TOTALLY BALANCED GAMES ARISING FROM
CONTROLLED PROGRAMMING PROBLEMS

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Abstract

A cooperative game in characteristic-function form
is obtained by allowing a number of individuals to exercise
partial control over the constraints of a (generally non-
linear) mathematical programming problem, either directly
or through committee voting. Conditions are imposed on
the functions defining the programming problem and the
control system which suffice to make the game totally
balanced. This assures a nonempty core and hence a stable
allocation of the full value of the programming problem among
the controlling players. In the linear case the core is
closely related to the solutions of the dual problem. Appli-
cations are made to a variety of economic models, including
the transferable utility trading economies of Shapley and
Shubik and a multi-shipper one-commodity transshipment model
with convex cost functions and concave revenue functions.
Dropping the assumption of transferable utility leads to a
class of controlled multi-objective or "Pareto programming"
problems, which again yield totally balanced games.

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1. Introduction

We consider in this paper a method of introducing players into the control structure of a mathematical programming problem in order to develop its game-theoretical properties. Specifically, in a _controlled programming problem_ (or CPP), some of the constraints have two possible states: their "constant" term (i.e., the right-hand side) has a designated positive value if the player who controls that constraint belongs to the operative coalition and is zero otherwise. These "variable constants" can be regarded as representing resources or skills that the player in question brings to the enterprise, and the optimized value of the objective function for each possible coalition is regarded as representing the "worth" of that coalition. The resulting set function, interpreted as the characteristic function of a cooperative game, can then be used as a guide in assigning credit for the optimized objective in accordance with the contributions of the different controllers.

More specifically, under the somewhat complicated set of conditions spelled out in Sec. 4, such a "CPP game" will be shown to be _totally balanced_ (Sec. 4, Theorem 1), and hence to have a nonempty _core_. This means that the optimum value of the CPP can be apportioned among the participating controllers in a way that is "stable" or "unobjectionable",

in the sense that no subset of them could operate the program by themselves and improve upon their allotted shares.

Secs. 2 and 3 are preparatory, and Sec. 5 contains a series of remarks about the model of Sec. 4. In Sec. 6 we extend the model to permit control by committees, making use of the theory of simple games. In order to obtain total balancedness in this case it is necessary to require that the committee associated with each constraint have at least one member with veto power. This is equivalent to saying that the simple game of each committee must be balanced. But it is interesting that total balancedness of the committees (which would trivialize the committees by giving every member a veto) is not required.

Not surprisingly, there is an intimate relationship between CPPs and trading economies with privately-owned production functions. The latter are known under quite general conditions to have characteristic functions that are totally balanced [1, 17, 21]. Our present model most naturally corresponds to the "transferable utility" case, since we work (until Sec. 11) with a single objective function which each controller is equally desirous of optimizing. In Secs. 8 and 9 we make this correspondence explicit in both directions. An interesting question, which we defer to a later occasion, is the extent to which the competitive equilibria of the economic model, whose allocations are known to form a subset of the core, can be given a direct interpretation in the programming problem.

The general CPP framework encompasses several special cases of interest in economics. The "assignment market" of Shapley and Shubik [19] is a simple, early example of a controlled linear programming problem, and many of its subsequent generalizations (e.g. [4, 10, 11]) also fit the CPP framework. The pure exchange economies ("market games")

of [6, 18, 20] also have simple CPP representations in which all the constraints are linear. Another special case is a multi-shipper one-commodity non-linear transshipment model that does not appear to have been previously studied. All these applications are described in Sec. 10 of this paper.

Finally, in Sec. 11 we give a brief sketch of the extension of our results to the nontransferable utility case, which requires a multi-objective "Pareto programming" model.

There is a broad similarity between this paper and some recent work of Kalai and Zemel [8, 9], also dealing with "controlled mathematical programming". While the two approaches have many applications in common, they were conceived and developed independently and there are several significant points of difference, namely (1) in the way in which the player controls are introduced, (2) in the conditions that are imposed in order to ensure total balance, and (3) in the extent to which nonlinear objectives and constraints can be accommodated. Considering also the fact that the two models have rather different, though overlapping, domains of natural application, it was concluded by both sets of authors (on comparing notes before publication) that neither approach "dominates" the other, and that both approaches are worth pursuing.

2. Notation

For any finite sets M and N, $M \subset N$ means nonstrict inclusion, $M \backslash N$ means Boolean subtraction, $|M|$ is the cardinal number of M, and 2^M is the collection of all subsets of M. Although we shall often for convenience use the natural numbers to name the elements of these sets, e.g., $M = \{1,\ldots,m\}$, $N = \{1,\ldots,n\}$, we do not mean to imply that they have elements in common. The symbol R denotes the real numbers or the real line, and R^M denotes the $|M|$-dimensional cartesian space whose coordinates are indexed by the elements of M. If $x \in R^M$ and $y \in R^M$ then $x \geq y$ means $x_i \geq y_i$ for each $i \in M$. The set of all $x \in R^M$ with $x \geq 0$ is denoted R^M_+, where "0" in this case (as is clear from the context) is the zero vector in R^M. For any $V \subset M$ we define

$$R^{M|V} = \{x \in R^M : x_i = 0 \text{ if } i \in M \backslash V\};$$

in other words, $R^{M|V}$ is the space R^V represented as a subspace of R^M. Finally, for any $x \in R^M$, x^V is the projection of x on $R^{M|V}$.

3. Games, Cores, Balancedness

A _game_ (more completely, "cooperative game with transferable utility") is denoted by an ordered pair (P,v), where $P = \{1,\ldots,p\}$ is a nonempty finite set and v is a function from 2^P to $\mathbb{R}$ satisfying $v(\emptyset) = 0$. The elements of P are called _players_, the elements of 2^P are called _coalitions_, and v is called the _characteristic function_ of the game. The values of $v(S)$ are meant to express in some sense the _worth_ or _profitability_ of the various coalitions $S \subset P$.

The _feasible payoffs_ of (P,v) are those vectors $x \in \mathbb{R}^P$ that satisfy

$$\sum_{i \in P} x_i \leq v(P).$$

The _core_ of (P,v) is defined as the set of feasible payoff vectors that cannot be improved upon by any coalition, i.e.:

$$\text{Core}(P,v) = \{x \in \mathbb{R}^P : \sum_{i \in P} x_i = v(P) \text{ and } \sum_{i \in P} x_i \geq v(S) \text{ for all } S \subset P\}.$$

If the core of (P,v) "exists" (i.e., is nonempty) it represents a region of stability in the space of all possible allocations of the game's total profit --- a region where the outcome is not likely to be disturbed by dissident coalitions.

A collection of coalitions $\beta = \{S_1,\ldots,S_q\} \subset 2^P$ is said to be _balanced_ (w.r.t. the finite set P) if there exist nonnegative _balancing weights_ $w_1,\ldots,w_q$ such that, for each $i \in P$

$$\sum_{\ell \,:\, i \in S_\ell} w_\ell = 1.$$

The game (P,v) is __balanced__ if for every such β, w,

$$\sum_{\ell=1}^{q} w_\ell v(S_\ell) \leq v(P).$$

The following result is well known $[3,16]$:

Theorem A. (Bondareva-Shapley). __The core of__ (P,v) __is nonempty if and only if__ (P,v) __is balanced.__

Each nonempty $Q \subset P$ defines a __subgame__ of (P,v) in the obvious way, namely, the game (Q,v_Q) where $v_Q(S) = v(S)$ for all $S \in 2^Q$. We say that (P,v) is __totally balanced__ if all its subgames are balanced. Total balancedness, first introduced in [18], is a characteristic feature of many economic models $[1, 6, 8, 9, 17, 20, 21]$.

