NONLINEAR DECISION WEIGHTS

WITH THE INDEPENDENCE AXIOM*

by

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Abstract

In this paper I give an axiomatic basis for an anticipated utility function of a preference relation between lotteries, which is a generalization of Expected Utility Theory. I prove that this function can explain the Allais paradox and the common ratio effect. Moreover, this function is compatible with both the Independence Axiom and the Reduction of Compound Lotteries Axiom. I show that when used together with the Independence Axiom, this function can explain some phenomena concerning two-stage lotteries, including the probabilistic insurance phenomenon.

1. Introduction

The optimal choice among a set of uncertain alternatives has been discussed since the early 18th century. Bernoulli (1738) suggested that people prefer the lottery yielding the greatest expected utility rather than the lottery with the highest expected value. According to this theory, the value of the lottery $(x_1,p_1;\ldots;x_n,p_n)$, which yields x_i dollars with probability p_i, $i = 1,\ldots,n$, is $\Sigma p_i u(x_i)$. Von Neumann and Morgenstern (1947) first presented a formal set of axioms implying maximization of expected utility, most important being the Independence Axiom (IA). This axiom states that lottery A is preferred to lottery B if and only if for every positive p and for every lottery C, $(A,p;C,1-p)$ is preferred to $(B,p;C,1-p)$. ($(A,p;C,1-p)$ constitutes a two-stage lottery where one participates with probability p in lottery A and with probability $1-p$ in lottery C). In addition to this, von Neumann and Morgenstern need the Reduction of Compound Lotteries Axiom (RCLA). This axiom states that a decision maker would be indifferent between a two-stage lottery and the one-stage lottery that yields the prizes of the original lottery with the corresponding probabilities. That is, if $A = (x_1,p_1;\ldots;x_n,p_n)$ and $B = (y_1,q_1;\ldots;y_m,q_m)$, then $(A,p;B,1-p)$ is equally preferred to the one-stage lottery $(x_1,pp_1;\ldots;x_n,pp_n;y_1,(1-p)q_1;\ldots;y_m,(1-p)q_m)$. Savage (1954) replaced the IA with the Sure Thing Principle (STP), where $(x,p;y,q;0,1-p-q)$ is preferred to $(x,p;y',q';0,1-p-q')$ if and only if $(x',p';y,q;0,1-p'-q)$ is preferred to $(x',p';y',q';0,1-p'-q')$.

Recent experimental evidence shows, however, that people do not always behave in accordance with this theory (see Allais (1953), Kahneman and Tversky (1979), MacCrimmon and Larsson (1979), Ronen (1971), and Snowball and Brown (1979)). To provide a descriptive theory consistent with these data, Kahneman and Tversky proposed Prospect Theory. They agreed with the STP, but suggested that the value of the lottery $(x,p;y,q;0,1-p-q)$ is

(1.1) $$\pi(p)u(x) + \pi(q)u(y).$$

However, the only π consistent with this axiom is the identity function, which reduces this theory to Expected Utility Theory (EU) (see Machina (1982) and Segal (1984)). The basic problem with this theory is that it contradicts first order stochastic dominance, where (a) $(x,p;x,q;y,1-p-q)$ and $(x,p+q;y,1-p-q)$ are equally preferred, and (b) if $x > y$, $p + q = p' + q' = r$, and $p > p'$, then $(x,p;y,q;z,1-r)$ is preferred to $(x,p';y,q';z,1-r)$.

Machina (1982) took a different approach. In his view, one cannot accept the IA and still behave in accordance with the Allais paradox (see Section 3 below). He assumed that the preferences are continuous and transitive, and that they can be represented by a Frechet-differentiable functional. Quiggin (1982) too claimed that if one wants to differ from EU without violating some fundamental assumptions such as transitivity or first order stochastic dominance, then the IA must be omitted or at least weakened.

This paper shows that observed violations of EU may be consistent with IA if RCLA is relaxed. Indeed, Kahneman and Tversky (1979), Ronen (1971) and Snowball and Brown (1979) presented empirical evidence suggesting that decision makers do not obey the RCLA. Kahneman and Tversky's experiments even indicate that people will accept the IA.

In Section 2 I present a set of axioms implying that the value of the lottery $(x_1,p_1;\ldots;x_n,p_n)$ where $x_1 < \ldots < x_n$ equals

(1.2) $$u(x_1) + [u(x_2) - u(x_1)]f(p_n+\ldots+p_2) + \ldots + [u(x_n) - u(x_{n-1})]f(p_n)$$

This function resembles Quiggin's function, therefore I follow him in calling it Anticipated Utility (AU). I discuss the differences between (1.2) and Quiggin's function in Section 9.

In Section 3 I show that (1.2) is compatible with the Allais paradox when f is convex. If, in addition, the elasticity of f is increasing, (1.2) is compatible with the common ratio effect and certain phenomena concerning two-stage lotteries. I discuss some generalizations of these phenomena in Section 4 and the IA and the RCLA in Sections 5 and 7. In Section 6 I show that AU can explain the probabilistic insurance phenomenon, even if the utility function is always concave, provided that f is convex. A discussion of AU and the Ellsberg Paradox will appear in a separate paper.

2. Representation of $\succsim$

Let L_1 be the family of all the bounded random variables. For every $A \in L_1$, define the cumulative distribution function F_A by $F_A(x) = \Pr(A < x)$. Let $A^o = C\ell\,\{(x,p) \in \mathbb{R} \times [0,1]:\ p > F_A(x)\}$, $A^+ = \inf\{x:\ F_A(x) = 1\}$ and $A^- = \sup\{x:\ F_A(x) = 0\}$ (see Figure 1).

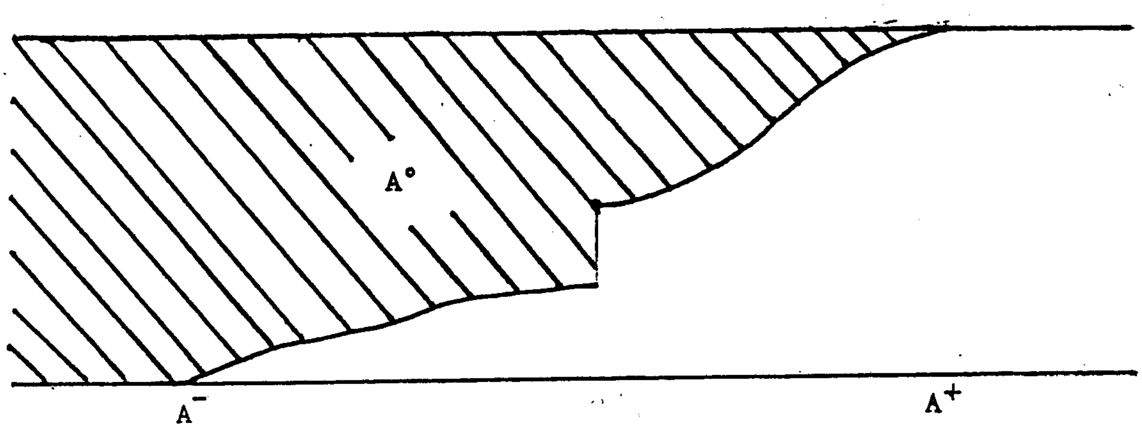

Figure 1

Let L_1^o be the family of all the closed sets A^o in $\mathbb{R} \times [0,1]$ that satisfy the following conditions:

1. If $(x,p) \in A^o$, $y < x$ and $p < q < 1$, then $(y,q) \in A^o$.

2. There exists x such that $(x,1) \notin A^o$.

3. There exists x such that $(x,0) \in A^o$.

Obviously, there is a one-to-one correspondence between L_1 and L_1^o.

Let L_1^* be the set of all the elements of L_1 for which the range of F_A is finite. Elements of L_1^*, called prospects, will be denoted by vectors of the form $(x_1,p_1;\ldots;x_n,p_n)$, where $x_1 < x_2 < \ldots < x_n$ and $\Sigma p_i = 1$. Such a vector represents a lottery yielding x_i dollars with probability p_i, $i = 1,\ldots,n$. Obviously, if $A = (x_1,p_1;\ldots;x_n,p_n)$, then

$$F_A(x) = \begin{cases} 0 & x < x_1 \\ \sum_{j=1}^{i} p_j & x_i \leq x < x_{i+1} \\ 1 & x \geq x_n \end{cases}$$

