A MODEL OF INVENTORY AND LAYOFF BEHAVIOR UNDER UNCERTAINTY

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Introduction

Fluctuations in inventory stocks and layoff rates of business firms are well-known to be prominent features of business cycles. Further, intuition suggests that movements in inventories and layoffs are apt to be closely interrelated. This is because firms respond to (say) a slump in demand by accumulating inventories and by reducing levels of output which are accomplished in part by laying off workers. Despite considerable heuristic discussion of this interaction in the literature, rather little formal analysis has been devoted to it.

In fact, the vast bulk of the theoretial literature analyzing inventory and layoff behavior has developed along two, quite independent lines. Buffer stock models of inventory behavior where the firm has some monopoly power over price have dominated developments in the literature on inventories. See Hay [1970], Zabel [1970, 1972], and more recently Amihud and Mendelson [1983], Blinder [1982], Maccini [1982], and Reagan [1982] for contributions to this literature.[1] In these models, higher inventories raise expected sales to the firm by reducing the chance that the firm will be caught out of stock if demand turns out to be high. These benefits must be balanced against the storage and interest costs of holding inventories. Essentially, the firm chooses price and output before demand is revealed, and then inventories act as a buffer by absorbing any random shocks to demand. In these models, however, no allowance is made for the possibility that the firm may use temporary layoffs as well to absorb sudden downswings in demand.

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[1]Other relevant models include those of Belsley [1969], Childs [1967] and Holt, et al [1960]. These models also use buffer stock motives for holding inventories but are based on the assumption that the firm minimizes costs subject to a given level of demand. These models also suffer from the limitation of the above models, namely, they ignore layoffs.

Implicit contract theory, on the other hand, has been used as a framework in which to analyze temporary layoffs.[2] Contributions to this literature include Azariadis [1975], Baily [1977], Burdett and Mortenson [1980], and Holmstrom [1983]. In these models, the firm contracts for its labor force prior to the realization of product demand. The "contract" consists of a wage-employment offer that specifies the wage and the probability of layoffs for each possible state of demand. Layoffs then occur in these models because the firm in general will have contracted for more workers than is optimal to fully employ when a slump in demand occurs. A limitation of these models is that inventories are ignored as a mechanism by which downward shocks to demand can be absorbed.

In this paper, we develop a model of the firm that possesses the opportunity to use both inventories and temporary layoffs to respond to fluctuations in demand. The model is essentially a blend of a buffer stock model of inventory behavior and an implicit contract model of layoff behavior. Our objective is to investigate the nature of the interaction of inventories and layoffs over the cycle and to explore the implications of this interaction for firm behavior.

That inventory and layoff behavior may be interrelated has received rather meagre attention in the literature. Brechling [1981], Crawford [1979] and Nadiri and Rosen [1973] have looked at this interrelationship. But, these are mainly empirical studies in which an explicit theoretical model to rigorously analyze the question was not developed. In an interesting recent paper,

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[2]Temporary layoffs refer to layoffs that result in the recall of a worker to the firm making the layoff when demand rebounds. The evidence indicates that about two-thirds of layoffs over the cycle are temporary in nature — See Lilien [1979].

Topel [1982] has proposed a model to consider the question. In his model, inventories are held to satisfy speculative motives and layoffs arise from perfectly anticipated fluctuations in demand. In contrast, in the model we will develop inventories are held to satisfy buffer stock motives and layoffs are due to unanticipated (i.e., random) changes in demand. As we will indicate below, these differences lead to important differences in results.

The paper proceeds as follows. In Section II, the optimization problem the firm uses to make decisions is set down. The problem can be separated into ex ante and ex post components. The ex ante problem focuses on decisions -- the choice of a price to set and a work force to employ -- that the firm must make prior to the revelation of demand. The ex post problem, on the other hand, focuses on decisions -- the level of layoffs and inventories -- that the firm can make after demand is known.

Section III is devoted to an analysis of the ex post problem. There, we establish a number of propositions regarding the conditions under which the firm will make layoffs and/or accumulate inventories in response to a decline in demand. We first identify firms as being either inventory-biased or layoff biased, depending on the relative costs of holding inventories or laying off workers. Inventory-biased (layoff-biased) firms respond to the initial stages of a slump solely by accumulating inventories (making layoffs). We then investigate the consequences of a deeper slump and focus on the effects of a non-linear cost structure and diminishing returns to labor. We find that if inventory holding costs and turnover costs rise at the margin firms will tend eventually to both accumulate inventories and layoff workers as the slump deepens. Diminishing returns to labor, on the other hand, give rise to a particularly interesting phenomenon. The incentive to smooth production induces an inventory-biased firm to postpone the onset of layoffs in a slump.

However, in a sufficiently severe slump inventory holding costs build to such high levels that the firm will eventually dump its inventories and make a "massive" layoff. These are results that cannot occur in standard inventory and layoff models.

Section IV takes up the firm's ex ante problem. There, we look at the response of price and the size of the work force to changes in exogenous variables, including initial inventories, the level of anticipated demand, and the parameters of the firm's cost structure. We focus in particular on the impact of the interaction between inventories and layoffs on the comparative static properties of the price and employment decisions. We find that permitting firms to have the opportunity to layoff workers as well as to accumulate inventories tends to strengthen the response of price and employment to changes in anticipated demand, but to weaken the response of these variables to changes in inventory holding costs. These results are of particular interest because they help to provide an explanation of empirical results that have appeared in the literature.

II. The Model

In this section, we outline the structure of the model that is to be developed in this paper. We first present some notation:

x = Beginning of period inventories of finished goods

w = Wage rate

L = Quantity of workers contracted for ex ante

u = Percentage of workers employed ex post

p = Price of output

V = Market determined contract value

K = Ex post opportunity cost of worker's time

n = Demand (orders)

S = Sales realized _ex post_

Z = End-of-period inventories

Ex ante choice variables, i.e., variables chosen before the random variable is realized, are p, w and L. _Ex post_ choice variables, i.e., variables chosen after the realization of the random variable, are u, S and Z.

Following the literature on buffer stock models of inventory behavior, we assume the representative firm has some monopoly power and faces a downward-sloping demand curve.[3] It has the following form:

$$n = m(p) + \varepsilon \qquad m' < 0,\ m'' \leq 0 \tag{1}$$

where $m(p)$, defined over $\bar{p} > p > 0$ is riskless demand and ε is a random variable with probability density function, $f(\varepsilon)$, defined over $\underline{\varepsilon} \leq \varepsilon < \infty$ (with $-m(\bar{p}) < \underline{\varepsilon} < 0$) and $E(\varepsilon) = 0$.

Following the implicit contract literature, the production function for the firm is

$$g(uL) \qquad g' > 0,\ g'' \leq 0 \tag{2}$$

Hours per worker are assumed to be fixed and normalized to unity; the consequences of relaxing this assumption will be analyzed below. Further, the ex post utilization rate of labor must satisfy:

$$u \leq 1 \tag{3}$$

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[3]As Mills [1962], among others, has pointed out, competitive firms will not hold inventories to satisfy buffer stock motives since by definition a competitive firm may buy or sell all it wants at the market price. In order to build a theory around a buffer stock motive for holding inventories, we thus assume that the typical firm has some monopoly power. The approach we take is consistent with the bulk of the inventory literature cited above; it contrasts with Topel [1982] who uses competitive firms and speculative motives. Furthermore, a buffer stock motive for holding inventories parallels the buffer stock motive for holding workers that is implied by the implicit contract features of the model.

Unlike the standard inventory model, the production of output by the firm has both ex ante and ex post dimensions. The utilization rate, u, is chosen after ε has been realized, but the labor force, L, is chosen before ε is revealed.

Like the standard implicit contract model, the firm offers a wage-employment contract to workers that is at least as good as is available elsewhere. At the outset, we assume that workers are risk neutral and thus evaluate a contract offer in terms of the expected income it provides; the effects of risk aversion on the part of workers will be considered below. Formally, this implies

$$\left(\int_{\underline{\varepsilon}}^{\infty} uf(\varepsilon)d\varepsilon\right) w + \left(1 - \int_{\underline{\varepsilon}}^{\infty} uf(\varepsilon)d\varepsilon\right) K \geq V, \qquad V \geq K \tag{4}$$

where V is the ex ante market determined value of the contract and K is the ex post opportunity cost of a worker's time. K consists of the income equivalent of the additional leisure time a worker acquires when laid off and any government financed unemployment benefits the worker may be entitled to. Workers are assumed to be perfectly immobile ex post so that K does not include any expected income from ex post employment opportunities.

In specifying the contract constraint, (4), we have assumed that the wage rate is chosen independently of ex post realized demand. This assumption can be made without loss of generality since workers are assumed to be risk neutral (see Burdett and Mortensen [1980]). Further, note that as long as $V > K$, the firm must compensate workers with a higher wage rate for a higher probability of being laid off.

The firm is assumed to incur turnover costs by which we mean the costs attached to making layoffs. These include the unemployment insurance taxes the firm must pay when it lays off workers, severance pay, and other separation costs.[4] Turnover costs are defined by

$$T = T((1-u)L) \qquad T' > 0,\ T'' \geq 0 \tag{5}$$

Unlike the standard layoff model, the firm need not sell its entire output. It may hold inventories. To capture this, we suppose that the firm faces the following constraints on its behavior:

$$S \leq m(p) + \varepsilon \tag{6}$$

$$S \leq x + g(uL) \tag{7}$$

The constraint, (7a) says that realized sales, S, must be no greater than demand, while (7b) states that realized sales must be no greater than "starting stock", i.e., initial inventories plus production.

If the firm holds inventories, it will of course incur holding costs. These include both the financial costs, namely the real interest charges, and the storage and insurance costs of holding inventories. The holding costs depend on end-of-period inventories, Z, and are defined by

$$H = H(Z) \qquad H' > 0,\ H'' \geq 0 \tag{8}$$

where $Z = x + g(uL) - S$.[5] Note that (7) implies that $Z \geq 0$.[6]

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[4]As we will indicate below, we will work with a one-period model in this paper. However, intertemporal turnover factors may be indirectly captured in the model if (5) is interpreted to include the potential hiring and training costs that arise when laid-off workers are not available for recall.

[5]Our model is a one period model in which the current benefits to holding inventories are embodied in the constraints on sales and the specification of expected revenue (to be specified below). Buffer stock motives provide incentives to the firm to hold inventories even in a one period model. However, there may also of course be intertemporal incentives. These may be approximately captured in a one period model by reinterpreting $H(Z)$ to be inventory holding costs net of the future benefits to holding inventories, the latter being positively related to end of period inventories. This is the interpretation given for example, in Mills [1962]. This interpretation may imply that $H' < 0$, at least over some range. However, in this latter case, $H'' > 0$ would still hold and accordingly the results that follow would be formally unaltered. The results would, however, require a slight change in interpretation.

[6]This presumes that orders that cannot be served result in lost sales, which is the most common assumption used in the inventory literature. An

The firm is assumed to have a one period horizon and expected profits, Π, are defined by

$$\Pi = \int_{\underline{\varepsilon}}^{\infty} [pS - wuL - H(Z) - T((1-u)L)]\, f(\varepsilon)d\varepsilon \tag{9}$$

The firm maximizes (9) subject to (3), (4), (6) and (7). Using the definition of Z, its choice variables are p, w, L, u and S.

In formulating this model, a key innovation that we have made is to reformulate the standard price-output-inventory model to make it amenable to integration with the standard implicit contract model. Essentially, the benefits to the firm to holding inventories are captured in our model in the specification of revenue and the constraints on realized sales, (7), while the costs are captured in (8). This structure, especially the benefits to holding inventories, may appear to be different from the standard inventory models of Mills [1962] and Zabel [1970], but it can be shown that if layoffs are prohibited our model yields equivalent optimality conditions to these models and is thus an equivalent formulation of inventory holding behavior. Since the proof of this assertion is somewhat lengthy, we relegate it to Appendix B.

We have assumed that the firm makes decisions on the basis of a one-period horizon. This is clearly a restrictive assumption. It can be justified on two grounds. First, the model is quite complex analytically. It is a model of firm behavior under uncertainty in which the firm uses two stocks — one for goods and one for workers — to absorb random fluctuations in demand. Working with a one-period, rather than a multi-period, model simplifies the mathematical analysis and enables us to draw insights into the interaction between inventories and layoffs that would be difficult to "see"

[Fn. 6 cont.]...alternative assumption is to assume that orders may be back-logged. Allowing for backlogging in the model is feasible but would not change our main results which focus on states where demand is low rather than high.

in a more complex framework.[7] Secondly, in our analysis we will focus on how taking into account the interaction between inventories and layoffs affects the implications of standard inventory and layoff models. The thrust of this analysis should carry-over to an intertemporal model, and thus a one-period model serves as a useful vehicle to isolate the basic nature of the interaction between inventories and layoffs.

Given the characterization of our maximization problem embodied in (9), it is easily demonstrated that the constraint (4) must hold as an equality at the optimum. Hence, we can use (4) to eliminate w from (9); this yields:

$$\int_{\underline{\varepsilon}}^{\infty}[pS - H(Z) - T((1-u)L) + K(1-u)L]f(\varepsilon)d\varepsilon - VL \tag{10}$$

The firm now maximizes (10) subject to (3), (6) and (7). In effect the contractual wage is determined residually by the expected income constraint (4) given the optimal choices of p, L, u and S.

