $n-1$.

Note that this implies a unit root of G.

We now claim that this is all: that (G.2) and (G.3) uniquely characterize $\bar{G}$ and that $\bar{g}$ is open. To prove this let p_o be in the left null space of $I - G_1 - G_2$: we think of p_o as lying in the manifold formed by identifying radially opposite points on the unit sphere. Thus, since $I - G_1 - G_2$ has rank $n-1$ p_o is a continuous function of G. To conclude our argument we must show how to locally map p_o and p continuously into the non-singular matrices as $y_2(p_o,p)$ such that $p'y_2(p_o,p) = p_o$. This, however, is obviously possible.

We conclude this section with a summary proposition:

Proposition (6-2): Let $\bar{G}$ be the space of (G,q) such that G has one unit root (counting left geometric multiplicity) and $Gq = \beta q$. Then the mapping of $\bar{D}^{NS}$ taking excess demand derivatives to coefficient matrices of the linearized system is continuous open and onto $\bar{G}$.

In particular G is a coefficient matrix of a linearized system of a steady state q if and only if G has one unit root and $Gq = \beta q$.

7. RESTRICTIONS ON EIGENVALUES

We now wish to examine the implication of the restrictions on G for its eigenvalues. It is convenient to work in the subspace $\bar{G}^*$ of $\bar{G}$ for which both $I - G_2 - G_1$ and $\beta^2 I - \beta G_2 - G_1$ have rank $n-1$. Since these are already generic in $\bar{D}$ they are generic also in $\bar{G}$. Let $\bar{S}$ be the manifold of eigenvalues of $2n \times 2n$ matrices: this is the subset of 2n-tuples of complex numbers in which complex numbers occur only in conjugate pairs and in which vectors which differ only by the order of componenets are identified. The map $\bar{\sigma}$ maps $2n \times 2n$ matrices to $\bar{S}$ and is known to be continuous. We now consider $\bar{S}^+$ the subset of $\bar{S} \times R_{++} = \{(s,\beta)\}$ which have a unitary component and a component equal to β and for which there exists p with

$(p,\beta) \in P_o^*$. Likewise we extend $\bar{\sigma}$ to $\bar{\sigma}^+$: $\bar{G}^* \to \bar{S}^+$. We claim that not only is $\bar{\sigma}^+$ continuous, but it is an open map onto an open dense subset of $\bar{S}^+$, and thus that the only non-trivial restriction on the eigenvalues of G are that one equal one and one β.

To show $\bar{\sigma}^+$ is open let $(s,\beta) = \bar{\sigma}^+(G,\beta)$, and suppose $(s^k,\beta^k) \to (s,\beta)$. We construct $G^k \to G$ with $\bar{\sigma}^+(G^k,\beta^k) = (s^k,\beta^k)$.

Set $G^k = H^k Q^k (H^k)^{-1}$. Given Q^k can we choose H^k so that G^k has the partitioned structure corresponding to a second order difference equation? Obviously G^k is the unique solution of $G^k H^k = H^k Q^k$. Writing this out in partitioned form we see

$$\begin{vmatrix} 0 & I \\ G_1^k & G_2^k \end{vmatrix} \begin{vmatrix} H_{11}^k & H_{12}^k \\ H_{21}^k & H_{22}^k \end{vmatrix} = \begin{vmatrix} H_{21}^k & H_{22}^k \\ * & * \end{vmatrix} =$$

$$\begin{vmatrix} H_{11}^k & H_{12}^k \\ H_{21}^k & H_{22}^k \end{vmatrix} \begin{vmatrix} Q_{11}^k & Q_{12}^k \\ Q_{21}^k & Q_{22}^k \end{vmatrix} \cdot = \begin{vmatrix} H_{11}^k Q_{11}^k + H_{21}^k Q_{21}^k & H_{11}^k Q_{12}^k + H_{12}^k Q_{23}^k \\ * & * \end{vmatrix}$$

from which it follows that G^k has the correct structure if and only if

$$\text{(7-1)}\quad \begin{aligned} H_{21}^k &= H_{21}(Q^k, H_{11}^k, H_{21}^k) = H_{11}^k Q_{11}^k + H_{21}^k Q_{21}^k \\ H_{22}^k &= H_{22}(Q^k, H_{11}^k, H_{21}^k) = H_{11}^k Q_{12}^k + H_{12}^k Q_{23}^k \end{aligned}$$