4. The Programming Game

We now make precise the notion of a "controlled programming problem" (CPP). We begin with an ordinary mathematical programming problem, as follows:

MAXIMIZE $\quad f(x)$ subject to $x \in R_+^N$, and

$$g_j(x) \leq a_j \qquad j \in C_1,$$

$$g_j(x) = a_j \qquad j \in C_2,$$

to which we append a system of <u>controls</u> given by a "control map"

$$\delta : C \to P \cup \{0\}.$$

In the above, the symbols N, C_1, C_2, C, and P represent the finite sets $\{1,\ldots,n\}$, $\{1,\ldots,c_1\}$, $\{c_1 + 1,\ldots,c\}$, $\{1,\ldots,c\}$ and $\{1,\ldots,p\}$, respectively, identifying the <u>variables</u>, <u>constraints</u>, and <u>controllers</u> of the problem, while f and the g_j are functions from R_+^N to R about which we shall have more to say presently.

First, let us describe how the control system operates. The quantities a_j are interpreted as "resources". If $\delta(j)$ is a positive integer, then the controller $\delta(j) \in P$ has a <u>veto</u> on the use of a_j. This means that he has the power unilaterally to replace the number a_j by 0 in the right-hand side of the j-th constraint. But if $\delta(j) = 0$, then the j-th constraint is fixed, and not subject to any such manipulation.[1]

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[1] In contrast, the formulation of Kalai and Zemel [8,9] has the players controlling the variables rather than the constraints; this is sometimes more natural, e.g., in network problems. However, it is usually possible to translate a given model from either control form to the other by putting in extra variables or constraints.

Using these ideas, we define the _S-reduced problem_ for each

$S \in 2^P$ to be the originally stated problem (above) with a_j replaced

by 0 whenever $\delta(j) \notin S \cup \{0\}$. Note that S may be the empty set. Let

$\Theta(S) \subset R_+^N$ denote the feasible set for the S-reduced problem. We wish

now to define a cooperative game (P,v), according to

$$(4.1) \qquad v(S) = \max\{f(x) : x \in \Theta(S)\}.$$

In other words, the _worth_ of S is the optimal value of the S-reduced

problem. Under the six conditions that we are about to impose, the maximum

in (4.1) will be well defined and $v(\emptyset)$ will be zero, so that (P,v)

will indeed be a game; moreover it will be totally balanced. The first

three conditions are as follows:

(I) $a_j \geq 0$, all $j \in C$.

(II) $a_j = 0$ if $\delta(j) = 0$.

(III) $v(S)$ is well-defined for each $S \in 2^P$; that is,

$\Theta(S)$ is nonempty and the maximum in (4.1) is achieved.

Before we state the other conditions, three special classes of

functions must be defined. First, for each player i, let A_i denote

the set of indices $k \in N$ for which x_k would be forced to be zero

if i withheld his resources, even if the other constraints in C did

not exist. Thus,

$$A_i = \{k \in N : \underline{if} \ g_j(x) \leq 0 \ \text{for all} \ j \in C_1 \cap \delta^{-1}(i),$$
$$g_j(x) = 0 \ \text{for all} \ j \in C_2 \cap \delta^{-1}(i), \ \text{and} \ x \in R_+^N,$$
$$\underline{then} \ x_k = 0\}$$

Denote the collection $(A_1,\ldots,A_p)$ by α. (Note that the sets A_i need not be disjoint and need not exhaust N.) We shall say that a function $h : R_+^N \to R$ is α-separable if there exist functions $h_i : R_+^{N|A_i} \to R$, $i = 1,\ldots,p$, such that

$$h = \sum_{i \in P} h_i.$$

If, furthermore, each h_i is a convex function, vanishing at 0, we shall say that h is convexly α-separable, or $(CS\,\alpha)$.[2]

For the two other special classes of functions, we shall write (CH) for the class of functions $h : R_+^N \to R$ that are convex and homogeneous (i.e., positively homogeneous of degree one), and (LH) for the class of homogeneous linear functions. We shall also write $h \in (-CS\,\alpha)$, etc., to mean that $-h \in (CS\,\alpha)$, etc. For example, we have $(LH) = (CH) \cap (-CH)$. Note that each of the classes defined is a convex cone, i.e., is closed under addition and multiplication by nonnegative scalars.

We can now complete our list of conditions.

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[2] It would be slightly stronger and superficially simpler to base our conditions on the collection α', given by

$$A_i' = \{k \in N : \text{ if } x \in \Theta(P\setminus\{i\}) \text{ then } x_k = 0\}.$$

Since $A_i' \supset A_i$ this would yield $(CS\,\alpha') \supset (CS\,\alpha)$ and result in a minor strengthening of Theorem 1. But this modified definition is undesirable because it depends too heavily and specifically on the data of the programming problem and, among other things, raises the possibility that the class of functions $(CS\,\alpha')$ might change as we move from one S-reduced problem to another. (See also Remark 4 in Sec. 5.)

(IV) each g_j for $j \in C_1$ is a sum of a (CS α) function and a (CH) function.

(V) each g_j for $j \in C_2$ is an (LH) function.

(VI) f is the sum of a (-CS α) function and a (-CH) function.

Thus, our mathematical programming problem is essentially a convex programming problem, although since we have adopted the "maximizing" point of view the objective function is concave.

Condition (III) tells us that (4.1) is well defined, but before we can claim to have defined a game we must also verify that $v(\emptyset) = 0$. Indeed, in the $\emptyset$-controlled problem the right hand side is all zero, by definition and by condition (II). Since the functions g_j all vanish at the origin (conditions (IV) and (V)), the $\emptyset$-controlled problem is feasible and its value is at least $f(0) = 0$. Now suppose there were a point $x \in \mathcal{Q}(\emptyset)$ with $f(x) > 0$. Let k be any index such that $x_k > 0$, and suppose that $k \in A_i$ for some $i \in P$. That means that replacing certain of the a_j by 0 would suffice to force $x_k = 0$. But we have "zeroed out" the entire right hand side and yet $x_k > 0$, so k does not belong to any of the A_i. This means that x lies in a subspace of R^N on which all the functions g_j as well as f are homogeneous. It follows that the value of the $\emptyset$-controlled problem is unbounded, since arbitrarily large positive multiples of $f(x)$ can be achieved. This contradicts condition (III). So we conclude that $v(\emptyset) = 0$, and that we do indeed have a well defined cooperative game.

