PRELIMINARY

COMPARATIVE STATICS AND PERFECT FORESIGHT IN INFINITE HORIZON ECONOMIES

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1. INTRODUCTION

Finite economies have the same number of equations as unknowns. Imposing assumptions of differentiability allows us to do regularity analysis. Almost all economies have equations that are locally independent at equilibria. This is important because it enables us to do comparative statics: First, equilibria are locally determinate. Second, small perturbations in the underlying parameters of the economy displace an equilibrium only slightly, and the displacement can be approximately computed by inverting a matrix of partial derivatives.

This paper considers whether infinite horizon economies have determinate perfect foresight equilibria. This question is of crucial importance. If instead equilibria are locally indeterminate, not only are we unable to make comparative static predictions, but the agents in the model are unable to determine the consequences of unanticipated shocks. The idea underlying perfect foresight is that agents' expectations should be the actual future sequence predicted by the model; if the model does not make determinate predictions, the concept of perfect foresight is meaningless.

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We restrict our attention to stationary pure exchange economies. No production, including the storage of goods between periods, can occur. These models are unrealistic, but are the easiest to study. We consider economies with both infinitely lived traders and with overlapping generations.

When there are a finite number of infinitely-lived trades, we argue that equilibria are generically determinate. This is because the effective number of equations determining equilibria are not infinite, but equal the number of agents minus one and must determine the marginal utility of income for all but one agent. Generically, near an equilibrium, these equations are independent and exactly determine the unknowns.

When there are infinitely many overlapping generations, this reasoning breaks down: An infinite number of equations is not necessarily sufficient to determine an infinite number of unknowns. We consider whether the initial conditions together with the requirement of convergence to a nearby steady state locally determine an equilibrium price path. Examples in which they do and examples in which they do not were constructed by Calvo (1978) in a related model.

We consider two alternative types of initial conditions. In the first the old generation in the initial period has nominal claims on the endowment of the young generation. In the second the old generation has real claims. In both cases there are many economies with isolated equilibria, many with continua of equilibria, and many with no equilibria at all. With two or more goods in every period not only can the price level be indeterminate but relative prices can be as well.

We also consider an alternative conceptual experiment in which agents use a forecast rule (which depends only on current prices) to predict next period prices. If the steady state is stable (and if we rule out a certain peculiar case) a perfect foresight forecast rule exists. If there is a continuum of equilibria, there may be a continuum of forecast rules. Even so, the derivative of such a rule (evaluated at steady state prices) is locally determinate. This makes it possible to do comparative statics in a neighborhood of the steady state despite the local non-uniqueness of equilibrium.

2. THE FINITE AGENT MODEL

We being by analyzing a pure exchange economy with a finite number of agents who consume over an infinite number of time periods. In each period there are n goods. Each of the m different consumers is specified by a utility function of the form $\Sigma_{t=0}^{\infty} \beta_i^t u_i(x_t^i)$ (where $1 > \beta_i > 0$ is a discount factor), and a vector of initial endowments w^i that is the same in every period. We make the following assumptions on u_i and w^i:

(a.1) (Differentiability) $u_i : R_{++}^n \to R$ is twice continuously differentiable.

(a.2) (Strict concavity) $D^2u_i(x)$ is negative definite for all $x \in R_{++}^n$.

(a.3) (Monotonicity) $Du_i(x) > 0$ for all $x \in R_{++}^n$.

(a.4) (Strictly positive endowments) $w^i \in R_{++}^n$, $i = 1,\ldots,m$.

(a.5) (Boundary) $\lim_{x_j \to 0} Du_i(x)x/\left|Du_i(x)\right| = 0$.

In another paper (1982a) we show that our results hold also for more general preferences which permit the possibility of intertemporal complementarity.

Let $p_t = (p_t^1,\ldots,p_t^n)$ denote the vector of prices prevailing in period t. When faced with a sequence $\{p_0,p_1,\ldots\}$ of strictly positive prices, agent i chooses a sequence of consumption vectors $\{x_0^i,x_1^i,\ldots\}$ that solves the problem

(2.1) $$\max \Sigma_{t=0}^{\infty} \beta_i^t u_i(x_t^i)$$

$$\text{subject to } \Sigma_{t=0}^{\infty} p_t'x_t^i \leq \Sigma_{t=0}^{\infty} p_t'w^i$$
$$x_t^i \geq 0.$$

The purpose of assumptions a.1 - a.5 is to ensure that, for any price sequence, this problem has a solution that is strictly positive and satisfies the budget constraint with equality. The necessary and sufficient conditions for $\{x_0^i,x_1^i,\ldots\}$ to solve 2.1 are

(2.2) $$\beta_i^t Du_i(x_t^i) = \mu_i p_t' \text{ for some } \mu_i > 0.$$

(2.3) $$\Sigma_{t=0}^{\infty} p_t'x_t^i = \Sigma_{t=0}^{\infty} p_t'w^i.$$

A (perfect foresight) equilibrium of this economy is defined to be a price sequence $\{p_0,p_1,\ldots\}$ and a sequence of consumption vectors $\{x_0^i,x_1^i,\ldots\}$ for each agent, $i = 1,\ldots,m$, that satisfies the following conditions:

(e.1) For each agent $\{x_0^i,x_1^i,\ldots\}$ solves 2.1.

