THE VOLUNTARY PROVISION OF PUBLIC GOODS —
DESCENDING-WEIGHT SOCIAL COMPOSITION FUNCTIONS

by

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Abstract

When public goods are provided by separate individuals subject to private cost-benefit calculations, the "social composition function" that determines the socially relevant aggregate X is often not the simple sum of the individual contributions, as has been traditionally assumed. For example, there are important instances where X is determined by the minimum of the individual contributions (Weakest-link case) or by the maximum (Best-shot case). The "descending-weight" social composition functions, ranging from simple Summation at one extreme to Weakest-link at the other, possess a diminishing-returns property that is the most usual situation. It is shown here that underprovision of the public good tends to be mitigated as the condition for social supply approaches the Weakest-link extreme, especially as community size increases. Underprovision does not entirely disappear, however, even in the Weakest-link case, unless all individuals are identical. Possible non-additivity in the social composition functions determining the social supply of public goods is analogous to problems like "crowding" that make for non-additivity on the demand side. Both demand-side and supply-side non-additivities may be important in applications of public-goods theory -- e.g., to the problem of "clubs" or optimal community size.

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I

A previous paper [Hirshleifer (1983)] introduced the idea of alternative social composition functions for public (collective) goods. Whereas public goods by definition have a peculiar technological feature on the demand side -- namely, that the quantity socially produced is available at no additional cost for concurrent consumption by all members of the community — it has traditionally been assumed that there is nothing at all special when it comes to the technology of the social supply of public goods. Thus, the standand assumption has been, the socially available aggregate of a public good is simply the sum of the amounts privately produced. In contrast, the earlier paper showed, the relevant technology may dictate that the amounts provided by separate individuals are compounded into a socially available aggregate supply in ways that deviate from the traditional additive formula.

Three especially simple social composition formulas were explicitly discussed in the original paper:

SOCIAL COMPOSITION FUNCTIONS

(1) $$X = \sum_i x_i$$ Summation

(2) $$X = \min_i x_i$$ Weakest-link

(3) $$X = \max_i x_i$$ Best-shot

Here the x_i are the separate amounts supplied by the individual $i=1,\ldots,N$ members of the community while X represents the socially available aggregate under the Summation, Weakest-link, and Best-shot composition functions respectively.

The Summation function (1) corresponds of course to the traditional model. To use the famous lighthouse example, imagine a long coastline with an indefinitely large number of private harbors. Each harbor-owner can, by providing a lighthouse, make his own port safe and usable. Then the social aggregate of the public good, the total number of harbors available to each and every mariner as user, is the sum of the number of lighthouses separately provided.

Continuing with nautical examples, for the Weakest-link function (2) imagine there is only a single communal harbor, whose topography is such that a series of sandbars blocks the way for low-draft ships. If each separate individual in the community is responsible for dredging a depth of channel through just one of the successive barriers, then the relevant social aggregate of the collective good — the overall depth of channel available to mariners — is not the sum but the minimum of the separate dredged depths. Here the shallowest barrier is the "weakest-link" which governs the capacity of the system. For an illustration of the Best-shot case (3), imagine once again a long coastline with multiple harbors, but now let each port be barred by its own separate sandbar. Assume also that any single harbor is locationally just as desirable as any other, and furthermore that each has indefinitely large capacity for handling additional traffic at constant costs. Then, from the users' point of view, the only thing that matters is the depth of channel at the single best harbor, i.e., the socially relevant supply of the public good is the maximum of the depths provided by the different private suppliers.

The previous paper showed that the tendency toward private underprovision of the public good is substantially mitigated (in comparison with the standard Summation case (1)) when the relevant social composition function is of the Weakest-link type (2),[1] but aggravated when the Best-shot formula (3) applies. Perhaps more importantly, underprovision relative to the efficient ideal tends to grow sharply with community size N in the standard Summation case (1), to remain substantially unchanged as N increases in the Weakest-link case (2), but to grow even more drastically with N in the Best-shot case (3).

In this paper I adopt a more general approach, in which all the social composition functions tabulated above are regarded as special cases of the more general form:

(4) $$X = w_i x_i$$

Here the x_i are as before the quantities individually provided, while the w_i are the corresponding weights entering into the social aggregate. I shall

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[1]I have come across an anticipation of this conclusion in Mueller (1979), pp. 13-14:

> One can envisage goods...in which the participation of all members of the community is necessary to secure any benefits. The crew of a sail boat, a two-man bobsled, are examples. With such goods... cooperative behavior is voluntarily forthcoming. Such cases undoubtedly make up a small portion of the set of all public goods, however, and almost always involve very small groups of individuals and constrained technological conditions.

My previous paper showed, in contrast, that the Weakest-link "technological condition" that Mueller has in mind here is by no means limited to relatively trivial examples like a two-man bobsled, nor to small numbers of participants. A considerable range of behavior takes place under conditions corresponding closely to the Weakest-link social composition condition — examples being a community struck by disaster or a military unit under heavy attack. An explanation is thereby provided, without calling upon "altruism," for the surprising degree of solidarity and mutual aid often observed under such conditions. I shall also be claiming further that public goods normally fall somewhere between the extremes of the Weakest-link situation and the standard textbook Summation case. However, I am happy to claim Mueller as an ally and predecessor in glimpsing this development.

henceforth be assuming that the individual inputs are ranked from the smallest to the largest. Then we can define three generalized classes of social composition functions that correspond to three special assumptions about weights, to wit:

(5) $1 = w_1 = w_2 = \ldots = w_N$ Constant-weight

(6) $1 = w_1 \geqq w_2 \geqq \ldots \geqq w_N \geqq 0$ Descending-weight

(7) $0 \leqq w_1 \leqq w_2 \leqq \ldots \leqq w_N = 1$ Ascending-weight

where, in (b) and (c), at least one inequality must hold strictly.

The constant-weight case (5) here is of course equivalent to the standard Summation formula (1). The Weakest-link summation function (2) can be seen to be a special extreme case of descending weights (6), to wit, the case where $w_1 = 1$ while $w_2 = \ldots = w_N = 0$. And similarly, the Best-shot function (3) is the extreme case of ascending weights (7), to wit, the case where $w_1 = w_2 = \ldots = w_{N-1} = 0$, while $w_N = 1$.

