GROVES MECHANISMS IN CONTINUUM ECONOMIES

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ABSTRACT: The equivalence in the finite agents case between the families of dominant strategy and Groves mechanisms is extended to continuum economies. The concept of an infinitesimal individual's marginal product is used to link the two families of mechanisms when agents are non-atomic.

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I. Introduction

This paper extends the characterization of dominant strategy mechanisms to models with a continuum of agents. This entails, as a preliminary, formulating an appropriate non-atomic analogue of the dominant strategy concept. As in the finite agents case (Vickrey [12], Clarke [2], Groves and Loeb [6], Green and Laffont [3], Walker [13], Holmstrom [7]), we show that they are equivalent to Groves mechanisms.

Research on economies with many agents has typically been divided into limit economies where each individual has zero weight or limiting economies consisting of an increasing sequence of finite agent models. Here, we combine the limit and limiting techniques in the notion of an "infinitesimal agent." An infinitesimal individual is the limit of a sequence of non-infinitesimal groups of individuals as their size diminishes to zero. This construct permits an idealized mirroring in the limit economy of asymptotic finite results.

The role of an infinitesimal individual, as distinct from a non-atomic agent, can be appreciated when one considers the problems in extending the definition of a Groves mechanism to non-atomic models. In a finite agent economy, a Groves mechanism gives each individual an allocation whose utility will the mechanism exhibit the dominant strategy property.

Besides filling a niche, what is the value added by establishing an equivalence between dominant strategy and Groves-marginal-product mechanisms with a continuum of agents that is not already present in the finite agents case? We believe that the extension leads to a change in perspective on what these results tell us about public versus private goods. Focusing exclusively on finite economies, there is a tendency towards overemphasis of the remarkable properties of Groves mechanisms and a corresponding underemphasis on the distinctions between public and private goods. This is because such mechanisms work equally well with public or private goods in the sense that they characterize the class of dominant strategy mechanisms and equally poorly in the sense that they generally lead to allocations that are infeasible and/or not fully Pareto-efficient (Vickrey [12], Groves and Ledyard [5], Walker [14]).

When the non-atomic setting is introduced, another issue emerges. While the purpose of this paper is to demonstrate that the same rule continues to characterize the dominant strategy property in finite and nonatomic economies — you have to pay individuals their marginal products — one begins to see that another important consideration is whether you can, through some feasible allocation, pay them their marginal products. In forthcoming work we shall maintain it is the "can" and not the "have to" that divides private from public goods models. But, before we go on to examine the "can" we must first establish the "have to".

$u(x(a),t(a))$ and $v(y(a),t(a))$.

The set of y-efficient outcomes in t is

$$PO_y(t) = \arg\max_Y \{\int_A v(y,t(a)d\lambda\};$$

and, for any $y \in PO_y(t)$,

$$g(t) = \int_A v(y,t(a))d\lambda$$

represents the maximum potential gains in t.

Let $T \subset \{t:A \to V\}$ be a set of possible populations. We shall assume that

(1) for each $t \in T$ there is a finite partition $\pi_t = \{E_t\}$ of A into intervals and finite set $\{v_t\} \subset V$ such that $\forall a \in E_t$, $t(a) = v \in \{v_t\}$.

This restricts the set of populations to type economies. Note that while the number of types in any t is finite, the number of types in T need not be uniformly bounded. Further, the set V of possible types may be infinite. Just as the simple functions are a dense subset of the integrable functions, so the set T satisfying (1) is "rich" in the set of measurable mappings $t:A \to V$ (assuming V has a Borel structure).

In addition, suppose that

(2) $$\forall t \in T, PO_y(t) \neq \emptyset.$$

Condition (2) is assumed rather than derived fromm more primitive assumptions about Y and V. It presumes that $\forall t \in T$ $\forall y \in Y$, $\int v(y,t(a))$ exists.

A mechanism is a mapping $f:T \to Y \times L$. Say that f is y-efficient, written $f \in PO_y[T]$, if $\forall t$, $f(t) \in PO_y(t)$.

says that each member of E_k makes the same representation (constant over k).