(e.2) $\Sigma_{i=1}^{m} x_t^i = \Sigma_{i=1}^{m} w^i$, $t = 0,1,\ldots$

To find the equilibria of this economy we utilize an approach developed by Negishi (1960) and Mantel (1971) for a model with a finite number of goods. Letting λ_i, $i = 1,\ldots,m$, be some strictly positive welfare weights, we set up the welfare maximization problem

$$\max \Sigma_{i=1}^{m} \lambda_i \, \Sigma_{t=0}^{\infty} \beta_i^t \, u_i(x_t^i)$$

(2.4)

$$\text{subject to} \quad \Sigma_{i=1}^{m} x_t^i \leq \Sigma_{t=1}^{m} w^i, \quad t = 0,1,\ldots$$
$$x_t^i \geq 0.$$

Again a.1 - a.5 guarantee that this problem has a solution that is strictly positive and satisfies the feasibility constraint with equality. The necessary and sufficient conditions for a solution are

(2.5) $$\lambda_i \, \beta_i^t \, Du_i(x_t^i) = p_t^i \quad , \quad i = 1,\ldots,m; \quad t = 0,\ldots$$
for some $p_t > 0$.

(2.6) $$\Sigma_{i=0}^{m} x_t^i = \Sigma_{i=1}^{m} w^i \quad , \quad t = 0,1,\ldots$$

An allocation sequence is pareto optimal if and only if it solves 2.4. Notice that e.2 and 2.6 are equivalent and, furthermore, if we set $\lambda_i = \frac{1}{\mu_i}$, that 2.2 and 2.5 are equivalent. In other words, a pareto optimal allocation and associated lagrange multipliers $\{p_0, p_1, \ldots\}$ satisfy all of our equilibrium conditions except, possibly, 2.3. The problem of finding an equilibrium therefore becomes one of finding the right welfare weights λ_i, $i = 1,\ldots,m$ so that 2.3 is satisfied.

Let $p_t(\lambda)$ and $x_t^i(\lambda)$ be the solutions to 2.5 and 2.6. The strict concavity of μ_i ensures that p_t and x_t^i are uniquely defined and continuous. For each agent we define the excess savings function

$$(2.7) \qquad s_i(\lambda) = \Sigma_{t=0}^{\infty} p_t(\lambda)' \, (w^i - x_t^i(\lambda)).$$

For this definition to make any sense we need to show that the sum on the right converges. Suppose that $\beta_1 > \beta_i$, $i = 2,\ldots,m$, it is clear that the sequence $\{x_0^1, x_1^1, \ldots\}$ involved in the solution to 2.4 cannot converge to any point on the boundary of R_+^n. Furthermore, the vector x_t^1 is bounded: $0 < x_t^1 < \Sigma_{i=1}^m w^i$. Thus since Du_1 is continuous and x_t^1 is in a compact subset of R_{++}^n $||Du_1(x_t^1)||$ must remain bounded. Since $0 < \beta_1 < 1$, this implies that the sum

$$(2.8) \qquad \Sigma_{t=0}^{\infty} p_t' = \lambda_1 \, \Sigma_{t=0}^{\infty} \beta_1^t \, Du_1(x_t^1)$$

must converge. Since x_t^i is bounded for all $i = 1,\ldots,m$ this, in turn, implies that $s_i(\lambda)$ is well-defined and continuous.

It is easy to verify that the functions $s_i(\lambda)$ are homogeneous of degree one and sum to zero. In fact, the functions $\frac{1}{\lambda_i} s_i(\lambda)$ have mathematical properties identical to the excess demand functions of a pure exchange economy with m goods. Standard arguments imply the existence of a vector of welfare weights λ such that

$$(2.9) \qquad s(\lambda) = 0.$$

We call this vector λ an equilibrium since our arguments above ensure that when we solve the welfare maximization problem 2.4 using λ for welfare weights the solution is an equilibrium.

We have reduced the equilibrium conditions for this model to a finite number of equations in the same finite number of unknowns: The homogeneity of s implies that one of the variables λ_i is redundant. That the $s_i(\lambda)$ sum to zero, however, implies that we can ignore one of the equations $s_i(\lambda) = 0$. Kehoe and Levine (1982a) give conditions on u_i that ensure s is continuously differentiable. They then define the concept of a regular economy as one for which $Ds(\lambda)$ has rank $m - 1$ at every equilibrium λ. This concept of regular economy is analogous to that developed by Debreu (1970) for pure exchange economies with a finite number of goods. If an economy is regular, the inverse function theorem implies that it has a finite number of isolated equilibria. The implicit function theorem implies also that these equilibria vary continuously with the parameters of the economy. The index theorem introduced by Dierker (1972) provides a valuable tool for counting the equilibria of such economies. The appeal of the concept of regularity is enhanced by its genericity: Kehoe and Levine (1982a) prove that almost all economies are regular in the sense that regular economies form an open dense set of full measure in the space of economies parameterized by endowment vectors.