II

In this paper I provide a generalized analysis of descending-weight social composition functions, i.e., the range of cases limited at one extreme by the Summation formula (1) and represented at the other extreme by the Weakest-link formula (2). Descending-weight social composition functions possesses a kind of "diminishing returns" characteristic: the minimum individual contribution x_1 enters fully into the social aggregate $(w_1 = 1)$, while all the larger-than-minimum individual contributions $x_2,\ldots,x_N$ are discounted to greater or lesser degree before incorporation into the socially available aggregate $(w_2,\ldots, w_N \leqq 1)$.

To illustrate, using the previous nautical example of a series of sandbars, imagine that, if some of the sandbars are dredged to greater depths than others, the "excess" depths may have some value. There may be local deep-draft traffic that does not literally have to traverse each and every successive barrier in the entire series. Or perhaps some of the through deep-draft traffic can, if need be and at additional but possibly non-prohibitive cost, be portaged around the shallowest barriers.

Getting away from nautical examples, consider "linear" technologies for public goods (see Hirshleifer [1983], p. 373) — as when each member of the community is responsible for defending a length of dike against a flood, or each soldier responsible for manning his sector of the front against enemy breakthrough. The Weakest-link assumption strictly holds, say in the dike case, if the social benefit (flood protection) depends entirely upon the minimum individual performance — i.e., the lowest stretch of dike determines the degree of protection. But we can imagine less extreme cases. In defending a military front line, for example, extra strength elsewhere can possibly (at some cost) be diverted to reinforce a weak point, or can be used to mount a distracting counterattack. More generally, we may say that the Weakest-link assumption corresponds to a fixed-proportions technology (with regard to the individual contributions), while the more general descending-weight formula corresponds to a diminishing-returns technology. It seems likely that diminishing-returns technologies — the class of descending-weight social composition functions, lying between the two extremes of the Summation formula (1) and the Weakest-link formula (2) — are the normal case whenever provision of a public good depends upon separate individual productive contributions.

In what follows I will illustrate the conditions for individual optimization and Nash-Cournot equilibrium: first for the standard Summation or constant-weight case, then for the general descending-weight case, and finally for the limiting Weakest-link model.

Figure 1 pictures the traditional Summation (constant-weight) social composition function. The axes x_A, x_B here represent the respective quantities of a public good produced by two individuals A and B comprising a community. Following the standard social composition formula, each has available for consumption the amount $X = x_A + x_B$. In producing the public good, furthermore, each individual trades off his x_i $(i = A,B)$ against amounts y_i that he could alternatively produce and consume of a private good Y. As will be seen shortly, the pattern of each person's indifference curves reflects this tradeoff, and also of course the fact that each party benefits from the other's production of the public good. The diagram shows: (i) A's indifference curves U_A (solid) and B's indifference curves U_B (dashed); (ii) A's Reaction Curve R_A drawn through his Nash-Cournot optimal reactions to B's choices (i.e., through the <u>horizontal</u> points on his U_A curves) and B's Reaction Curve R_B through his Nash-Cournot optimal reaction positions (i.e., through the <u>vertical</u> points on the U_B curves); and (iii) the Nash-Cournot equilibrium point E. (The curve CD, representing the locus of mutual indifference-curve tangencies, will be discussed later on.)

For the generalized <u>descending-weight</u> case, the individuals continue respectively to produce x_A and x_B, but now each consumes $X = w_A x_A + w_B x_B$, where $w_i = 1$ for the smaller while $0 < w_j < 1$ for the larger of x_A and x_B. Figure 2a indicates the shape of A's preference map. Here, in the region where $x_A < x_B$ (northwest of the 45° line), the indifference curves are relatively steep (in the negative sense). With a kink at the 45° line (due to the abrupt shift of weights), the indifference curves turn sharply less negative, or more positive, as they enter the region where $x_A >$