The maximization problem characterized by (10) and the constraints (3), (6), and (7) can be decomposed into <u>ex ante</u> and <u>ex post</u> components. We first undertake an analysis of the <u>ex post</u> problem. In this problem, the firm is faced with a realized value of the random variable, ε, and takes as given p and L which are determined <u>ex ante</u>; it then makes an optimal choice of S and u. This yields a solution for S and u as a function of ε, p, and L. Next, we consider the firm's <u>ex ante</u> problem. The firm makes an optimal choice of p and L, given the derived functions for S and u. It

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[7]It is important to note in this regard that it is common in both the inventory literature and the implicit contract literature to begin with analyses of one period models to gain basic insights and then to extend the models to multi-period settings once this is complete — see Mills [1962] and Zabel [1970] on inventory models and Azariadis [1975] on implicit contract models.

is important to emphasize that this decomposition of the maximization problem into two components is for expositional purposes only. It does not imply an opportunity for the firm to exert ex post monopsony power over its workers and thereby violate the expected income constraint.

III. The Ex Post Problem

A. Some Preliminary Results

In the ex post problem, the firm takes p and L to be given and is faced with a realization of ε; it then chooses S and u to maximize:

$$pS - H(Z) - T((1-u)L) + K(1-u)L \tag{11}$$

subject to (3), (6) and (7). The optimality conditions for this problem are:

$$p + H'(Z) - \lambda_0 - \lambda_1 = 0 \tag{12}$$

$$(\lambda_1 - H'(Z))g'(uL) = K - T'((1-u)L) + \lambda_2 \tag{13}$$

$$\lambda_0 [m(p) + \varepsilon - S] = 0, \ \lambda_0 \geq 0 \tag{14}$$

$$\lambda_1 [x + g(uL) - S] = 0, \ \lambda_1 \geq 0 \tag{15}$$

$$\lambda_2 [L - uL] = 0, \ \lambda_2 \geq 0 \tag{16}$$

where λ_0, λ_1, and λ_2 are the Kuhn-Tucker multipliers associated with the constraints (6), (7), and (3) respectively. λ_0 is the shadow value of having an additional unit of demand; it is the value of what is gained in sales by having a marginally higher demand (through either a marginally higher ε or a marginally lower p). Further, λ_1 is the shadow value of having an additional unit of starting stock, i.e., of initial inventories or production. Finally, λ_2 is the shadow value of having an additional worker in the contractual labor pool. Given this interpretation, it is clear that $\lambda_2 = 0$ when some of the contractual workers are unemployed.

Our objective is to characterize how the firm's ex post decisions on sales, layoffs, and inventories depend upon the size of the demand shock and the firm's cost structure. The following proposition begins this task.

Proposition 1: Let $b = x + g(L) - m(p)$. If $\varepsilon > b$, then the firm is faced with a "boom," and thus the labor force is fully utilized $(u = 1)$, and inventories are depleted $(Z = 0)$. But if $\varepsilon < b$, then the firm is faced with a "slump" and thus either layoffs occur $(u < 1)$ or inventories are accumulated $(Z > 0)$, or both.

Proof: Provided in Appendix A.

This proposition essentially defines the conditions under which the firm faces a relatively high or low state of demand — a "boom" or a "slump" respectively -- and indicates broadly the firm's optimal response to these conditions. Using the definition of b, the condition that $\varepsilon > b$ implies that $m(p) + \varepsilon = n > x + g(L)$. That is, when $\varepsilon > b$, the realized value of the random variable is so high that realized demand exceeds the firm's potential starting stock, which is initial inventories plus maximum production.[8] In this sense, the firm is faced with "boom" conditions, and under these circumstances it is quite natural that the firm will fully utilize its workforce and deplete its inventories. On the other hand, when $\varepsilon < b$, the firm is caught in a "slump" in the sense that the realized value of ε is sufficiently low that realized demand is below potential starting stock. Under these conditions optimal behavior dictates that the firm absorb the demand shock either by making layoffs or by holding inventories or both.

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[8]Recall that x is an initial condition and p and L are determined *ex ante* and are thus fixed when the firm makes *ex post* decisions. Hence, whether $n \gtrless x + g(L)$ is determined solely by the realized value of ε.

either by making layoffs or by holding inventories or both.

The next task is to identify more precisely how the firm divides its response to a slump between layoffs and inventory accumulation. To do this, it is useful to make a distinction between what we will call inventory-biased and layoff-biased firms.

Proposition 2: Consider a small neighborhood of b with $\varepsilon < b$. If

$$H'(0)\ g'(L) < T'(0) - K \tag{17}$$

then the firm will accumulate inventories $(Z > 0)$ but fully utilize its labor force $(u = 1)$. A firm of this type will be referred to as an Inventory-Biased firm. Alternatively, if

$$H'(0)\ g'(L) > T'(0) - K \tag{18}$$

then the firm will make layoffs $(u < 1)$ but will avoid accumulating inventories $(Z = 0)$. A firm of this type will be referred to as a Layoff-Biased firm.

Proof: Provided in the appendix.

This proposition considers a state where realized demand dips just below potential starting stock; it is what we will loosely refer to as the initial stages of a slump. At the onset of a slump, the firm has a choice as to whether to accumulate inventories or to layoff workers when demand falls. This choice will be resolved on the basis of its cost structure. The term, $H'(0)\ g'(L)$, is the increase in inventory holding costs (evaluated at $Z^{\varepsilon} = 0$) that arises from using an additional unit of labor services which in turn adds to output and thus to inventories. The term, $T'(0) - K$, is marginal turnover costs (evaluated at $u=1$), net of any subsidy borne by the government or the worker. When (17) holds for a firm, marginal net turnover costs exceed marginal inventory holding costs just before the slump is to begin. If this is so, it is not surprising that the firm's immediate response to a slump is to accumulate inventories rather than to layoff workers. In this sense, it is inventory-biased. Alternatively, when (18) holds, marginal inventory holding costs exceed marginal net turnover costs at the onset of the slump, and thus the firm is layoff-biased in the sense that its immediate response to a decline in demand is to make layoffs rather than to allow inventories to accumulate.

Given the behavior of inventory-biased and layoff-biased firms in the initial stages of a slump, it is natural to ask whether an inventory-biased firm will eventually make layoffs and whether a layoff-biased firm will eventually accumulate inventories if demand were to decline to lower levels. To answer this question, it is helpful to take up two subcases that are distinguished by different assumptions regarding inventory-holding costs, turnover costs, and production conditions. Furthermore, for the sake of brevity, we will hereafter confine our analysis in the body of the paper to inventory-biased firms. The results for layoff-biased firms generally parallel those for inventory-biased firms, and thus to avoid repetition we will relegate a discussion of these to footnotes.

To facilitate this discussion, it is useful to refine the firm's <u>ex post</u> optimization problem still further. Observe first of all that in a slump ($\varepsilon < b$) sales equal demand ($S = m(p) + \varepsilon$) and thus the constraint (7) can be written as $Z = x + g(uL) - m(p) - \varepsilon \geq 0$. Inverting this constraint yields

$$u \geq \frac{g^{-1}(m(p) + \varepsilon - x)}{L} = \bar{u}(\varepsilon,p,L,x) \tag{19}$$

where in particular $\bar{u}$ falls as ε falls. In other words, in a slump the constraint that inventories be non-negative is equivalent to placing a lower bound on the utilization rate. Furthermore, in a slump, since sales are set equal to demand, maximizing <u>ex post</u> profits, (11), reduces to minimizing total variable costs. Hence, using the definition of Z, (3) and (19), the firm's *ex post* optimization problem may be written as

$$\min_{u} TVC = H(x+g(uL) - m(p) - \varepsilon) + T((1-u)L) - K(1-u)L \tag{20}$$

subject to

$$\bar{u}(\varepsilon,p,L,x) \leq u \leq 1. \tag{21}$$

B. The Effects of Non-linearities in the Cost Structure

We consider first the case where the production function is linear, but inventory holding costs and turnover costs may be non-linear. This helps to illuminate the role played in the analysis by non-linearities in the firm's cost structure.

Proposition 3: Consider an inventory-biased firm so that $H'(0)\beta < T'(0) - K$. Assume that $g(uL) = \beta uL$, and $H'' > 0$, then

(i) if $\varepsilon^h \leq \varepsilon < x + \beta L - m(p)$, then $u = 1$, $Z > 0$, and $\partial Z/\partial\varepsilon = -1$;

(ii) if $\varepsilon < \varepsilon^h$, then $u < 1$, $Z > 0$, and $-1 < \partial Z/\partial\varepsilon < 0$,

where ε^h is defined by[9]

$$H'(x+\beta L-m(p)-\varepsilon^h)\beta = T'(0) - K \tag{22}$$

Proof: Provided in Appendix A.

Recall that an inventory-biased firm responds to the initial stages of a slump by accumulating inventories and fully utilizing its labor force. Proposition 3 establishes that such a firm will eventually make layoffs if the demand shock is sufficiently severe and if inventory holding costs rise at the margin. In terms of the firm's *ex post* optimization problem, if $g(uL) = \beta uL$ and $H'' > 0$, *ex post* total variable costs, (20), are strictly convex

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[9] We are assuming here that $\varepsilon^h > \underline{\varepsilon}$ where $\underline{\varepsilon}$ is the lowest level of demand.

in u, and thus eventually an interior solution for u is optimal in which case the firm makes layoffs in addition to accumulating inventories.[10] This result contrasts rather dramatically with standard inventory models under linear production conditions in which layoffs are precluded — see, e.g., Mills [1962], Zabel [1970], etc. In these models, inventories absorb entirely the demand shock no matter how severe it is.

Another interesting aspect of Proposition 3 is that, once layoffs begin, a unit decline in demand gives rise to a less than a proportionate increase in inventories. This result is in contrast to that of the standard inventory model which predicts that a unit decline in demand gives rise to a proportionate increase in inventories. Out model thus extends standard inventory models by allowing explicitly for ex post production revisions in the decision process.[11]

Given the convexity of total variable costs, it is perhaps not surprising that whether and when an inventory-biased firm makes layoffs depends on the cost at the margin of making layoffs relative to the cost of accumulating inventories. The advantage of an explicit model is that it defines precisely

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[10]A layoff-biased firm differs from an inventory-biased firm only in the initial stages of a slump. Such a firm responds to the onset of a slump by laying-off workers rather than accumulating inventories. However, analogously to an inventory-biased firm, it can be shown that if the decline in demand is sufficiently severe a layoff-biased firm will accumulate inventories as well as make layoffs. This extends the standard layoffs model (e.g., Azaradis [1975], Baily [1977]), to allow for inventory accumulation.

[11]The notion that firms can make ex post production revisions has of course been recognized when buffer stock models of inventory behavior have been used as a basis for empirical work. Numerous authors, e.g., Lovell [1961], have rationalized inventory investment equations with buffer stock models. These equations are then typically modified to allow for ex post production revisions that firms make in response to sales "surprises". There are, however ad hoc adjustments to the underlying theoretical model. The model developed here provides an explicit rationale for these revisions based on the temporary layoffs that emerge from implicit contracts.

what the relevant costs are and identifies the critical value of demand at which layoffs begin. The latter is defined by $\varepsilon = \varepsilon^h$ and is determined by (22). It depends on the firm's cost structure and demand conditions. In particular, straightforward calculations indicate that ε^h will be higher, i.e., layoffs will begin to occur sooner, the higher is initial inventories, the lower is the level of riskless demand, the greater is the <u>ex post</u> opportunity cost of a worker's time, and the faster marginal inventory holding costs rise relative to marginal turnover costs.

[Place Figure 1 About Here.]