To illustrate some of these concepts, we can consider a simple example of an economy with two agents, and one good in every period. Suppose that $w^1 = w^2 = 1$ and $u_1(x) = u_2(x) = \log x$. The only difference between the two consumers is in their discount rates, $1 > \beta_1 > \beta_2 > 0$. In this example the welfare maximization problem 2.4 is

$$\text{(2.11)} \qquad \max \lambda_1 \Sigma_{t=0}^{\infty} \beta_1^t \log x_t^1 + \lambda_2 \Sigma_{t=0}^{\infty} \beta_2^t \log x_t^2$$

$$\text{subject to } x_t^1 + x_t^2 \leq 2, \ t = 0,1,\ldots$$
$$x_t^i \geq 0.$$

Solving conditions 2.5 and 2.6, we obtain

$$\text{(2.12)} \qquad x_t^1(\lambda) = \frac{2\lambda_1\beta_1^t}{\lambda_1\beta_1^t + \lambda_2\beta_2^t}$$

$$\text{(2.13)} \qquad x_t^2(\lambda) = \frac{2\lambda_2\beta_2^t}{\lambda_1\beta_1^t + \lambda_2\beta_2^t}$$

$$\text{(2.14)} \qquad p_t = \frac{1}{2}(\lambda_1 \beta_1^t + \lambda_2 \beta_2^t)$$

The savings functions are

$$\text{(2.15)} \qquad s_1(\lambda) = \Sigma_{t=0}^{\infty} p_t(\lambda)(1 - x_t^1(\lambda)) = \frac{\lambda_2}{1-\beta_2} - \frac{\lambda_1}{1-\beta_1}$$

$$\text{(2.16)} \qquad s_2(\lambda) = \frac{\lambda_1}{1-\beta_1} - \frac{\lambda_2}{1-\beta_2}.$$

As promised, the savings functions are continuous, are homogeneous of degree one, and sum to zero. Imposing the restriction $\lambda_1 = 1$, we can solve 2.16 to find the equilibrium welfare weights

$$\text{(2.17)} \qquad \lambda_1 = 1, \quad \lambda_2 = \frac{1-\beta_2}{1-\beta_1}.$$

We can substitute back into 2.12 - 2.14 to find the equilibrium values of p_t, x_t^1, and x_t^2.

3. THE OVERLAPPING GENERATIONS MODEL

We now analyze an economy with an infinite number of finitely lived agents, a stationary overlapping generations model similar to that introduced by Samuelson (1958). Again there are n goods in each time period. Each generation $0 \leq t < \infty$ is identical and consumes in periods t and $t + 1$. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t,p_{t+1})$ in period t and $z(p_t,p_{t+1})$ in period $t + 1$. The vector $p_t = (p_t^1,\ldots,p_t^n)$ denotes the prices prevailing in period t. Let Q R_+^{2n} be a closed convex cone with non-empty interior and a boundary that is smooth except of the origin. Also assume that $q \varepsilon Q_0 \equiv Q \{0\}$ has $q >> 0$. Excess demand is assumed to satisfy

(A.1) (Differentiability) $y, z : Q_0 \rightarrow R^n$ are smooth functions.

(A.2) (Walras's Law) $p_t' y(p_t,p_{t+1}) + p_{t+1}' z(p_t,p_{t+1}) = 0$.

(A.3) (Homogeneity) y and z are homogeneous of degree zero.

A.1 has been shown by Debreu (1972) and Mas-Colell (1974) to entail relatively little loss of generality. A.2 implies that there is some means of contracting between generations so that each consumer faces an ordinary budget constraint in the two periods of his life. As we show later, this means the economy is equivalent to one with a constant stock of fiat money.

Note that we consider only pure exchange economies and two period lived consumers. We do, however, allow many goods and types of consumers, and the multi-period consumption case can easily be reduced to the case we consider:

If consumers live m periods, we simply redefine generations so that consumers born in periods $1, 2, \ldots, m-1$ are generation 1, consumers born in periods $m, m+1, \ldots, 2m-2$ are generation 2, and so on. In this reformulation each generation overlaps only with the next generation.

The economy begins in period 1. The excess demand of old people (generation 0) in period 1 is $z_0(a,p_1)$ where a is a vector of parameters representing the past history of the economy. A (perfect foresight) equilibrium of an economy (z_0,y,z) starting at a is defined to be a price sequence $\{p_1,p_2,\ldots\}$ that satisfies the following conditions:

$$z_0(a,p_1) + y(p_1,p_2) = 0. \tag{E.1}$$

$$z(p_{t-1},p_t) + y(p_t,p_{t+1}) = 0, \ t > 1. \tag{E.2}$$

Once p_1 and p_2 are determined E.2 acts as a non-linear difference equation determining all future prices. Our major focus is on the extent to which E.1 determines initial prices p_1 and p_2. Subsequent sections study the role of initial conditions z_0 and a. Let us now ignore E.1, however, and focus attention on the difference equation E.2.