We shall repeatedly refer to a t, $\{E_k\}$ and $\{t|t_k'\}$ which will be implicitly understood to satisfy (respectively) $t \varepsilon T$, (3) holds for $\{E_k\}$, and (4) holds for $\{t|t_k'\} \in T$.

For the population t, the utility function of the group E_k over outcomes is

$$U_k(x,t) = \int_{E_k} u(x,t(a))d\lambda.$$

(The existence of $U_k(.,t)$ is presumed.)

Let $\delta_k(t|t_k',t)$ be a measure of the amount by which the mechanism fails to achieve the dominant strategy property for the group E_k when the true population is t and the announcement by E_k is $t|t_k'$. The measure satisfies

$$(5) \qquad \delta_k(t|t_k',t) \geq U_k(f(t),t) - U_k(f(t|t_k'),t).$$

Relative to the mechanism f, E_k can improve its payoff by at most $|\delta_k|$ units of money if it changes the population from t to $t|t_k'$.

The mechanism f has <u>the dominant strategy property for infinitesimal individuals</u>, written $f \varepsilon DS[T]$, if $\forall t \, \forall\{E_k\} \forall\{t|t_k'\}$, there exists $\delta_k(t|t_k',t)$ satisfying (5) such that

$$(6) \qquad \lim \frac{\delta_k(t|t_k',t)}{\lambda_k} = 0 \qquad (\lambda_k \equiv \lambda(E_k))$$

When added to (5), condition (6) says that the <u>per capita</u> advantage of misrepresentation goes to zero faster than the size of E_k. Thus, any satisfies the Radon-Nikodym condition if $\forall t\ \forall\{E_k\}\ \forall\{t|t'_k\}$,

$$\text{(7)} \qquad \lim \frac{U_k(f(t|t'_k),t)}{\lambda_k} \text{ exists.}$$

This restriction means that the per capita payoff to E_k cannot oscillate indefinitely throughout the sequence $\{t|t'_k\}$. When $\cap E_k = \{a\}$, the definition of $u(f(t|t'(a)),t(a))$ is given by the limit in (7).

An interpretation of condition (†) for infinitesimal individuals can now be given in terms of a derivative. Regard $\delta_k(t|t'_k,t)$ as a function of λ_k. The slope of δ_k at $\lambda_k = 0$ measures the mechanism's departure from incentive compatibility for an infinitesimal agent of type $t(a)$. For a dominant strategy mechanism, $\left.\frac{d\delta_k(t|t'_k,t)}{d\lambda_k}\right|_{\lambda_k} = 0$. Similarly, regarding $U_k(f(t|t'_k),t)$ as a function λ_k, (7) says that $\left.\frac{dU_k(f(t|t'_k),t)}{d\lambda_k}\right|_{\lambda_k=0}$ exists. Thus, (†) can be written as

$$0 = \left.\frac{d\delta_k(t|t'_k,t)}{d\lambda_k}\right|_{\lambda_k=0} > \left.\frac{dU_k(f(t|t'_k),t)}{d\lambda_k}\right|_{\lambda_k=0} - \left.\frac{dU_k(f(t),t)}{d\lambda_k}\right|_{\lambda_k=0}.$$

small group E_k's deviation from t and varies not at all in the limit for any infinitesimal individual. Thus, for infinitesimal individuals the quantity H_k is a lump sum.

Theorem 1. $G[T] \subset DSPO_y[T]$.

Proof: If $f \in G[T]$, then $\forall t$,

$$U_k(f(t),t) = \int_{E_k} v(y(t),t(a) + M_k(t) = \int_{E_k} v(y(t),t(a)) + g^k(f(t),t) + H_k(t)$$

$$= g(t) + H_k(t).$$

The last inequality follows from $f(t) \in PO_y(t)$. Similarly,

$$U_k(f(t|t_k'),t) = g(y(t|t_k'),t) + H_k(t|t_k'),$$

where $g(y,t) = \int_A v(y,t)$.