Our main result in this section is:

Theorem 1. <u>Suppose conditions (I)-(VI) hold in a CPP. Then (P,v) is totally balanced.</u>

Lemma 1. <u>Suppose conditions (I), (II), (IV) and (V) hold. Let $\mathcal{B} = (S_1, \ldots, S_q)$ be a balanced collection of coalitions (w.r.t. P), with weights $(w_1, \ldots, w_q)$. Then</u>

$$w_1 \, \Theta \, (S_1) + \cdots + w_q \, \Theta \, (S_q) \subset \Theta \, (P).$$

Proof. Put $Q = \{1, \ldots, q\}$ and $Q^i = \{\ell \in Q : i \in S_\ell\}$ and $\tilde{Q}^i = Q \backslash Q^i$. Choose $x^\ell \in \Theta \, (S_\ell)$ for each $\ell \in Q$, and let

$$\hat{x} = \sum_{\ell \in Q} w_\ell x^\ell .$$

We must show that

$$\hat{x} \in \Theta \, (P)$$

First, let $j \in C_1$ and assume that g_j satisfies (IV). We shall prove that

$$(4.2) \qquad g_j(\hat{x}) \leqq \sum_{\ell \in Q} w_\ell g_j(x^\ell).$$

First, if g_j is (CH) let $\bar{w} = \sum_{\ell \in Q} w_\ell$. Then

$$(4.3) \qquad g_j(\hat{x}) = \bar{w} g_j \left(\sum_{\ell \in Q} (w_\ell / \bar{w}) x^\ell \right)$$

$$\leq \bar{w} \sum_{\ell \in Q} (w_\ell / \bar{w}) g_j(x^\ell)$$

$$= \sum_{\ell \in Q} w_\ell g_j(x^\ell),$$

which verifies (4.2) if g_j is (CH). (The first line in (4.3) is due to homogeneity, the second to convexity.)

Next, if g_j is (CS a), say $g_j = g_j^1 + \cdots + g_j^p$ with $g_j^i(x) = g_j^i(x^{A_i})$ for all $x \in R_+^N$ and all $i \in P$, then for each i,

$$(4.4) \qquad g_j^i(\hat{x}^{A_i}) = g_j^i\left(\sum_{\ell \in Q^i} w_\ell (x^\ell)^{A_i} + \sum_{\ell \in \tilde{Q}^i} w_\ell (x^\ell)^{A_i}\right)$$

$$= g_j^i\left(\sum_{\ell \in Q^i} w_\ell (x^\ell)^{A_i}\right)$$

$$\leq \sum_{\ell \in Q^i} w_\ell g_j^i((x^\ell)^{A_i})$$

$$= \sum_{\ell \in Q^i} w_\ell g_j^i((x^\ell)^{A_i}) + \sum_{\ell \in \tilde{Q}^i} w_\ell g_j^i((x^\ell)^{A_i})$$

$$= \sum_{\ell \in Q} w_\ell g_j^i((x^\ell)^{A_i}).$$

The inequality in (4.4) follows from the g_j^i being convex. The reasons for the equalities are ...

first: The projection map $x \to x^{A_i}$ is linear;

second: $(x^\ell)^{A_i} = 0$ if $\ell \in \tilde{Q}^i$;

third: $g_j^i(0) = 0.$

Summing (4.4) over $i \in P$, we continue

$$g_j(\hat{x}) = \sum_{i \in P} g_j^i(\hat{x}^{A_i})$$

$$\leq \sum_{i \in P} \sum_{\ell \in Q} w_\ell g_j^i((x^\ell)^{A_i})$$

$$= \sum_{\ell \in Q} w_\ell \sum_{i \in P} g_j^i((x^\ell)^{A_i})$$

$$= \sum_{\ell \in Q} w_\ell g_j(x^\ell).$$

This verifies (4.2) if g_j is (CS a), and hence also if g_j is a sum of a (CS a) and a (CH) function. So (4.2) has been established for all g_j satisfying condition (IV).

In order to complete the proof of Lemma 1 we must verify that $\hat{x}$ satisfies each constraint $j \in C$.

Suppose first that $j \in C_1$. Put [3]

$$Q_j = Q^{\delta(j)} = \{\ell \in Q : \delta(j) \in S_\ell\},$$

and put $\tilde{Q}_j = Q \backslash Q_j$. Then

$$(4.5) \qquad g_j(x^\ell) \leq \begin{cases} a_j & \text{if} \quad \ell \in Q_j \\ \\ 0 & \text{if} \quad \ell \in \tilde{Q}_j \end{cases}$$

But

$$g_j(\hat{x}) \leq \sum_{\ell \in Q_j} w_\ell g_j(x^\ell) + \sum_{\ell \in \tilde{Q}_j} w_\ell g_j(x^\ell) \qquad \text{(by (4.2))}$$

$$= \left(\sum_{\ell \in Q_j} w_\ell \right) a_j \qquad \text{(by (4.5))}$$

$$= a_j \qquad \text{(by balancedness).}$$

Finally, suppose that $j \in C_2$. Since g_j is now (LH) by condition (V), we can go through the same argument as just given with " $=$ " in place of " $\leq$ ". So all constraints are satisfied, and we have $\hat{x} \in \Theta (P)$. This completes the proof of Lemma 1.

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[3]Thus, Q^i names the sets in β that contain player i, while Q_j names the sets in β that contain the controller of constraint j.

Proof of Theorem 1. First we check that (P,v) is balanced. Let $\mathcal{B} = \{S_1,\ldots,S_q\}$ be balanced (w.r.t. P), with the weights $(w_1,\ldots,w_q)$. Our task is to show that

$$(4.6) \qquad v(P) \geq w_1 v(S_1) + \cdots + w_q v(S_q).$$

Condition (III) guarantees for each $\ell \in Q$ the existence of $x^\ell \in \Theta(S_\ell)$ such that $v(S_\ell) = f(x^\ell)$. Put

$$\hat{x} = w_1 x^1 + \cdots + w_q x^q;$$

then $\hat{x} \in \Theta(P)$ by Lemma 1. Moreover, the same argument that established (4.2) may be applied to the function $-f$ if condition (VI) holds, so we have

$$f(\hat{x}) \geq \sum_{\ell \in Q} w_\ell f(x^\ell).$$

But then

$$v(P) \geq f(\hat{x})$$

$$\geq \sum_{\ell \in Q} w_\ell f(x^\ell)$$

$$= \sum_{\ell \in Q} w_\ell v(S_\ell),$$

showing that (P,v) is a balanced game.

Finally, to show that (P,v) is totally balanced, we observe that all of its subgames are also "controlled programming games", arising from convex programs that differ from the original convex program only in their right-hand sides, and employing control maps that differ from the original function δ only in that they map into 0 the constraints of the players who are not present in the subgame. Conditions (I)-(VI) are therefore preserved and all the subgames of (P,v) are balanced. This completes the proof of Theorem 1.

5. Discussion

Remark 1. Dummy players.

By an argument similar to our proof that $v(\emptyset) = 0$, it can be demonstrated (as one would expect) that $v(D) = 0$ for any coalition D of players who are "dummies", in the sense that $\delta(j) \in D$ implies $a_j = 0$.

Remark 2. Duality in the linear case.

In the case of a purely linear program, a direct proof of Theorem 1 using duality is available. Indeed, the dual LP solutions of the P-controlled problem yield points in the core of the game, though not all core points are dual solutions in general.

Assuming linearity, the S-controlled problem may be written

$$\text{MAXIMIZE} \quad \sum_{k \in N} d_k x_k$$

subject to $x \in R_+^N$, and

$$\sum_{k \in N} b_{kj} x_k \leq a_j(S), \qquad j \in C_1,$$

$$\sum_{k \in N} b_{kj} x_k = a_j(S), \qquad j \in C_2,$$

where we define

$$a_j(S) = \begin{cases} a_j & \text{if} & \delta(j) \in S \\ 0 & \text{if} & \delta(j) \notin S. \end{cases}$$

The LP dual of the S-controlled problem is

$$\text{MINIMIZE} \quad \sum_{k \in N} a_j(S) y_j$$

subject to $y^{C_1} \in R^{C_1}$, $y^{C_2} \in R^{C_2}$, and

$$\sum_{j \in C} b_{kj} y_j \geq d_k, \qquad k \in N.$$

Note that the feasible set of the dual does not depend on S.