Now let H be a basis for R^{2n} such that $Q = H^{-1}GH$ is in real canonical form. Thus $\bar{\sigma}(Q) = \bar{\sigma}(G) = s$. Hirsch/Smale [1974] show how to construct a sequence of real matrices $Q^k \to Q$ with $\sigma(Q^k) = s^k$. Set $H_{11}^k = H_{11}, H_{21}^k = H_{21}$ and H_{21}^k, H_{22}^k as defined above. By continuity $H^k \to H$

and is eventually non-singular, so G^k is well-defined and by construction has the right structure. Furthermore since components of s^k are one and βG^k has them as eigenvalues. Observe that since G^k has a unit root $I - G_1^k - G_2^k$ is singular, but since $G^k \to G$ it has rank $n-1$. Next, by the structure of G^k there is an eigenvector corresponding to β^k that has the form $q^k = (p^k, \beta^k p^k)$. We think of this as lying on the unit sphere with radial identification and thus being unique. Further, since $G^k \to G$ p^k is the unique component in the right null space of $\beta^2 I - \beta G_2 - G_1$ and thus converges to p. Thus $\bar{G}^*$ $(G^k, p^k, \beta^k) \to (G, p, \beta)$, and the map is open.

Finally we want $\bar{\sigma}^+(\bar{G}^*)$ to be open dense in $\bar{S}^+$. Only density remains to show; we do so by giving an open dense subset of $\bar{S}^+$ denoted $\bar{S}^*$ such that $\bar{\sigma}^+(\bar{G}^*)$ $\bar{S}^*$. This set is defined by having distinct eigenvalues and such that if r_i, r are any real roots and $r_k + ic_k$ is a complex root then

$$(*) \quad \det \begin{vmatrix} r_k - r_i & -c_k \\ c_k & r_k - s_i \end{vmatrix} \neq 0.$$

Let $s \in \bar{S}^*$. By arranging diagonal blocks we may construct a block diagonal matrix $Q = \mathrm{diag}(Q_1, Q_2)$ in real canonical form with $\bar{\sigma}(Q) = s$ and where the first diagonal entry of Q_1 is β. Let p be such that $(p, \beta p) \in p_o^*$ and let H_{11} be a non-singular matrix with first column equal to p. Let $H_{12} = H_{11}$. From (7-1) $H_{21} = H_{11}Q_1$ and $H_{22} = H_{12}Q_2$.

A calculation shows that (*) implies that H is nonsingular. Further by construction $(HQH^{-1}, p, \beta) \in \bar{G}^*$.

8. NOMINAL DYNAMICS

Until now we have largely combined the study of real and nominal steady states. However the dynamics near each type of steady state are rather different so we begin by studying the nominal case. In this case we know only that G has one unitary root.

In studying the dynamics it is useful to define the money supply $m(q_t) \equiv p_t' z(p_{t-1}, p_t)$. This is homogeneous of degree one. By Walras's law this equals $-p_{t-1}' y(p_{t-1}, p_t)$ and the equilibrium condition implies $p_t' z(p_{t-1}, p_t) = p_t' y(p_t, p_{t+1})$. Thus $m(q_t) = m(g(q_t))$: the money supply is constant along paths. At a nominal steady state q $\mu \equiv m(q) \neq 0$. The homogeneity condition shows that if $m(q_t) = \mu$ $D\mu(q_t)q_t = \mu \neq 0$ and thus $m(q_t) = \mu$ defines a global $2n-1$ submanifold $\bar{Q}_o^{\mu}$ transverse to the steady state ray which is invariant under g. We denote the restriction of g to $\bar{Q}_o^{\mu}$ by g^{μ}.

All interest focuses on g^{μ}. If $\operatorname{sgn} \mu' = \operatorname{sgn} \mu$ then g^{μ} and $g^{\mu'}$ exhibit the same dynamics except that the price level is increased by a factor of μ'/μ.