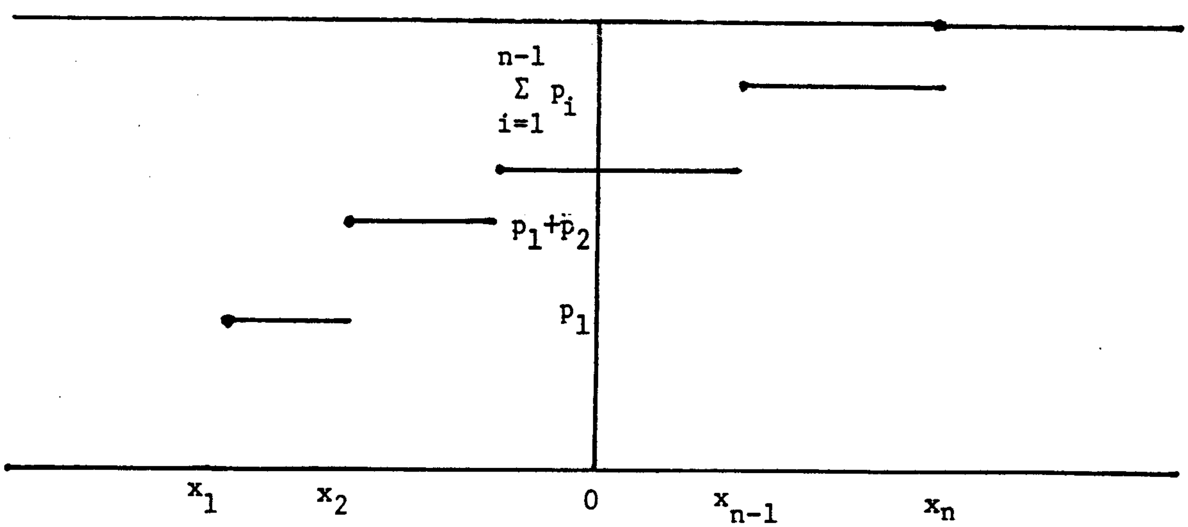

Figure 2

On L_1 (and L_1^o) there exists a complete and transitive binary relation, $\succsim$. $A \sim B$ iff $A \succsim B$ and $B \succsim A$, and $A \succ B$ iff $A \succsim B$ but not $B \succsim A$. I assume that $\succsim$ satisfies the following assumptions:

(a) Continuity: $\succsim$ is continuous in the topology of weak convergence. That is, if $A, B, B_1, B_2, \ldots \in L_1$, for all i $A \succsim B_i$, and if at each continuity point x of F_B, $F_{B_i}(x) \to F_B(x)$, then $A \succsim B$. Similarly, if $A, B, A_1, A_2, \ldots \in L_1$, for all i $A_i \succsim B$, and if at each continuity point x of A, $F_{A_i}(x) \to F_A(x)$, then $A \succsim B$.

(b) First Order Stochastic Dominance: If, for every x, $F_A(x) \leq F_B(x)$, then $A \succsim B$.

Remark: The representation function (1.1) suggested by Kahneman and Tversky (1979) contradicts this assumption, unless π is linear.

Let Δ be the set of all the bounded closed sets in $\mathbb{R} \times [0,1]$, and let

$$\psi = \{(A^o, \delta) \in L_1^o \times \Delta\colon \text{Int } A^o \cap \text{Int } \delta = \emptyset,\ A^o \cup \delta \in L_1^o\}$$

Define $\oplus\colon \psi \to L_1^o$ by $A^o \oplus \delta = A^o \cup \delta$.

(c) Cancellation: $A^o \oplus \delta \succsim B^o \oplus \delta$ iff $A^o \succsim B^o$.

Define on Δ orders R_A by $\delta_1 R_A \delta_2$ iff $A^o \oplus \delta_1 \succsim A^o \oplus \delta_2$.

<u>Lemma 2.1</u>: For every A and B, R_A and R_B do not contradict each other.

<u>Proof</u>: All the proofs appear in the Appendix.

Let $R = \bigcup_A R_A$. That is, $\delta_1 R \delta_2$ iff there exists A such that $\delta_1 R_A \delta_2$. It can be proved that R is acyclic. That is, $\delta_1 R \delta_2 R \ldots R \delta_t R \delta_1$ imply $\delta_1 R \delta_t R \ldots R \delta_2 R \delta_1$. Let $\succsim^*$ be the transitive closure of R: $\delta_1 \succsim^* \delta_2$ iff there exist $\delta_3, \ldots, \delta_t$ such that $\delta_1 R \delta_3 R \ldots R \delta_t R \delta_2$ and $\succsim^*$ is obviously complete and transitive.

(d) For every $x < y < z$ and $0 \leq p < q < r \leq 1$, $[y,z] \times [p,q] \succsim^* [y,z] \times [q,r]$ iff $[x,y] \times [p,q] \succsim^* [x,y] \times [q,r]$.

For the reasoning behind this assumption, consider Figure 3. Assume that $\delta_5 \sim^* \delta_1$ and $\delta_5 \cup \delta_6 \sim^* \delta_2$. That is, $A^o \cup \delta_5 \sim A^o \cup \delta_1$ and $A^o \cup \delta_1 \cup \delta_5 \cup \delta_6 \sim A^o \cup \delta_1 \cup \delta_2$. $A^o \cup \delta_5 \cup \delta_6 \succsim A^o \cup \delta_5$, hence $\delta_5 \cup \delta_6 \succsim^* \delta_5$ and $\delta_2 \succsim^* \delta_1$. Since the projections of the rectangles δ_1 and δ_2 on the prizes axis are the same, the $\succsim^*$ order between them is determined by their projections on the probabilities axis. The projection of δ_3 and δ_4 on the prizes axis is also the same, hence the $\succsim^*$ order between them is defined by their projections on the probabilities axis. Since the probabilities axis projections of δ_3 and δ_4 equal those of δ_1 and δ_2 respectively, $\delta_4 \succsim^* \delta_3$ iff $\delta_2 \succsim^* \delta_1$.

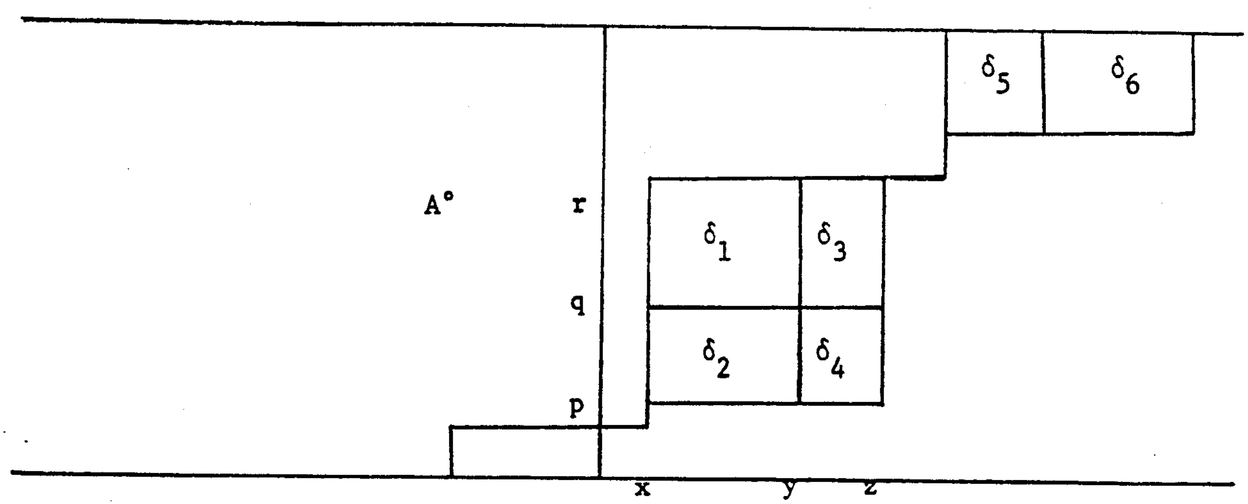

Figure 3

<u>Theorem 2.2</u>: There exists a function $u: \mathbb{R} \to \mathbb{R}$, unique up to positive linear transformations, and a unique function $f: [0,1] \to [0,1]$, satisfying $f(0) = 0$ and $f(1) = 1$, such that $A \succsim B$ iff

$$(2.1)\quad u(A^-) + \int_{A^-}^{A^+} \int_{F_A(x)}^{1} u'(x)f'(1-p)\,dp\,dx \geq u(B^-) + \int_{B^-}^{B^+} \int_{F_B(x)}^{1} u'(x)f'(1-p)\,dp\,dx$$

Conclusion: On L_1^* the relation $\succsim$ can be represented by the function

$$V(x_1,p_1;\ldots;x_n,p_n) =$$

(2.2) $$u(x_n)f(p_n) + u(x_{n-1})[f(p_n+p_{n-1}) - f(p_n)] + \ldots + u(x_1)[1 - f(p_n+\ldots+p_2] =$$

(2.3) $$u(x_1) + [u(x_2) - u(x_1)]f(p_n+\ldots+p_2) + \ldots + [u(x_n) - u(x_{n-1})]f(p_n)$$

Remark: If $f(p) = p$, then (2.1) - (2.3) reduce to the EU representation function.

3. Unravelling of "Paradoxes"

In this section I show that some behavioral patterns, although inconsistent with Expected Utility Theory (EU), may agree with AU theory. In each case I will present the behavioral pattern, explain why it contradicts EU, and show that it may be consistent with AU theory. (MacCrimmon and Larsson (1979) discussed these patterns and their relationship to EU hypotheses in detail.) As before, $u(0) = 0$.