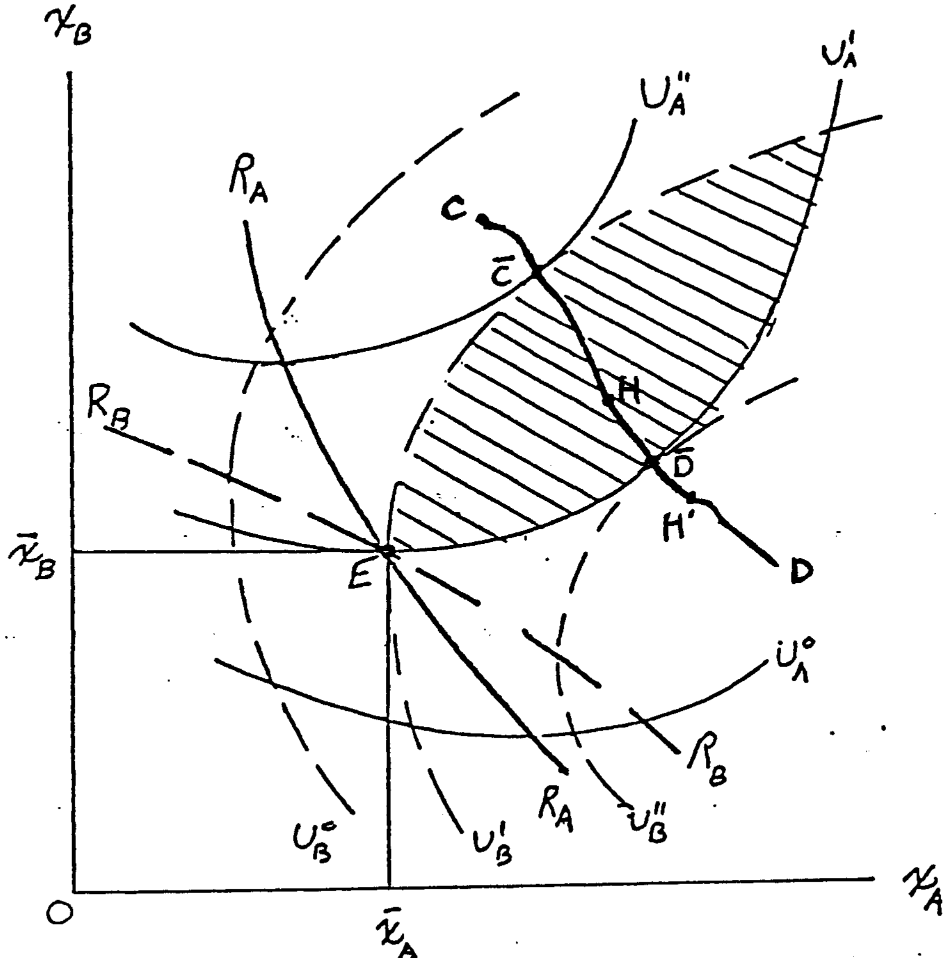

FIGURE 1: Efficient and equilibrium solutions, Summation Composition Function

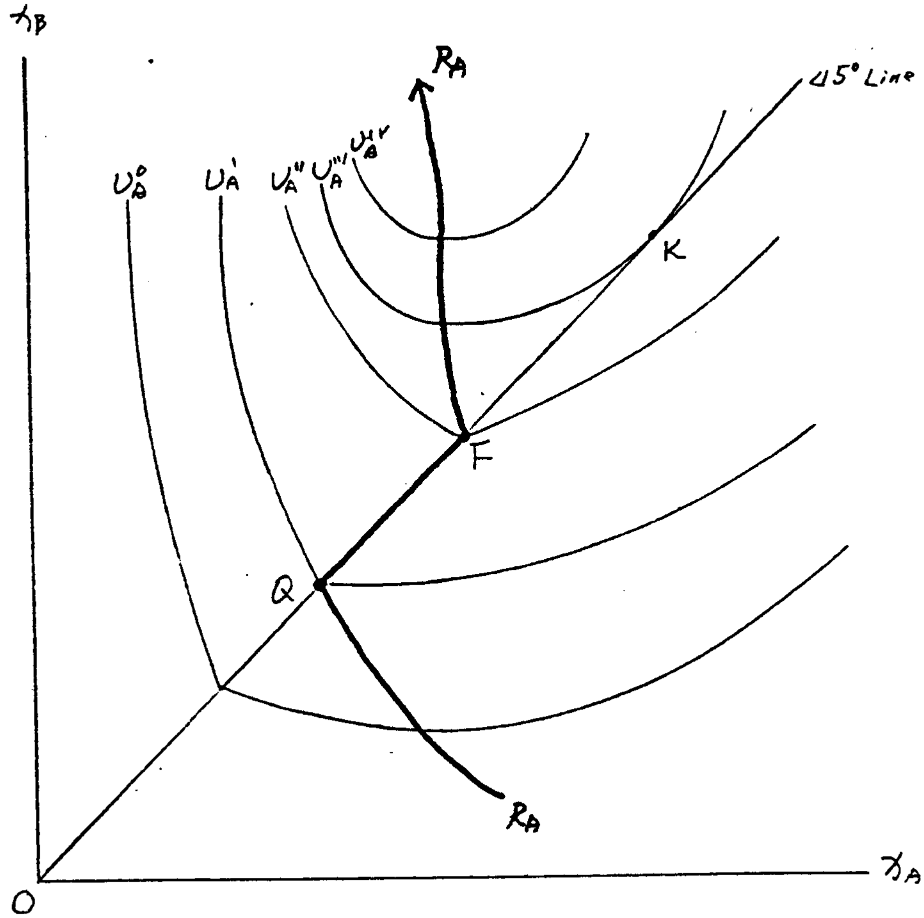

FIGURE 2a: Descending-weight function, individual A

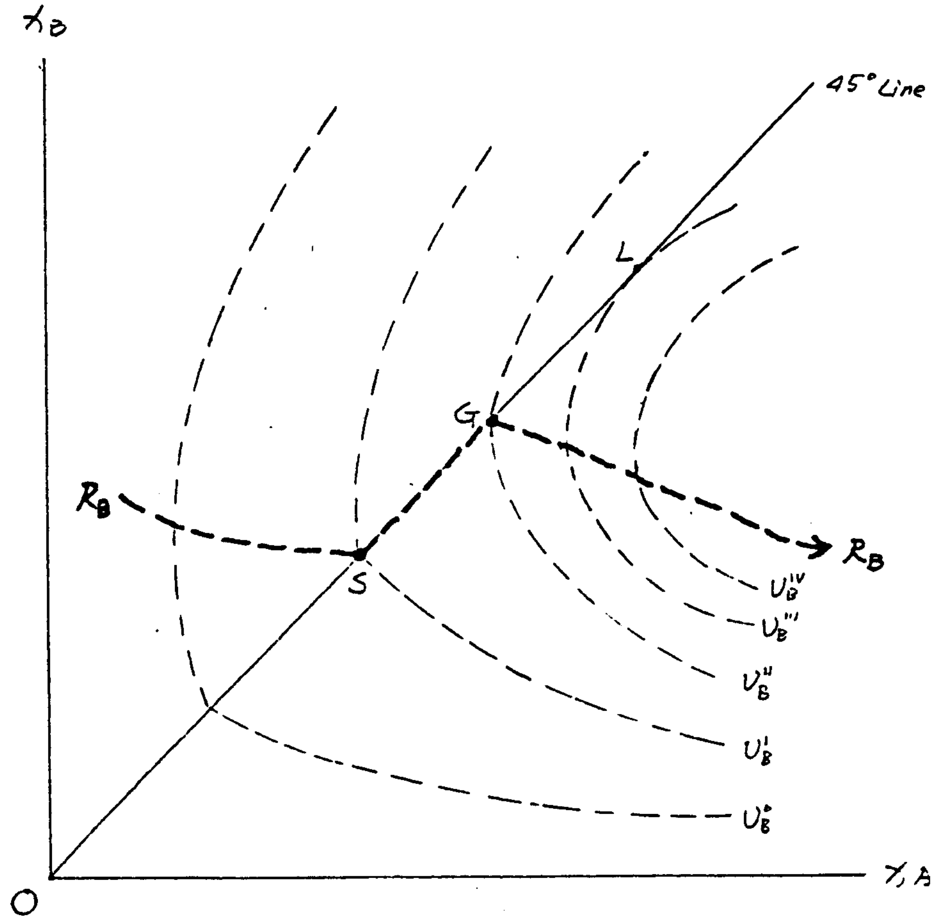

FIGURE 2b: Descending-weight function, individual B

x_B southeast of the 45° line.