We define a steady state of E.2 to be a vector $p \in R^n_+$ and growth rate $\beta > 0$ with $(p,\beta p) \in Q_0$ and

$$\begin{aligned} z(p,\beta p) + y(\beta p,\beta^2 p) &= \\ z(p,\beta p) + y(p,\beta p) &= 0 \end{aligned} \tag{3.1}$$

That is, if the relative prices p prevail forever and the price level grows by β each period markets would always clear. In the generic case Kehoe and

Levine (1982b) show that up to price level indeterminacy there are finitely many steady states.

Our interest in this paper is in what happens near a steady state. Let (p,β) be a steady state, and let $U \quad R^n_+$ be an open cone containing p. We call a path $(p_1,p_2,\ldots)$ satisfying E.1 and E.2 locally stable with respect to (p,β) and U if $p_t \varepsilon U$, $t > 1$, and $\lim_{t\to\infty} p_t/|p_t| = p/|p|$. The question we are trying to answer is whether or not there is a determinate price path that satisfies E.1 and E.2 and is locally stable.

One reason for this restriction is that it is the easiest case to study. Stable price paths are also the most plausible perfect foresight equilibria. If prices are converging to a nearby steady state, then traders can compute future prices by using only local information. If prices are not going to the steady state, then traders need global information and very large computers to compute future prices.

Note that, if equilibrium is indeterminate in our restricted sense so that a continuum of equilibria converge to the steady state from a single initial condition, it is indeterminate in the broader sense as well. On the other hand, even if equilibrium is determinate in the restricted sense there may be a continuum of equilibria which leave the neighborhood of the steady state.

Before studying the initial condition E.1, we examine the behavior of paths satisfying E.2 near a steady state (p,β). We can linearize E.2 around a steady state as

$$(3.2) \qquad D_1 z(p_{t-1}-\beta^{t-1}p) + (D_2 z + \beta^{-1}D_1 y)(p_t - \beta^t p) + \beta^{-1}D_2 y(p_{t+1} - \beta^{t+1}p) = 0$$

Here all derivatives are evaluated at $(p,\beta p)$ and we use the fact that the derivatives of excess demand are homogeneous of degree minus one. Our homogeneity assumption A.2 allows us to rewrite 3.2 as

$$D_1 z p_{t-1} + (D_2 z + \beta^{-1} D_1 y) p_t + \beta^{-1} D_2 y p_{t+1} = 0. \tag{3.3}$$

If the following regularity condition is satisfied, then 3.3 defines a second order linear difference equation.

(R.1) $D_2 y(p,\beta p)$ is non-singular at all steady states (p,β).

Letting $q_t = (p_t, p_{t+1})$, we can write out 3.3 as the first order equation $q_t = Gq_{t-1}$ where

$$G = \begin{vmatrix} 0 & I \\ -\beta D_2 y^{-1} D_1 z & -D_2 y^{-1}(\beta D_2 z + D_1 y) \end{vmatrix}. \tag{3.4}$$

Homogeneity implies that $Gq = \beta q$; in other words, G has a root equal to β. Walras's law implies that $p'[D_2 y - \beta D_1 z]G = p'[D_2 y - \beta D_1 z]$; in other words G has a root equal to one. Let us assume that G also satisfies the following regularity condition.

(R.2) G is non-singular and has distinct eigenvalues; furthermore eigenvalues have the same modulus if and only if they are complex conjugates.

Let n^s be the number of eigenvalues of G that lie inside the circle with radius β, that is, whose moduli are less than β. Let $q \equiv (p,\beta p)$. A standard theorem on linear difference equations implies that the set of initial conditions q_1 such that $q_t = Gq_{t-1}$ has $\lim_{t\to\infty} q_t/|q_t| = q/|q|$ is a $n^s + 1$ dimensional subspace V_s of R^{2n} (see Irwin (1980)).

The implicit function theorem implies that (if R.1 is satisfied) we can solve E.2 to find a non-linear difference equation $q_t = g(q_{t-1})$ defined for an open cone U q. Naturally, $Dg(q) = G$. Let W_s be the subset of initial conditions $q_1 \epsilon U$ such that $\lim q_t/|q_t| = q/|q|$. In other words, given (p_1,p_2) we can find a path in U that converges to the ray proportional to p and only if $(p_1,p_2) \epsilon W_s$. The relationship between V_s and W_s is given in the following theorem.

PROPOSITION 2.1: W_s is an $n^s + 1$ dimensional manifold with tangent space at q equal to V_s.

This result is proven in Kehoe and Levine (1982b). That V_s is the tangent space of W_s at q justifies our intuition about 3.4 as a linear approximation to E.2: It says that the best linear approximation to W_s at q is affine set $V_s + \{q\}$.

To establish Proposition 2.1 we need the regularity conditions R.1 - R.2. These can be justified by showing that they hold for almost all economies. We do this in (1982b). This means that any regular economy can be approximated by one that satisfies R.1 - R.2 and that any slight perturbation of an economy that satisfies R.1 - R.2 still satisfies them.