Noting that $g(y(t|t_k'),t) < g(t)$, let

$$\delta_k(t|t_k',t) = H_k(t|t_k') - H_k(t) > [g(y(t|t_k'),t) - g(t)] + [H_k(t|t_k') - H_k(t)]$$

$$= U_k(f(t|t_k'),t) - U_k(f(t),t).$$

Thus, E_k satisfies (5) and dividing by λ_k, (9) implies (6). ||

B. The Replacement of Groves Mechanisms by Marginal Product Mechanisms

If f is a Groves mechanism, the proof of Theorem 1 shows that $U_k(f(t),t) = g(t) + H_k(t)$, i.e., the utility received by E_k equals the entire gains from trade in t plus a lump sum. In finite economies, this marginal product. The mechanism $f \in MP[T]$ is a marginal product (MP) mechanism if $f \in PO_y[T]$ and $\forall t\ \forall\{E_k\}\ \forall\{t|t_k'\}$,

$$\text{(11)} \qquad \lim \frac{I_k(t|t_k') - I_k(t)}{\lambda_k} = 0$$

With the definition of I_k in (10), condition (11) says that small groups receive a per capita utility that differs from their marginal product by an amount that does not vary very much with their characteristics. So, in any MP mechanism, every infinitesimal individual always gets his/her marginal product plus a lump sum.

Theorem 2. $G[T] = MP[T]$.

Proof: Suppose $f \in G[T]$. Then by the argument in the proof of Theorem 1 $\forall t\ \forall\{E_k\}\ \forall\{t|t_k'\}$,

$$\begin{aligned} U_k(f(t|t_k'),t|t_k') &= g(t|t_k') + H_k(t|t_k') \\ &= [g(t|t_k') - g^k(t|t_k')] + [g^k(t|t_k') + H_k(t|t_k')] \\ &= MP_k(t|t_k') + [g^k(t|t_k') + H_k(t|t_k')] \\ &= MP_k(t|t_k') + I_k(t|t_k'). \end{aligned}$$

Since $g^k(t|t_k') = g^k(t)$, condition (9) on H_k implies (11) on I_k.

The converse just runs the argument in reverse.||

REMARK 3: Within the class of Groves mechanisms for finite economies, the pivot mechanism (see Green and Laffont [4], p. 42]) is the one which rewards mechanisms are the only ones in $DSPO_y[T]$.

Throughout the following discussion, t, $\{E_k\}$, and $\{t|t'_k\}$ are fixed. Consider a one-dimensional parameterized family of populations $\{t|t'_k(\alpha)\}$, where $\alpha \in [0,1]$. Say that $\{t|t'_k(\alpha)\}$ connects t with $\{t|t'_k\}$ if for all α and k,

(12) (i) $t|t'_k(\alpha)(a) = t(a)$ if $a \in A\backslash E_k$

(ii) $t|t'_k(\alpha)(a) = t(a) \in V$ if $a \in E_k$

(iii) $t|t'_k(0) = t$

(iv) $t|t'_k(1) = t|t'_k$

Restrictions 12(i) and (ii) imply that $\{t|t'_k(\alpha)\}$ exhibits the same condition (3) as we imposed on $\{t|t'_k\}$. Restrictions 12(iii) and (iv) say that for each k the family of preferences start at t and end at $t|t'_k$. Merely by putting $t|t'_k(\alpha) = t$ when $0 \leq \alpha < 1$ and $t|t'_k(1) = t|t'_k$ we have a trivial example of (12). We shall need more.

For a mechanism f, let $y(t|t'_k(\beta))$ be the y-outcome when individuals in E_k announce that their preferences are of parameterized type β, or simply type β, where $\beta \in [0,1]$. Define

$$g_k(\beta,\alpha) = g(y(t|t'_k(\beta), t|t'_k(\alpha))$$

as the total gains when E_k announces that it is of type β when it is actually of type α. Because t and $t|t'_k$ are fixed, we have suppressed the functional dependence of g_k on them.