By condition (III), the primal has an optimal solution $\hat{x}(S)$ and a value $v(S)$:

$$v(S) = \sum_{k \in N} d_k \hat{x}_k(S).$$

Therefore the dual has an optimal solution $\hat{y}(S)$ and the <u>same</u> value:

$$v(S) = \sum_{j \in C} a_j(S) \hat{y}_j(S).$$

Define, for each $i \in P$,

$$u_i = \sum_{j \in C} a_j(\{i\}) \hat{y}_j(P) = \sum_{j \in \delta^{-1}(i)} a_j \hat{y}_j(P) .$$

We claim that the vector $u = (u_1, \ldots, u_p)$ is in the core of the game (P, v).

Indeed, by condition (II) we have $a_j = 0$ for $j \in \delta^{-1}(0)$, so

$$\sum_{i \in P} u_i = \sum_{j \in \delta^{-1}(P)} a_j \hat{y}_j(P) = \sum_{j \in C} a_j \hat{y}_j(P) = v(P).$$

Also, since $y(P)$ is feasible in the dual of the S-controlled problem for every $S \subset P$, we have

$$\sum_{i \in S} u_i = \sum_{j \in \delta^{-1}(S)} a_j \hat{y}_j(P) = \sum_{j \in C} a_j(S) \hat{y}_j(P)$$

$$\geq \sum_{j \in C} a_j(S) \hat{y}_j(S) = v(S).$$

Hence u is in the core, and it follows that the game is balanced.
That it is in fact totally balanced follows as in the last paragraph
of the proof of Theorem 1.

In some special cases of interest (see (II) in Sec. 10) the
dual solutions yield the entire core. In general, however, they
describe only a subset of the core. For further discussion of this
point see Owen [11], Rosenmüller [12], and Samet and Zemel [13].

Remark 3. Convexity is not enough.

The following example with $P = \{1,2\}$ shows that convex separa-
bility cannot be replaced in Theorem 1 by simple convexity.

$$\text{MAXIMIZE} \quad 20x_1 - x_1^2 - x_2^2 ,$$

subject to $x \in R^2$, and

$$\delta(j):$$

$$x_1 - x_2 \leq 1 \qquad\qquad 1$$

$$x_2 \leq 2 \qquad\qquad 2$$

$$-x_1 + x_2 \leq 0 \qquad\qquad 0$$

Then $A_1 = \emptyset$ and $A_2 = \{2\}$, so the objective function is concave
but not (CS $\mathcal{A}$). One easily calculates

$$v(P) = 47, \quad v(\{1\}) = 19, \quad v(\{2\}) = 32, \quad v(\emptyset) = 0.$$

Since $v(\{1\}) + v(\{2\}) > v(P)$, the game is not balanced.

Remark 4. A simple test for (CS $\mathcal{A}$).

Condition (IV) is complicated by the fact that it acts not on
the individual functions g_j but on all of them together, since the
definition of (CS $\mathcal{A}$) involves the sets $A_i \in \mathcal{A}$. In many cases,
however, this complication can be avoided, as we can learn enough
about $\mathcal{A}$ by a simple inspection of the individual g_j's.

For instance, it may happen that g_j, in addition to being convex and vanishing at 0, is strictly increasing in all the variables x_k on which it actually depends --- that is, for all $k \in T$ where $T \subset N$ is such that $g_j(x) \equiv g_j(x^T)$. If this is true, then the single constraint $g_j(x) \leq 0$ will force x^T to be 0. So we have $T \subset A_{\delta(j)}$ (assuming $\delta(j) \neq 0$) and hence $g_j \in (CS\ \mathcal{Q})$, without further ado.

6. Committee Controls

Our model already provides a way to represent the control of a resource by a "committee" of players $T \subset P$ in one special case, namely, when the committee's rules require a unanimous vote before the resource will be made available. This is accomplished by writing the constraint in question $|T|$ times, mapping it by δ each time to a different member of T.

To represent more general types of committee control, however, we must make an extension of the model. We eliminate the function $\delta : C \to P \cup \{0\}$, and, in its place, assign to each $j \in C$ a collection $\mathbb{w}_j \subset 2^P$ of pairwise independent coalitions,[4] each one having authority over the use of the j-th resource. The generalized S-controlled problem is then defined by putting $a(S)$ for the right hand side, where

$$
a_j(S) = \begin{cases} a_j & \text{if} \quad \mathbb{w} \cap 2^S \neq \emptyset. \\[2em] 0 & \text{if not.} \end{cases}
$$

In other words, a_j can be used only if some subset of S has the power to authorize it.

One may regard the elements of $\mathbb{w}_j$ as the __minimal winning coalitions__ of a simple game [15]. Minimal winning coalitions are typically assumed to be non-disjoint, and in fact our present purpose will require that there is at least one player common to them all. Define

$$
V(j) = \bigcap \{S : S \in \mathbb{w}_j\},
$$

$$
V^{-1}(i) = \{j : i \in V(j)\}.
$$

---

[4] Two sets S and T are "independent" if neither $S \subset T$ nor $T \subset S$.

Here the letter V is meant to suggest "veto". It is well known that a simple game has a nonempty core if and only if there is at least one veto player.

Let $v(S)$ denote the value of the generalized S-controlled problem just defined. We wish to show that the game (P,v) is totally balanced. First, however, we must adjust our conditions (I)-(VI) to the extent that they involve the function δ. Specifically, we replace condition (II) by

$$(II') \qquad a_j = 0 \quad \text{if} \quad V(j) = \emptyset.$$

(In other words, committees with no veto members do not control anything.) In addition, we shall replace a with $a' = (A'_1,\ldots,A'_p)$, where A'_i is the set of all indices $k \in N$ such that the three conditions

$$\left\{ \begin{array}{lll} g_j(x) \le 0 & \text{for all} & j \in C_1 \cap v^{-1}(i), \\[2mm] g_j(x) = 0 & \text{for all} & j \in C_2 \cap v^{-1}(i), \quad \text{and} \\[2mm] x \ge 0 & & \end{array} \right.$$

together imply that $x_k = 0$. (In words, player i can force x_k to be zero by exercising all his vetoes.) This modification leads to new conditions (IV') and (VI'), with (CS a') and (-CS a') replacing (CS a) and (-CS a).[5]

Theorem 2. Suppose conditions (I), (II'), (III), (IV'), (V), and (VI') hold in a generalized (committee-controlled) CPP. Then (P,v) is totally balanced.

---

[5]To see that our generalized model includes the original, merely set $w_j = \{\{\delta(j)\}\}$ for $j \in \delta^{-1}(P)$ and $w_j = \emptyset$ for $j \in \delta^{-1}(0)$.

Proof. Let $P^\natural$ be any subset of P. For each $j \in C$, define

$$v^\natural(j) = \cap \{S : S \in 2^{P^\natural} \cap W_j\}$$

with the convention that $\cap \emptyset = P$. Intuitively, $v^\natural(j)$ consists of those individuals in $P^\natural$ who are __essential__ to any attempt by the group $P^\natural$ to control j. For any $S \subset P^\natural$, we now define an auxiliary "S-problem" by taking the right-hand side to be $a^\natural(S)$, where

$$a^\natural_j(S) = \begin{cases} a_j & \text{if} \quad v^\natural(j) \subset S, \\ \\ 0 & \text{if not.} \end{cases}$$

We see that this control structure is of the special "unanimous" form described in the first paragraph of this section. The controls can therefore be "individualized", and after a routine verification of the conditions of Sec. 4, Theorem 1 can be applied to show that the game $(P^\natural, v^\natural)$ is totally balanced, where $v^\natural(S)$ denotes the optimal value of the S-problem. By Theorem A, $\text{Core}(P^\natural, v^\natural)$ is not empty.