To illustrate Proposition 3, consider Figure 1. The curve labelled MNTC represents marginal net turnover costs; it is defined by

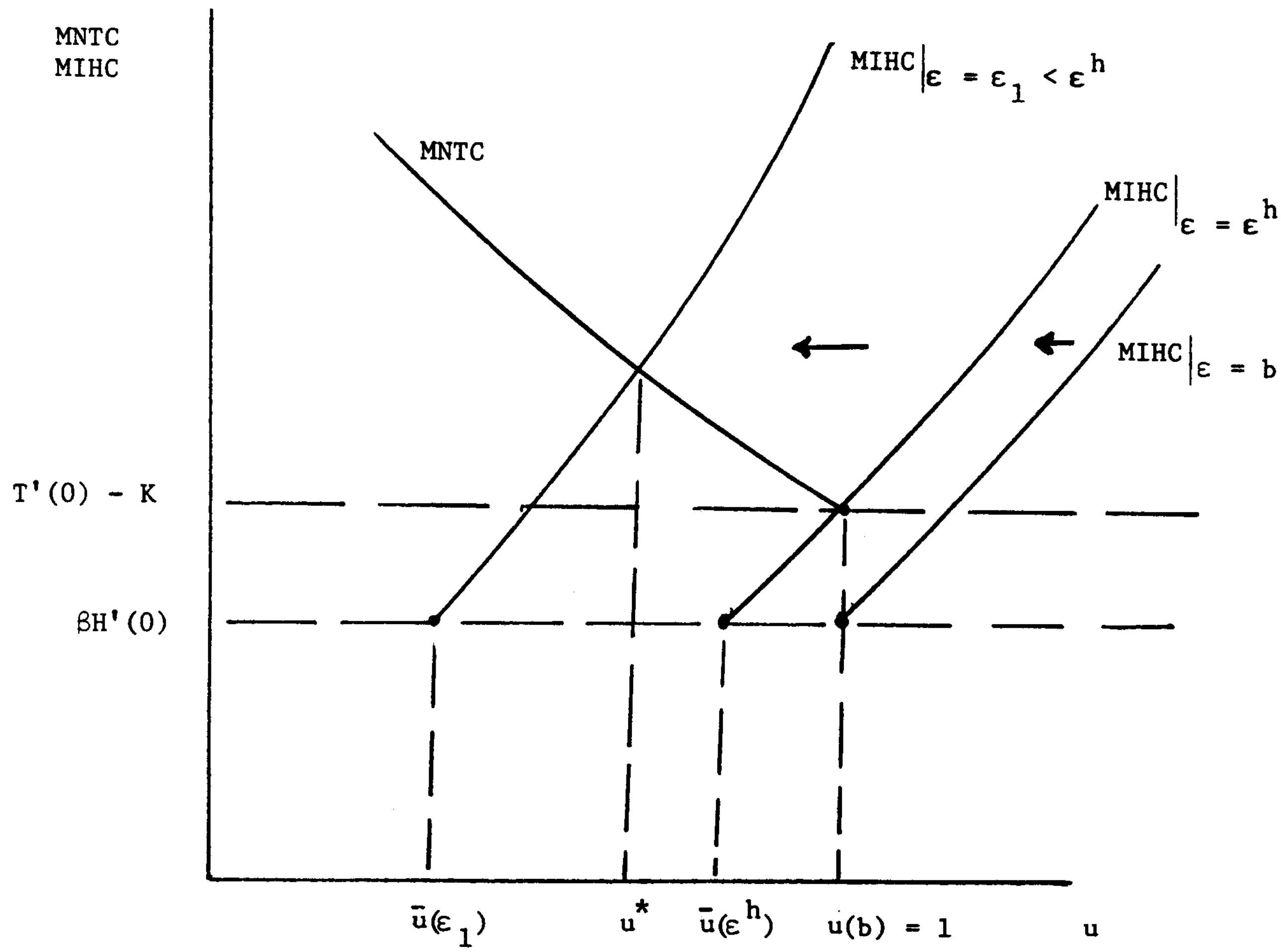
$$MNTC = T'((1-u)L) - K.$$

It takes the value, $T'(0) - K$ at $u = 1$, and, assuming that $T'' > 0$, rises as u falls, i.e., as layoffs occur. The curve labeled MIHC, on the other hand, represents marginal inventory holding costs expressed as a function of u; given the definition of Z, it is defined by

$$MIHC = \beta H'(x + \beta uL - m(p) - \varepsilon).$$

The curve begins at the point $\beta H'(0)$, where $\bar{u}$ is the layoff rate given ε, that keeps inventories equal to zero. Given that $H'' > 0$, the curve rises as u rises. Furthermore, MIHC shifts to the left as ε falls, other things equal.

Clearly, for an inventory-biased firm, when the slump begins (i.e., for ε just below b), MIHC are below MNTC so the firm begins to accumulate inventories and to utilize its work force fully $(u = 1)$. As the slump deepens (i.e., as ε falls), and inventories pile up, the MIHC curve shifts



MNTC
MIHC
MNTC
MIHC|ε = ε1 < εh
MIHC|ε = εh
MIHC|ε = b
T'(0) - K
βH'(0)
ū(ε1)
u*
ū(εh)
u(b) = 1
u

Figure 1

to the left. At $\varepsilon = \varepsilon^h$, inventories have accumulated to the point where MIHC equal MNTC. As the slump deepens still further (as $\varepsilon < \varepsilon^h$), the firm begins to layoff workers and thus to cut back ex post production as well as to accumulate inventories, and an equality between MIHC and MNTC is maintained.

Finally, it is worthwhile to note that our characterization of the interaction of inventories and layoffs contrasts sharply with that given by Topel [1982]. The reasons for this are that Topel emphasizes speculative motives for holding inventories and ignores turnover costs. In Topel's model, firms will only begin making layoffs when inventory stocks are completely exhausted. In contrast, as indicated in Proposition 3, in our model inventory-biased firms will begin making layoffs when inventory stocks have become sufficiently large that marginal inventory holding costs are at least as great as marginal turnover costs. Moreover, unlike Topel's model, firms in our model will not in general deplete inventory stocks as they make layoffs, but rather will simultaneously accumulate inventories and make layoffs.

C. The Effects of Production Smoothing

We now take up the case where the production function of the firm is strictly concave but inventory holding costs and turnover costs are linear. This helps to isolate the role that production-smoothing plays in the analysis.

Proposition 4: Assume that $H(Z) = h_1 Z$, $T((1-u)L) = t_1(1-uL)$, $g'' < 0$, and that $h_1 g'(L) < t_1 - K$, so that the firm is inventory-biased. Then:

(i) if $\varepsilon^c < \varepsilon < x + g(L) - m(p)$, then $u = 1$, $Z > 0$, and $\partial Z/\partial\varepsilon = -1$;

(ii) if $\varepsilon < \varepsilon^c$, then $u < 1$, $Z = 0$, and $\partial Z/\partial\varepsilon = 0$;

where ε^c is defined by:[12]

$$x + g(u^c L) - m(p) - \varepsilon^c = 0$$

$$h_1(x+g(L)-m(p)-\varepsilon^c) = (t_1-K)(1-u^c)L$$

Proof: Provided in Appendix A.

This is a particularly interesting and surprising result. It states that the combination of a strictly concave production function and relatively low marginal inventory holding costs induce the firm to use inventories alone to absorb a small downswing in demand. Production-smoothing incentives make layoffs unattractive in the initial stages of a slump. However, eventually, inventory holding costs at full employment rise above the net turnover costs associated with zero inventories. At this point, it is optimal for the firm to undertake a "massive" layoff -- a layoff so large that the firm depletes its inventories and absorbs the demand shock with a large cutback in production through layoffs. Further declines in demand are then absorbed entirely by layoffs with no inventory accumulation.[13]

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[12]We are assuming here that $\varepsilon^c > \underline{\varepsilon}$ where $\underline{\varepsilon}$ is the lowest level of demand. If $\varepsilon^c < \underline{\varepsilon}$, then the firm will keep $u = 1$, and layoffs will never occur. This is essentially what is assumed in standard inventory models and absorbs the demand shock with a large cutback in production through layoffs.

[13]Under the conditions of Proposition 4, an analogous result for a layoff-biased firm does not arise. A layoff-biased firm will respond to a slump by laying off workers no matter how severe the decline in demand. In effect, our model reverts to the standard layoff model in this case. The reason is that marginal inventory holding costs are too high to take advantage of any production-smoothing incentives associated with he concave production function. In essence, this is a situation in which ex post total variable costs in a slump are always minimized at the corner

$$\bar{u}(\varepsilon) = 0, \text{ i.e., } Z = 0.$$

To see what is going on here, it is helpful once again to consider the firm's _ex post_ optimization problem. When the production function is strictly concave and when inventory-holding costs and turnover costs are both linear, _ex post_ total variable costs (20), are _strictly concave_ in u. Hence, while an interior solution for u exists (i.e., satisfies the first-order conditions), it is a _cost maximum_ rather than a minimum. Therefore, in the present case, the cost minimum occurs at one of the two corners, namely, at either $u = 1$ or $u = \bar{u}$ (equivalently, $Z = 0$).

[Place Figure 2 About Here.]

This case is pictured in Figure 2. There, we plot TVC as a function of u. As the diagram indicates, it is defined only for $\bar{u}(\varepsilon) \leq u < 1$ for a given ε, and given our assumptions in this case it is strictly concave. Further, as ε falls, TVC shifts upward and $\bar{u}(\varepsilon)$ falls. Now, since we are looking at an inventory-biased firm, when demand begins to fall, the firm will accumulate inventories, and in accordance with production-smoothing motives will seek to maintain $u = 1$ for as long as possible. However, there exists a sufficiently large decline in demand, namely, when $\varepsilon = \varepsilon^c$, such that it is optimal for the firm to switch from a corner at $u = 1$ to one where $u = \bar{u}(\varepsilon^c)$ and $Z = 0$. It is at this point that the firm dumps its inventories and undertakes a massive layoff. For declines in demand in which $\varepsilon < \varepsilon^c$, it is optimal for the firm to continue to layoff workers, as this lowers TVC.

What is surprising about this result is that the non-convexity of _ex post_ total variable costs arises despite the fact that we have made widely-used, standard assumptions about production conditions and the individual components

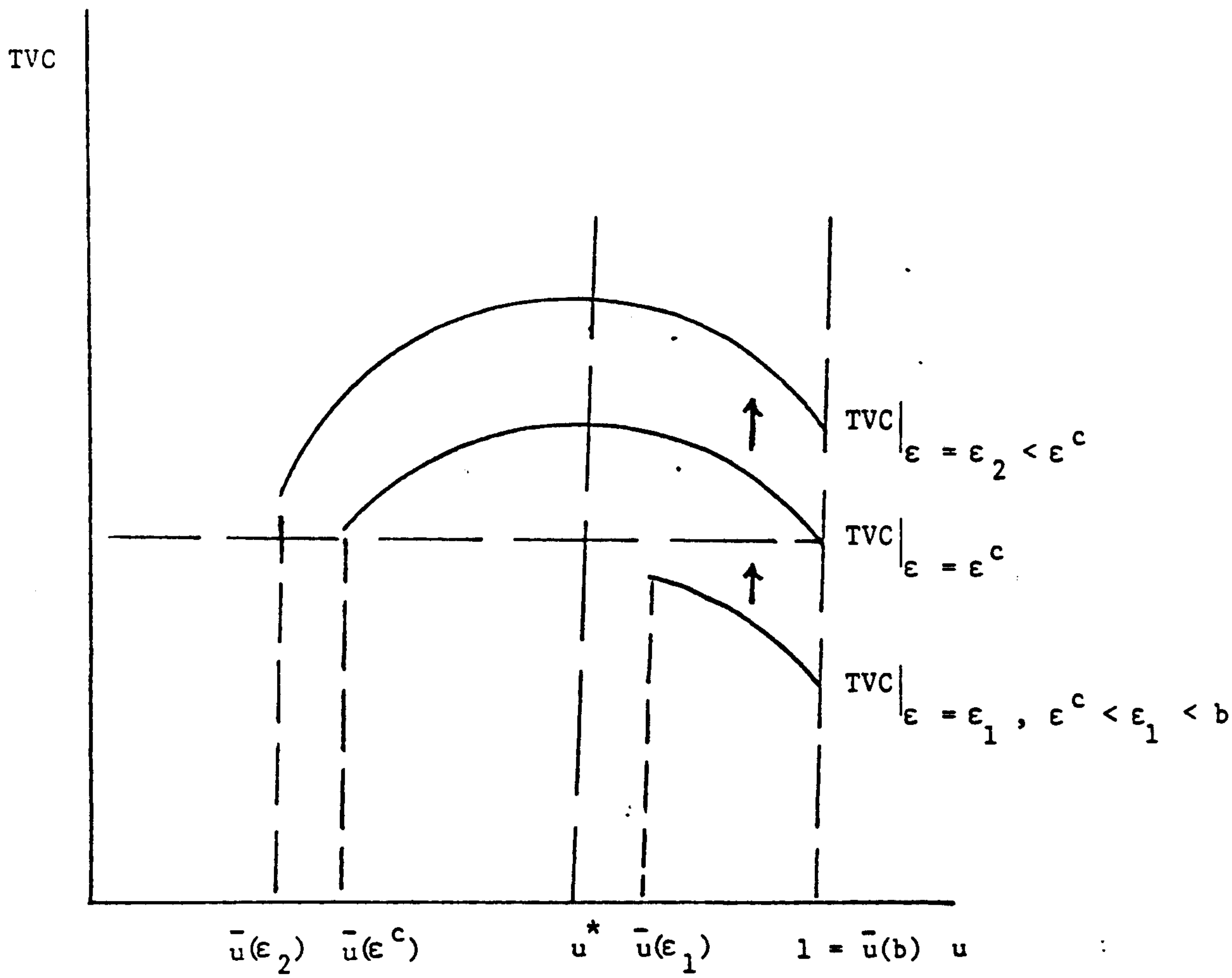
TVC
TVC$|_{\varepsilon = \varepsilon_2 < \varepsilon^c}$
TVC$|_{\varepsilon = \varepsilon^c}$
TVC$|_{\varepsilon = \varepsilon_1,\ \varepsilon^c < \varepsilon_1 < b}$
$\bar{u}(\varepsilon_2)$
$\bar{u}(\varepsilon^c)$
u^*
$\bar{u}(\varepsilon_1)$
$1 = \bar{u}(b)$
u

Figure 2

of the firm's cost structure. In particular, the production function is strictly concave, and inventory holding costs and turnover costs are convex. Nevertheless, the interaction between these "well-behaved" individual relationships produces non-convex total variable costs (in terms of u).