3.1 The Paradox of Allais (1953)

Problem 1: Choose between

$$A_1 = (0,0.9;\ 5000000,0.1) \quad \text{and} \quad B_1 = (0,0.89;\ 1000000,0.11)$$

Problem 2: Choose between

$$A_2 = (0,0.01;\ 1000000,0.89;\ 5000000,0.1) \quad \text{and} \quad B_2 = (1000000,1).$$

Let 1M denote 1000000. According to EU, $A_1 \succsim B_1$ iff $0.1u(5M) > 0.11u(1M)$ iff $0.1u(5M) + 0.89u(1M) > u(1M)$ iff $A_2 \succsim B_2$. However, most people prefer A_1 to B_1, but B_2 to A_2.

Using (2.3) yields $A_1 \succ B_1$ iff

$$(3.1) \qquad u(5M)f(0.1) > u(1M)f(0.11)$$

and $B_2 \succ A_2$ iff

$$(3.2) \qquad u(1M)f(0.99) + [u(5M) - u(1M)]f(0.1) < u(1M)f(1).$$

(3.1) and (3.2) together imply

$$u(1M)[f(1) - f(0.99)] > u(5M)f(0.1) - u(1M)f(0.1) > u(1M)[f(0.11) - f(0.1)].$$

Thus, if f is convex and if

$$\frac{f(1) - f(0.99) + f(0.1)}{f(0.1)} > \frac{u(5M)}{u(1M)} > \frac{f(0.11)}{f(0.1)}$$

then the choices $A_1 \succ B_1$ and $B_2 \succ A_2$ are compatible with (2.3).

The following data come from Kahneman and Tversky (1979):

Problem 3: Choose between

$$A_3 = (0,0.67;\ 2500,0.33) \quad \text{and} \quad B_3 = (0,0.66;\ 2400,0.34)$$

Problem 4: Choose between

$$A_4 = (0,0.01;\ 2400,0.66;\ 2500,0.33) \quad \text{and} \quad B_4 = (2400,1).$$

According to EUT, $A_3 \succsim B_3$ iff $A_4 \succsim B_4$. Most people, however, prefer A_3 to B_3, but B_4 to A_4. These results are compatible with AU theory if f is convex and if

$$\frac{f(1) - f(0.99) + f(0.33)}{f(0.33)} > \frac{u(2500)}{u(2400)} > \frac{f(0.34)}{f(0.33)}.$$

3.2 The Common Ratio Effect

Problem 5: Choose between

$$A_5 = (1000000,1) \quad \text{and} \quad B_5 = (0,0.2;\ 5000000,0.8)$$

Problem 6: Choose between

$$A_6 = (0,0.95;\ 1000000,0.05) \quad \text{and} \quad B_6 = (0,0.96;\ 5000000,0.04).$$

According to EU, $A_5 \succsim B_5$ iff $u(1M) > 0.8u(5M)$ iff $0.05u(1M) > 0.04u(5M)$

iff $A_6 \succsim B_6$. Most people prefer A_5 to B_5, but B_6 to A_6.

By (2.3), $A_5 \succ B_5$ and $B_6 \succ A_6$ iff

$$\left.\begin{array}{l} u(1M)f(1) > u(5M)f(0.8) \\ \\ u(5M)f(0.04) > u(1M)f(0.05) \end{array}\right\} \Rightarrow \frac{f(1)}{f(0.8)} > \frac{u(5M)}{u(1M)} > \frac{f(0.05)}{f(0.04)} .$$

A sufficient condition for $f(1)/f(0.8) > f(0.05)/f(0.04)$ is that for every $\alpha > 1$, $f(\alpha p)/f(p)$ is increasing with p. (In this example $\alpha = 5/4$). This occurs iff

$$\alpha f'(\alpha p)f(p) > f(\alpha p)f'(p) \iff \frac{\alpha p f'(\alpha p)}{f(\alpha p)} > \frac{pf'(p)}{f(p)} .$$

The elasticity of a function f is defined as $xf'(x)/f(x)$. Thus, if the elasticity of f is increasing, then choosing A_5 and B_6 is compatible with (2.3).

MacCrimmon and Larsson (1979) investigated a more general form of this decision problem:

Problem 5^*: Choose between

$$A_5^* = (0,1-p;\ x,p) \quad \text{and} \quad B_5^* = (0,1-0.8p;\ 5x,0.8p)$$

By (2.4), $A_5^* \succsim B_5^*$ iff $u(x)f(p) > u(5x)f(0.8p)$ iff

$$\frac{f(p)}{f(0.8p)} > \frac{u(5x)}{u(x)}$$

MacCrimmon and Larsson found that the preference for A_5^* and B_5^* is increasing with x and with p. One obtains these results if the elasticity of f is increasing and the elasticity of u, decreasing.

Kahneman and Tversky (1979) observed similar patterns. For example:

Problem 7: Choose between

$$A_7 = (3000,1) \quad \text{and} \quad B_7 = (0,0.2;\ 4000,0.8)$$

Problem 8: Choose between

$$A_8 = (0,0.75;\ 3000,0.25) \quad \text{and} \quad B_8 = (0,0.8;\ 4000,0.2).$$

Most people prefer A_7 to B_7, but B_8 to A_8. Increasing elasticity of f may explain this phenomenon.

4. The Convexity of f

In this section I discuss some properties of the preference relation $\succsim$ resulting from the assumption that f is a convex function.

Definition: F_B is said to differ from F_A by a simple compensated spread if $A \sim B$ and if there exists a point x^* such that for every $x < x^*$ $F_B(x) > F_A(x)$ and for every $x > x^*$ $F_B(x) < F_A(x)$ (Machina (1982, p. 281)).

Generalized Common Ratio Effect (GCRE): Let $A, B, C, D \in L_1$ such that C and D stochastically dominate A and B respectively, and $F_D - F_C \equiv \xi(F_B - F_A)$ for some $\xi > 0$. If F_B differs from F_A by a simple compensated spread, then $C \succsim D$, and if F_D differs from F_C by a simple compensated spread, then $B \succsim A$ (Machina (1982, p. 305)).

The common ratio effect (3.2 above) constitutes a special case of the GCRE. Let $A = (0,1-p;\ x,p)$ and $B = (0,1-q;\ y,q)$ such that $0 < x < y$, $1 > p > q$ and $A \sim B$. By definition, B differs from A by a simple compensated spread. Let $1 < \lambda < \frac{1}{p}$ and let $C = (0,1-\lambda p;\ x,\lambda p)$, $D = (0,1-\lambda q;\ y,\lambda q)$. C and D stochastically dominate A and B respectively, and $F_D - F_C \equiv \lambda(F_B - F_A)$. The GCRE requires that $C \succsim D$, as MacCrimmon and Larsson (1979) found.

If $\succsim$ can be represented by an EU function, then it satisfies the GCRE assumption because $B \sim A$ iff $D \sim C$. I now prove that EU function is the only function satisfying assumptions (a) - (d) and the GCRE. This functional form cannot resolve the Allais paradox and the common ratio effect. Hence, if a decision maker behaves in accordance with assumptions (a) - (d) and the Allais paradox or the common ratio effect, then he cannot satisfy the GCRE.

Theorem 4.1: If $\succsim$ satisfies assumptions (a) - (d) and the GCRE, then it can be represented by an EU function. In other words, AU reduces to EU.[1]

Although AU cannot satisfy GCRE (unless $f(p) = p$), it satisfies some modifications of this assumption.

Generalized Allais Paradox (GAP): Let $A, B, C, D \in L_1$ such that C and D stochastically dominate A and B respectively, and $F_D - F_C \equiv F_B - F_A$. Assume, moreover, that B differs from A by a simple compensated spread, and let x^* be such that for $x < x^*$ $F_B(x) > F_A(x)$ and for $x > x^*$ $F_B(x) < F_A(x)$. If for $x > x^*$ $F_C(x) = F_A(x)$ (and $F_D(x) = F_B(x)$), then $C \succsim D$.

To obtain the Allais paradox, let $A = (0,0.89; 1000000,0.11)$, $B = (0,0.9; 5000000,0.1)$, $C = (1000000,1)$, $D = (0,0.01; 1000000,0.89; 5000000,0.1)$, and $x^* = 1000000$.

It is reasonable to assume that decision makers obey the GAP. Let $A = (0,1-p; x,p)$, $B = (0,1-q; y,q)$, $C = (0,1-p-r; x,p+r)$ and $D = (0,1-q-r; x,r; y,q)$ such that $0 < x < y$, $p > q$ and $A \sim B$. C may be understood as A plus an r chance of receiving x, while D equals B plus an r chance of receiving x. Note, however, that with the shift from A to C the probability of 0 is reduced relatively more than with the

[1]Mark Machina pointed out to me that Quiggin's theory of AU is consistent with GCRE iff it reduces to EUT.

shift from B to D. Since $A \sim B$ and $F_C - F_A \equiv F_D - F_B$, C should be preferred to D, as predicted by the GAP.

Theorem 4.2: Assume that $\succsim$ can be represented by (2.1). $\succsim$ satisfies the GAP iff f is convex.

