INPUT VERSUS OUTPUT INCENTIVE SCHEMES*

by

Eric Maskin** and John Riley***

UCLA Discussion Paper #354

December 1984

---

*An earlier version of this paper was entitled "Income Versus Leisure Taxes." We thank Ephraim Sadka and two referees for very helpful comments and the National Science and Sloan Foundations for financial support.

**Harvard University.

***University of California, Los Angeles.

It is an obvious proposition that the more instruments a principal uses, the better the incentive scheme he can design for a group of agents -- at least if he can commit himself to the scheme he announces. Each instrument used, however, typically requires monitoring some aspect of agents' behavior, and monitoring is often costly. As a step towards a theory of optimal monitoring, one may ask, therefore, which subsets of instruments are the most effective by themselves, assuming that it is prohibitively expensive to monitor everything. We attempt to answer that question in this paper for a simple two good model.

We consider a population of agents with identical preferences for two goods. Agents differ according to their productive capabilities. One good can be consumed directly or used as an input in the production of the other. We suppose that the principal's objective is to raise a prescribed level of revenue at a minimum social cost. Equivalently, he seeks to maximize revenue subject to the constraint that agents' welfare is held at a prescribed level. Clearly, the principal can do better (or, at least, no worse) to make the payment scheme depend on both input and output rather than just one. But which of the two would be selected, if the principal had to choose between monitoring input and output, may not be obvious.

We shall show below that, allowing for general nonlinear incentive schemes, monitoring output is superior under plausible assumptions on preferences and technology. Nonetheless, there remain cases in which input monitoring dominates. We provide conditions that characterize when each of the instruments is better.

In Section 1 we set up the model and, in Section 2, discuss the case where both input and output can be monitored. In Section 3 we take up the cases in which only one of the two instruments is monitored. Finally, in

Section 4, we compare the performance of the two instruments.

1. The Model

Label the two goods "1" and "2". An agent of type θ, where $\theta \in [0,1]$, can produce y_2 units of good 2 with an input of $c(y_2,\theta)$ units of good 1. The cost function c is twice continuously differentiable, increasing and convex in y_2 and decreasing in θ if $y_2 > 0$. The value of the parameter θ is private information. It is known to the individual agent but to no one else. Let $F(\theta)$ be the cumulative distribution function of θ. We assume that F is differentiable. All agents share the utility function $u(x_1,x_2)$, where u is concave and twice differentiable and x_i is the agent's consumption good i net of endowment.

One interpretation of the model is that of a tax authority trying to raise revenue subject to a welfare constraint. Good 1, which cannot be traded, is leisure, whereas good 2 is a consumption good. The agents are consumers. The cost function is $c(y_2,\theta) = y_2/\theta$, that is, θ is the marginal product of a consumer of type θ.

Let $T(\theta)$ be the tax revenue, in units of the consumer good, raised from type θ. The tax authority then seeks to maximize total tax revenue

(1) $$\int T(\theta)\, dF(\theta),$$

subject to a welfare constraint. Below we consider two social welfare constraints in particular: (a) the "utilitarian" criterion

(2) $$\int u(x_1(\theta),x_2(\theta))\, dF(\theta) \geq \bar{u},$$

and (b) the maximin (or "Rawlsian") criterion,

(3) $$\min_{\theta} u(x_1(\theta),x_2(\theta)) \geq \bar{u}.$$

The case in which only output is monitored is precisely that considered by Mirrlees [1971].

A second interpretation of the model is that of a monopsonist who hires workers with different input costs, $c(y_2,\theta)$, of producing y_2 units of output (cf., Maskin and Riley [1984]). Each agent seeks to maximize consumer surplus

$$u(x_1,x_2) = x_1 - c(x_2,\theta).$$

Reinterpreting $T(\theta)$, now measured in units of commodity 1, as the profit extracted from type θ, the monopsonist seeks to maximize (1) subject to the constraint that no agent can be coerced into working. With $\bar{u}$ now interpreted as the reservation utility level, the participation constraint is simply condition (3).