We remarked that G has a root equal to β and a unit root. Are we justified in assuming it satisfies no other restrictions? Might it not be the case, as for example in optimal control, that half the eigenvalues of G lie inside the unit circle and half lie outside? Calvo (1978) has constructed examples in a related model for which this is not the case. More strongly, Kehoe and Levine (1982b) show that for any n^s satisfying $2n - 1 > n^s > 0$, there exists an open set of economies in that have a steady state with n^s roots inside the circle of radius β and $2n - n^s - 1$ outside the circle with radius β. Furthermore the work of Mantel (1974) and Debreu (1974) show that for any excess demands (y,z) we can find consumers with well behaved preferences whose aggregate excess demands are exactly (y,z).

4. DETERMINANCY OF EQUILIBRIUM

The excess demand of generation 0 in period 1 is $z_0(a,p_1)$. The vector a represents the history of the system. This is our conceptual experiment: Prior to $t = 1$ the economy is on some price path. Suddenly, after generation 0 has made its savings decisions, but before p_1 is determined, an unanticipated shock occurs. No further shocks occur, and hereafter expectations are fulfilled, although there is no reason why generation 0's expectations of p_1 should be. Do the equilibrium conditions E.1 and E.2 determine a unique path to the new steady state, at least locally? If so, we can do comparative statics, evaluating the impact of the unanticipated shock. If not, it is questionable that traders can deduce which of the many perfect foresight paths they would be on.

Note that this is not the only question we could ask. We might enquire whether given a perfect foresight path stretching back to minus infinity there is a unique extension to plus infinity. We believe that the answer to this question is in general yes. Or we might ask whether the family of price paths $\{\ldots,p_{-1},p_0,p_1,\ldots\}$ that are perfect foresight are locally unique. We believe that there is a "large" set of economies for which the answer to this question is yes, and an equally "large" set for which it is no. We feel that the question we have posed is the most interesting one, however, and, of these questions, the only one relevant for applied work. Another relevant question is, of course, how to handle price paths that are not near steady states. As we have mentioned, however, it is not clear that perfect foresight is a good hypothesis in such cases.

With this conceptual experiment in mind, we can now see the role played by the vector a: It represents the claims on current consumption owed to old people based on their savings decisions made in period 0. Define the money supply $M = p_1'z_0(a,p_1)$ to be the nominal claims of old people. Observe that in equilibrium $p_1'y(p_1,p_2) = -M$; by Walras's law $p_2'z(p_1,p_2) = M$ in equilibrium $p_2'y(p_2,p_3) = -M$ and so forth. Thus M is the fixed nominal net savings of the economy for all time (i.e. there is no government intervention in money markets). In the steady state we have $\beta^n p'z = M$ and $\beta^n p'y = -M$. There are two cases of interest. The "nominal" case has $M \neq 0$. In this case it must be that $\beta = 1$. Gale (1973) called steady states of this type "golden rule" steady states. This is because for excess demand functions derived from utility maximization nominal steady states maximize a weighted sum of individual utilities subject to the constraint of stationary consumption over time. Alternatively in the "real" case $M = 0$. Gale referred to steady states of this type as "balanced" steady states. In this case if $\beta = 1$ then $y + z = 0$ and $p'y = 0$ which are typically n equations in the n-1 unknowns p and $\beta = 1$ is merely coincidental. Thus when $M = 0$ the most interesting case is $\beta \neq 1$. In (1982b) we show that there are generically an odd number of steady states of each type.

We suppose first that claims are denoted in nominal terms. Thus we cannot assume that excess demand by the old $z_0(a,p_1)$ is homogeneous of degree zero in p_1. We do assume, however, that a is an element of an open subset A of a finite dimensional vector space and that

(I.1) (Differentiability) $z_0\colon A \times P \to R^n$ is a smooth function.

Let $q = (p,\beta p)$ be the steady state before the shock. We assume

(I.2) (Steady state) $z_0(0,p) + y(p,\beta p) = 0$.

That is, when $a = 0$ we are at a steady state. Our goal is to analyze what happens when $|a|$ is small, that is, when a small shock occurs.

To analyze the impact of the shock observe that prices (p_1,p_2) are determined by E.1. We can linearize E.1 around the steady state to find

$$(4.1) \qquad (D_2z_0 + \beta^{-1}D_1y)p_1 + D_1z_0a + \beta^{-1}D_2yp_2 = 0.$$

R.1 implies that we can solve 4.1 for p_2 as

$$(4.2) \qquad p_2 = D_2y^{-1}(\beta D_1z_0 + D_1y)p_1 + \beta D_2y^{-1}D_2z_0a,$$

or, introducing, as before, $q_1 = (p_1,p_2)$,

$$(4.3) \qquad q_1 = L\begin{vmatrix} a \\ p_1 \end{vmatrix} = \begin{vmatrix} -D_2y^{-1}(\beta D_2z_0 + D_1y) & \beta D_2y^{-1}D_1z_0 \\ I & 0 \end{vmatrix}\begin{vmatrix} a \\ p_1 \end{vmatrix}$$