Machina (1982) defined the local utility function U(x,F) and proved that $A \succsim B$ whenever A stochastically dominates B iff U(x,F) is nondecreasing in x for every cumulative distribution function F. Also, $A \succsim B$ whenever B differs from A by a mean preserving increase in risk iff U(x,F) is a concave function of x for every F.[2] One can prove that if $\succsim$ is represented by (2.1), then the local utility function $U(\cdot,F)$ is given by

$$U(x,F) = \int^x u'(s)f'(1-F(s))ds.$$

Differentiating twice with respect to x implies

$$u_1(x,F) = u'(x)f'(1-F(x)) \tag{4.1}$$

$$u_{11}(x,F) = u''(x)f'(1-F(x)) - u'(x)f''(1-F(x))F'(x) \tag{4.2}$$

Because u and f are increasing functions, by (4.1), U is nondecreasing in x. $\succsim$ indeed satisfies the first order stochastic dominance axiom (Section 2, assumption (b)). If u is concave and f is convex, then by (4.2), U is a concave function of x. According to Machina's theorem, if B differs from A by a mean preserving increase in risk, then $A \succsim B$.

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[2]The local utility function is defined in Machina (1982), Section 3.1. For the definition of mean preserving increase in risk see Rothschild and Stiglitz (1970).

5. Two-Stage Lotteries and the Independence Axiom

A two-stage lottery is a vector $(A_1,p_1;\ldots;A_m,p_m)$, where $A_1,\ldots,A_m$ are prospects in L_1^*. In such a lottery there is a p_i probability that the decision maker will participate in lottery A_i, $A_i = (x_1^i,p_1^i;\ldots;x_{n_i}^i,p_{n_i}^i)$, $i = 1,\ldots,m$. Let $L_2^{*'} = \{A_1,p_1;\ldots;A_m,p_m): A_1,\ldots,A_m \in L_1^*, p_1,\ldots,p_m > 0, \Sigma p_i = 1\}$, let $L_2^* = L_2^{*'} \cup L_1^*$, and assume that the decision maker has a complete and transitive preference relation $\succsim$ on L_2^*, which, when restricted to L_1^*, satisfies assumptions (a) - (d) (and can therefore be represented on L_1^* by (2.2) - (2.3)).

Ttwo independent axioms concern the transformation of a two-stage lottery into a one stage lottery.

1. Reduction of Compound Lotteries Axiom (RCLA):

Let $A_i = (x_1^i,p_1^i;\ldots;x_{n_i}^i,p_{n_i}^i)$ $i = 1,\ldots,m$.

$$(5.1) \qquad (A_1,p_1;\ldots;A_m,p_m) \sim (x_1^1,p_1p_1^1;\ldots;x_{n_1}^1,p_1p_{n_1}^1;\ldots;x_1^m,p_mp_1^m;\ldots;x_{n_m}^m,p_mp_{n_m}^m)$$

2. Independence Axiom (IA): For $p > 0$, $(A,p;C,1-p) \succsim (B,p;C,1-p)$ iff $A \succsim B$.

Conclusion: Let $CE(A_i)$ be the certainty equivalence of A_i, given implicitly by $(CE(A_i),1) \sim A_i$, and explicitly by $CE(A_i) = u^{-1}(V(A_i))$. Assume without loss of generality that $CE(A_1) < \ldots < CE(A_m)$. If $\succsim$ satisfies IA, then

$$(5.2) \qquad (A_1,p_1;\ldots;A_m,p_m) \sim (CE(A_1),p_1;\ldots;CE(A_m),p_m) = (u^{-1}(V(A_1)),p_1;\ldots;u^{-1}(V(A_m)),p_m).$$

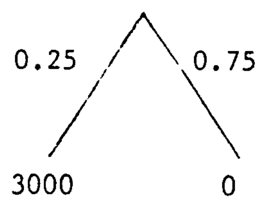
EU employs both axioms. Quiggin (1982) accepted only the RCLA and a weak version of the IA (see Section 8 below).

Kahneman and Tversky examined the validity of the RCLA.

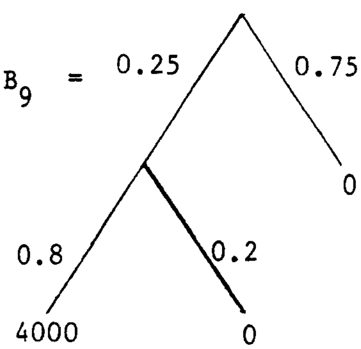
Problem 9: Choose between

$$A_9 = (0,0.75;\ 3000,0.25) \quad \text{and} \quad B_9 = (0,0.75;\ (0,0.2;\ 4000,0.8),0.25)$$

A_9 =



B_9 =



According to the RCLA, $B_9 \sim B_8$, hence $A_8 \succsim B_8$ iff $A_9 \succsim B_9$. Most people prefer A_9 to B_9, but B_8 to A_8 in accordance with the IA. $A_9 = ((0,1),0.75;\ A_7,0.25)$, $B_9 = ((0,1),0.75;\ B_7,0.25)$, and indeed, $A_7 \succ B_7$ and $A_9 \succ B_9$.

Other empirical studies, like Ronen (1971) and Snowball and Brown (1979), also showed violations of the RCLA.

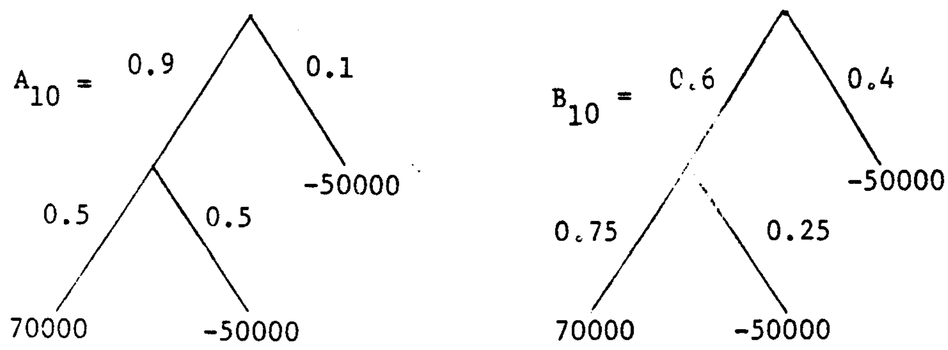
Problem 10 (Ronen (1971)): Choose between

$$A_{10} = (-50000,0.1;\ (-50000,0.5;\ 70000,0.5),0.9)$$

$$B_{10} = (-50000,0.4;\ (-50000,0.25;\ 70000,0.75),0.6)$$

By the RCLA, $A_{10} \sim B_{10} \sim (-50000,0.55;\ 70000,0.45)$. Most people prefer, however, A_{10} to B_{10}.

AU theory, as developed so far, is compatible with both the RCLA and the IA. Since p_i^j in (5.1) may equal zero, we may assume without loss of generality that the set of prizes in each of the prospects $A_1,\dots,A_m$ is the



same, denote these prizes by $x_1,\ldots,x_\ell$, and assume that $x_1 < \ldots < x_\ell$. By the RCLA and (5.1)

$$(A_1,p_1;\ldots;A_m,p_m) \sim (x_1,\sum_{j=1}^{m} p_j p_1^j;\ldots;x_\ell,\sum_{j=1}^{m} p_j p_\ell^j)$$

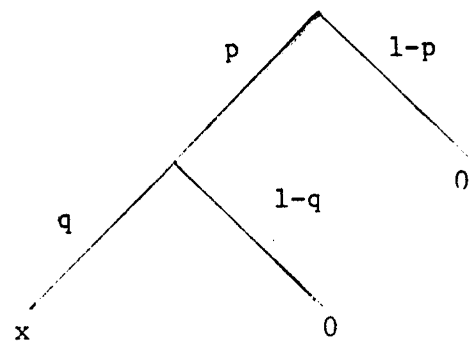
hence by (2.3)

$$(5.3) \qquad V(A_1,p_1;\ldots;A_m,p_m) = u(x_1) + [u(x_2) - u(x_1)]f(\sum_{i=2}^{\ell}\sum_{j=1}^{m} p_j p_i^j) + \ldots + [u(x_\ell) - u(x_{\ell-1})]f(\sum_{j=1}^{m} p_j p_\ell^j)$$

If the IA is employed, then by (5.2) and (2.3)

$$(5.4) \qquad V(A_1,p_1;\ldots;A_m,p_m) = V(A_1) + [V(A_2) - V(A_1)]f(p_m+\ldots+p_2) + \ldots + [V(A_m) - V(A_{m-1})]f(p_m)$$

In particular, it follows from (5.4) that if $\succsim$ satisfies the IA, then the value of the two-stage lottery $(0,1-p;\ (0,1-q;x,q),p)$ equals $u(x)f(q)f(p)$. Applying this function to problem 9 shows that $A_9 \succ B_9$ iff $u(3000)f(0.25) > u(4000)f(0.8)f(0.25)$, while $B_8 \succ A_8$ iff $u(4000)f(0.2) > u(3000)f(0.25)$. These two inequalities are consistent only if



$\sim (u^{-1}\left[u(x)f(q)f(p)\right],1)$

$$\frac{f(1)}{f(0.8)} > \frac{f(0.25)}{f(0.2)}.$$

Increasing elasticity of f implies this inequality. Ronen's results ($A_{10} \succ B_{10}$) may also be explained in the same way. $V(A_{10}) = u(-50000) + [u(70000) - u(-50000)]f(0.5)f(0.9)$ while $V(B_{10}) = u(-50000) + [u(70000) - u(-50000)]f(0.75)f(0.6)$, therefore $V(A_{10}) > V(B_{10})$ only if

$$\frac{f(0.9)}{f(0.6)} > \frac{f(0.75)}{f(0.5)}.$$

Increasing elasticity of f implies this inequality.