More formally, the absolute slope of A's indifference curves in the standard <u>Summation</u> composition function (Figure 1) is given by:

$$(8) \qquad -\left.\frac{dx_B}{dx_A}\right|_{U_A} \equiv \frac{\partial U_A/\partial x_A}{\partial U_B/\partial x_B} \equiv \frac{\frac{\partial U_A}{\partial X}\frac{\partial X}{\partial x_A} + \frac{\partial U_A}{\partial y_A}\frac{dy_A}{dx_A}}{\frac{\partial U_A}{\partial X}\frac{\partial X}{\partial x_B}}$$

Since $(\partial U_A/\partial X)/(\partial U_A/\partial y_A)$ can be identified with MRS_A, A's Marginal Rate of Substitution in Consumption between the private and public good, while $-dy_A/dx_A$ is his private Marginal Cost MC_A of producing the collective good, A's absolute indifference-curve slopes on x_A, x_B axes can be written more intuitively as:[2]

$$(9) \qquad -\left.\frac{dx_B}{dx_A}\right|_{U_A} \equiv \frac{MRS_A(X) - MC_A(x_A)}{MRS_A(X)}$$

For the <u>descending-weight</u> social composition function (Figure 2a), the indifference-curve slopes in the two different regions can be expressed as:

$$(10) \qquad -\left.\frac{dx_B}{dx_A}\right|_{U_A} \equiv \begin{cases} \dfrac{MRS_A(X) - MC_A(x_A)}{w_B \quad MRS_A(X)} & \text{for } x_A < x_B \text{ (where } w_B < 1) \\[2ex] \dfrac{w_A \quad MRS_A(X) - MC_A(x_A)}{MRS_A(X)} & \text{for } x_A > x_B \text{ (where } w_A < 1) \end{cases}$$

As (10) indicates, the kink in the U_A slopes at the 45° line is due to the fact that the weight attached to the individual's own contribution shifts discontinuously in crossing this line. Northwest of the 45° line, where $x_A <$

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[2] See Hirshleifer (1983), fn 2.

x_B, A's marginal production of the public good is fully translated, unit for unit, into the relevant social aggregate X; southeast of the 45° line, where $x_A > x_B$, his contribution is discounted by the multiplicative factor $w_A < 1$.

In Figure 2a, as in Figure 1, A's Nash-Cournot Reaction Curve R_A cuts through all the minima of his U_A indifference curves. But, because of the slope discontinuity along the 45° line, R_A now overlies the 45° line over a finite range (QF in the diagram). And in that range, of course, R_A will necessarily have positive slope instead of being, as before, everywhere negatively sloped.

Figure 2b shows the corresponding picture for B's indifference curves U_B and Reaction Curve R_B (both shown as dashed). Here R_B overlies the 45° line in the finite range SG. Finally, Figure 2c puts the two pictures together, eliminating unessential elements for purposes of clarity. As will be evident there are three possible classes of Nash-Cournot equilibria: the Reaction Curves R_A and R_B might intersect southeast of the 45° line, northwest of it, or along the 45° line itself. This last, the most interesting case, is the one pictured in the diagram.

Intersections along the 45° line will generally have R_A and R_B overlying one another in a finite range -- the distance SF in the diagram. All the points in this range meet the conditions for a Nash-Cournot equilibrium. However, the previous paper argued, given adequate knowledge (as by learning, or by mutual visibility), under plausible dynamic processes the parties will be led to an equilibrium at the upper limit of this range -- point F in the diagram.[3]

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[3]If A chooses first, for example, he will (given adequate knowledge) select the x_A-coordinate of point F as his contribution. For, he knows, in

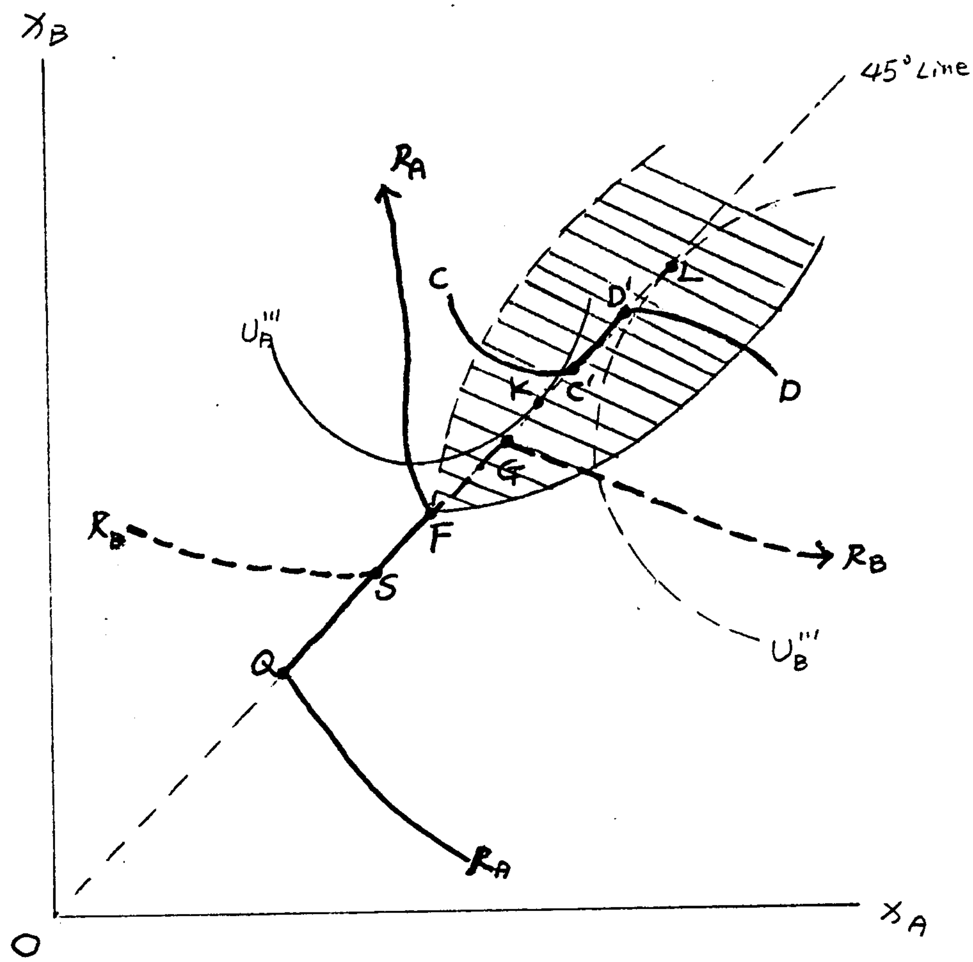

FIGURE 2c: Descending-weight function, efficient and equilibrium solutions

What happens now as we proceed to the limiting case -- the strict Weakest-link formula (2) that is the extreme of the class of descending-weight social composition functions?