The implicit function theorem implies that in a neighborhood of the steady state we get a corresponding solution of the non-linear equation E.1, $q_1 = \ell(a,p_1)$, defined for $p_1 \in U_1$, $a \in A$, with $D\ell(0,p) = L$. We ask whether for given $a \varepsilon A$, is there a unique initial (p_1, p_2) that satisfies E.1 and has an extension to a price path $\{p_1, p_2, \ldots\}$ in U that satisfies E.2 and converges to some point on the new steady state ray. The results of the last section imply that the corresponding mathematical question is whether, for given a, is there a unique p_1 such that $\ell(a, p_1) \varepsilon W_s$.

Let us consider the linear problem first. For fixed $a \varepsilon A$ 4.3 defines an n dimensional affine subspace of R^{2n}. The linearized version of W_s is V_s, which is $n^s + 1$ dimensional. We would expect, in general, that these spaces intersect in an $n + (n^s + 1) - 2n = n^s + 1 - n$ dimensional linear at space. Suppose, in fact, that L satisfies

(IR.1) L has rank $2n$.

Note that this implies that A is at least n dimensional, in other words, tht there are at least n independent ways to shock the economy. The transversality theorem of differential topology (see Guillemin and Pollack (1974), pp. 67-69) can be translated into the following result.

PROPOSITION 4.1: Let S_a denote the set of $p_1 \varepsilon U_1$ such that $\ell(a, p_1) \varepsilon W_s$. For almost all $a \varepsilon A$ the set S_a, if it is non-empty, has dimension $n^s + 1 - n$.

In other words, what we expect in general of the linear system is almost always true of the non-linear system. Here we use almost all to mean an open dense subset of A whose complement has measure zero. If $n^s + n - 1 < 0$, this means there is no $p_1 \varepsilon U_1$ with $\ell(a, p_1) \varepsilon W_s$. If $(n^s + 1 - n) > 0$, however, S_a can either have this dimension or be the empty set. I.2 implies that S_0 is non-empty. If we can ensure that ℓ is transversal to W_s at q, then the structural stability of transversality would imply that S_a is non-empty for all a close enough to 0. We assume

$$\text{(IR.2)} \qquad [-D_2y^{-1}(\beta D_1 z_0 + D_1 y) \;\; q \;\; v_1 \cdots v_{n^s}] \qquad \text{has full rank.}$$

where $v_1,\ldots,v_{n^s}$ span V_s. For $n^s + 1 - n > 0$ this says that ℓ is transversal to W_s at q.

Thus, under IR.1 and IR.2, we can distinguish three cases:

I. $n^s < n - 1$. In this case, for almost all a, S_a is empty. In other words, there are no stable paths locally. We call such a (p,β) an unstable steady state. For most initial conditions the asymptotic behavior of the system is to not reach the steady state. Such steady states are not very interesting; they are unreachable.

II. $n^s = n - 1$. In this case, locally stable equilibrium paths are locally unique and, in a small enough neighborhood actually unique. This is the case where we can do comparative statics and in which perfect foresight is a plausible description of behavior. This is called the determinate case.

III. $n^s > n - 1$. In this case there is a continuum of locally stable paths. Equilibrium is indeterminate. Comparative statics is impossible and perfect foresight implausible.

There are "large" sets of economies (open sets of economies) that have steady states of any desired type: unstable, determinate or indeterminate. Thus none of these possibilities is in any way degenerate.

As a final note we consider the argument that we get indeterminacy because we ask too much: Because z_0 is not homogeneous we demand that the price level be determined by initial conditions. Is it possible that this is the only possible form of indeterminacy? No. If $n^s + 1 - n > 1$, S_a has two or more dimensions implying that there must be relative price indeterminacy.

Now we consider the case of real initial conditions. The change in conceptual experiment lies in z_o: it is homogeneous of degree zero in p_1 and satisfies Walras's law $p_1'z_o(a,p_1) = 0$. Since $M = 0$ the initial price vector must satisfy $p_1'z(p_1,p_2) = 0$. Provided at the steady state $(z' + p'D_1z, D_2z)$ doesn't vanish the condition $M = 0$ defines a $2n - 1$ dimensional manifold in Q_o which we will refer to as the "real" manifold Q_r. The stability of the system is determined by the roots of G/β. This has one unit root which is irrelevant and one root equal to $1/\beta$. Furthermore, algebraic manipulation shows that the root $1/\beta$ determines the behavior of the system <u>outside</u> of Q_r: If $\beta < 1$ no path with nominal initial condition can ever approach the real steady state. Thus we let $\bar{n}^s$ be the number of roots of G/β excluding the root $1/\beta$ and the root 1, which lie inside the unit circle.