Following Holmstrom, T is smoothly connected if $\forall t\ \forall\{E_k\}\ \forall\{t|t'_k\}$, there exists a parameterized family satisfying (12) and $\forall\beta,\alpha \in [0,1]$ and $\forall k$, there is an M such that

Note that $\lambda_k mp_k(\alpha,\alpha) = MP_k(t|t_k'(\alpha))$. Further, since $\partial g^k/\partial\alpha = 0$, condition (13) defining smooth connectedness implies that

(14) $\forall k\ \forall\alpha,\beta \in [0,1]$, $\partial mp_k(\beta,\alpha)/\partial\alpha$ exists and is uniformly bounded by M.

To show that the dominant strategy property requires that infinitesimal agents must be rewarded with their marginal products, we shall have to assume that such marginal products exist. Say that T <u>is regular with respect to</u> f if $f \in PO_y[T]$ and $\forall t\ \forall\{E_k\}\ \forall\{t|t_k'\}$, there is a function $mp:[0,1] \times [0,1] \to \mathbb{R}$ differentiable in its second argument such that $\forall\beta,\alpha \in [0,1]$

(15) (i) $\lim mp_k(\beta,\alpha) = mp(\beta,\alpha)$

(ii) $$\lim \frac{\partial mp_k(\beta,\alpha)}{\partial\alpha} = \frac{\partial mp(\beta,\alpha)}{\partial\alpha}$$

Let $RNDSPO_y[T]$ be those mechanisms in $DSPO_y[T]$ that also satisfy the Radon-Nikodym condition (7). We can now state and prove our main result.

<u>Theorem 3</u>: <u>Assume</u> $f \in RNDSPO_y[T]$ <u>and that</u> T <u>is smoothly connected and regular.</u> <u>Then,</u> $f \in MP[T]$.

<u>Proof</u>: Suppose $f \in RNDSPO_y[T]$ and consider an arbitrary t, $\{E_k\}$, and $\{t|t_k'\}$. Without loss of generality, we can assume that for all α,β, and k there exists a $h_k:[0,1] \to \mathbb{R}$ such that

(16) $$\frac{U_k(f(t|t_k'(\beta)),t|t_k'(\alpha))}{\lambda_k} = mp_k(\beta,\alpha) + h_k(\beta).$$

To verify, note that

(18) $$\alpha \in \arg\max_{\beta} mp(\beta,\alpha).$$

Finally, observe that (14) and (15) imply that $\forall \alpha,\beta$,

(19) $$\left| \frac{\partial mp(\beta,\alpha)}{\partial \alpha} \right| < M.$$

Given (17), (18) and (19), we can apply the principal lemma in Holmstrom [7].

__Lemma__: __Let__ $\psi:[0,1] \times [0,1] \to \mathbb{R}$ __and__ $\nu:[0,1] \to \mathbb{R}$ satisfy $\forall \alpha,\beta \in [0,1]$,

(a) $$\alpha \in \arg\max \psi(\beta,\alpha) + \nu(\beta),$$

(b) $$\alpha \in \arg\max_{\beta} \psi(\beta,\alpha),$$

(c) $$\left| \frac{\partial \psi(\beta,\alpha)}{\partial \alpha} \right| < M.$$

__Then,__ ν __is constant on__ $[0,1]$.

Substituting $mp(\beta,\alpha)$ for $\psi(\beta,\alpha)$ and $h(\beta)$ for $\nu(\beta)$, we conclude that h must be constant on $[0,1]$. ||

[12] Vickrey, W.: "Counterspeculation, Auctions, and Competitive Scaled Tenders," Journal of Finance, 16 (1961), 8-37.

[13] Walker, M.: "A Note on the Characterization of Mechanisms for the Revelation of Preferences," Econometrica, 46 (1978), 147-152.

[14] __________: "On the Existence of a Dominant Strategy Mechanism for Making Optimal Public Decisions," Econometrica, 48 (1980), 1521-1540.