Consider the situation for individual A. When $x_A < x_B$, A makes his productive tradeoff decision between x_A and y_A knowing that any additional marginal contribution on B's part will have no effect on the social aggregate X of the public good. Thus, in Figure 3, in the zone northwest of the 45° line A's indifference curves U_A are all vertical.[4] This is confirmed by equation (10) since (in the upper line) w_B in the denominator goes to zero. Furthermore, in this zone there will be one single best vertical indifference curve for A (labelled $\hat{U}_A$ in the diagram). B's production of the public good being irrelevant in this zone, X becomes in effect a private good in A's calculations. Then there will be some optimal $\hat{x}_A$ such that $MC_A(\hat{x}_A) = MRS_A(\hat{x}_A)$.[5] As for the zone of Figure 3 to the southeast of the 45° line, here A's indifference curves will generally be curved. However, since it is always irrational for A to choose $x_A > x_B$ (so long as his Marginal Cost MC_A is positive), A's indifference-curve map is shown in Figure 3

(fn. 3 cont.)...that case B is motivated to go along and choose as his contribution the x_B-coordinate of point F. B in his turn can be confident that A will not be motivated thereafter to renege on his initial choice.

[4]Regrettably, my presentation in the previous paper (Hirshleifer [1983], p. 378) erred in showing normally curved rather than vertical indifference curves even for the limiting (Weakest-link) case. The indifference curves do retain curvature for the general descending-weight class of composition formulas, but in the limit they lose their curvature and become straight lines.

[5]Note that the two distinct verticals both labelled U'_A represent the same level of utility. The more westerly vertical represents producing too little of the public good, the easterly vertical represent producing too much -- either way there is a loss of satisfaction for A.

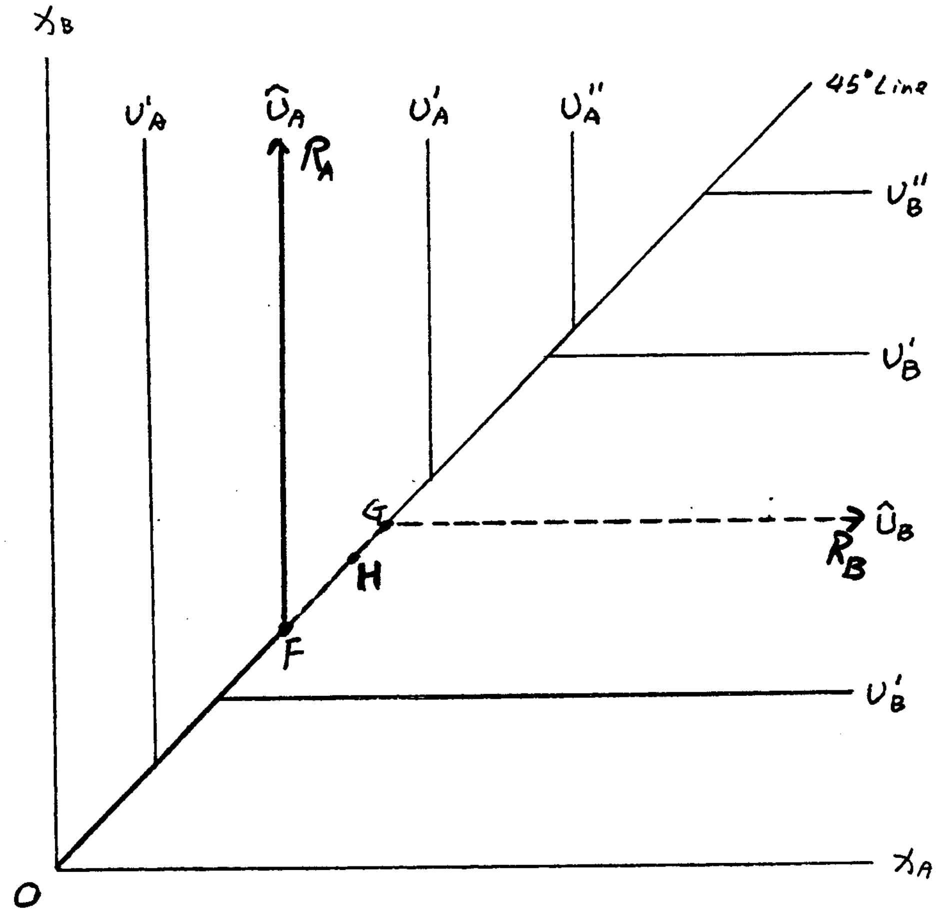

FIGURE 3: Weakest-link function, efficient and equilibrium solutions

northwest of the 45° line only.

Corresponding arguments of course hold for individual B. His indifference curves are shown only southeast of the 45° line, where they take the form of horizontal lines. In this zone he will have a single best indifference curve $\hat{U}_B$, where the amount of the public good he produces $\hat{x}_B$ satisfies the condition $MC_B(\hat{x}_B) = MRS_B(\hat{x}_B)$.

It will also be evident that A's Reaction Curve R_A will overlie the 45° line from the origin until point F, after which it will run vertically along his optimal indifference curve $\hat{U}_A$. Similarly, B's Reaction Curve R_B will run along the 45° line until point G, and then horizontally along his optimal indifference curve $\hat{U}_B$. The Reaction Curves therefore again intersect over a finite range along the 45° line, in this case the distance OF. By the same argument used previously, the equilibrium position is taken to be the upper limit of this common range, to wit, the point F.

III

I now turn to a comparison of equilibrium versus efficiency conditions, to show how underprovision of the public good varies over the different social composition functions. And I especially want to analyze in each case how the extent of underprovision changes parametrically -- first, in response to the number of individuals N comprising the community, and second, in response to degree of inter-individual heterogeneity (in tastes, endowments, and productive opportunities).

Summation (constant-weight) social composition function:

Returning to Figure 1 that pictured the standard Summation or constant-weight formula for aggregating individual contributions, we have seen that the equilibrium is at point E where the two Reaction Curves intersect. This solution is obviously inefficient, in that the shaded area in the diagram pictures a "region of mutual advantage" such that both parties can gain from increased production of the collective good.