Because of homogeneity (including that of z_o) the price level is indeterminate and we can reduce everything by one dimension by a price normalization. In this reduced space Q_r has $2n - 2$ dimensions, while the initial condition $z_0(a,p_1) + y(p_1,p_2) = 0$ determines an $n - 1$ dimensional submanifold. The intersection of stable manifold W_s with Q_r has dimension $\bar{n}^s$; thus the intersection of S_a and W_s has dimension $(\bar{n}^s + n - 1) - (2n - 2) = \bar{n}^s - n + 1$. Thus there are the same three possibilities in the real case as in the nominal case, although in the real case $0 < \bar{n}^s < 2n - 2$ while in the nominal case $0 < n^s < 2n - 1$.

It might be conjectured that in the case where excess demand is derived from consumer optimization of well-behaved preferences that the pareto inefficiency of paths is related to the indeterminary of equilibrium. A moments reflection on the real case shows this is not true. If $\beta < 1$ prices along paths converging to the steady state decline exponentially in the limit; this means that the value of all traders' endowments is finite and by the standard argument for finite economies in Arrow (1951) all these paths are efficient. But $\beta < 1$ implies only that no path with $M \neq 0$ ever approaches the real steady state; it places no restrictions on n^s. Thus if $n > 1$ indeterminary is possible. Conversely if $\beta > 1$ all convergent paths are inefficient, but there is still no restriction on the possible types of steady states.

Perhaps the case $\beta < 1$ is the most puzzling of all: here if $n > 1$ we can have indeterminacy among equilibria converging to the steady state, yet all these paths are pareto efficient and all mimic the finite dimensional case in that Walras's law is satisfied even by the initial generation.

We conclude this section by noting that there are six possible types of steady states: real or nominal each of which may be unstable, determinate or indeterminate. If there are two or more goods each period then there are open sets of economies with each possible combination. The case with one good each period (which has been studied most extensively) is exceptional however: instability is impossible and in the real case indeterminacy is also impossible.

5. FORECASTING

In this section we examine the case of nominal initial conditions in more detail. We focus on the neighborhood of a stable steady state p with $n^s > n - 1$, and we assume that all regularity conditions are satisfied. Our focus is on how agents forecast future prices. One possibility is that they use the dynamic equation E.2; equivalently, they forecast $q_{t+1} = g(q_t)$. Note that unless $n^s = 2n - 1$ this is actually an unstable dynamical system: Small perturbations can cause the path to depart from the steady state.

We now investigate the alternative possibility that traders forecast future prices solely as a function of current prices. This type of closed-loop forecasting leads to convergence to the steady state. Surprisingly, it also is locally determinate: This restriction on forecasting rules is sufficient to eliminate most of the indeterminacy we found in the previous section, making comparative statics possible.

A closed-loop forecast rule is a function $p_{t+1} = f(p_t)$ giving prices next period as a function of current prices. We assume that f satisfies the following assumptions:

(F.1) (Differentiability) f is a smooth function defined on an open cone U that contains the steady state relative prices p.

(F.2) $f(p) = p$.

(F.3) (Homogeneity) f is homogeneous of degree one.

(F.4) (Perfect foresight) $z(p,f(p)) + y(f(p), f^2(p)) = 0$.

(F.5) (Convergence) $\lim_{t\to\infty} f^t(p)/|f^t(p)| = p/|p|$.

F.2 insists that at the steady state the forecast rule pick out the steady state. F.4 is the perfect foresight assumption: If forecast are realized, markets indeed clear. F.5 says we are interested only in forecast rules that permit convergence to the steady state, in other words, are stable.

We begin by asking whether, for $n^s > n - 1$, there actually exists a forecast rule that satisfies F.1 - F.5. As usual, we consider the linearized problem first. To construct a forecast rule we choose $v_1,\ldots,v_{n-1}$, q to be independent eigenvectors in V_s, the stable subspace of the linearized system. It is important that we be able to choose $v_1,\ldots,v_{n-1}$ so that complex vectors appear in conjugate pairs. This can always be done if n^s is even. It can also always be done if $n^s = n - 1$ since $v_1,\ldots,v_{n-1}$ includes all of the eigenvectors corresponding to eigenvalues inside the circle of radius β and such eigenvectors necessarily show up in complex conjugates. In the peculiar case where n^s is odd and there are no real eigenvalues inside the unit circle, and hence no real eigenvectors in V_s we cannot make this choice of $v_1,\ldots,v_{n-1}$. This is no accident: This is the only case in which there are no stable perfect foresight forecasting rules.

Let V_* be the real vector space spanned by $v_1,\ldots,v_{n-1}$, q; it is n dimensional. What we suggest is, for given p_t, choose p_{t+1} so that $(p_{t+1} - \beta^{t+1}p, p_t - \beta^t p)$ is an element of V_*. From the structure of g there exists a unique choice of p_{t+1} provided

(FR) $v_1^2,\ldots,v_{n-1}^2,p$ are independent vectors.

Where v_i^2 $i = 1,\ldots,n - 1$ are the final n components of the v_i. Provided that FR holds, we can find a unique matrix F, which depends on $v_1,\ldots,v_{n-1}$, so that

$$(5.1) \quad (p_{t+1} - \beta^{t+1}p) = F(p_t - \beta^t p)$$

is our linear forecast rule.