We now compare this auxiliary game $(P^\natural, v^\natural)$ with the subgame $(P^\natural, v_{P^\natural})$ of (P, v), obtained in the usual way be restricting the domain of v to the subsets of $P^\natural$ (see the last paragraph of Sec. 3). First we claim that

$$a_j(S) \leq a^\natural_j(S), \quad \text{all} \quad S \subset P^\natural .$$

If $a_j(S) = 0$, this is immediate. If $a_j(S) \neq 0$ then there is a subset Q of S that belongs to W_j by the definition of $a_j(S)$. Hence $v^\natural(j) \subset Q \subset S$ by the definition of $v^\natural(j)$. Hence $a^\natural_j(S) = a_j = a_j(S)$ by the definition of $a^\natural_j(S)$, and the claim is proved. It follows that all of the S-problem's constraints are (if anything) less constraining that those of the S-problem. Hence

$$v_{P^\natural}(S) = v(S) \leq v^\natural(S), \quad \text{all} \quad S \subset P^\natural .$$

On the other hand, the $P^\natural$-problem and the $P^\natural$-problem are identical. So we have

$$v_{P^\natural}(P^\natural) = v(P^\natural) = v^\natural(P^\natural),$$

and hence

$$\text{Core}(P^\natural, v_{P^\natural}) \supset \text{Core}(P^\natural, v^\natural) \neq \emptyset.$$

By Theorem A, the subgame $(P^\natural, v_{P^\natural})$ is balanced. Since this holds for all $P^\natural \subset P$, the game (P, v) is totally balanced.

7. A Canonical Separable Form

We return to the case of an individualized control function δ.

In Sec. 8 it will be useful to have a way of eliminating the $(-CH)$ part of the objective, leaving it purely $(-CS\ \alpha)$. (See condition (VI) in Sec. 4.) Accordingly, consider a CPP in which the objective has the form

$$f(x) \equiv f^1(x^{A_1}) + \cdots + f^p(x^{A_p}) + h(x),$$

where h is $(-CH)$, $h \neq 0$, and $f - h$ is $(-CS\ \alpha)$. Introduce pn new variables y_k^i, $i \in P$, $k \in N$, and write y^i for $(y_1^i,\ldots,y_n^i)$ and y for $(y^1,\ldots,y^p)$. Define a new CPP with $c + p + n$ constraints as follows:

$$\text{MAXIMIZE} \quad \hat{f}(x,y) = \sum_{i=1}^{p} (f^i(x^{A_i}) + h(y^i)),$$

subject to $x \in R_+^N$, $y \in R_+^{N \times P}$, and

$$g_j(x) \leq a_j, \qquad j \in C_1,$$

$$g_j(x) = a_j, \qquad j \in C_2,$$

$$\sum_{k \in N} y_k^i \leq B, \qquad i \in P,$$

$$x_k - \sum_{i \in P} y_k^i = 0, \qquad k \in N.$$

Here B is a positive constant, chosen so large that the constraints in which it appears are ineffective (that is, until B is replaced by 0). The new control δ function is given by

$$
\begin{cases}
\hat{\delta}(j) \equiv \delta(j), & j \in C, \\
\hat{\delta}(c + i) = i, & i \in P, \\
\hat{\delta}(c + p + k) = 0, & k \in N.
\end{cases}
$$

Each of the new sets $\hat{A}_i$, $i \in P$, therefore consists of just the variables in the original A_i, together with the n new variables y_k^i, $k \in N$. It follows that $\hat{f}$ is $(-CS\ \hat{a})$, as desired. Moreover, if the functions g_j originally satisfied condition (IV), then they continue to do so with $(CS\ \hat{a})$ in place of $(CS\ a)$. Of course, all the new constraints also satisfy condition (IV).

The equivalence of the old and new CPPs may be seen from the fact that when we maximize the new objective $\hat{f}(x,y)$, the concavity of h and the symmetrical form of the constraints as they involve the "y" variables permits us to assume w.l.o.g. that the y^i are all identical. Then, by the homogeneity of h,

$$\sum_{i \in P} h(y^i) = ph(y^1) = h(py^1) = h(\sum_{i \in P} y^i) = h(x),$$

so

$$\max \hat{f}(x,y) = \max f(x),$$

and the two programs have the same value. In the S-problem we can similarly take the y^i, $i \in S$ to be equal, with the same result. So the characteristic functions of the two CPPs coincide.

8. The Programming Problem as a Production Market

The next two sections explicate the connection between controlled programming problems (CPP) and transferable utility production markets (TUPM). As we shall see, either model can be translated under certain conditions into the terms of the other, in a way that not only preserves the characteristic functions of the associated cooperative games, but also reproduces more or less faithfully the detailed modelling of production, trade, and consumption.

We first formulate the TUPM model.[6] We start with a finite set T of __agents__ and a finite set L of __commodities__, and equip each agent $i \in T$ with an __initial endowment__ $\omega^i \in \mathbb{R}^L_+$, a set of __production possibilities__ $Y^i \subset \mathbb{R}^L$, and a __utility function__ $u^i : \mathbb{R}^L_+ \to \mathbb{R}$.[7] We assume

(a) that each u^i is continuous, concave, and nondecreasing;

and

(b) that each Y^i is convex and closed, with $Y^i \cap \mathbb{R}^L_+ = \{0\}$.

----

[6] This model may be regarded either as a generalization of the pure exchange TU model of [18] and [20] or as a specialization of the more general non-TU production models of [1] or [17].

[7] Depending on the particular role of the agent i -- producer, supplier, consumer, etc., some of these items may be trivial, e.g. $u^i = \text{const.}$ or $Y^i = \{0\}$.

For nonempty $S \subset T$, define [8]

$$Y^S = \sum_{i \in S} Y^i$$

and

$$w(S) = \max \left\{ \sum_{i \in S} u^i(x) : x \in \mathbb{R}^L \text{ and } \sum_{i \in S} (x - \omega^i) \in Y^S \right\} .$$

Intuitively, $w(S)$ represents the best result possible for the members of S if they pool their resources and production capabilities and distribute the output among themselves to maximize their combined utilities. The game (T,w) is known to be totally balanced, and so invites comparison with the programming game of Theorem 1.

Consider a CPP satisfying conditions (I)-(VI). In view of Sec. 7, condition (VI) can be strengthened w.l.o.g. to the requirement that the objective function be $(-CS \, \mathcal{Q})$, so the problem takes the following form:

$$\text{MAXIMIZE} \quad f(x) = f^1(x^{A_1}) + \cdots + f^p(x^{A_p})$$

subject to $x \in \mathbb{R}^N_+$, and

$$g_j(x) \leq a_j, \qquad j \in C_1$$

$$g_j(x) = a_j, \qquad j \in C_2,$$

and governed by the control function

$$\delta : C \to P \cup \{0\}.$$

---

[8] More generally, we could equip each <u>coalition</u> S with its own production set $Y^S \subset \mathbb{R}^L$ and make the natural requirement that the ensemble $\{Y^S : S \subset T, \; S \neq \emptyset\}$ be "totally balanced", in the sense that for each $S \subset P$,

$$Y^S \supset \bigcup \sum_k w_k Y^{S_k},$$

where the union runs over all balanced collections of subsets of S and their respective balancing weights. (see Sec. 3).