6. Probabilistic Insurance

Problem 11 (Kahneman and Tversky (1979, p. 269)): Suppose you consider the possibility of insuring some property against damage, e.g., fire or theft. After examining the risks and the premium you find that you have no clear preference between the options of purchasing insurance or leaving the property uninsured. [Denote the possible loss by x, its probability by p, and the insurance premium by k.]

It is then called to your attention that the insurance company offers a new program called probabilistic insurance. In this program you pay half of the regular premium. In case of damage, there is a 50 per cent chance that you pay the other half of the premium and the insurance company covers all the losses; and there is a 50 per cent chance that you get back your insurance payment and suffer all the losses.

Let $A_{11} = (-x,p;-k,1-p)$ (no insurance), $B_{11} = (-k,1)$ (full insurance), and $C_{11} = ((-x,\frac{1}{2};-k,\frac{1}{2}),p; -\frac{k}{2},1-p)$ (probabilistic insurance). Kahneman and Tversky found that given $A_{11} \sim B_{11}$ most subjects preferred not to buy the probabilistic insurance.

By EU, $A_{11} \sim B_{11} \succ C_{11}$ iff

$$pu(-x) = u(-k) > \frac{p}{2}u(-x) + \frac{p}{2}u(-k) + [1-p]u(-\frac{k}{2}) \iff$$

$$1 < \frac{1}{2} + \frac{p}{2} + [1-p]\frac{u(-\frac{k}{2})}{u(-k)} \iff$$

(6.1) $$\frac{u(-\frac{k}{2})}{u(-k)} > \frac{1}{2}.$$

A risk averse EU maximizer has a concave utility function. By (6.1), $B_{11} \succ C_{11}$ implies that u cannot be concave. In Kahneman and Tversky's words, "This is a rather puzzling consequence of the risk aversion hypothesis of utility theory, because probabilistic insurance appears intuitively riskier than regular insurance, which entirely eliminates the element of risk" (p. 270). Kahneman and Tversky suggested instead that u for losses is convex.

In this section I show that AU can explain the probabilistic insurance phenomenon even if u is concave, using RCLA or IA. For the general case define $C^*_{11} = ((-x,1-q;-k,q),p;-qk,1-p)$. Note that for $q = 0$, $C^*_{11} = A_{11}$; for $q = 1$, $C^*_{11} = B_{11}$; and for $q = \frac{1}{2}$, $C^*_{11} = C_{11}$. For the sake of simplicity I assume in this section that there exists $\beta > -\infty$ such that for every $A \in L_1$, $A^- > \beta$. In particular, $-x, -k \gtrsim \beta$.

6.1 Probabilistic Insurance and RCLA

If $\succsim$ satisfies RCLA, then by (5.1), $C_{11}^{*} \sim (-x,(1-q)p;\ -k,qp;\ -qk,1-p)$. By (2.2), $A_{11} \sim B_{11} \succ C_{11}^{*}$ iff

$$u(-x)[1 - f(1-p)] = u(-k) >$$

$$u(-x)[1 - f(1-p+qp)] + u(-k)[f(1-p+qp) - f(1-p)] + u(-qk)f(1-p) \iff$$

$$1 < \frac{1 - f(1-p+qp)}{1 - f(1-p)} + f(1-p+qp) - f(1-p) + \frac{u(-qk)}{u(-k)} f(1-p) \iff$$

(6.2) $$\frac{f(1-p+qp) - f(1-p)}{1 - f(1-p)} < \frac{u(-qk)}{u(-k)}$$

If $u(x) = x$, then (6.2) holds for every convex function f. There are therefore strictly concave u and convex f satisfying (6.2).

6.2 Probabilistic Insurance and IA

If $\succsim$ satisfies the IA, then by (5.2)

(6.3) $$C_{11}^{*} \sim (CE(-x,1-q;-k,q),p;\ -qk,1-p).$$

By (2.2), $CE(-x,1-q;-k,q) = u^{-1}(u(-x)[1 - f(q)] + u(-k)f(q))$, hence by (6.3) and (2.2), $A_{11} \sim B_{11} \succ C_{11}^{*}$ iff

$$u(-x)[1 - f(1-p)] = u(-k) >$$

$$(u(-x)[1 - f(q)] + u(-k)f(q))[1 - f(1-p)] + u(-qk)f(1-p) \iff$$

$$1 < 1 - f(q) + f(q)[1 - f(1-p)] + \frac{u(-qk)}{u(-k)} f(1-p) \iff$$

(6.4) $$f(q) < \frac{u(-qk)}{u(-k)}$$

Assume that $u(\beta) > -\infty$. Define $g(q) = \min\left\{\frac{u(-qk)}{u(-k)} : 0 < k < -\beta\right\}$. By the L'Hospital's rule,

(6.5) $$\lim_{k \to 0} \frac{u(-qk)}{u(-k)} = q.$$

It thus follows that $g(0) = 0$, $g(1) = 1$, and g is strictly increasing.

For every function u (concave or convex) there therefore exists a convex function f that satisfies (6.4).

7. Further Remarks on the IA

As proved in the last section, AU and the IA do not depend on each other, even for a preference relation on two-stage lotteries. Quiggin (1982) suggested that AU must contradict the IA, but, as proved above, this holds only if one accepts the RCLA. As mentioned above, some evidence sugests that decision makers obey the IA and not the RCLA (problems 7 - 10). In this section I discuss some possible arguments against the IA.

Machina (1983) suggested that the Allais paradox and the common ratio effect violate the IA. Let $A = C = (1000000,1)$, $B = (0,\frac{1}{11}; 5000000,\frac{10}{11})$ and $D = (0,1)$. By the IA, $(D,0.89; A,0.11) \succsim (D,0.89; B,0.11)$ iff $(C,0.89; A,0.11) \succsim (C,0.89; B,0.11)$, while most people prefer the second lottery to the first, and the third lottery to the fourth (p. 64).

This argument proved valid only if one accepts the RCLA. Without this assumption, there is no reason to assume that $(D,0.89; B,0.11) \sim (0,0.9; 5000000,0.1)$ or that $(C,0.89; B,0.11) \sim (0,0.01; 1000000,0.89; 5000000,0.1)$. Similarly, the common ratio effect violates the IA only if one assumes the RCLA. Indeed, IA and RCLA together imply EU, which is inconsistent with the Allais paradox and the common ratio effect. However, in the absence of the RCLA, these phenomena do not contradict the IA.

A stronger objection to the use of the IA with AU is that if a decision maker behaves in accordance with the IA, he should also accept Savage's Sure Thing Principle, which in turn implies EU. Savage (1954) outlined his assumption on a space of lotteries, where the winning of each prize depends on the occurrence of a certain event. The adoption of this principle to prospects (where the prizes are not necessarily ordered) is as follows:

Sure Thing Principle (STP): If $\Sigma p_i = \Sigma q_i$, $\Sigma r_i = \Sigma s_i$ and $\Sigma p_i + \Sigma r_i = 1$, then

$$(7.1) \quad (x_1,p_1;\ldots;x_n,p_n;z_1,r_1;\ldots;z_\ell,r_\ell) \succsim (y_1,q_1;\ldots;y_m,q_m;z_1,r_1;\ldots;z_\ell,r_\ell)$$
$$\Leftrightarrow \quad (x_1,p_1;\ldots;x_n,p_n;w_1,s_1;\ldots;w_k,s_k) \succsim (y_1,q_1;\ldots;y_m,q_m;w_1,s_1;\ldots;w_k,s_k)$$

As Savage proved, this assumption, together with transitivity and continuity, implies EU. Note that the STP applies to one-stage lotteries and therefore is in no way a consequence of the IA unless one assumes the RCLA. It may be argued, however, that the reasoning behind the IA and the STP are very much the same. By the IA, if A, B, and C are lotteries and $p > 0$, then

$$(7.2) \qquad (A,p;\ C,1-p) \succsim (B,p;\ C,1-p) \quad \Leftrightarrow \quad A \succsim B$$

One possible justification for the IA is that the induced order between A and B should not depend on the common outcome C. Similarly, the induced order between $(x_1,p_1;\ldots;x_n,p_n)$ and $(y_1,q_1;\ldots;y_m,q_m)$ of (7.1) should not depend on the common outcome $(z_1,r_1;\ldots;z_\ell,r_\ell)$ or $(w_1,s_1;\ldots;w_k,s_k)$.