Proposition 4 is of considerable interest when compared with the implications of the standard inventory model. In the latter, since layoffs are not permitted, the firm simply continues to accumulate inventories as demand falls and in this manner smooths production. In our model, production-smoothing motives are present too, and it is these motives that give rise to corner solutions for u. But, given the relationship between inventory holding costs and turnover costs, production-smoothing motives will be compromised by a possibly dramatic reduction in the level of production if demand falls far enough.[14]

A further implication of interest of Proposition 5 is that it raises the possibility that ex post production varies more widely than what is predicted from standard inventory models with strictly concave production functions. In the latter, the production-smoothing incentives suggest that production should vary less than sales. But, in fact, as Blinder [1983] has indicated, production seems to vary more than sales in most industries. This clearly causes uneasiness about buffer stock inventory models. Our model suggests that the integration of a model of temporary layoffs with a buffer stock model may help to explain the volatility of production that seems to exist in reality.

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[14]Note that this result whereby the firm switches from inventory accumulation to massive layoffs contrasts not only with the standard inventory and standard layoff models but with Topel [1982] as well. That is, this type of inventory-layoff interaction cannot occur in his model.

IV. The Ex Ante Decision Problem

A. Optimality Conditions

We now turn to an analysis of the firm's ex ante decision problem. This problem consists of the firm's choice of a price to set and a work force to employ where these decisions must be made before the random variable is revealed. Further, these decisions must remain in force for the duration of the decision period.

The above analysis of the ex post decision problem yields optimal solutions for S and u for every ε and for given p, L, and x. These may be expressed as $S^* = S^*(p,L,x,\varepsilon)$ and $u^* = u^*(p,L,x,\varepsilon)$. By substituting these relationships into (10), using Propositions 1 and 2, and rearranging terms the firm's ex ante choice problem can be written as

$$\max_{p,L} (\Pi) = \max_{p,L} \{p[m(p) - D(p,L,x)] - C(p,L,x) - VL\}$$

where

$$D(p,L,x) = \int_b^\infty [\varepsilon - (x+g(L) - m(p))]\ f(\varepsilon)d\varepsilon$$

$$C(p,L,x) = \int_{\underline{\varepsilon}}^b [H(Z^*) + T((1-u^*)L) - K(1-u^*)L]\ f(\varepsilon)d\varepsilon$$

$$Z^* = x + g(u^*L) - m(p) - \varepsilon.$$

The first term of expected profits is expected revenue, which is the product of price and expected sales. Expected sales is riskless demand, $m(p)$, minus expected shortages, or stockouts, $D(p,L,x)$. The latter is the sales that the firm can expect to lose as a result of having too few inventories; this captures the benefits to the firm to holding inventories. $C(p,L,x)$ is expected <u>ex post</u> variable cost, consisting of both inventory holding costs and net turnover costs. The final term, VL, is simply the firm's wage bill.

The optimality conditions for this problem are

$$\Pi_p = m(p) - D(p,L,x) + p(m'(p) - Dp(p,L,x)) - Cp(p,L,x) = 0 \qquad (23)$$

$$\Pi_L = - pD_L(p,L,x) - C_L(p,L,x) - V = 0 \qquad (24)$$

The first condition requires that the marginal revenue for changing price be equal to its marginal cost. The marginal cost of changing price is positive which derives from the fact that a higher price lowers riskless demand and raises the possibility that the firm will be caught with higher inventories, thus absorbing higher expected inventory holding costs. The second condition equates the marginal revenue to the firm from changing its work force to the marginal cost. The former is positive, i.e., higher L raises expected revenue, because a higher work force raises ex ante production and thus starting stock and reduces the chance that the firm will suffer a "stock-out". The marginal cost from additional employment is also positive; it consists of both the value of the contract the firm must absorb when it hires an additional worker together with the higher expected inventory holding costs from having another worker on hand. The latter arises because a higher L increases ex ante production which raises the possibility the firm will be caught with higher inventories if demand turns out to be low.

B. Comparative Statics

We now undertake an analysis of the comparative static properties of the optimal price and work force of the firm. For this purpose, we assume the riskless demand function and the inventory holding cost and turnover cost functions have the following form:

$$m(p) = \bar{m} + \hat{m}(p) \qquad \hat{m}' < 0 \quad \hat{m}'' \leq 0$$

$$H(Z) = h_1 Z + (h_2/2)Z^2 \qquad h_1 > 0 \quad h_2 \geq 0$$

$$T((1-u)L) = t_1(1-u)L + (t_2/2)\ ((1-u)L)^2 \qquad t_1 > 0 \quad t_2 \geq 0$$

These assumptions are made purely to facilitate an analysis of the model. Equivalent results could be obtained in a more cumbersome manner by allowing the demand and cost functions to change with shift parameters.

A key feature of our model is that the firm has the opportunity to use inventories or layoffs or both to cover downward shocks to demand. An interesting question then is to ask how the comparative static properties of our model compare with models in which the firm does not have this flexibility; that is, it can only use layoffs or inventories but not both to buffer shocks. To gain some insight into this question, we decompose the comparative static results into "conventional effects" and "substitution effects". To illustrate the meaning of this decomposition, consider the price response to a change in an exogenous variable for an inventory-biased firm. The "conventional effect" of the response represents the change in price that would result from a change in an exogenous variable, if the firm was prohibited from making layoffs. The "substitution effect" represents the additional change in price that results solely due to the ability to make layoffs <u>ex post</u>. The sign of the substitution effect thus provides insight into how the capacity to use layoffs as well as inventories changes the sensitivity of the firm's price decision to changes in exogenous variables.

The main idea we wish to advance here is a general one which holds for a variety of circumstances concerning costs and production conditions, though the details differ from one case to another. To avoid repetition and to keep the analysis brief, we will state and prove the main result for an inventory-biased firm with linear production conditions and a non-linear cost structure. Using the same procedure it is easy to show that an identical result holds for a layoff-biased firm under these conditions. Further, the main result holds as well for a firm with a strictly concave production function and a linear cost structure. Here, however, the details differ, and thus a proposition for such a firm is stated and proved in Appendix C.

Proposition 5: Assume that the production function is linear, $g(uL) = \beta uL$, that both inventory holding costs and turnover costs rise at the margin, $h_2 > 0$, $t_2 > 0$, and that expected profits are strictly concave in p and L.[15] Then, the following comparative static results hold for an inventory-biased firm: (note: CE is the conventional effect and SE is the substitution effect):

(i) $\frac{\partial p}{\partial x} = CE_{p,x} + SE_{p,x} = 0$

$CE_{p,x} = SE_{p,x} = 0$

(ii) $\frac{\partial L}{\partial x} = CE_{L,x} + SE_{L,x} = -\frac{1}{\beta} < 0$

$CE_{L,x} = -\frac{1}{\beta} < 0$ and $SE_{L,x} = 0$

(iii) $\frac{\partial p}{\partial \bar{m}} = CE_{p,\bar{m}} + SE_{p,\bar{m}} > 0$

$CE_{p,\bar{m}} > 0$ and $SE_{p,\bar{m}} > 0$

(iv) $\frac{\partial L}{\partial \bar{m}} = CE_{L,\bar{m}} + SE_{L,\bar{m}} > 0$

$CE_{L,\bar{m}} > 0$ and $SE_{L,\bar{m}} > 0$

(v) $\frac{\partial p}{\partial h_1} = CE_{p,h_1} + SE_{p,h_1} < 0$

$CE_{p,h_1} < 0$ and $SE_{p,h_1} > 0$

(vi) $\frac{\partial L}{\partial h_1} = CE_{L,h_1} + SE_{L,h_1} < 0$

$CE_{L,h_1} < 0$ and $SE_{L,h_1} > 0$

Proof: Provided in Appendix A.

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[15] A sufficient condition for the concavity of Π in p and L is that $-pm'(p)f(b) \geq 1 - F(b)$ which essentially requires that demand be sufficiently elastic. This assumption is similar to those commonly imposed in standard inventory models to guarantee the concavity of profits. See Zabel [1970, 1972] for a discussion of concavity restrictions in inventory models.

First, consider the implications of the signs of the total effects provided in Proposition 5. Price is unrelated to initial inventories, and the size of the work force is inversely related to the reciprocal of the marginal product of labor which is marginal production cost. These results may seem surprising. They are solely a consequence of the linear production function which implies that marginal production cost is constant, and they are completely consistent with results which emerge from standard inventory models which assume constant marginal production costs (see, e.g., Zabel [1970], p. 213).

The response of price and the work force to shifts in anticipated, i.e., riskless, demand are straightforward. The results for changes in marginal inventory-holding costs are more interesting. An increase in h_1 reduces both price and employment. When h_1 rises, since it is now more expensive at the margin to hold inventories, the firm reduces price to raise riskless demand to get rid of existing inventories and reduces the potential work force and thus potential output to avoid accumulating inventories.[16]

These comparative static results contrast sharply with those of Topel [1982] who has developed a model that does consider the interrelationship between inventories and layoffs. The reasons for this are that Topel emphasizes speculative, rather than buffer stock, motives for holding inventories and layoffs. More specifically, in the Topel model, firms plan on accumulating inventories during periods when prices are rising in order to accommodate periods of high future anticipated demand. With this speculative motive, if there is a parametric increase in inventory holding costs, firms

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[16]The effects of a shift in net marginal turnover costs are analogous to that of a shift in marginal inventory holding costs, and thus for the sake of brevity are ignored here.

become more reluctant to accumulate inventories. Hence, firms will increase work force capacity in order to reduce the necessity for speculative inventory accumulation. This prediction is in direct conflict with the prediction that emerges from our model. As Proposition 5 indicates, in our model a parametric increase in inventory holding costs leads to a decrease in optimal work force capacity. This is because when inventory holding costs rise, the firm in our model will alter its plans to reduce the likelihood of inventory accumulation. Since inventories act as a buffer stock absorbing unexpectedly low realizations of demand in our model, inventories are more likely to be accumulated, the greater is *ex ante* production capacity. The firm can reduce the likelihood of inventory accumulation by reducing the attached labor force and thereby reducing *ex ante* productive capacity. Hence, the reason for the difference in results is that in our model inventories are accumulated to buffer unanticipated changes in demand whereas in Topel inventories are accumulated to take advantage of anticipated increases in demand.

Next, let us consider the implications of the decompositions given in Proposition 5. As the proposition indicates, the substitution effects may reinforce or partially offset the conventional effects. In the case of a change in anticipated demand, the substitution effects are positive. They thus *strengthen* the conventional response of price and employment to changes in anticipated demand. The basic reason is that a firm that has the flexibility to layoff workers as well as to build-up inventories when demand falls needs to be less cautious in terms of avoiding inventory accumulation. It will therefore raise its price, i.e., cut mean demand, and raise employment, i.e., raise *ex ante* production, more so than a firm that lacks the capacity to use layoffs to absorb demand shocks.

The substitution effects for a change in marginal inventory holding costs are also positive. But in this case, the substitution effects partially offset the conventional effects and thus weaken the response of price and employment to changes in inventory holding costs. In the standard inventory model, an increase in h_1 induces the firm to reduce both price and employment in order to hold down inventory stocks and thus holding costs. But, if the firm can layoff workers as an alternative to accumulating inventories, it can be less cautious, and it thus needs less of a reduction in p and L to avoid the higher inventory holding costs.

C. Empirical Implications

These results have particular interest for the interpretation of empirical results. Historically, empirical studies of employment and production decisions, and, to a lesser extent, studies of price decisions, have displayed a strong response of these decision variables to changes in anticipated demand. The responses of output, employment and prices, however, to changes in inventory holding costs, especially, real interest rates, have been extremely weak. Further, an empirical puzzle that has stirred considerable interest in recent years is the finding that inventory investment tends to be strongly related to anticipated demand or sales but very weakly related to real interest rates.

These empirical findings can be readily interpreted in the present model. When firms have the ability to layoff workers as well as to accumulate inventories in response to a slump, the substitution effects, generally, tend to reinforce the conventional effects of a change in anticipated demand. This gives rise to a strong response of employment and output, and to a lesser extent price, to changes in expected demand measures. At the same time, the substitution effects of a change in marginal inventory holding costs tends to offset to some degree the conventional effects; thereby giving rise to a weak response of price, output, and employment to changes in real interest rates. Moreover, viewed ex ante, when the firm sets p and L, it is implicitly planning to build up (draw down) inventories to serve as a buffer stock of potential output exceeds (falls below) riskless demand. In this sense, the firm is undertaking planned inventory investment (I^P) defined by:

$$I^P = g(L) - m(p)$$

Clearly, the above results for p and L imply that one would expect a strong response of inventory investment to changes in expected demand, but a weak response to changes in real interest rates.