First we check the linearized system 5.1 satisfies the linearized versions of F.2 - F.5. Since $q \in V_*$, $(\beta p, p) \in V_*$ and, consequently, $Fp = \beta p$. Since $v_1,\ldots,v_{n-1}$, q are eigenvectors of g, V_* is invariant under the dynamical system g, which implies that if $q_t \in V_*$ then $gq_t \in V_*$. Finally, since V_* V_s and $(p_{t+1},p_t) \in V_*$, we must have $\lim_{t\to\infty} p_t/|p_t| = p/|p|$.

It is natural to conjecture that we can thus find an f with $Df(p) = F$ that satisfies F.1 - F.5; this follows from Hartmann's smooth linearization theorem in Irwin (1980). Because g is homogeneous of degree one f may also be chosen to be homogeneous of degree one. If $n^s = n - 1$, then f is unique. This is well known when f is linear (see, for example, Blanchard and Kahn (1980)). If, however, $n^s > n - 1$ f may not be unique nor even locally unique. Furthermore, in the case where $n - 1$ is odd and all the eigenvalues of G that lie inside the unit circle are complex, f does not even exist. The derivative $Df(p) = F$ at the steady state is locally unique, however; there are only finitely many possibilities. To see this write F.4 as $(f^2(p_t), f(p_t)) = g(f(p_t),p_t)$. Differentiating this at p we see that

$$(5.2) \qquad \begin{vmatrix} F^2 \\ F \end{vmatrix} = g \begin{vmatrix} F \\ I \end{vmatrix} .$$

Writing F in Jordan form as $F = H\Lambda H^{-1}$, we see that

$$(5.3) \qquad \begin{vmatrix} H\Lambda^2 \\ H\Lambda \end{vmatrix} = g \begin{vmatrix} H\Lambda \\ H \end{vmatrix},$$

which implies that Λ is diagonal with diagonal entries equal to eigenvalues of G and that the columns of $\begin{vmatrix} H\Lambda \\ H \end{vmatrix}$ are the corresponding eigenvectors of G. Since G has only finitely many eigenvalues, there are only finitely many choices of F; indeed our original construction is the only way to get solutions that satisfy the stability requirement F.5.

Notice that, if $n^s > n + 1$, there are in general many possible choices of $v_1,\ldots,v_{n-1}$, and, consequently, of F. The important fact is that there are only a finite number of choices. Furthermore, under our regularity assumptions, F varies smoothly with small changes in the parameters of (y,z). When doing comparative statics faced with a choice of finitely many forecast rules, we choose the unique F that corresponds to the forecast rule being used before the shock.

Finally, let us check on the initial condition; it is now

$$(5.4) \qquad z_0(a,p_1) + y(p_1,f(p_1)) = 0.$$

We can locally solve for p_1 as

$$(5.5) \qquad p_1 = - [D_2z_0 + D_1y + D_2yF]^{-1}D_1z_0a.$$

REFERENCES

K. J. Arrow (1951). "An Extension of the Basic Theorems of Classical Welfare Economies," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman, ed., UC Press, 507-532.

O. J. Blanchard and C. M. Kahn (1980), "The Solution of Linear Difference Models Under Rational Expectations," Econometrica, 48, 1305-1312.

G. A. Calvo (1978), "On the Indeterminancy of Interest Rates and Wages with Perfect Foresight," Journal of Economic Theory, 19, 321-337.

G. Debreu (1970), "Economies with a Finite Set of Equilibria," Econometrica, 38, 387-392.

_________ (1972), "Smooth Preferences," Econometrica, 40, 603-612.

_________ (1974), "Excess Demand Functions," Journal of Mathematical Economics, 1, 15-23.

D. Gale (1973), "Pure Exchange Equilibrium of Dynamic Economic Models," Journal of Economic Theory, 6, 12-36.

E. Dierker (1972), "Two Remarks on the Number of Equilibria of an Economy," Econometrica, 40, 951-953.

V. Guillemin and A. Pollack (1974), Differential Topology. Englewood Cliffs, New Jersey: Prentice Hall.

M. C. Irwin (1980), Differentiable Dynamical Systems. New York: Academic Press.

T. J. Kehoe and D. K. Levine (1982a), "Economies with a Finite Number of Agents," unpublished manuscript.

___________________________ (1982b), "Regularity in Overlapping Generations Exchange Economies," unpublished manuscript.

R. R. Mantel (1971), "The Welfare Adjustment Process: Its Stability Properties," International Economic Review, 12, 415-430.

____________ (1974), "On the Characterization of Aggregate Excess Demand," Journal of Economic Theory, 7, 348-353.

A. Mas-Colell (1974), "Continuous and Smooth Consumers: Approximation Theorems," Journal of Economic Theory, 8, 305-336.

T. Negishi (1960), "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," Metroeconomica, 12, 92-97.

P. Samuelson (1958), "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," Journal of Political Economy, p6, 467-482.