Careful consideration of the IA and the STP proves, however, that one may accept the IA without the STP. The IA states that if A is preferred to B in an existing preference relation, then whenever A replaces B, the decision maker is better off. The natural version of this assumption for one-stage lotteries suits lotteries over a set of prizes X, where there exists a preference relation $\succsim^{\bullet}$. This relation, defined on the prizes themselves, does not depend on the existence of the preference relation $\succsim$ for lotteries. By using the reasoning behind the IA, we may assume that for every $x,y,z \in X$ and $p > 0$, $(x,p;\ z,1-p) \succsim (y,p;\ z,1-p)$ iff $x \succsim^{\bullet} y$ (see assumption (e) in

Section 8 below). This argument cannot justify the STP because there exists no a priori order between $(x_1,p_1;\ldots;x_n,p_n)$ and $(y_1,q_1;\ldots;y_m,q_m)$ of (7.1). Indeed, as AU predicts, the induced relation need not be independent of the rest of the lottery.

Finally, notice that the IA does not imply that the value of lottery A in (7.2) does not depend on lottery C. By (2.2) - (2.3) and (5.4) it follows that the "value" of A is a function of A, p, and C. What one may deduce from the IA is that if the values of A and B equal (i.e., $A \sim B$), then their "values" will equal in the presence of the alternative lottery C.

8. The Case of Non-Money Prizes

The construction of the representation function in Section 2 depends on the assumption that the set of prizes is an ordered set. When the prizes are bundles of commodities, one must explicitly outline this assumption. For the sake of simplicity I deal only with prospects. Let $L_1^* = \{(\mathbf{x}_1,p_1;\ldots;\mathbf{x}_n,p_n)\colon \mathbf{x}_1,\ldots,\mathbf{x}_n \in \mathbf{R}^k,\ p_1,\ldots,p_n > 0,\ \Sigma p_i = 1\}$ and let $\succsim$ be a complete and transitive continuous preference relation on L_1^*. Define on $\mathbf{R}^k$ a preference relation $\succsim^\bullet$ by $\mathbf{x} \succsim^\bullet \mathbf{y}$ iff $(\mathbf{x},1) \succsim (\mathbf{y},1)$, and assume from now on that if $(\mathbf{x}_1,p_1;\ldots;\mathbf{x}_n,p_n) \in L_1^*$, then $\mathbf{x}_n \succsim^\bullet \cdots \succsim^\bullet \mathbf{x}_1$. As stated already in Section 7, the natural interpretation of the IA to L_1^* is:

(e) If $\mathbf{x}_i \sim^\bullet \mathbf{x}_i'$, then $(\mathbf{x}_1,p_1;\ldots;\mathbf{x}_i,p_i;\ldots;\mathbf{x}_n,p_n) \sim$
$(\mathbf{x}_1,p_1;\ldots;\mathbf{x}_i',p_i;\ldots;\mathbf{x}_n,p_n)$.

If, as assumed, $\succsim$ is continuous, then so is $\succsim^\bullet$. Hence, $\succsim^\bullet$ can be represented by a real, order-preserving function (Debreu (1954)). For the time being, I arbitrarily choose one such function, v. Later, I will show that the representation of $\succsim$ does not depend on v. By using the function v, elements of L_1^* may be represented as lotteries over utilities of the form $(v(x_1),p_1;\ldots;v(x_n),p_n)$. On such lotteries assumptions (a) - (d) of Section 2 imply that $\succsim$ on L_1^* can be represented by

$$\bar{u}(v(x_1)) + [\bar{u}(v(x_2)) - \bar{u}(v(x_1))]f(p_n+\ldots+p_2) + \ldots + [\bar{u}(v(x_n)) - \bar{u}(v(x_{n-1}))]f(p_n)$$

Let $u = \bar{u} \circ v$ and obtain that $\succsim$ can be represented by

(8.1) $$V = u(x_1) + [u(x_2) - u(x_1)]f(p_n+\ldots+p_2) + \ldots + [u(x_n) - u(x_{n-1})]f(p_n)$$

where $f(0) = 0$, $f(1) = 1$, f is unique, and v is unique up to positive linear transformations.

Lemma 8.1: u and f do not depend on the choice of v.

9. Some Remarks On the Literature

Quiggin (1982) first presented AU. He presented a set of axioms implying that the value of the lottery $(x_1,p_1;\ldots;x_n,p_n)$ where $x_1 < \ldots < x_n$ equals

(9.1) $$\sum_i [\bar{f}(\sum_{j=1}^{i} p_j) - \bar{f}(\sum_{j=1}^{i-1} p_j)]u(x_i), \quad \bar{f}(0) = 0, \quad \bar{f}(1) = 1, \quad \bar{f}(1/2) = 1/2.$$

To obtain the functional form of (2.2), let $\bar{f}(p) = 1 - f(1-p)$. Obviously, f is convex iff $\bar{f}$ is concave. Although this function seems the same as (2.2), it is not because an essential part of Quiggin's theory is that $\bar{f}(1/2) = 1/2$. In particular, $\bar{f}$ is not concave. It follows from Theorem 4.2 that such a function cannot satisfy the GAP unless $\bar{f}(p) = p$. The results of Section 6 also suggest that $\bar{f}(\frac{1}{2}) > \frac{1}{2}$. Substitute into (6.2) $p = 1$, $q = \frac{1}{2}$, $k \to 0$, and substitute into (6.4) $q = \frac{1}{2}$, $k \to 0$. It follows from (6.5) that in both cases $f(\frac{1}{2}) < \frac{1}{2}$.

Quiggin suggested that $\bar{f}$ is concave on $[0,\frac{1}{2}]$ and convex on $[\frac{1}{2},1]$.

Problem 12: Choose between

$$A_{12} = (0,0.51;\ 1000000,0.39;\ 5000000,0.1) \quad \text{and}$$

$$B_{12} = (0,0.5;\ 1000000,0.5)$$

MacCrimmon and Larsson (1979) found that about 31% of their subjects preferred A_1 to B_1 (see Section 3), but B_{12} to A_{12}. By (9.1), $A_1 \succ B_1$ iff

(9.2) $$[\bar{f}(1) - \bar{f}(0.9)]u(5M) > [\bar{f}(1) - \bar{f}(0.89)]u(1M)$$

and $B_{12} \succ A_{12}$ iff

(9.3) $$[\bar{f}(1) - \bar{f}(0.5)]u(1M) > [\bar{f}(0.9) - \bar{f}(0.51)]u(1M) + [\bar{f}(1) - \bar{f}(0.9)]u(5M).$$

From (9.2) and (9.3) it follows that

$$\bar{f}(0.51) - \bar{f}(0.5) > \bar{f}(0.9) - \bar{f}(0.89)$$

in contradiction to the assumption that $\bar{f}$ is convex on $[\frac{1}{2},1]$. Note, however, that choosing A_1 and B_{11} is consistent with (2.2) - (2.3), provided that f is convex.

In addition to RCLA, continuity, completeness, and transitivity, Quiggin assumed:

Q1: If $(x_1,1) \succsim (x_2,1)$, then $(x_1,1) \succsim (x_2,p;\ x_1,1-p)$, and from his theorem it follows that $(x_2,p;\ x_1,1-p) \succsim (x_2,1)$.

Q2: Let $A = (x_1,p_1;\ldots;x_n,p_n)$, $B = (y_1,p_1;\ldots;y_n,p_n)$, $z_i = CE(x_i,p;\ y_i,1-p)$ (that is, $(z_i,1) \sim (x_i,p;\ y_i,1-p)$), $x = CE(A)$, and $y = CE(B)$. Let $C = (z_1,p_1;\ldots;z_n,p_n)$ and $D = (x,p;y,1-p)$. For $p = 1/2$, $C \sim D$.

Assumption (e) of Section 8 follows directly from these axioms. Let $(x_i,1) \sim (x_i',1)$, $A = (x_1,p_1;\ldots;x_i,p_i;\ldots;x_n,p_n)$, and $B = (x_1,p_1;\ldots;x_i',p_i;\ldots;x_n,p_n)$. By Q1, $(x_i,1) \sim (x_i,1/2;\ x_i',1/2)$ and obviously for every $j \neq i$, $(x_j,1) \sim (x_j,1/2;\ x_j,1/2)$. By Q2, $A \sim (CE(A),1/2;\ CE(B),1/2)$. Hence, by Q1, $CE(A) \sim CE(B)$. By the transitivity assumption, $A \sim B$.

In the same way that one proves Quiggin's theorem, one can prove that if Q2 holds for $p = p_0$, then $\bar{f}(p_0) + \bar{f}(1-p_0) = 1$ (that is why $\bar{f}(1/2)$ must equal 1/2). I believe that if one accepts Q2 for $p = 1/2$, then one should accept it for every $0 < p < 1$. Moreover, by Lemma 9.1, $\succsim$ actually satisfies Q2 for every p provided that B stochastically dominates A or vice versa. Thus, generalizing this axiom for every A and B does not require much more than already assumed. In such a case, $\succsim$ violates the Allais paradox, because $f(p) + f(1-p) = 1$ implies $f'(0.99) = f'(0.01)$ (see Section 3.1).

<u>Lemma 9.1</u>: If $\succsim$ satisfies Q2 for $p = 1/2$, then it satisfies Q2 for every p provided that B stochastically dominates A, or A stochastically dominates B.