V. Extensions of the Basic Model

A. Risk Aversion

The basic model assumed that workers are risk neutral. However, since much of the early contract literature (e.g., Azariadis [1975], Baily [1974]) focused on the influence of risk aversion on the optimal contract, it is of interest to consider the effects of risk aversion on the part of workers in our model. Assuming that workers are risk averse changes the contract constraint (5). Formally, the constraint becomes:

$$\left(\int_{\underline{\varepsilon}}^{\infty} uU(w)f(\varepsilon)d\varepsilon\right) + \left(1-\int_{\underline{\varepsilon}}^{\infty} uf(\varepsilon)d\varepsilon\right)U(K) \geq V^* \qquad (5')$$

where U is a strictly concave utility function and V^* represents the ex ante market determined expected value of the contract in terms of expected utility. The formulation of the contract constraint as in (5) follows that of Azariadis [1975] by assuming that the firm is prohibited from paying laid off workers directly. The firm now maximizes (9) subject to (4), (7) and the new contract constant, (5'). It is easily demonstrated that in this instance the optimal wage strategy involves a fixed wage independent of the state of the world. Hence, with risk aversion, the state independent (real) wage is derived rather than assumed. Marginal inventory holding costs are unaffected by risk aversion, but marginal net turnover costs become:

$$T'((1-u)L) - [w + \frac{U(K)-U(w)}{U'(w)}]$$

By the strict concavity of the utility function, $K > [w + \frac{U(K)-U(w)}{U'(w)}]$; this implies that marginal net turnover costs are higher for every level of layoffs the more risk averse are workers. This makes intuitive sense because, the more risk averse are workers, the greater the required compensating differential in wages that the firm will have to provide for fluctuations in employment. These considerations mean that, in general, the greater the degree of risk aversion of workers, the greater will be the tendency for firms to use inventories rather than layoffs to absorb declines in demand.[17]

B. Variable Hours

The basic model assumed that hours per worker are fixed. It is of interest to relax this assumption since variation in hours worked may be an important instrument for firms in responding to cyclical variations in demand. In this section, we briefly consider the ramifications of relaxing this assumption.

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[17]While this result is intuitively appealing, it does depend on the assumption that firms cannot pay their laid off workers directly. If this latter assumption is relaxed, then in the face of worker risk aversion, it is easily demonstrated that the firm will find it optimal to completely insure workers against variations in their income. Under this perfect risk shifting wage strategy, marginal inventory holding costs and marginal net turnover costs will be unaffected by the degree of worker risk aversion and thus all the results derived from the basic model will hold.

Accordingly, we assume that the production function takes the following form:

$$g(ruL,\ r) \quad g_1 > 0,\ \ g_2 > 0$$

where r is the number of hours worked per worker. This specification presumes that there are increasing returns to hours worked per worker, a presumption that is consistent with empirical evidence — see, e.g., Feldstein [1967] on this point — and seems to be an important factor for understanding the relationship between hours and employment variation.[18]

Allowing hours per worker to vary changes the nature of the implicit contract since workers will be concerned not only with the wage and their employment status but also with the number of hours to be worked. Formally, the contract constraint (5) is changed to:

$$\int_{\underline{\varepsilon}}^{\infty} u\{wr - D(r)\} f(\varepsilon)d\varepsilon + \left(\int_{\underline{\varepsilon}}^{\infty}(1-u)f(\varepsilon)d\varepsilon\right)K > V \qquad (5'')$$

where $D(r)$ is the income equivalent of the disutility of hours worked and is such that $D' > 0$ and $D'' > 0$.[19]

Allowing for hours variation means that firms may be hours-biased as well as layoff-biased or inventory-biased depending upon the production conditions, the cost structure of the firm and the utility function of the worker.

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[18]If alternatively $g_2 = 0$ so that production depends solely on labor services, the model implies that a firm in a slump will always reduce r to some lower bound before laying off workers or accumulating inventories. This is because in this case there are costs to making layoffs or accumulating inventories, but not to reducing hours worked. See Baily [1977] for further discussion of this point in the context of a model of temporary layoffs. Allowing for increasing returns to variation in hours creates an incentive for the firms to consider hours variation in consort with layoffs and inventory accumulation.

[19]This specification of the contract constraint presumes that the worker's utility function is additively separable in income and leisure and that the worker is risk neutral with respect to variations in his income.

Moreover, using procedures analogous to those used above, it can be shown that the firm will be less likely to use hours reductions as opposed to layoffs or inventory accumulation in response to an unanticipated decline in demand, the smaller are inventory holding costs and net turnover costs, the smaller the increase in utility associated with the reduction in hours, and the greater is the degree of increasing returns in hours per worker. Overall, incorporating the possibility of variable hours into the basic model yields results that are consistent with the results derived above but are richer in the details of the relative substitutability of layoffs, inventories, and hours variations.

VI. Conclusions and Extensions

In this paper we have developed a model of firm behavior designed to study the interaction of inventories and temporary layoffs. Buffer stock motives provide an incentive for the firm to hold inventories while implicit contracts provide a rationale for the firm to make temporary layoffs. The purpose of the model is to study the consequences for firm behavior permitting the firm the opportunity to use both inventories and temporary layoffs to absorb fluctuations in demand.

The model yielded three main findings. First, if inventory holding costs and turnover rise at the margin the firm will have an incentive to respond eventually to a slump by both laying off workers and accumulating inventories. Exactly when this will happen depends on the relative marginal cost of using each of the response mechanisms. Secondly, diminishing returns to labor induce an inventory-biased firm to accumulate inventories and fully utilize its workforce in the initial stages of a slump in an effort to smooth production. But, if the slump becomes deep enough, the firm will sharply cut back production, undertake a massive layoff, and dump its inventories — in effect switching from an inventory to a layoff policy. Finally, we find that permitting firms to layoff workers as well as to accumulate inventories in the face of a slump tends to strengthen the price and employment responses to changes in anticipated demand, but to weaken the responses of these variables to changes in inventory holding costs — results which appear to have empirical content.

The most important direction in which the model needs to be extended is to permit the firm to make decisions over a multi-period horizon. This would enrich the analysis in several respects. For one, temporary layoffs are to some degree an intertemporal phenomenon so that a complete analysis requires a multi-period specification so that the potential sequence of a worker initially being employed, laid off, and recalled can be analyzed. For another, although single-period inventory models of the buffer stock variety do incorporate the current benefits to holding inventories, they have the limitation that future benefits can only be treated in an ad hoc manner. Moreover, only in an intertemporal framework will it be possible to understand differences between buffer stock and speculative motives for holding inventories.

Finally, and most importantly, the present model has abstracted from permanent layoffs. Permanent layoffs are layoffs in which neither the worker nor the firm expects recalls to take place. Extending the model to allow for intertemporal decision-making requires that the firm be concerned with the evolution of the attached workforce through permanent layoffs and new hires. An intertemporal model would thus permit an analysis of the interaction of permanent layoffs, temporary layoffs and inventory adjustments in response to changes in demand.

REFERENCES

Amihud, Yakov and H. Mendelson, 1983, "Price Smoothing and Inventory," Review of Economic Studies, January.

Azariadis, C., 1975, "Implicit Contracts and Underemployment Equilibria," Journal of Political Economy, December, 1183-1202.

Baily, M.N., 1977, "On the Theory of Layoffs and Unemployment," Econometrica, July, 1043-1063.

Belsley, D.A., 1969, Industry Production Behavior: The Order-Stock Distinction, North-Holland, Amsterdam.

Blinder, A., 1982, "Inventories and Sticky Prices: More on the Microfoundations of Macroeconomics," American Economic Review, June, 334-349.

________, 1983, "Can the Production-Smoothing Model of Inventory Behavior Be Saved?" unpublished manuscript, December.

Brechling, F., 1981, "Layoffs and Unemployment Insurance," in S. Rosen (ed.) Studies in Labor Markets, Chicago: U. of Chicago Press.

Burdett, K. and D. Mortensen, 1980, "Search, Layoffs, and Labor Market Equilibrium," Journal of Political Economy, 88, August, 652-672.

Childs, G., 1967, Inventories and Unfilled Orders, North-Holland, Amsterdam.

Crawford, R., 1979, "Expectations and Labor Market Adjustments," Journal of Econometrics, 207-32.

Feldstein, M., 1967, "The Specification of Labor Input in the Aggregate Production Function," Review of Economic Studies, October.

Hay, G.A., 1970, "Production, Price and Inventory Theory," American Economic Review, September.

Holmstrom, B., 1983, "Equilibrium Long Term Contracts," Quarterly Journal of Economics, supplement.

Holt, C., Modigliani, F., Muth, J., and Simon, H., 1960, Planning, Production, Inventories, and Work Force (Englewood Cliffs, Prentice-Hall).

Lilien, D.M., 1979, "The Cyclical Pattern of Temporary Layoffs in United States Manufacturing," Review of Economics and Statistics,

Lovell, M., 1961, "Manufacturer's Inventories, Sales Expectations, and the Acceleration Principle," Econometrica, 293-314.

Maccini, L.J., 1982, "The Interrelationship Between Price and Output Decisions and Investment Decisions: Microfoundations and Aggregate Implications," forthcoming in the Journal of Monetary Economics.

Mills, E.S., 1962, Price, Output, and Inventory Policy (New York: John Wiley and Sons).

Mortensen, D., 1978, "On the Theory of Layoffs," Discussion Paper No. 322, The Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Nadiri, M.I. and S. Rosen, 1973, A Disequilibrium Model of the Demand for Factors of Production (New York: Columbia University Press).

Reagan, P., 1982, "Inventory and Price Behavior," Review of Economic Studies, January.

Topel, R., 1982, "Inventories, Layoffs, and the Short-Run Demand for Labor," American Economic Review, September.

Zabel, E., 1970, "Monopoly Under Uncertainty," Review of Economic Studies, April.

________, 1972, "Multiperiod Monopoly Under Uncertainty," Journal of Economic Theory, December.

APPENDIX A

Proof of Proposition 1: Suppose $\varepsilon > b$ but $Z > 0$ or $S < x + g(uL)$. This implies with (14) and (15) that $\lambda_0 = \lambda_1 = 0$. Yet, by (12), this yields a contradiction. Hence $S = x + g(uL)$ and $Z = 0$.

Now suppose $u < 1$ when $\varepsilon > b$. This implies by (10) that $\lambda_2 = 0$. Also, it implies by (14) that $\lambda_0 = 0$ and hence by (12) $\lambda_1 - H'(Z) = p$. Hence, condition (13) becomes:

(A1) $$pg'(uL) = K - T'((1-u)L)$$

Since $T' > 0$ and $T'' > 0$, $K - T'(0) > K - T'((1-u)L)$ which implies by (A1):

(A2) $$pg'(uL) < K - T'(0).$$

Given (A2), the concavity of g implies that:

(A3) $$pg'(L) < K - T'(0)$$

Given that $\lambda_2 > 0$ only if $u = 1$, condition (A3), (12) and (13) together imply that $\lambda_2 = 0$ $(\forall\varepsilon)$. However, it is easily demonstrated using the optimality conditions for p and L (see equations (23) and (24)) that λ_2 must satisfy

(A4) $$\int_{\underline{\varepsilon}}^{\infty} \lambda_2 \, f(\varepsilon)d\varepsilon = V - K + \int_{\underline{\varepsilon}}^{\infty} T'(1-u)L) \, f(\varepsilon)d\varepsilon > 0$$

By (A4), $\lambda_2 > 0$ for at least one ε. Hence, we have a contradiction and $u = 1$.

Now, suppose $\varepsilon < b$, which implies $m(p) + \varepsilon < x + g(L)$. Suppose further that $u = 1$, then since $S < m(p) + \varepsilon$, $x + g(L) > S$ which implies $Z > 0$. Alternatively, suppose $Z = 0$, then $x + g(uL) = S$; this requires

$u < 1$ since $S = x + g(uL) < m(p) + \varepsilon < x + g(L)$ and thus $g(uL) < g(L)$. Finally, it is possible that $u < 1$ and $Z > 0$ since $S < m(p) + \varepsilon < x + g(uL) < x + g(L)$ is consistent with the presumption that $\varepsilon < b$ and the optimality conditions. Q.E.D.

<u>Proof of Proposition 2</u>: First, suppose (17) holds. Since $\varepsilon < b$, by Proposition 1, either $Z > 0$ or $u < 1$ or both. Suppose $Z = 0$. This implies $u < 1$. If $u < 1$, then by (16), $\lambda_2 = 0$. Hence, in this event, (13) becomes:

(A5) $$(\lambda_1 - H'(0))\, g'(uL) = K - T'((1-u)L)$$

Since $\varepsilon < b$ and $Z = 0$, this implies $S < m(p) + \varepsilon$ and hence by (14), $\lambda_o = 0$. By (12), this implies $p = \lambda_1 - H'(Z) = \lambda_1 - H'(0) > 0$. Since $T'' \geq 0$ and $g'' < 0$, this implies with (17) that

$$(\lambda_1 - H'(0))\, g'(L) < K - T'(0)$$

Given that $\lambda_1 > 0$ in this case, this latter equation is inconsistent with (17). Hence, $Z > 0$.