We wish to translate this CPP into a TUPM.

Some further notation will be helpful. For each $i \in P$, let $a(i)$ denote the vector in R_+^C whose j-th term is a_j if $\delta(j) = i$ and 0 otherwise. (In the notation of Sec. 6, this would be called $a(\{i\})$.) Also, for any $y \in R_+^C$, let $\Theta(P,y)$ denote the feasible set for the above program with $(a_1,\ldots,a_c)$ replaced by $(y_1,\ldots,y_c)$. Finally, define

$$Z = \{(-y,x) : y \in R_+^C \text{ and } x \in \Theta(P,y)\};$$

this is a closed, convex subset of $R^C \times R^N$, containing the origin.

We can now make the appropriate identifications:[9]

$$T \iff P$$
$$L \iff C \cup N$$
$$\left.\begin{array}{l} \omega^i \iff (a^{(i)},0) \in R^{C \cup N} \\[8pt] u^i(y,x) \iff f^i(x^{A_i}) \\[8pt] Y^i \equiv Y \iff Z \end{array}\right\} \text{ for all } i \in P = T$$

It is now immediate that the two games (T,w) and (P,v) are equal. Note that the CPP makes a sharp distinction between raw materials and consumer goods. The more general "production possibility set" approach of the TUPM accommodates this distinction but does not require it. Note also that the fact that different players control different inputs in the CPP gives them different practical production possibilities, despite the fact that the Y^i are all equal.

----

[9] It is understood that the sets C and N have no members in common, despite the convention of using overlapping sets of positive integers to name their members.

9. The Production Market as a Programming Problem

We now reverse the viewpoint of the preceding section, and attempt to represent the TUPM as a CPP. Since no extra trouble is involved, we shall do this in terms of the more general coalitional production market, described in the second footnote of Sec. 8, in which the coalition-controlled production possibility sets Y^S may be larger that what would arise from merely adding up the individual sets Y^i, $i \in S$.

In order to carry out the translation of the TUPM into a CPP we have to introduce a regrettably large array of new variables $z = (z_k^{i,S})$ and $x = (x_k^{S,i})$, where $i \in T$, $S \in 2^T \setminus \{\emptyset\}$, and $k \in L$. We shall interpret $z_k^{i,}$ as the quantity of commodity k given by agent i to the coalition S, before the production move, and $x_k^{S,i}$ the quantity of commodity k given by the coalition S to agent i, after the production move.

For each nonempty $S \subset T$, define $h_S : R^L \to R$ by

$$h_S(x) = \min_{y \in Y^S} d(x,y),$$

where d denotes Euclidean distance. Note that h_S is a convex function vanishing at 0, by condition (b) in Sec. 8.

We can now write out the programming problem

$$\text{MAXIMIZE} \quad \sum_{i \in T} u^i \left(\sum_{S \in 2^T \setminus \{\emptyset\}} x^{S,i} \right)$$

subject to $x \geq 0$, $z \geq 0$, and

$$h_S \left(\sum_{i \in T} x^{S,i} - \sum_{i \in T} z^{i,S} \right) \leq 0 \quad \left. \right\}$$

$$\sum_{k \in L} \sum_{i \in T} z_k^{i,S} \leq B \quad \left. \right\} \quad \text{all} \quad S \in 2^T \backslash \{\emptyset\}$$

$$\sum_{S \in 2^T \backslash \{\emptyset\}} z^{i,S} \leq \omega^i \quad \left. \right\}$$

$$\sum_{k \in L} \sum_{S \in 2^T \backslash \{\emptyset\}} x_k^{S,i} \leq B \quad \left. \right\} \quad \text{all} \quad i \in T$$

As before, B is a sufficiently large constant.

Let us now describe the controls for the four classes of constraints above. Those of the first class are uncontrolled, i.e., are mapped by δ into 0. Those of the second are controlled by the unanimous vote of their respective coalitions, as described in the first paragraph of Sec. 6. (Technically, each constraint must be repeated $|S|$ times to accomplish this by a singled-valued function δ.) The third and fourth classes are controlled by the respective agents $i \in T$

The objective function of this CPP is clearly $(-CS \; \mathcal{Q})$. Likewise, the h_S are $(CS \; \mathcal{Q})$ --- indeed, their domains are $R_+^{A_S}$, where $A_S = \cap \{A_i : i \in S\}$. It follows that all the conditions (I)-(VI) of Sec. 4 are satisfied. That the CPP has the same characteristic function as the TUPM we started with is obvious from the construction.

To what extent are our two translations

$$\text{TUPM} \rightarrow \text{CPP} \quad \text{and} \quad \text{CPP} \rightarrow \text{TUPM}$$

"faithful" in their preservation of structure? Consider their composition:

(9.1)   TUPM → CPP → TUPM.

Comparing the two TUPMs that appear in (9.1) will reveal to us just
how much structural detail, if any, is "last in translation".

In each step the characteristic function is preserved, and
hence the core. But the original TUPM is obviously not recovered
unchanged. Thus, while we start with $2^t - 1$ production sets Y^S,
possibly all different, we end with essentially just one set, Y.
(More precisely, we end with $t = |T|$ identical copies of Y.) Also,
in place of the original $\ell = |L|$ commodities, we wind up with
$2t(2^t - 1)\ell$ commodities. So our two transformations are certainly
not exact inverses of one another!

The proliferation of commodities in (9.1) is more apparent
than real, however. In order to gain the desired (CS α) form for
the constriants and objective in the CPP, we had to distinguish each
commodity according to its ownership, before, during, and after
production. For fixed k and S, the t commodities $(z_k^{i,S})$ are
perfect substitutes in production; and for fixed k and i, the
$2^t - 1$ commodities $(x_k^{S,i})$ are perfect substitutes in consumption.

Moreover, the production set Y is in a very high-dimensional
space, and includes representations of all the original production
sets Y^S. A little reflection reveals that while a coalition S (in
the CPP) has __formal__ access to all the original production sets, it
has practical access only to those $Y^{S'}$ with $S' \subset S$, since in the
S-problem it does not control the inputs required to operate the other
production sets.

In short, while the composition of the two transformations
necessarily yields a TUPM of a special and rather unwieldy form, it

does not lose any structural information of significance. Indeed it
is not difficult to establish, though we shall not do so here, that
the __competitive equilibrium prices__ of the two TUPMs are the same,
with due allowance for the multiplicity of labels worn by each of the
original commodities in the second model.

10. Other Applications

In this section we present three applications of our very general formulation to more particular economic models.

(I) The TU Pure Exchange Market.

If we eliminate production, the economic models considered in Sec. 8 and 9 reduce to the simple trading economies with transferable utility that were introduced in [18] in order to provide an economic characterization of totally balanced cooperative games. (See also [6, 20].) The characteristic function of such a "TU exchange market" (TUEM) is given by

$$v(S) = \max\left\{ \sum_{i \in S} u^i(x^i) : x \in R_+^{L \times P} \quad \text{and} \quad \sum_{i \in S} (x^i - \omega^i) = 0 \right\};$$

this is known to be totally balanced (see [18]).