What are the requirements for efficiency? It is useful to distinguish what I shall call a mutual-tangency condition and an absolute condition. The mutual-tangency condition is pictured in Figure 1 by the curve CD, of which the portion $\overline{CD}$, is included within the shaded region of mutual advantage. CD is the locus of points satisfying

$$\text{(11)} \qquad \frac{MRS_A(X) - MC_A(x_A)}{MRS_A(X)} = \frac{MRS_B(X)}{MRS_B(X) - MC_B(x_B)}$$

Notice that this equation relates the ratio of Marginal Cost MC to Marginal Rate of Substitution MRS as between the two individuals. The absolute condition adds the requirement that the two Marginal Costs be equal, since otherwise production of the public good could efficiently be reallocated from one party to the other:

$$\text{(12)} \qquad MC_A(x_A) = MC_B(x_B)$$

Putting the two conditions together leads to the familiar formula for optimal provision of a public good:

$$\text{(13)} \qquad MC_A(x_A) = MC_B(x_B) = \sum_i MRS_i(X) \quad [i = A,B]$$

Unfortunately, our diagrammatics on x_A, x_B axes lacks the dimensionality required to show satisfaction of the absolute condition (12) constructively, and therefore to locate the point or points satisfying the full optimality expression (13). Let us assume that the functional forms are "nice" enough to guarantee a unique solution. Even so, the solution point might either lie along CD within the region of mutual advantage (like point H) or, more surprisingly, along CD but outside the region of mutual advantage (like point H').[6] The interpretation would be as follows. Attaining an efficient point like H requires only that the parties mutually agree that each will increase his production of the public good X, by specified amounts, at the expense of his own production of the private good Y. But if the efficient point is located like H' outside the region of mutual advantage, its achievement would in addition require a compensating transfer of the private good between the two parties.[7] This is clearest in the case where, due to differing comparative advantages in production, achievement of the absolute condition (12) would require one of the parties to almost fully specialize in production of the public good X. Such an optimum could only be Pareto-superior to the equilibrium point E if the individual specializing in producing the public good were compensated by a transfer of the private good Y from the other party.

The equilibrium conditions that define point E in the diagram correspond to Nash-Cournot behavior on both sides:

$$(14) \qquad \begin{cases} MC_A(x_A) = MRS_A(X) \\ MC_B(x_B) = MRS_B(X) \end{cases}$$

Thus, for efficiency, (13) indicates that each individual should set his

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[6]My previous paper was erroneous in implying that the solution points would necessarily be like H, located within the region of mutual advantage.

[7]When the compensating transfer is allowed for, both parties will of course end up better off in comparison with the equilibrium at E. I.e., H' would lie in the region of mutual advantage with respect to the four dimensions x_A, x_B, y_A, y_B. The paradox of the efficient solution lying outside the region of mutual advantage is resolved when we appreciate that the shaded area in the diagram is really a projection of the true 4-dimensional region onto the x_A, x_B plane.

Marginal Cost equal to the sum of the Marginal Rates of Substitution. But, for equilibrium, (14) indicates that each would set his MC simply equal to his own individual MRS. It will immediately be evident that, in this standard Summation case, underprovision tends to increase proportionately with community size N, at least as a first approximation.[8] Heterogeneity, on the other hand, tends to reduce the extent of underprovision, as can intuitively be seen if we go to the limit. If one person has practically all the social weight (possesses almost all the resources, and therefore is responsible for almost all the effective demand), he will consider the public good X almost as if it were just as much a private good as Y, and hence would produce approximately the efficient amount.

Weakest-link social composition function

Before turning to the general descending-weight case, it will be useful to skip to the opposite limit represented by the Weakest-link social composition formula (2). We have seen in Figure 3 that the Reaction Curves RC_A and RC_B intersect over the whole range OF along the 45° line, but that under plausible dynamic protocols of interaction the equilibrium will actually

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[8]It is not clear whether this first approximation is likely to be on the high side (in which case underprovision does not quite grow proportionately with N) or on the low side (so that underprovision grows more than proportionately with N). First, as is well-known, equilibrium provision does tend to increase with N. The reason is that a new entrant's provision of X, while a perfect substitute for others' production of X, also tends to enrich all previous members of the community. Hence, if X and Y are normal superior goods, these previous members will cut back their production of X, but not quite 1:1. (This tendency of aggregate equilibrium provision of X to grow with N tapers off for large N.) As for efficient provision, as N increases in (13) each separate individual should produce more and, in addition, there are more individuals. But this double effect tends to be counterbalanced by diminishing-returns considerations in the MC and MRS functions. Hence the final net balance is unclear as to whether efficient provision rises more than or less than linearly with N, and therefore as to whether underprovision grows more or less than proportionately with N.

lie at the upper limit of this range — point F.

Formalizing this, we have:

$$(15)\quad \begin{cases} MC_A(x_A) = MRS_A(X), & \text{where A is the first to reach the equality} \\ MC_B(x_B) \lesseqq MRS_B(X), & \text{where } x_B = x_A \end{cases}$$

Of course, $X \equiv \min(x_i)$ here equals the common value of x_A and x_B.

The efficiency conditions can most clearly be understood in the following way. The mutual-tangency conditions specify, first, that the parties must be on the 45° line, and second that mutual agreement to move away from any member of the efficient set cannot be achieved. Thus, recalling that $X \equiv \min(x_i)$:

$$(16)\quad x_A = x_B$$

$$(17)\quad \begin{cases} MC_A(x_A) < MRS_A(X) \text{ and } MC_B(x_B) > MRS_B(X) \\ \text{or} \\ MC_A(x_A) > MRS_A(X) \text{ and } MC_B(x_B) < MRS_B(X) \end{cases}$$

These combine to specify only the range FG along the 45° line — since, in the range below point F both parties find that $MC_i < MRS_i$ while the reverse holds in the range above point G. Notice that the equilibrium solution point F lies at the lower limit of this range.

The absolute condition here takes the form:

$$(18)\quad \sum_i MC_i = \sum_i MRS_i$$

Incorporating this condition will identify a subset of points, or in nice cases a single point like H along the 45° line, as the efficient solution.