Next, we prove that there exists a neighborhood of b with $\varepsilon < b$ where not only is $Z > 0$ but also $u = 1$. Consider the ramifications of (17). Given (17), there must exist an ε^* where $\varepsilon^* < b$ and is such that:

(A6) $$H'(x + g(L) - m(p) - \varepsilon^*)\, g'(L) < T'(0) - K$$

and

(A7) $$H'(0)\, g'(\bar{u}^* L) < T'((1-\bar{u}^*)L) - K$$

where $\bar{u}^*$ is defined by $x + g(\bar{u}^* L) - m(p) - \varepsilon^* = 0$. Given the definition of $\bar{u}^*$, the optimal u associated with ε^* must be such that $\bar{u}^* < u < 1$. This implies with (13), (A6) and (A7) that $\lambda_2 > 0$ if $\varepsilon = \varepsilon^*$. Hence, $u = 1$ if $\varepsilon = \varepsilon^*$. Following similar arguments for all ε such that $b > \varepsilon > \varepsilon^*$,

$u = 1$ and consequently $Z > 0$. This completes the proof for the inventory-biased case. The proof of the layoff biased cased follows similar arguments.

Q.E.D.

<u>Proof of Proposition 3</u>: Suppose $b > \varepsilon > \varepsilon^h$ and $u > 1$. Since $\varepsilon < b$, $S = m(p) + \varepsilon$. Since $u < 1$, $\lambda_2 = 0$ by (16). Hence, (13) and $T'' > 0$ imply that:

(A8) $$(\lambda_1 - H'(Z))\beta = K - T'(1-u)L) < K - T'(0).$$

Define $Z^h = x + \beta L - m(p) - \varepsilon^h$, then since $\varepsilon > \varepsilon^h$ and $u < 1$, $Z^h - Z = \beta(1-u)L - (\varepsilon^h - \varepsilon) > 0$. Hence, using the definition of ε^h and $H'' > 0$ we have:

(A9) $$K - T'(0) = -\beta H'(Z^h) < -\beta H'(Z) < \beta(\lambda_1 - H'(Z))$$

The last inequality follows from $\lambda_1 > 0$. But, (A8) and (A9) involve a contradiction and thus $u = 1$.

Since $u = 1$, $Z = x + \beta L - m(p) - \varepsilon > 0$ by Proposition 1. Further, since $Z = x + \beta L - m(p) - \varepsilon$, $\frac{\partial Z}{\partial \varepsilon} = -1$.

Now suppose $\varepsilon < \varepsilon^h$ and $u = 1$. This implies $Z - Z^h = \varepsilon^h - \varepsilon > 0$, which since $Z^h > 0$ implies $Z > 0$. But using the definition of ε^h and given that $H'' > 0$, this implies:

(A10) $$-\beta H'(Z) < -\beta H'(Z^h) = K - T'(0).$$

But, when $Z > 0$, which implies $\lambda_1 = 0$, and $u = 1$, (13) becomes:

$$-\beta H'(Z) = K - T'(0) + \lambda_2$$

Since $\lambda_2 > 0$, this requires that

(A11) $$K - T'(0) < -\beta H'(Z)$$

(A10) and (A11) involve a contradiction. Hence, $u < 1$.

Further, when $\varepsilon < \varepsilon^h$ and $u < 1$, $Z = x + \beta uL - m(p) - \varepsilon > 0$. To see this, suppose $Z = 0$ when $\varepsilon < \varepsilon^h$ and $u < 1$. Then (13) becomes:

$$(\lambda_1 - H'(0))\ \beta = K - T'((1-u)L)$$

Since $\lambda_1 > 0$ and $T'' > 0$, this implies:

$$-\beta H'(0) < (\lambda_1 - H'(0))\beta = K - T'((1-u)L) < K - T'(0).$$

But, this violates the assumption that $\beta H'(0) < T'(0) - K$. Hence, $Z > 0$.

Finally, when $\varepsilon < \varepsilon^h$, $u < 1$, and $Z = x + \beta uL - m(p) - \varepsilon > 0$, we have that $\frac{\partial Z}{\partial \varepsilon} = \beta L \frac{\partial u}{\partial \varepsilon} - 1$. Further, (13) becomes:

(A14) $$-\beta H'(x + \beta uL - m(p) - \varepsilon) = K - T'((1-u)L)$$

Differentiating (A14) implicitly yields:

$$\frac{\partial u}{\partial \varepsilon} = \frac{\beta H''}{(\beta^2 H'' + T'')L}$$

and thus

$$0 < \beta L \frac{\partial u}{\partial \varepsilon} = \frac{1}{1 + \frac{T''}{\beta^2 H''}} < 1$$

Hence, $-1 < \frac{\partial Z}{\partial \varepsilon} < 0$ which establishes (ii-c).

Q.E.D.

Proof of Proposition 4: Recall that for $\varepsilon < b$, the ex post problem reduces to choosing u so as to minimize ex post total variable costs which in this case are given by:

$$TVC = h_1(x + g(uL) - m(p) - \varepsilon) + (t_1 - K)\ (1-u)L$$

The two constraints on the choice of u are $u \leq 1$ and $x + g(uL) - m(p) - \varepsilon \geq 0$ or, equivalently, by the definition of $\bar{u}$, $u \geq \bar{u} = \bar{u}(\varepsilon)$.[20] Observe that since $h_1 g''(uL)L^2 < 0$, in this case TVC are strictly concave in u. This implies that an interior solution to this cost minimization problem defines a cost maximum rather than a cost minimum. Hence, the cost minimum must be at one of the two corners, either $u = 1$ or $u = \bar{u} = \bar{u}(\varepsilon)$. Observe that if $u = 1$, then $TVC = h_1[x + g(L) - m(p) - \varepsilon]$ whereas if $u = \bar{u} = \bar{u}(\varepsilon)$ then $TVC = (t_1-K)(1-\bar{u})L$. Differentiating TVC with respect to u yields:

$$\frac{\partial(TVC)}{\partial u} = L[h_1 g'(uL) + (K-t_1)]$$

Define u^* to be the value of u that maximizes TVC. That is, it satisfies $h_1 g'(u^*L) + K - t_1 = 0$. Observe that u^* is independent of ε. Then clearly this implies that $\frac{\partial(TVC)}{\partial u} < 0$ for $u > u^*$, and $\frac{\partial(TVC)}{\partial u} > 0$ for $u < u^*$. Further, given u^*, define ε^* as the value of ε such that $x + g(u^*L) - \varepsilon^* = 0$ so that for $\varepsilon = \varepsilon^*$, $u^* = \bar{u}(\varepsilon^*)$.

(i) If $\varepsilon > \varepsilon^* > \varepsilon^c$, then by previous argument $\bar{u}(\varepsilon) > u^*$ and hence TVC is minimized at $u = 1$.

Now, suppose $\varepsilon^c < \varepsilon < \varepsilon^*$. Note that at $\varepsilon = \varepsilon^c$, $\bar{u} = u^c$ and

$$TVC(u=\bar{u}) = (t_1-K)(1-u^c)L = TVC(u=1) = h_1[x + g(L) - m(p) - \varepsilon^c].$$

Now consider a slight increase in ε from $\varepsilon = \varepsilon^c$. We have:

$$\frac{\partial[TVC(u = \bar{u})]}{\partial \varepsilon} = -\frac{(t_1-K)}{g'(\bar{u}L)} L$$

and

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[20] We suppose the fact $\bar{u}(\varepsilon)$ depends also on p, L, and x.

$$\frac{\partial[TVC(u=1)]}{\partial \varepsilon} = -h_1 L$$

Since $\varepsilon^c < \varepsilon^*$, this implies that:

$$0 > \frac{\partial[TVC(u = \bar{u}, \bar{u}=u^c)]}{\partial \varepsilon} > \frac{\partial[TVC(u=1, \varepsilon=\varepsilon^c)]}{\partial \varepsilon}$$

Hence evaluated at $\varepsilon = \varepsilon^c$, an increase in ε leads to a larger (in magnitude) decrease in $TVC(u = 1)$ than in $TVC(u = \bar{u})$. Hence, for ε marginally higher than ε^c, $TVC(u = 1) < TVC(u = \bar{u})$. Given the earlier result, this implies for $\varepsilon^c < \varepsilon < b$, $u = 1$. By Proposition 1, this implies that $Z > 0$. Further, since $Z = x + g(L) - m(p) - \varepsilon$, this implies $\frac{\partial Z}{\partial \varepsilon} = -1$.

(ii) Following the same arguments as before, since:

$$0 > \frac{\partial[TVC(u=\bar{u}, \bar{u}=u^c)]}{\partial \varepsilon} > \frac{\partial[TVC(u = 1, \varepsilon=\varepsilon^c)]}{\partial \varepsilon}$$

this implies that, evaluated at $\varepsilon = \varepsilon^c$, a decrease in ε leads to a larger (in magnitude) <u>increase</u> in $TVC(u = 1)$ than in $TVC(u = \bar{u})$. Hence, for $\varepsilon < \varepsilon^c$, $TVC(u = 1) > TVC(u = \bar{u})$. This implies that for $\varepsilon < \varepsilon^c$, $u = \bar{u}$. By the definition of $\bar{u}$, this implies that $Z = 0$. Q.E.D.

<u>Proof of Proposition 5</u>: First, consider the total effect. Taking the total differential of (23) and (24) and applying Cramer's rule yields:

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial L}{\partial x} = -\frac{1}{\beta} < 0$$

$$\frac{\partial p}{\partial \bar{m}} = \frac{\beta(pf(b) - C_{L,\bar{m}}/\beta)}{\Delta} > 0$$

$$\frac{\partial L}{\partial \bar{m}} = \frac{-\eta(pf(b) - \frac{1}{\hat{m}'}\frac{C_{L,p}}{\beta}) + F(b)(1-F(b))}{\Delta} > 0$$

$$\frac{\partial p}{\partial h_1} = \frac{D_L\ C_{L,h_1}/\beta}{\Delta} < 0$$

$$\frac{\partial L}{\partial h_1} = \frac{(C_{L,h_1}/\beta)\ [\eta - D_p]}{\Delta} < 0$$

where $\Delta = -\beta[(pf(b) - \frac{1}{\hat{m}'}\frac{C_{L,p}}{\beta})\eta + (1-F(b))^2] > 0$

$$\eta = 2\hat{m}' + \hat{m}''(p - \frac{V}{\beta}) < 0 \qquad \frac{\hat{C}_{L,p}}{\beta} = -\hat{m}'\ [h_1 f(b) + h_2 F(b)] > 0$$

$$\frac{C_{L,p}}{\beta} = \frac{\hat{C}_{L,p}}{\beta} + \hat{m}' h_2 \alpha > 0 \qquad \frac{\hat{C}_{L,L}}{\beta} = \beta[h_1 f(b) + h_2 F(b)] > 0$$

$$\frac{C_{L,L}}{\beta} = \frac{\hat{C}_{L,L}}{\beta} - \beta h_2 \alpha > 0 \qquad \frac{\hat{C}_{L,\bar{m}}}{\beta} = -\ [h_1 f(b) + h_2 F(b)] < 0$$

$$\frac{C_{L,\bar{m}}}{\beta} = \frac{\hat{C}_{L,\bar{m}}}{\beta} + h_2 \alpha < 0 \qquad \frac{\hat{C}_{L,h_1}}{\beta} = F(b) > 0$$

$$\frac{C_{L,h_1}}{\beta} = \frac{\hat{C}_{L,h_1}}{\beta} - \alpha > 0 \qquad \alpha = (\frac{\beta^2 h_2}{\beta^2 h_2 + t_2})\ F(\varepsilon^h) > 0$$

The signs of the total effects follow directly from the signs of the individual terms given above. Note that the terms with a superscript "^" represent the partial derivatives of the marginal cost function C_L, keeping the number of layoffs constant. Given this interpretation, it is easily demonstrated that the conventional effects are given by:

$$CE_{p,x} = 0 \qquad CE_{L,x} = -\frac{1}{\beta} < 0$$

$$CE_{p,\bar{m}} = \frac{\beta(pf(b) - (\frac{\hat{C}_{L,\bar{m}}}{\beta}))}{\hat{\Delta}} > 0$$

$$CE_{L,\bar{m}} = \frac{-\eta(pf(b) - \frac{1}{\hat{m}'}(\frac{\hat{C}_{L,p}}{\beta})) + F(b)(1-F(b))}{\hat{\Delta}} > 0$$

$$CE_{p,h_1} = \frac{D_L(\hat{C}_{L,h_1}/\beta)}{\hat{\Delta}} < 0$$

$$CE_{L,h_1} = \frac{(\hat{C}_{L,n_1}/\beta)\ [\eta - D_p]}{\hat{\Delta}} < 0$$

where $\hat{\Delta} = -\beta\ [(pf(b) - \frac{1}{\hat{m}'}\frac{\hat{C}_{L,p}}{\beta})\eta + (1-F(b))^2] > 0$. The signs of the conventional effects follow directly from the signs of the individual terms. The substitution effects are derived by simply subtracting the conventional effects from the total effects. This yields:

$$SE_{p,x} = 0 \qquad SE_{L,x} = 0$$

$$SE_{p,\bar{m}} = \frac{\beta^2 h_2 \alpha\ (1-F(b))^2}{\Delta\hat{\Delta}} > 0$$

$$SE_{L,\bar{m}} = \frac{-(\eta\alpha\ \beta h_2)\ F(b)(1-F(b)) - h_2\alpha\ \beta(1-F(b))^2(\eta - \hat{m}')}{\Delta\hat{\Delta}} > 0$$

$$SE_{p,h_1} = \frac{D_L[\alpha[\beta(pf(b) + h_1 f(b))\eta + (1-F(b)^2]}{\Delta\hat{\Delta}} > 0$$

$$SE_{L,h_1} = \frac{(\eta - D_p)[\alpha[\beta(p+h_1)f(b)n + (1-F((b)^2]}{\Delta\hat{\Delta}} > 0$$

The signs of the substitution effects follow directly from the signs of the individual terms.