The corresponding CPP requires far fewer variables than the production version in Sec. 9. Let

$$q = (q^{ij} : i \in T, \ j \in T),$$

with q_k^{ij} representing the quantity of commodity k shipped by trader i to trader j. We then see without difficulty that the following CPP represents the TUEM (as a special case of the representation in Sec. 9), and yields the characteristic function v given above.

$$\text{MAXIMIZE} \quad \sum_{i \in T} u^i \left(\sum_{j \in T} q^{ji} \right)$$

$$\text{subject to} \quad q \in R_+^{L \times T \times T}, \quad \text{and}$$

$$\left.\begin{array}{l} \displaystyle\sum_{j\in T} q^{ij} \leq \omega^i \\[2em] \displaystyle\sum_{j\in T} q^{ji} \leq B \end{array}\right\} \quad \text{all} \quad i \in T$$

with each constraint controlled by the respective trader i. (Since these are vector inequalities there are $2|T|\,|L|$ constraints in all, and B is now a vector of large constants.)

The relation of this simplified, production-free version to the general form in Sec. 9 is given by the identifications

$$Y^i \equiv Y^S \equiv \{0\}$$

$$q^{ij} = \sum_{S\in 2^T\setminus\{\emptyset\}} \left(z^{i,S} - \sum_{\ell\neq j} x^{S,\ell} \right)$$

$$= \sum_{S\in 2^T\setminus\{\emptyset\}} \left(x^{S,j} - \sum_{\ell\neq i} z^{\ell,S} \right),$$

as one can easily verify.

(II) The Optimal Assignment Game.

This model was first presented in [19]; see also [4, 10, 11]. There are two types of agents, T and U, thus

$$T \neq \emptyset, \quad U \neq \emptyset, \quad T \cap U = \emptyset, \quad T \cup U = P.$$

It is assumed that any agent can enter into an exclusive, bilateral contract with at most one agent of the opposite type, and together they will enjoy a nonnegative profit a_{ij}, $i \in T$, $j \in U$. The potential profitability of a coalition $S \subset P$ is therefore

$$\max \{a_{i_1 j_1} + \cdots + a_{i_k j_k}\} = v(S)$$

where $i_1, \ldots, i_k$ are distinct members of $S \cap T$ and $j_1, \ldots, j_k$ are distinct members of $S \cap U$, and

$$k = \min (|S \cap T|, |S \cap U|).$$

This will be recognized as a form of the well known optimal assignment problem, which can be expressed as a linear programming problem despite its discrete character. For $S = P$ we have

$$\text{MAXIMIZE} \quad \sum_{i \in T} \sum_{j \in U} a_{ij} x_{ij},$$

subject to $\quad x \in R_+^{T \times U}$, and

$$\sum_{j \in U} x_{ij} \leq 1, \qquad \text{all } i \in T$$

$$\sum_{i \in T} x_{ij} \leq 1, \qquad \text{all } j \in U.$$

This has always a solution in integers, and if we interpret $x_{ij} = 1$ to mean that i and j have entered into a contract, we see that its optimal value is precisely $v(P)$, as defined above. To obtain a CPP with v as its characteristic function, we merely introduce a control function δ that maps each constraint onto its respective $i \in T$ or $j \in U$.[10]

Since everything is linear, the argument of Remark 2 in Sec. 5 applies. The solutions of the LP-dual problem, which is

$$\text{MINIMIZE} \quad \sum_{i \in T} t_i + \sum_{j \in U} u_j,$$

subject to $\quad t \in R_+^T$, $u \in R_+^U$, and

$$t_i + u_j \geq a_{ij}, \quad i \in T, \quad j \in U,$$

----

[10] Replacing the right-hand side by arbitrary nonnegative integers would yield the job-matching game treated by Crawford and Knoer [4].

are therefore outcomes in the core of the game. In fact, as was shown in [19], the core and the set of dual solutions coincide exactly in this case.

(III) A transshipment game.

Let (V,E) denote a transportation network consisting of a finite set V of <u>nodes</u> and a finite set $E \subset V \times V$ of (directed) <u>arcs</u>. A single commodity is to be shipped over this network, and the shippers (players) will control the events at some of the nodes. For simplicity, assume that each shipper controls precisely one node;[11] this enables us to treat the players P as a subset of V. Denote the set of uncontrolled nodes $V \backslash P$ by J. With each node $i \in P$ we associate ...

     a   <u>supply capacity</u> $S_i \geq 0$,

     a   <u>demand capacity</u> $D_i \geq 0$,

     a   <u>purchase cost</u> function $p_i : [0, S_i] \to R_+$,   and

     a   <u>sales revenue</u> function $r_i : [0, D_i] \to R_+$

With each arc $(i,j) \in E$ we associate ...

     a   <u>flow capacity</u> $C_{ij} \geq 0$,   and

     a   <u>shipping cost</u> function $c_{ij} : [0, C_{ij}] \to R_+$.

The p_i and c_{ij} are concave, the r_i convex, and all these functions are continuous and vanish at 0.

The "rules of the game" allow any set S of shippers, if they wish to act independently of the others, to purchase, transship, and sell the commodity, provided they obey the capacity constraints and avoid the nodes belonging to the other players. In other words, they are restricted to the subnetwork (V_S, E_S), where

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[11] As previously, more complex systems of control could be considered here.

$$V_S = S \cup J$$

$$E_S = E \cap (V_S \times V_S).$$

The "worth" $v(S)$ of the coalition S is just the maximum profit (i.e. total revenue minus total cost) that it can achieve in this subnetwork.

This model can be expressed as a CPP. In the following, z_i represents the supply at node i and x_{ij} the amount shipped from node i to node j.

MAXIMIZE
$$\sum_{i \in P} \left[r_i \left(z_i - \sum_{j:(i,j) \in E} x_{ij} + \sum_{j:(j,i) \in E} x_{ji} \right) - p_i(z_i) \right] - \sum_{ij \in E} c_{ij}(x_{ij})$$

subject to $z \in R_+^V$, $x \in R_+^E$, and

$$\left. \begin{array}{c} z_i - \sum\limits_{j:(i,j) \in E} x_{ij} + \sum\limits_{j:(j,i) \in E} x_{ji} \leq D_i \\ z_i \leq S_i \end{array} \right\} \quad i \in P$$

$$x_{ij} \leq C_{ij}, \qquad (i,j) \in E \setminus (J \times J).$$

The first two types of constraints are controlled by the corresponding players $i \in P$. The constraints of the third type are controlled by

i and[12] j	if	$i \in P$, $j \in P$,
i alone	if	$i \in P$, $j \in J$,
j alone	if	$i \in J$, $j \in P$.

With this control structure the required separability is attained and the CPP satisfies conditions (I)-(VI) of Sec. 4. We conclude that this transshipment game has a core, i.e., a way of distributing

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[12] See the first paragraph of Sec. 6.

the profits among the shippers which will not leave any subset of them in a position to improve their payoffs by independent action.

Special cases of this transshipment model include the classical linear transportation problem, the maximum flow problem, and the optimal assignment problem already discussed.

11. The Non-TU Case. Controlled Pareto-Programming Problems

If the utilities of the players are not transferable we can no longer work with a single objective function; instead, there will be a finite set of non-comparable objective functions to be "Pareto-optimized." Nevertheless, it is still possible to show that the non-TU game arising from a "controlled Pareto-programming problem" is totally balanced, and hence has a non-empty core. We briefly outline this.

Let M be a finite set. Call a subset X of R^M <u>comprehensive</u> if $y \in X$ and $z \leq y$ imply $z \in X$. Also let $\overline{X}$ denote the closure of X and let $\hat{X}$ denote its <u>comprehensive hull</u>, i.e., the smallest comprehensive set that contains X.