The crucial implication here is that, necessarily, H will lie along the 45° line between points F and G. Thus, as in the standard Summation case, there will also in general be underprovision in the Weakest-link case. One interesting difference emerges if we recall that, in the Summation analysis, the efficient point might lie either inside or outside the region of mutual advantage relative to the equilibrium solution. In the Weakest-link case, in contrast, there is no region of mutual advantage relative to the equilibrium point F: moving northwest from F necessarily reduces A's utility while increasing B's. So the efficient solution must lie outside the region of mutual advantage on x_A, x_B axes. Any such solution could not be attained merely by negotiating as to the amounts x_A and x_B of the public good to be produced by the two parties -- some kind of compensating transfer of Y would have to be involved as well.[9]

Turning now to parametric changes, we get a very neat result in allowing the degree of heterogeneity to change. In particular, if A and B are identical then points F and G coincide in Figure 3. Since F is the equilibrium point, and the efficient solution H must lie between F and G, it follows that as heterogeneity decreases, in the Weakest-link case underprovision tends to disappear.

What about the effect of increasing community size N? There is a slight trend toward reduced equilibrium provision as N increases, since a new entrant k might possibly reach the equality in (15) at a lower x_k than any of the individuals previously comprising the community. On the other hand, there is no systematic tendency for the efficient provision to either rise or fall with N. All we can say in general is that the efficient point H lies between F and G. Increasing community size N will thus tend to enlarge the range FG at both ends, with no clear net effect. (Also, either change becomes decreasingly likely as N grows.)

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[9]Once again, of course, the efficient solution is in the region of mutual advantage if a diagram could be drawn in the four dimensions x_A, x_B, y_A, y_B.

Summing up, then: In the Weakest-link case, there is no underprovision unless individuals are heterogeneous. Given heterogeneity, underprovision increases but only relatively slightly, as community size N increases.

Descending-Weight Social Composition Function (General Case)

The implications of generalized descending-weight social composition functions should now be reasonably clear, being intermediate between those holding for the standard Summation case pictured in Figure 1 and for the limiting Weakest-link case of Figure 3.

Recall, however, that in the general case equilibrium could fall in any of three different zones of Figure 2c: on the 45° line, or off to either side. Figure 2c pictures the most characteristic case (the one likely to be achieved unless the individuals are very heterogeneous) in which equilibrium occurs at point F on the 45° line, i.e., at the upper limit of the range where the Reaction Curves RC_A and RC_B overlie one another. The equations of equilibrium are identical with (15) that held for the extreme Weakest-link case, except that X must now be interpreted as a descending-weight sum of the individual contributions in accordance with (6).

For equilibrium along the 45° line, (6) here takes the special form:

$$X = w_1\underline{x} + w_2\underline{x} = \underline{x}(1+w_2) \tag{19}$$

where $\underline{x}$ represents the common magnitude of the individual contributions. Since $w_1 = 1$ and $w_2 < 1$ for the descending-weight function, here $\underline{x} < X < 2\underline{x}$ — the social aggregate X is greater than the equalized individual contributions, but is less than their unweighted sum.

Turning now to the efficiency conditions, in Figure 2c there is now a region of mutual advantage relative to the equilibrium point F. The mutual-tangency condition now determines a curve like CD in the diagram, overlying the 45° line along a range C'D' — which range lies between the respective contact points of A's indifference curve with the 45° line (point K) and B's (point L). The CD curve will of course be described by different equations in its various ranges, to wit:

(20)

$$\frac{MRS_A - MC_A}{w_B MRS_A} = \frac{MRS_B}{w_B MRS_B - MC_B}, \quad \text{for} \quad x_A < x_B$$

$$\frac{w_A MRS_A - MC_A}{MRS_A} = \frac{w_A MRS_B}{MRS_B - MC_B}, \quad \text{for} \quad x_B < x_A$$

Of course, for the sector C'D' along the 45° line the equation is $x_A = x_B$. Over this range the mutual-tangencies occur at the respective indifference-curve kinks, the slopes in either case being between the limits represented by the upper and the lower expressions for slopes along U_A (the left-hand expressions) and along U_B (the right-hand expressions).

Now, as heterogeneity parametrically decreases, points F and G tend to merge in the <u>equilibrium</u> solution. Similarly, points K and L (and therefore also C' and D') tend to merge in the <u>efficient</u> solution. Since G and K are separated by a finite gap along the 45° line, some underprovision will in general persist even in the case of completely identical individuals. Only as the weights w_i approach the Weakest-link condition (i.e., the more powerful is the diminishing-returns effect that discounts greater-than-minimum contributions) will underprovision tend to disappear in the identical-individual case.

As for population size, we saw that underprovision tended to grow proportionately with N in the Summation case that represents one end of the descending-weight spectrum, but only very slightly with N in the Weakest-link case that represents the other end. Thus, in the general descending-weight case, the population-size effect will be intermediate between these two extremes.

IV

Since the idea of social composition functions is a relatively new one, I believe it will be useful to discuss certain additional features of different production technologies for public goods, to see how these features are reflected in the form of the composition function.

1. Inequality

In considering descending-weight composition functions I have allowed for individuals who are heterogeneous with respect to tastes, productive advantages, and endowments. But there is one respect in which I have implicitly assumed everyone to be on a basis of equality: to wit, each has equal responsibility for the public good. In the case of a chain, each individual was assumed responsible for one single link — though varying personal circumstances might incline some of them to produce links of greater or lesser strength than others choose to provide.

This equality assumption is by no means necessary for the analysis. We could easily generalize by allowing for unequal responsibilities, for example, letting individuals have control over varying numbers of links in the chain. I will not attempt to provide the generalized equations here, but it can intuitively be seen that qualitatively similar results continue to hold. Imagine a two-person community using a chain of 100 links (perhaps for the purpose of supporting a bridge for common use), and suppose a "little man" A is responsible for only one link while a "big man" B is responsible for the other 99. If the technology is still of the strict Weakest-link form, despite his small share of responsibility A still has "veto power" over the strength of the chain.

Weakest-link social composition functions are of course somewhat analogous to unanimity-rule voting schemes, and the veto power of the small individual in this example suggests the "strategic holdout" problem that arises in unanimity voting.[10] However, there is an important contrast here. While it is true that the "little man" A has veto power, his willingness to exercise that veto is weakened by the fact that his costs of production are low. At one extreme we might imagine that the "big man" B derives no more benefit than A from any given level of X. For example, A and B might use the chain bridge with equal frequency. Here the individual benefit-cost calculus makes the bridge relatively much more desirable for the little man, hence it is B who has greater strategic bargaining power. Or, it may be that the big man derives higher benefit roughly in proportion with his higher cost, e.g., suppose B also uses the bridge 99 times as frequently as A. Here there is no particularly greater bargaining advantage one way or the other, at least in proportionate terms.