Examining the linearization we see that Dg^{μ} is G restricted to $Dm(q)q_t = 0$. Since $\bar{Q}_o^{\mu}$ is invariant and transverse to the steady state ray it follows that the generalized eigenspace of G excluding the eigenvector q spans the space $Dm(q)q_t = 0$ and that G restricted to this space has the eigenvalues of G excluding the one unit root known <u>a priori</u> to exist. Furthermore the results of the previous section show these remaining eigenvalues to be unrestricted. Letting n^s be the number of these eigenvalues inside the unit circle it follows from standard results such as those in Irwin [1980] that

<u>Proposition (8-1)</u>: There is an open dense set of economies which locally satisfy at all nominal steady states

A. g^{μ} is a local diffeomorphism (G is non-singular).

B. g^μ has no roots on the unit circle (is hyperbolic).

C. g^μ has an n^s-dimensinoal stable manifold W^s of $q_o \varepsilon \bar{Q}^\mu_o$ with $g^{\mu t}(q_o) \to q$.

D. g^μ has a $2n-n^s-1$-dimensional unstable manifold W^μ of $q_o \varepsilon \bar{Q}^\mu_o$ with $g^{\mu-t}(q_o) \to q$.

E. [Hartmann's theorem] there is a smooth coordinate change $c(q)$ such that $c^{-1} o g^\mu o c^{-1} = G$ on W^s and for a residual set of economies this holds on all of $\bar{Q}^\mu_o$ (and thus $\bar{Q}_o$).

9. REAL DYNAMICS

We now study the neighborhood of a steady state $q = (p,\beta p)$ with $m(q) = 0$ and $\beta \neq 1$. In this case prices are not stationary of a steady state, but grow (or decline) exponentially. Let h be a price index such that $h(q) = 1$. We can normalize prices to focus on the convergence of relative prices. Define g^h on $\bar{Q}^h_o \equiv \{q_t \varepsilon \bar{Q}_o \mid h(q_t) = 1\}$ by $g^h(q_t) = g(q_t)/h(g(q_t))$. By homogeneity $h(g^h(q_t)) = 1$. Thus we say that a path q_t converges to q if $q_t/h(q_t) \to q$. This is true of a path beginning at q_o if and only if the path under g^h starting of $q_o/h(q_o)$ converges to q.

What is the linear approximation to g^h? It is $(I - q'H)G/\beta$ $(H \equiv Dh(q))$ restricted to $Hq_t = 0$. Choosing H so that $Hq_t = 0$ defines the generalized eigenspace of G in which the eigenvector q is excluded we see that the eigenvalues of g^h are those of G/β excluding the value 1 that arises from the eigenvalue β corresponding to q. One of these values is equal to $1/\beta$ the remaining $2n-2$ are unrestricted. Let n^s be the number of these remaining eigenvalues inside the unit circle and let $n^\beta = 1$ if $\beta > 1$, 0 otherwise. Then g^h generically is hyperbolic with an

$n^s + n^\beta$ dimensional stable and $2n - n^s - n^\beta - 1$ dimensional unstable manifold. Furthermore g^h is linearizable by a smooth coordinate change on the stable manifold.

It is useful also to distinguish between initial conditions with $m(q_o) = 0$ (real initial conditions) and those with $m(q_o) \neq 0$ (nominal initial conditions). Observe that $Dm(q) = (-p'\beta z_1, p'y_2)$ which by examining the space $\bar{D}$ we see generically doesn't vanish. Thus generically $Dm(q_t) = 0$ defines a $2n-1$ cone $\bar{Q}_o^o$ invariant under g. This is transverse to $\bar{Q}_o^h$ and thus intersects it in a $2n-2$ manifold $\bar{Q}_o^{oh}$ invariant under g^h. Furthermore a simple computation shows that $\bar{Q}_o^o$ is tangent to the eigenvectors of g <u>except</u> the one having a unit root; thus $\bar{Q}_o^{oh}$ is tangent to the eigenvectors of g^h <u>except</u> the eigenvector with root $1/\beta$. Since $\bar{Q}_o^{oh}$ is invariant and for $q_t \in \bar{Q}_o^{oh}$ $m(q_t) = 0$ nominal initial q_o (those with $m(q_o) \neq 0$) can approach $\dot{q}$ only if $\beta > 1$; otherwise if $\beta < 1$ nominal paths can't approach the real steady state. On the other hand in $\bar{Q}_o^{oh}$ the linearized system has the eigenvalues of G/β <u>except</u> 1 and $1/\beta$. The purely real system on the invariant manifold $\bar{Q}_o^{oh}$ is thus generally hyperbole and has an n^s dimensional stable and $2n - n^s - 2$ dimensional unstable manifold and is linearizable on the stable manifold.

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