Q.E.D.

APPENDIX B

The Relationship Between The Basic Model And Standard Inventory Models

Introduction

In the text, we noted that the approach to inventory-holding behavior proposed here may appear to be different from the standard approach, as evidenced by the Mills model. We claimed, however, that our approach was in fact equivalent to the standard approach. The purpose of this appendix is to prove this claim. To do this, we show that our approach yields optimality conditions that are equivalent to those of the standard approach.

Proposed Approach to Inventory-Holding Behavior

To see more clearly the approach to inventory holding proposed here, and to permit a precise comparison between our proposed approach and the standard one, it is useful to eliminate the layoff-wage decision from the above model. To do this, suppose in (3) that $u^{\varepsilon} \equiv 1$, and thus the production function may be written as

$$q = g(L) \tag{i}$$

Further, assume that the firm is a price-taker in labor markets so that the wage rate is now exogenous to the firm. Then, inverting the production function, labor costs may be written as

$$WL = Wg^{-1}(q) = C(q) \tag{ii}$$

$$C' > 0 \qquad C'' > 0$$

where W, which is now a parameter, has been absorbed into the cost function. Finally, ignore the implicit contract constraint, (4), and turnover costs, (5), since these are no longer applicable.

For simplicity, suppose that inventory holding costs are linear[21] so that (8) may be written as

$$H = h_1 Z \tag{iii}$$

where $Z = x + q - S$ and $h_1 > 0$.

The firm is assumed to choose p, q and S to maximize expected profits; that is,

$$\max_{p,q,S} E[\Pi] = \int_{\underline{\varepsilon}}^{\infty} [pS - c(q) - h_1 Z] f(\varepsilon)d\varepsilon \tag{iv}$$

subject to

$$S \leq m(p) + \varepsilon \tag{va}$$

$$S \leq x + q \tag{vb}$$

where $Z = x + q - S$.

Define λ_o and λ_1 as Kuhn-Tucker multipliers. Then the Lagrangean is:

$$A = \int_{\underline{\varepsilon}}^{\infty} [pS - c(q) - h_1(x + q - S)] f(\varepsilon)d\varepsilon$$

$$+ \int_{\underline{\varepsilon}}^{\infty} \lambda_o[m(p) + \varepsilon - S] f(\varepsilon)d\varepsilon + \int_{\underline{\varepsilon}}^{\infty} \lambda_1 [x + q - S] f(\varepsilon)d\varepsilon$$

The optimality conditions are[22]:

$$\frac{\partial A}{\partial S} = p + h_o - \lambda_o - \lambda_1 = 0 \quad \forall \varepsilon \tag{vi}$$

$$\frac{\partial A}{\partial P} = \int_{\underline{\varepsilon}}^{\infty} [S + \lambda_o m'(p)] f(\varepsilon)d\varepsilon = 0 \tag{vii}$$

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[21]This assumption is by no means necessary; it merely simplifies the comparison of optimality conditions from the two approaches.

[22]The assumptions on riskless demand and the cost function insure that the Lagrangean is concave in the choice variables which guarantees that optimality conditions are both necessary and sufficient for a maximum.

$$\frac{\partial A}{\partial q} = \int_{\underline{\varepsilon}}^{\infty} [-c'(q) - h_1 + \lambda_1] \; f(\varepsilon)d\varepsilon = 0 \tag{viii}$$

$$\lambda_o \frac{\partial A}{\partial \lambda_o} = \lambda_o [m(p) + \varepsilon - S] = 0, \quad \lambda_o \geq 0 \quad \forall \; \varepsilon \tag{ix}$$

$$\lambda_1 \frac{\partial A}{\partial \lambda_1} = \lambda_1 \; [x + q - S] = 0 \quad \lambda_1 \geq 0 \quad \forall \; \varepsilon \tag{x}$$

For later purposes, it is useful to observe that there are three regimes:

Regime I: Demand is greater than starting stock, i.e., $x + q < m(p) + \varepsilon$ or $\varepsilon > x + q - m(p)$. This implies that

$$S = x + q \tag{xi.a}$$

$$S < m(p) + \varepsilon \tag{xi.b}$$

Hence, by (ix), (x) and (vi),

$$\lambda_o = 0 \tag{xii.a}$$

$$\lambda_1 > 0 \tag{xii.b}$$

$$p + h_1 = \lambda_1 \tag{xii.c}$$

Regime II: Demand equals starting stock, i.e., $x + q = m(p) + \varepsilon$ or $\varepsilon = x + q - m(p)$. This implies

$$x + q = S = m(p) + \varepsilon \tag{xiii}$$

Hence, by (ix), (x) and (vi)

$$\lambda_o > 0 \tag{xiv.a}$$

$$\lambda_1 > 0 \tag{xiv.b}$$

$$p + h_1 = \lambda_o + \lambda_1 \tag{xiv.c}$$

Regime III: Demand is less than starting stock, i.e., $x + q < m(p) + \varepsilon$ or $\varepsilon > x + q - m(p)$. This implies

$$S = m(p) + \varepsilon \tag{xv.a}$$

$$S < x + q \tag{xv.b}$$

Hence, by (ix), (x), and (vi),

$$\lambda_o > 0 \tag{xvi.a}$$

$$\lambda_1 = 0 \tag{xvi.b}$$

$$p + h_1 = \lambda_o \tag{xvi.c}$$

<u>The Standard Model</u>

The standard model that we will use is essentially that of Mills [1962] in which demand is additive in the random variable. The model consists of three components:

(1) Revenue. The demand function and the properties of the distribution of the random variable are still given by (1). Define

Surplus: $n \leq x + q$ or $\varepsilon \leq x + q - m(p)$

Shortage: $n > x + q$ or $\varepsilon > x + q - m(p)$

Then, when surpluses arise, revenue R,

$$R = pn \quad \text{for} \quad n \leq x + q \quad \text{or} \quad \varepsilon \leq x + q - m(p) \tag{xvii.a}$$

and, when shortages develop, it is

$$R = p(x+q) \quad \text{for} \quad n > x + q \quad \text{or} \quad \varepsilon > x + q - m(p) \tag{xvii.b}$$

Using (1) and (xvii), expected revenue is then

$$E[R] = \int_{\underline{\varepsilon}}^{b(p,x+q)} p(m(p) + \varepsilon)\, f(\varepsilon)d\varepsilon + \int_{b(p,x+q)}^{\infty} p(x+q)f(\varepsilon)d\varepsilon \tag{xviii}$$

where $b = b(p,x+q) = x + q - m(p)$.

Next, add and subtract $\int_{b(p,x+q)}^{\infty} p(m(p)+\varepsilon)f(\varepsilon)d\varepsilon$ from (xviii), and combine terms to get

$$E[R] = \int_{\underline{\varepsilon}}^{\infty} p(m(p)+\varepsilon)f(\varepsilon)d\varepsilon - \int_{b(p,x+q)}^{\infty} p(m(p) + \varepsilon - x - q)f(\varepsilon)d\varepsilon$$

which using the properties of the distribution of ε becomes

$$= pm(p) - pD(p,x+q) \qquad \text{(xix)}$$

where

$$D(p,x+q) = \int_{b(p,x+q)}^{\infty} [\varepsilon - (x+q-m(p)]\ f(\varepsilon)d\varepsilon$$

is "average stockouts" or "average shortages". The latter represents the sales that are lost to the firm as a result of having too few inventories.

(2) Production Costs. Production costs are the same as in our approach and are thus given by (vi). Since q is not random, expected production costs and production costs are one and the same.

(3) Inventory Holding Costs. As above, inventory holding costs are assumed to be linear; they are identical in form to (iii) except that we now add the condition that $h_1 = 0$ when $Z < 0$.[23] Computing expected inventory holding costs, we have that

$$E[H] = \int_{\underline{\varepsilon}}^{bp,x+q)} h_o(x + q - m(p) - \varepsilon)\ f(\varepsilon)d\varepsilon$$

$$= h_1 k(p,x+q) \qquad \text{(xx)}$$

where

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[23]This condition is required because there is no constraint, like (5-b), in this model that prevents Z from being negative.

$$k(p,x+\varepsilon) = \int_{\underline{\varepsilon}}^{b(p,x+q)} (x + q - m(p) - \varepsilon)\ f(\varepsilon)d\varepsilon.$$

The firm in this approach chooses p and q so as to maximize expected profits, $E[\Pi]$. Using (xix), (ii), and (xx), this means

$$\max_{p,q} E[\Pi] = pm(p) - pD(p,x+q) - c(q) - h_1k(p,x+q)$$

The optimality conditions for this problem are[24]

$$m(p) - D(p,x+q) + p(m'(p) - D_1(p,x+q)) - h_1k_1(p,x+q) = 0 \qquad \text{(xxi)}$$

$$- pD_2(p,x+q) - c'(q) - h_1k_2(p,x+q) = 0 \qquad \text{(xxii)}$$

Further, from the definitions of $D(p,x+q)$ and $k(p,x+q)$ and the properties of the distribution of demand, it can be shown that

$$D_1(p,x+q) = m'(p)(1 - F(x+q - m(p)))$$

$$D_2(p,x+q) = - (1 - F(x+q - m(p)))$$

$$k_1(p,x+q) = - m'(p)F(x+q - m(p))$$

$$k_2(p,x+q) = F(x+q - m(p))$$

Using these results, the optimality conditions (xxi) and (xxii) can be rewritten as

$$m(p) - D(p,x+q) + (p+h_1)m'(p)F(x+q - m(p)) = 0 \qquad \text{(xxiii)}$$

$$p - C'(q) = (p+h_1)F(x+q - m(p)) \qquad \text{(xxiv)}$$

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[24]The assumptions made on the riskless demand function and the holding cost function insure that expected profits are concave in p and q. This guarantees that the first order conditions are both necessary and sufficient for a maximum.

Equivalence of the Two Approaches

To prove that the approach to inventory holding behavior proposed here and the standard approach are equivalent, we show that the optimality conditions for our approach, equations (vi)-(x), are equivalent to those of the standard approach, equations (xxiii) and (xxiv).

Consider first condition (vii) of the optimality conditions of our proposed model. It may be written as

$$\int_{\underline{\varepsilon}}^{\infty} [S + \lambda_o\, m'(p)]\ f(\varepsilon)d\varepsilon = \int_{\underline{\varepsilon}}^{b(p,x+q)} [S + \lambda_o\, m'(p)]\ f(\varepsilon)d\varepsilon + \int_{b(p,x+q)}^{\infty} [S + \lambda_o\, m'(p)]\ f(\varepsilon)d\varepsilon \qquad \text{(xxv)}$$

The first integral on the right-hand side coincides essentially with the conditions of Regime III and the second coincides essentially with the conditions of Regime I. Then, applying conditions (xi)-(xii) and (xv)-(xvi) respectively, (xxv) becomes

$$= \int_{\underline{\varepsilon}}^{b(p,x+q)} [m(p) + \varepsilon + (p+h_1)m'(p)]\ f(\varepsilon)d\varepsilon + \int_{b(p,x+q)}^{\infty} (x+q)\ f(\varepsilon)d\varepsilon \qquad \text{(xxvi)}$$

Note that the conditions of Regime II apply at the upper limits of integration of the first integral and at the lower limit of the second integral, but given that they apply only at the limits of integration, they do not affect the value of the integrals.

Now, adding and subtracting $\int_{b(p,x+q)}^{\infty} (m(p) + \varepsilon)\ f(\varepsilon)d\varepsilon$ to (xxvi) gives

$$= \int_{\underline{\varepsilon}}^{\infty} [m(p) + \varepsilon] \ f(\varepsilon)d\varepsilon + m'(p)(p+h_1) \int_{\underline{\varepsilon}}^{b(p,x+q)} f(\varepsilon)d\varepsilon$$

$$+ \int_{b(p,x+q)}^{\infty} [x + q - m(p) - \varepsilon] \ f(\varepsilon)d\varepsilon$$

which using the properties of the distribution of demand and the definition of $b(p,x+q)$ becomes

$$= m(p) + m'(p) \ (p+h_1) \ F(x + q - m(p)) - D(p,x+q) = 0 \qquad \text{(xxvii)}$$

where $D(p,x+q) = \int_{x+q-m(p)}^{\infty} [\varepsilon - (x+q-m(p))]f(\varepsilon)d\varepsilon$ is "average stockouts". Condition (xxvii), which was derived using the optimality condition (vi) together with (v), (ix) and (x) from the proposed model, is identical to equation (xxiii) which is one of the optimality conditions of the standard model.