2. Amount of public good in relation to community size

There are possible ambiguities about the meaning of the social aggregate of X when N varies. As a specific metaphor for the Weakest-link case, let us return to the earlier example of a port whose topography is such that ship traffic has to pass a series of sandbars. Each individual member of the community is responsible for dredging a depth of channel through one sandbar. The socially available amount X of the public good is the minimum depth of channel. For simplicity, we will also assume here that the members of the community are the only beneficiaries — the channel is not available to outsiders.

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[10]See, for example, Black [1958], p. 147. Of course, strategic behavior is only relevant if negotiations prior to final decision are permitted, such negotiations being generally ruled out in public-good models like ours.

What do we have in mind when we say that community size N grows, parametrically? We must not think of the ratio of population to sandbars as increasing -- this changes the nature of the problem. Rather, we want to compare another community where "everything else is equal" in some relevant sense while N grows larger -- and we want to ask how this change in size affects both equilibrium provision and efficient provision of X.

An interpretation consistent with the previous diagrams and equations would be that the larger community in terms of N is located in a port with a correspondingly larger number of sandbars. Then, as was argued above, the efficient solution shows no systematic tendency for a larger or smaller X to be provided as N increases. For, in (18), any additional individual represents an additional element in the Marginal Cost summation on the left-hand-side of the equation (the cost of dredging another foot of depth through one additional sandbar) but also an additional element in the MRS summation on the right-hand-side (the value of another foot of depth for a member of the enlarged community). As for the equilibrium provision, we saw that (given a degree of heterogeneity) equilibrium provision tends to fall, though perhaps slowly, as N grows. The reason is that a larger population is likely to have a more extreme "minimal individual" who hits the limit in equation (15) earlier, in terms of willingness to dredge the sandbar for which he is responsible.

3. Social composition functions, "impure" public goods, and clubs

Public goods may be said to be "impure" when crowding occurs as consumption is extended over more individuals. More explicitly, when it is no longer possible to extend consumption of the public good without imposing some loss of benefit to previous consumers. A clear instance is the physical crowding of a theater as audience size grows. When public goods are impure, therefore, there is a kind of decreasing returns in the social demand for the public good -- in the sense that, for any aggregate quantity X assumed to be available, the sum of the willingnesses to pay grows less than proportionately with community size N.

The theory of social composition functions discussed here deals with the technology of social supply of public goods. The key point, as we have seen, is that the socially available quantity X may not be represented by the simple summation of the quantities provided by the separate individuals, but may instead follow the Weakest-link, the Best-shot, or some other law of social composition. Thinking in terms of a Marshallian scissors, the degree of "impurity" affects the aggregation of individual demands for a public good, while the social composition functions governs the aggregation of individual productive contributions into a socially available supply.

An interesting application of public-goods theory, that involves both demand-side and supply-side considerations, is "the theory of clubs" (Buchanan [1965]). The theory of clubs inquires into the determination of optimal community size N. Following the original argument of Buchanan, it is necessary to balance an impurity or crowding effect leading to a range of diminishing returns (possibly after an initial range of increasing returns) on the demand side, against some kind of technological function showing how cost responds to N on the supply side. With regard to the latter, however, Buchanan's assumption was relatively uninteresting, the dominant feature being the saving due to simple cost-spreading as N increases. That is, given any amount X of the public good, he in effect assumed that the aggregate cost was a fixed magnitude to be divided over the members of the community. Thus, Buchanan was implicitly assuming a completely collectivist production function, subject to a joint community decision. In contrast, the analysis here assumed separable individual production functions, with voluntary private decisions as to the amounts of the public good each person provides. We might say that Buchanan's analysis refers to the optimal size of a "tight" or collectivist community, while the discussion here would be applicable to a "loose" or voluntarist community. (Of course, optimal community size is only one of the many questions that can be addressed by such a generalized approach that takes account both of demand-side and supply-side considerations.)

V

I can sum up briefly. Assuming that individuals voluntarily provide quantities of the public good in response to personal cost-benefit calculations, there are many important instances where the socially relevant aggregate is not the simple <u>sum</u> of the individual contributions. For example, the relevant magnitude for consumption purposes may be determined by the <u>minimum</u> of the individual contributions (Weakest-link) or the <u>maximum</u> (Best-shot). More generally, individual contributions are aggregated into available quantities of the public good via a "social composition function" that can take any of a variety of forms. I argue that the most usually observed instances tend to fall into the class of <u>descending-weight</u> social composition functions, which include the Weakest-link case at one extreme and the standard Summation case at the other. This class has a kind of diminishing-returns property with regard to the number of individuals comprising the community.

The main concern of the paper was to show how <u>underprovision</u> of the public good — the divergence between the ideally efficient and the equilibrium quantity — varied as descending-weight social composition functions approach the Weakest-link condition at one extreme or the Summation condition at the other. Also, I investigated how the degree of underprovision responds to certain shifts — in particular, to greater or lesser individual heterogeneity (with respect to endowments, tastes, and productive capacities) and to larger or smaller or community size N.

Some of the specific results arrived at are:

(i) Underprovision is of course normal in all public-good situations, but (for any given community size) underprovision tends to become less serious as the social composition function approaches the Weakest-link case.

(ii) Underprovision disappears entirely (i.e., the equilibrium result is efficient) only where (a) the strict Weakest-link condition applies, and (b) heterogeneity is absent (all individuals are identical).

(iii) Underprovision tends to increase proportionately with community size in the standard Summation case, but only relatively slowly with community size in the Weakest-link case.

Finally, I pointed to a certain logical complementarity between the analysis here of how the socially relevant supply of a public good is compounded from individual contributions, and recent discussions as to how "crowding" phenomena affect the aggregation of individual desires into a socially relevant demand for public goods. Specifically, for an applied problem like the "theory of clubs," simple summation is often not the relevant composition function governing social supply of a public good, just as simple summation may not always be the way to derive social demand.

References

Black, D. The Theory of Committees and Elections (Cambridge: Cambridge Univ. Press, 1958).

Buchanan, J.M., "An Economic Theory of Clubs," Economica, v. 32 (Feb. 1965).

Hirshleifer, J., "From Weakest-link To Best-Shot: The Voluntary Provision of Public Goods," Public Choice, v. 41 (1983), pp. 371-386.

Mueller, Dennis C., Public Choice (Cambridge: Cambridge Univ. Press, 1979).