Schmeidler (1984) suggests a similar axiom. An act maps states of the world to outcomes. Two acts ϕ and ψ are comonotonic if it never happens that $\phi(s) \succ \phi(t)$ but $\psi(t) \succ \psi(s)$.

Comonotonic Independence (CI): For all pairwise comonotonic acts ϕ, ψ, and θ, and for all $0 < \alpha < 1$, $\phi \succ \psi$ implies $\alpha\phi + (1-\alpha)\theta \succ \alpha\psi + (1-\alpha)\theta$.

Obviously, CI is weaker than IA. Schmeidler suggests that the statement "If the space is partitioned into k symmetric events, then the probability of each event is $\frac{1}{k}$" should be accepted only for $k = 2$. This of course agrees with Quiggin's results, namely that $\bar{f}(\frac{1}{2}) = \frac{1}{2}$.

One of the tools for analyzing the behavior of individuals in uncertainty situations is the risk aversion measure. The Arrow-Pratt measures are determined by the utility function and their properties are based on the assumption that decision makers maximize expected utility. However, some behavioral patterns cannot be explained through the common definition of risk aversion, although they seem to depend on the decision maker's attitude toward risk. To solve this problem, Yaari (1984) defines risk aversion through the decision weights rather than through the utility function. Let $(0,1-p-q;\ y,q;\ x,p)$ be a lottery such that $x > y > 0$. This lottery may be broken up into two stages. First, the decision maker participates in the lottery $(0,1-p-q;\ y,p+q)$, then in the lottery $(0,1-p;\ x-y,p)$. Given this assumption, Yaari proves that the value of the lottery $(x_1,p_1;\ldots;x_n,p_n)$ when $x_1 < \ldots < x_n$ equals

(9.4) $$x_1 + [x_2 - x_1]f(p_2+\ldots+p_n) + \ldots + [x_n - x_{n-1}]f(p_n).$$

Note that this function coincides with (2.3) if u is linear. From Yaari's assumptions it follows that the utility function is linear. This result enables us to define a risk aversion measure through the decision weights, since a linear utility function is risk neutral.

Yaari demonstrates that a decision maker is risk averse if f is convex. The results of this paper agree with this definition. The GAP, which implies the convexity of f (Theorem 4.2), seems acceptable because with the shift from A to C the probability of the risky prize (0) is reduced relatively more than with the shift from B to D. Probabilistic insurance, which appears riskier than normal insurance, is rejected if f is convex. The convexity of f also implies that $A \succsim B$ whenever B differs from A by a mean preserving increase in risk.

An interesting use of AU was suggested by Karni and Safra (1984). They proved that AU, together with RCLA can explain the so-called preference reversal phenomenon. Since they employ an example as proof, no specific properties of f can be deduced from their paper.

Appendix

Proof of Lemma 2.1

Let A_1^o, $A_2^o \in L_1^o$, such that $A_i^o \oplus \delta_j$ are defined, $i,j = 1,2$. Let $A^o = A_1^o \cap A_2^o$. Obviously, $A^o \in L_1^o$ and $A^o \oplus \delta_j$ $j = 1,2$ are defined. $C\ell(A_1^o \setminus A^o)$, $C\ell(A_2^o \setminus A^o) \in \Delta$, hence by the cancellation assumption $A_1^o \oplus \delta_1 \succsim A_1^o \oplus \delta_2$ iff $A^o \oplus \delta_1 \succsim A^o \oplus \delta_2$ iff $A_2^o \oplus \delta_1 \succsim A_2^o \oplus \delta_2$.

Proof of Theorem 2.2

Let $u(0) = 0$, $u(1) = k$, $f(0) = 0$, $f(1) = 1$. According to the continuity assumption, there exist p, q, and x such that $[1,x] \times [q,1] \sim^* [0,1] \times [p,q]$ (Figure 4). By the same assumption, there exist x^1 and z_1^1 such that $[1,x^1] \times [q,1] \sim^* [0,z_1^1] \times [p,q]$ and $[x^1,x] \times [q,1] \sim^* [0,z_1^1] \times [p,q]$. Since $A^o \oplus [0,1] \times [p,q] \sim A^o \oplus [1,x] \times [q,1] \sim A^o \oplus [1,x] \times [q,1] \oplus [0,z_1^1] \times [p,q]$, we obtain that $[1,x^1] \times [q,1] \sim^* [x^1,x] \times [q,1] \sim^* [z_1^1,1] \times [p,q]$. Let $u(z_1^1) = k/2$.

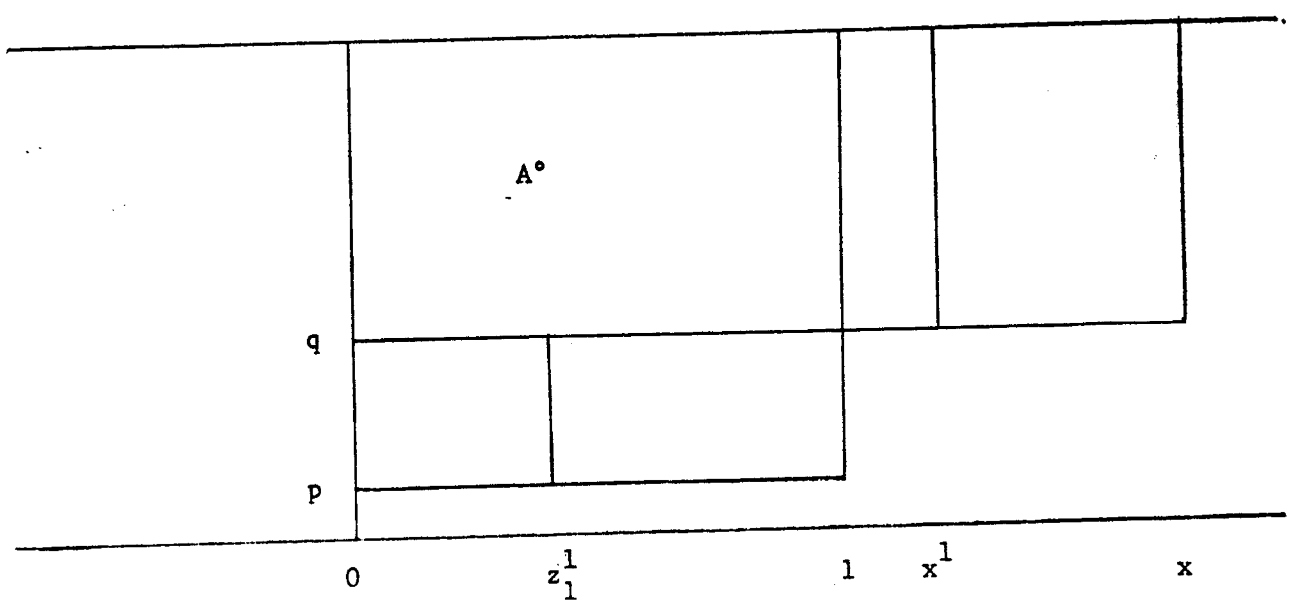


Figure 4

Similarly, there exist $x^2,\ldots,x^m,\ldots;$ $z_1^2,z_2^2,z_3^2,\ldots,z_1^m,\ldots,z_{2^m-1}^m,\ldots$ such that $[1,x^m] \times [q,1] \sim^* [x^m,x^{m-1}] \times [q,1] \sim^* [0,z_1^m] \times [p,q] \sim^* \ldots \sim^*$ $[z_{2^m-1}^m,1] \times [p,q]$. Obviously, $z_{2i}^{2m} = z_i^m$. Let $u(z_i^m) = ik/2^m$ and $u(z) =$ $\sup \{u(z_i^m): z_i^m < z\}$. By the continuity assumption, $\{z_i^m\}$ is dense in $[0,1]$, hence u is continuous.

Using the same methods, u can now be defined on every segment $[a,b]$ where $a < 0$ and $b > 1$. Moreover, if $u(0) = 0$ and $u(1) = k$, then this can be done in only one way.

For every $0 < p < 1$ there exist $x < y < z$ such that $[y,z] \times$ $[p,1] \sim^* [x,y] \times [0,p]$. Thus one may define f_p on $[0,p]$. This can be done for every p, and in such a way that if $q > p$, then on $[0,p]$ f_q and f_p coincide. Moreover, f_p can be defined such that $\lim_{p \to 1} f_p(p) = 1$. By asumption (d), one sees that $\succsim^*$ can be represented by the product of u' and f'. Hence, $\succsim$ can be represented by (2.1).