A <u>cooperative game without transferable utility</u> is an ordered triple (P,F,D). Here P is a finite set, F is a closed subset of R^P, and D is a map from the nonempty subsets of P to nonempty, open, comprehensive, proper subsets of R^P, satisfying three further conditions:

(i) $D(P) \subset \hat{F}$;

(ii) if $x \in D(S)$ and $x^S = z^S$, then $z \in D(S)$;

(iii) $\{y^S : y \in \overline{D(S)} \setminus \bigcup_{i \in S} D(\{i\})\}$ is bounded and nonempty. [13]

We now interpret these symbols. P is the set of <u>players</u>, as before; R^P is the set of <u>payoff vectors</u>, as before; F is the set of payoff vectors that are <u>feasible</u>; and $D(S)$ is the set of payoff vectors that coalition S <u>can improve upon</u>. Consequently, the <u>core</u> of (P,F,D) is just the set

---

[13]There are many variations on this "NTU characteristic-function form" in the literature, all more or less equivalent. For a discussion of this specific version, see [17].

$$F \setminus \bigcup \{D(S) : S \in 2^P \setminus \{\emptyset\}\}.$$

Following Scarf [14], we define the game (P,F,D) to be __balanced__ if, for every balanced family $\{S_1, \ldots, S_q\}$ of coalitions we have

$$\bigcap_{\ell=1}^{q} D(S_\ell) \subset \hat{F} \,.$$

It is __totally balanced__ if each of its subgames (R, F_R, D_R) is balanced, where $\emptyset \neq R \subset P$, and F_R and D_R are defined in the obvious way. The following generalization of the "sufficiency" half of Theorem A is due to Scarf [14]; see also [17].

Theorem B. __If__ (P,F,D) __is balanced, its core is non-empty.__

In a __controlled Pareto-programming problem__ (or CPPP), the constraints, the control map, and the associated sets $A_i \subset N$ are exactly as in Sec. 4, i.e., are subject to conditions (I), (II), (IV), and (V). Let $u_i : R^{N|A_i} \to R$ be the utility function for player i. We further assume that

   (iv) u_i is quasi-concave and continuous for each $i \in P$,

   (v) $\Theta(S) \cap \underset{i \in S}{\times} R^{N|A_i}$ is compact for each $S \in 2^P \setminus \{\emptyset\}$,

where $\Theta(S)$ is defined to be $\overline{D(S)}$, in effect the feasible set of the "S-problem" in this new setting.

For any $S \in 2^P \setminus \{\emptyset\}$, put

$$V(S) = \{z \in R^P : \text{for some } x \in \Theta(S), \ z_i = u^i((x)^{A_i}) \text{ for all } i \in S\}.$$

The game (P,F,D) that arises from this CPPP is given by the identifications:

$$F \iff \overline{\hat{V}(P)} \quad \text{and} \quad D(S) \iff \text{interior of } \hat{V}(S) .$$

It is easy to check that, given (iv) and (v) and these identifications, conditions (i), (ii), and (iii) are also satisfied.

To check that (P,F,D) is balanced, let $\{S_1,\ldots,S_q\}$ be a balanced collection with weights $(w_1,\ldots,w_q)$. Take any $x \in \cap \{D(S_\ell) : \ell = \ell,\ldots,q\}$. We must show that $x \in F$. By the definition of D, there exist $x^\ell \in \Theta(S_\ell)$, $\ell = 1,\ldots,q$, such that

$$u^i((x^\ell)^{A_i}) > u^i((x)^{A_i}) \quad \text{for all} \quad i \in S.$$

Consider the point

$$\hat{x} = \sum_{\ell=1}^{q} w_\ell x^\ell .$$

By Lemma 1, $\hat{x}$ is in $\Theta(P)$. But for any $i \in P$,

$$u^i((\hat{x})^{A_i}) = u^i\left(\sum_{\ell : i \in S_\ell} w_\ell (x^\ell)^{A_i} + \sum_{\ell : i \notin S_\ell} w_\ell (x^\ell)^{A_i} \right)$$

$$= u^i\left(\sum_{\ell : i \in S_\ell} w_\ell (x^\ell)^{A_i} \right)$$

$$\geq \min_{\ell : i \in S_\ell} u^i((x^\ell)^{A_i})$$

$$> u^i((x)^{A_i}) .$$

(The first inequality follows from the quasi-concavity of u^i and the fact that the sum of the weights $w_\ell : i \in S_\ell$ is 1.) It follows from this that $s \in F$, since F is comprehensive and contains $V(P)$. Thus (P,F,D) is balanced. But since each S-problem of a CPPP is itself a CPPP (compare the last step in the proof of Theorem 1), we conclude:

Theorem 3. <u>Assume that conditions</u> (I), (II), (IV), (V), (iv), and (v)

<u>hold for a CPPP. Then the derived non-TU cooperative game is totally</u>

<u>balanced</u>.

As in the TU case, a production economy can be associated with

any controlled multi-objective programming problem of this type, and

vice versa. We omit the details.

Acknowledgment.

Footnotes

1. In contrast, the formulation of Kalai and Zemel [8,9] has the players controlling the variables rather than the constraints; this is sometimes more natural, e.g., in network problems. However, it is usually possible to translate a given model from either control form to the other by putting in extra variables or constraints.

2. It would slightly stronger and superficially simpler to base our conditions on the collection $\mathcal{C}'$, given by

$$A_i' = \{k \in N : \text{ if } x \in \Theta(P \setminus \{i\}) \text{ then } x_k = 0\} .$$

Since $A_i' \supset A_i$ this would yield $(CS\,\mathcal{C}') \supset (CS\,\mathcal{C})$ and result in a minor strengthening of Theorem 1. But this modified definition is undesirable because it depends too heavily and specifically on the data of the programming problem and, among other things, raises the possibility that the class of functions $(CS\,\mathcal{C}')$ might change as we move from one S-reduced problem to another. (See also Remark 4 in Sec. 5.)

3. Thus, Q^i names the sets in $\mathcal{B}$ that contain player i, while Q_j names the sets in $\mathcal{B}$ that contain the controller of constraint j.

4. Two sets S and T are "independent" if neither $S \subset T$ nor $T \subset S$.

5. To see that our generalized model includes the original, merely set $\mathbb{b}_j = \{\{\delta(j)\}\}$ for $j \in \delta^{-1}(P)$ and $\mathbb{b}_j = \emptyset$ for $j \in \delta^{-1}(0)$.

6. This model may be regarded either as a generalization of the pure exchange TU model of [18] and [20] or as a specialization of the more general non-TU production models of [1] or [17].

7. Depending on the particular role of the agent i -- producer, supplier, consumer, etc., some of these items may be trivial, e.g. u^i = const. or $Y^i = \{0\}$.

8. More generally, we could equip each _coalition_ S with its own production set $Y^S \subset R^L$ and make the natural requirement that the ensemble $\{Y^S : S \subset T, S \neq \emptyset\}$ be "totally balanced", in the sense that for each $S \subset P$,

$$Y^S \supset \bigcup_k \sum w_k Y^{S_k} \, ,$$

where the union runs over all balanced collections of subsets of S and their respective balancing weights. (See Sec. 3).

9. It is understood that the sets C and N have no members in common, despite the convention of using overlapping sets of positive integers to name their members.

10. Replacing the right-hand side by arbitrary nonnegative integers would yield the job-matching game treated by Crawford and Knoer [4].

11. As previously, more complex systems of control could be considered here.

12. See the first paragraph of Sec. 6.

13. There are many variations on this "NTU characteristic-function form" in the literature, all more or less equivalent. For a discussion of this specific version, see [17].

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