Next, consider equation (viii) of the optimality conditions of our proposed model. It can be rewritten as

$$\int_{\underline{\varepsilon}}^{\infty} [-c'(q) - h_1 + \lambda_1] \ f(\varepsilon)d\varepsilon$$

$$= \int_{\underline{\varepsilon}}^{b(p,x+q)} - c'(q) - h_1 + \lambda_1] \ f(\varepsilon)d\varepsilon \qquad \text{(xxviii)}$$

$$+ \int_{b(p,x+q)}^{\infty} [-c'(q) - h_1 + \lambda_1] \ f(\varepsilon)d\varepsilon$$

Again, the first integral coincides essentially with the conditions of Regime III and the second with Regime I. Then, applying (xi)-(xii) and (xv)-(xvi) respectively, (xxviii) becomes

$$= - \int_{\underline{\varepsilon}}^{b(p,x+q)} [c'(q) + h_1] \ f(\varepsilon)d\varepsilon + \int_{b(p,x+q)}^{\infty} [p - c'(q)] \ f(\varepsilon)d\varepsilon$$

which, combining terms where possible, becomes

$$= - c'(q) \int_{\underline{\varepsilon}}^{\infty} f(\varepsilon)d\varepsilon - h_1 \int_{\underline{\varepsilon}}^{b(p,x+q)} f(\varepsilon)d\varepsilon + p \int_{b(p,x+q)}^{\infty} f(\varepsilon)d\varepsilon$$

which, using the properties of the distribution of demand and the definition of $b(p,x+q)$, yields

$$= - c'(q) - h_1 F(x + q - m(p)) + p[1 - F(x + q - m(p))] = 0 \qquad \text{(xxix)}$$

Condition (xxix), which was derived using the optimality condition (viii) together with (v), (ix) and (x) of the proposed model, is identical to condition (xxiv) of the standard model.

Since the optimality conditions of the two approaches are equivalent, the two approaches are equivalent ways of looking at price-output-inventory behavior.

APPENDIX C

In Proposition 5, comparative static properties of the optimal price and work force of the firm were demonstrated for an inventory-biased firm with linear production conditions and a non-linear cost structure. As we suggested in the text, the main result demonstrated by Proposition 5 holds as well for a firm with a strictly concave production function and a linear cost structure. In the latter case, however, the details differ. The following proposition and the ensuing discussion characterizes the similarities and differences in the results.

Proposition 6: Assume that the production function is strictly concave and that inventory-holding costs and turnover costs are linear, i.e., $t_2 = h_2 = 0$. Then the following results hold for an inventory-biased firm:

$$\frac{\partial p}{\partial x} = CE_{p,x} + SE_{p,x} \gtrless 0 \qquad \frac{\partial L}{\partial x} = CE_{L,x} + SE_{L,X} < 0$$

$$CE_{p,x} < 0 \quad SE_{p,x} \gtrless 0 \qquad CE_{L,x} < 0 \quad SE_{L,x} \lessgtr 0$$

$$\frac{\partial p}{\partial \bar{m}} = CE_{p,\bar{m}} + SE_{p,\bar{m}} \gtrless 0 \qquad \frac{\partial L}{\partial \bar{m}} = CE_{L,\bar{m}} + SE_{L,\bar{m}} > 0$$

$$CE_{p,\bar{m}} > 0 \quad SE_{p,\bar{m}} \gtrless 0 \qquad CE_{L,\bar{m}} > 0 \quad SE_{L,\bar{m}} \gtrless 0$$

$$\frac{\partial p}{\partial h_1} = CE_{p,h_1} + SE_{p,h_1} \gtrless 0 \qquad \frac{\partial L}{\partial h_1} = CE_{L,h_1} + SE_{L,h_1} \gtrless 0$$

$$CE_{p,h_1} < 0 \quad SE_{p,h_1} \gtrless 0 \qquad CE_{L,h_1} < 0 \quad SE_{L,h_1} \gtrless 0$$

Proof of Proposition C: The total effects are computed by totally differentiating (23) and (24) under the assumed conditions of the proposition together with the assumption that profits are strictly concave in p and L and that riskless demand is linear. This yields

$$\frac{\partial p}{\partial x} = \frac{1}{\Delta}\left\{\frac{1}{\hat{m}'}\Delta - (1+F)\Pi_{LL} + \frac{g'}{\hat{m}'}(1-F)\Pi_{pL}\right\} \gtrless 0$$

$$\frac{\partial L}{\partial x} = \frac{1}{\Delta}\left\{-(1-F)\frac{g'}{\hat{m}'}\Pi_{pp} + (1+F)\Pi_{pL}\right\} < 0$$

$$\frac{\partial p}{\partial \bar{m}} = \frac{1}{\Delta}\left\{-\frac{1}{\hat{m}'}\Delta + F\Pi_{LL} - \frac{g'}{\hat{m}'}(1-F)\Pi_{pL}\right\}$$

$$\frac{\partial L}{\partial \bar{m}} = \frac{1}{\Delta}\left\{\frac{g'}{\hat{m}'}(1-F)\Pi_{pp} - F\Pi_{pL}\right\} > 0$$

$$\frac{\partial p}{\partial h_1} = \frac{\Pi_{LL}}{\Delta}\left\{\Pi_{Lh_1}\left[\frac{\Pi_{pL}}{\Pi_{LL}} + \frac{\hat{m}'}{g'}\right] + \hat{m}'(1-\delta)Z^c f(\varepsilon^c)\right\} \gtrless 0$$

$$\frac{\partial L}{\partial h_1} = \frac{-\Pi_{pL}}{\Delta}\left\{\Pi_{Lh_1}\left[\frac{\Pi_{pp}}{\Pi_{LL}} + \frac{\hat{m}'}{g'}\right] + \hat{m}'(1-\delta)Z^c f(\varepsilon^c)\right\} \gtrless 0$$

where $\Delta = \Pi_{pp}\Pi_{LL} - \Pi_{pL}^2 > 0$, $\Pi_{pp} < 0$ and $\Pi_{LL} < 0$ by concavity, and $\Pi_{pL} = g'[(p+h_1)\hat{m}'f + 1 - F] - \hat{m}'[h_1 g'(L) - t_1 + K] < 0$.

The conventional effects are given by

$$CE_{p,x} = -\frac{1}{\hat{\Delta}}\frac{g''}{(g')^2}(V+\hat{C}_L)\hat{\Pi}_{pL} < 0$$

$$CE_{L,x} = \frac{1}{\hat{\Delta}}\left\{-(1-F)\frac{g'}{\hat{m}'}\hat{\Pi}_{pp} + (1+F)\hat{\Pi}_{pL}\right\} < 0$$

$$CE_{p,\bar{m}} = \frac{1}{\hat{\Delta}}\left\{-\hat{\Pi}_{LL} + \frac{g''}{(g')^2}(V+\hat{C}_L)\hat{\Pi}_{pL}\right\} > 0$$

$$CE_{L,\bar{m}} = \frac{1}{\hat{\Delta}}\left\{\frac{g'}{\hat{m}'}(1-F)\hat{\Pi}_{pp} - F\hat{\Pi}_{pL}\right\} > 0$$

$$CE_{p,h_1} = \frac{1}{\hat{\Delta}} \left\{-\hat{m}' \frac{g''}{g'} (V+\hat{C}_L)F - (g')^2F(1-F)\right\} < 0$$

$$CE_{L,h_1} = \frac{1}{\hat{\Delta}} \left\{(1+F)\hat{m}'g'F\right\} < 0$$

where the conventional efforts are computed by totally differentiating (23) and (24) holding the firm's layoff path constant. The substitution effects are then computed by subtracting the conventional effects from the total effects. This yields

$$SE_{p,x} = \frac{1}{\Delta\hat{\Delta}} \Big\{(\hat{\Delta}-\Delta)\Big[-(1+F)\hat{\Pi}_{LL} + \frac{g'}{m'}(1-F)\hat{\Pi}_{pL}\Big]$$

$$+ \hat{\Delta}\Big[(1+F)\hat{\hat{C}}_{LL} - \frac{g'}{\hat{m}'}(1-F)\hat{\hat{C}}_{Lp}\Big]\Big\} > 0$$

$$SE_{L,x} = \frac{1}{\Delta\hat{\Delta}} \Big\{(\hat{\Delta}-\Delta)\Big[-\hat{\Pi}_{pp}\frac{g'}{m'}(1-F) + \hat{\Pi}_{pL}(1+F)\Big]$$

$$+ \hat{\Delta}\Big[\hat{\hat{C}}_{pp}\frac{g'}{\hat{m}'}(1-F) - \hat{\hat{C}}_{pL}(1+F)\Big]\Big\} < 0$$

$$SE_{p,\bar{m}} = \frac{1}{\Delta\hat{\Delta}} \Big\{(\hat{\Delta}-\Delta)\Big[F\hat{\Pi}_{LL} - \frac{g'}{\hat{m}'}(1-F)\hat{\Pi}_{pL}\Big]$$

$$+ \hat{\Delta}\Big[-F\hat{\hat{C}}_{LL} + \frac{g'}{\hat{m}'}(1-F)\hat{\hat{C}}_{Lp}\Big] < 0$$

$$SE_{L,\bar{m}} = \frac{1}{\Delta\hat{\Delta}} \Big\{(\hat{\Delta}-\Delta)\Big[\frac{g'}{\hat{m}'}(1-F)\hat{\Pi}_{pp} - F\hat{\Pi}_{pL}\Big]$$

$$+ \hat{\Delta}\Big[-\hat{\hat{C}}_{pp}\frac{g'}{\hat{m}'}(1-F) + F\hat{\hat{C}}_{pL}\Big] > 0$$

$$SE_{p,h_1} = \frac{1}{\Delta\hat{\Delta}} \Big\{\hat{\Delta}\Pi_{LL}\Big[\Pi_{Lh_1}\Big(\frac{\hat{m}'}{g'} + \frac{\Pi_{pL}}{\Pi_{LL}}\Big) + \hat{m}'(1-\delta)Z^c f(\varepsilon^c)\Big]$$

$$+ \Delta[\hat{m}' \frac{g''}{g'} (V+\hat{C}_L)F + (g')^2 F(1-F)]\} > 0$$

$$SE_{L,h_1} = \frac{1}{\hat{\Delta}\Delta} \{-\hat{\Delta}\Pi_{pL}[\Pi_{Lh_1}(\frac{\Pi_{pp}}{\Pi_{pL}} + \frac{\hat{m}'}{g'}) + \hat{m}'(1-\delta)Z^c f(\varepsilon^c)]$$

$$- \Delta(1+F)\hat{m}'g'F\} > 0$$

where $\hat{\hat{C}}_{pp}$ captures the effect of a price change on C_p when the firm is permitted to switch from inventory accumulation to layoffs. The signs on the substitution effects presume that the conditions (C1) and (C2), stated below are satisfied.

Consider first the total effects of the response of price and employment to changes in exogenous variables. In this case, the total effects display more ambiguity than in Proposition 5. It is possible to show that employment will be inversely related to initial inventories and positively related to anticipated demand. Other results, however, cannot in general be signed unambiguously.

To see why this is so, it is useful to once again decompose the total effects into conventional effects and substitution effects. The conventional effects refer to the response of price and employment to changes in exogenous variables, assuming that the firm is prohibited from altering its layoff path. The substitution effects then capture the additional effects on price and employment of changes in exogenous variables when the firm is permitted to substitute layoffs for inventory accumulation when demand falls.

As one can see from the proposition, the conventional effects are well-defined and are consistent with standard inventory models. In particular, both price and employment will rise with a decline in initial inventories, an increase in anticipated demand, and a decline in inventory holding costs.

Note that the results for a change in initial inventories differ from the previous case of Proposition 5. The results here reflect the effects of production-smoothing.

It is the substitution effects that give rise to the ambiguities. In general, these may re-inforce or offset the conventional effects. With further assumptions, however, results analogous to those of Proposition 5 can be obtained. In particular, if

$$\hat{\hat{C}}_{LL} < 0 \tag{C1}$$

where $\hat{\hat{C}}_{LL}$ is the effect of another attached worker on the marginal cost of employment when the firm is permitted to switch from inventories to layoffs, then $SE_{p,x} > 0$, $SE_{L,x} < 0$, $SE_{p,\bar{m}} < 0$ and $SE_{L,\bar{m}} > 0$. These results imply that the substitution effects re-inforce the conventional effects for employment but offset the conventional effects for price. In other words, if (C1) is satisfied, permitting the firm to have the flexibility of using either layoffs or inventory accumulation to absorb declines in demand strengthens the response of employment to changes in initial inventories and anticipated demand, but weakens the response of prices to changes in these variables.

With a change in inventory holding costs, if

$$-\Pi_{Lh_1}\left[\frac{\Pi_{pp}}{\Pi_{pL}} + \frac{\hat{m}'}{g'}\right] < m'(1-\delta)Z^c\ F(\varepsilon^c) < -\ \Pi_{Lh_1}\left[\frac{\Pi_{pL}}{\Pi_{LL}} + \frac{\hat{m}'}{g'}\right] \tag{C2}$$

where $\delta = [g'(u^cL)(h_1g'(L)-t_1+K)]/[g'(L)(h_1g'(u^cL)-t_1+K)]$, then $S_{p,h_1} > 0$ and $S_{L,h_1} > 0$. In this case, the substitution effects for both price and employment offset the conventional effects and thus the standard responses are weakened.