<u>Proof of Theorem 4.1</u>

Let $0 < x < y$, $p > q$ be such that $(0,1-p;\, x,p) \sim (0,1-q;\, y,q)$. By the GCRE, it follows that for every $p < p' < 1$ and for every $0 < r < p'$, $(0,1-p';\, x,p'-r;\, y,r) \succsim (0,1 - [q(p'-r) + rp]/p;\, y,[q(p'-r) + rp]/p)$. Since $\succsim$ satisfies assumptions (a) - (d), it can be represented by (2.3). Hence

$$u(x)f(p) = u(y)f(q)$$

$$u(x)f(p') + [u(y) - u(x)]f(r) > u(y)f\left(\frac{q(p'-r) + rp}{p}\right)$$

and it follows that

$$\frac{f(p)}{f(q)} = \frac{u(y)}{u(x)} < \frac{f(p') - f(r)}{f\left(\frac{q(p'-r) + rp}{p}\right) - f(r)}$$

Because f is increasing, it is almost everywhere differentiable. If f is differentiable at p', then by the L'Hospital's Rule

$$\frac{f(p)}{f(q)} \leq \lim_{r\to p'} \frac{f(p') - f(r)}{f(\frac{q(p'-r) + rp}{p}) - f(r)} =$$

$$= \lim_{r\to p'} \frac{-f'(r)}{(\frac{p-q}{p})f'(\frac{q(p'-r) + rp}{p}) - f'(r)} = \frac{p}{q}$$

Since f is a continuous function, it follows that

(4.3) $$p > q \Rightarrow \frac{f(p)}{p} \leq \frac{f(q)}{q} .$$

Let $0 < x < y$, $p > q$ be such that $(0,1-p;\ x,p) \sim (0,1-q;\ y,q)$. By the GCRE, $(x,1) \succsim (0,p-q;\ x,1-p;\ y,q)$. By (2.3),

$$u(x)f(p) = u(y)f(q)$$

$$u(x) \geq u(x)f(1-p+q) + [u(y) - u(x)]f(q)$$

hence $1 \geq f(1-p+q) + f(p) - f(q)$. For $q = 0$ we obtain

(4.4) $$f(p) + f(1-p) \leq 1.$$

Since $f(1) = 1$, it follows from (4.3) and (4.4) that $f(1/2) = 1/2$. Let $1/2 < p < 1$. By (4.3),

$$1 - \frac{f(1/2)}{1/2} \geq \frac{f(p)}{p} \geq \frac{f(1)}{1} = 1$$

hence $f(p) = p$.

Let $0 < p < 1/2$. By (4.4) $f(p) + 1 - p \leq 1$, thus $f(p) \leq p$. By (4.3)

$$\frac{f(p)}{p} \geq \frac{f(1/2)}{1/2} = 1$$

hence $f(p) = p$.

<u>Proof of Theorem 4.2</u>

Let A, B, C, and D be as in the definition of the GAP and assume that $\succsim$ can be represented by (2.1) with a convex function f. $A \sim B$ implies that

(4.5) $u(A^-) + \int_{A^-}^{A^+} \int_{F_A(x)}^{1} u'(x)f'(1-p)dpdx = u(B^-) + \int_{B^-}^{B^+} \int_{F_B(x)}^{1} u'(x)f'(1-p)dpdx$

Obviously, $C^- > A^-$, $A^- > B^-$ and $D^- > C^-$. By (2.1), $C \succsim D$ iff

(4.6) $u(A^-) + \int_{A^-}^{C^+} \int_{F_C(x)}^{1} u'(x)f'(1-p)dpdx > u(B^-) + \int_{B^-}^{D^+} \int_{F_D(x)}^{1} u'(x)f'(1-p)dpdx$

iff (by (4.5))

$$\int_{B^-}^{x^*} \int_{F_C(x)}^{F_A(x)} u'(x)f'(1-p)dpdx > \int_{B^-}^{x^*} \int_{F_D(x)}^{F_B(x)} u'(x)f'(1-p)dpdx.$$

According to the definition of the GAP, $F_D - F_B \equiv F_C - F_A$ and on $[B^-,x^*]$, $F_D > F_C$. Since f is convex, for every r $f'(1 - F_D(x) + r) < f'(1 - F_C(x) + r)$, hence by (4.6), $C \succsim D$.

Assume now that $\succsim$ satisfies the GAP. Let $0 < x < y$ and $p > q$ such that $(0,1-p;\ x,p) \sim (0,1-q;\ y,q)$. Hence

(4.7) $$u(x)f(p) = u(y)f(q).$$

By the GAP, for every $0 < r < 1 - p$, $(0,1-p-r;\ x,p+r) \succsim (0,1-q-r;\ x,r;\ y,q)$. By (2.3), this preference holds iff $u(x)f(p+r) > u(x)f(q+r) + [u(y) - u(x)]f(q)$ iff (by (4.7)) $f(p+r) - f(p) > f(q+r) - f(q)$. Hence $f'(p) > f'(q)$ and f is convex.

<u>Proof of Lemma 8.1</u>

Assume that the relation $\succsim$ on L_1^* can be represented by (8.1) and by

(8.2) $$V^* = u^*(\mathbf{x}_1) + [u^*(\mathbf{x}_2) - u^*(\mathbf{x}_1)]f^*(p_n+\ldots+p_2) + \ldots + [u^*(\mathbf{x}_n) - u^*(\mathbf{x}_{n-1})]f^*(p_n)$$

where $f(0) = f^*(0) = 0$, $f(1) = f^*(1) = 1$, $u(\mathbf{0}) = u^*(\mathbf{0}) = 0$. Therefore, there exists a continuous function h such that $V^* = h(V)$ (see (8.1) - (8.2)). In particular, for every $\mathbf{x}$ and p

(8.3) $$u^*(\mathbf{x})f^*(p) = h(u(\mathbf{x})f(p))$$

Substitute $p = 1$ and obtain

$$u^*(\mathbf{x}) = h(u(\mathbf{x}))$$

Substitute $\mathbf{x}$ such that $u(\mathbf{x}) = 1$ and obtain

$$h(1)f^*(p) = h(f(p)).$$

Hence, by (8.3),

$$\frac{h(u(\mathbf{x}))h(f(p))}{h(1)} = h(u(\mathbf{x})f(p)).$$

The solution of this functional equation is $h(\alpha) = a\alpha^b$ (see Aczel (1966)), hence $u^*(\mathbf{x}) = a[u(\mathbf{x})]^b$ and $f^*(p) = [f(p)]^b$.

I now prove that $b = 1$. Let $\mathbf{x} \succ' \mathbf{y}$ and $X = u(\mathbf{x})$, $Y = u(\mathbf{y})$ and $P = f(p)$. There exists $\mathbf{z}$ such that $(\mathbf{z},1) \sim (\mathbf{x},p; \mathbf{y},1-p)$, hence

$$u(\mathbf{z}) = u(\mathbf{y}) + [u(\mathbf{x}) - u(\mathbf{y})]f(p) = Y + [X - Y]P.$$

By (8.2) we obtain

$$u^*(\mathbf{z}) = u^*(\mathbf{y}) + [u^*(\mathbf{x}) - u^*(\mathbf{y})]f^*(p)$$

Since $u^*(\mathbf{z}) = a[u(\mathbf{z})]^b$, it follows that

(8.4) $$[Y + (X - Y)P]^b = Y^b + (X^b - Y^b)P^b$$

Diferentiating with respect to X we obtain, for $P \neq 0$,

$$[Y + (X - Y)P]^{b-1} = (XP)^{b-1}.$$

For $b \neq 1$, this last equality implies that $Y(1-P) = 0$, in contradiction to the assumption that (8.4) holds true for every X, Y, and P. It follows that $b = 1$. This completes the proof.

Proof of Lemma 9.1

Let $A = (x_1,p_1;\ldots;x_n,p_n)$ and $B = (y_1,p_1;\ldots;y_n,p_n)$ such that for every i, $x_i < y_i$. Also, let $x^* = CE(A)$ and $y^* = CE(B)$. By Q2 and (9.1), and since $\bar{f}(1/2) = 1/2$,

$$(u^{-1}(\tfrac{1}{2}\,u(x_1) + \tfrac{1}{2}\,u(y_1)),p_1;\ldots;u^{-1}(\tfrac{1}{2}\,u(x_n) + \tfrac{1}{2}\,u(y_n)),p_n) \sim$$

$$(x^*,1/2;\ y^*,1/2) \sim$$

$$(u^{-1}(\tfrac{1}{2}\,u(x^*) + \tfrac{1}{2}\,u(y^*)),1)$$

Assume now that for every $m < \ell$ and for every $0 < k < 2^m$,

(9.5)
$$(u^{-1}(\frac{k}{2^m}\,u(x_1) + [1 - \frac{k}{2^m}]u(y_1)),p_1;\ldots;u^{-1}(\frac{k}{2^m}\,u(x_n) +$$

$$[1 - \frac{k}{2^m}]u(y_n)),p_n \sim (u^{-1}(\frac{k}{2^m}\,u(x^*) + [1 - \frac{k}{2^m}]u(y^*)),1)$$

To prove that (9.5) also holds for $\ell + 1$, let $0 < k < 2^{\ell+1}$ be an uneven number. By the induction hypothesis, (9.5) holds for $\frac{k-1}{2}$ and ℓ, and for $\frac{k+1}{2}$ and ℓ. By Q2 it follows that (9.5) also holds for k and $\ell + 1$. By the continuity assumption, (9.5) holds for every $0 < \alpha < 1$. Since by (9.1), $CE(x_i,p;\ y_i,1-p) = u^{-1}(u(x_i)\bar{f}(p) + u(y_i)[1 - \bar{f}(p)])$, Q2 holds for A, B, and every p.

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