Yet another interpretation of the model is that of a regulatory authority intervening to reduce inefficiency in a monopolized industry (cf., Baron and Myerson [1982]). Let $B(x_2)$ be the benefit from the production of x_2 units at a cost of $c(x_2,\theta)$, and let $R(\theta)$ be the revenue that a type θ monopolist receives from society -- all measured in units of good 1. Then society maximizes $\int(B-R)\,dF(\theta)$, whereas the monopolist maximizes $R-c$. If we take $T = B - R$ the monopolist maximizes

$$B - T - c,$$

which corresponds to a utility function,

$$u(x_1,x_2) = B(x_2) + x_1.$$

The utilitarian constraint, (2), then corresponds to an _ex ante_ expected profit guarantee, whereas the maximin constraint ensures the monopolist of making a certain minimum amount.

Rather than maintain the more abstract principal-agent terminology we henceforth speak of the problem as an issue in optimal taxation. Of course, our conclusions hold as well for the other two interpreatations, since, as we have just argued, they can be viewed as special cases of our model.

Since it will prove more convenient at some points in our argument, we next note that the cost function

$$z = c(x_2, \theta)$$

can be inverted and expressed as a "production" function

$$x_2 = q(z, \theta) \equiv c^{-1}(z, \theta)$$

By assumption c is convex in x_2 and hence q is concave in z.

In the following sections we also make use of three further restrictions.

<u>Assumption 1</u>: Good 1 is nowhere inferior.

<u>Assumption 2</u>: Under lump sum taxation an individual with a higher θ supplies a larger input.

<u>Assumption 3</u>: Under lump sum taxation an individual with a higher θ produces a larger output.

The mathematical implications of these three asumptions are summarized in the following Lemmas.

<u>Lemma 1</u>: Assumption 1 holds if and only if

(4) $$u_2 u_{12} - u_1 u_{22} > 0.$$ [1]

<u>Proof</u>: Good 1 is nowhere inferior if and only if

---

[1] Subscripts of u, q and c denote partial derivatives.

$$\frac{\partial}{\partial x_2}\left(\frac{u_1(x_1,x_2)}{u_2(x_1,x_2)}\right) = \frac{u_2u_{12}-u_1u_{22}}{u_2^2} > 0 \qquad \text{Q.E.D.}$$

Lemma 2: Assumption 2 holds if and only if

(5) $$q_{12} + q_2\,\frac{(u_1u_{22}-u_2u_{12})}{u_2^2} > 0$$

Proof: Under lump sum taxation a type θ individual chooses z^* to solve

$$\underset{z}{\text{Max}}\ u(-z,q(z,\theta)-T)$$

Since u is concave and q is a concave function of z the first order condition

(6) $$\frac{\partial}{\partial z}u = -u_1(-z,q(z,\theta)-T) + u_2(-z,q(z,\theta)-T)\ q_1(z,\theta) = 0$$

is both necessary and sufficient. Differentiating (6) by θ we obtain

$$\frac{\partial^2}{\partial z^2}u\,\frac{dz^*}{d\theta} + \frac{\partial^2}{\partial\theta\partial z}u = 0$$

Thus z^* increases with θ if and only if

$$\frac{\partial}{\partial z}u = 0 \Rightarrow \frac{\partial^2}{\partial\theta\partial z}u > 0$$

From (6)

$$\frac{\partial^2}{\partial\theta\partial z}u = -u_{12}q_2 + u_{22}q_1q_2 + u_2q_{12}.$$

But if (6) holds we can substitute for q_1 to obtain

$$\frac{\partial^2}{\partial\theta\partial z}u = u_2\left[q_{12} - \frac{u_{12}}{u_2}q_2 + \frac{u_{22}u_1}{u_2^2}q_2\right] \qquad \text{Q.E.D.}$$

By an almost identical argument we can also prove

Lemma 3: Assumption 3 holds if and only if

(7) $$u_2u_{11}c_1c_2 - u_1u_2c_{12} - u_1u_{12}c_1c_2 > 0.$$

2. Input and Output Both Monitored

When input and output can both be observed, the tax authority can deduce the θ of an agent producing x_2 units of good 2 using $c(x_2,\theta)$ units of good 1. Therefore, if a θ-agent is confined to produce according to the cost function $c(.,\theta)$, the tax authority can raise the same level of revenue as though it could observe each agent's θ directly. That is, the first-best is attainable. However, it is probably more plausible to assume that a θ-agent can incur any costs not less than $c(.,\theta)$. Equivalently, a θ-agent can produce any point in the production set

$$\{(-y_1,y_2) \mid y_2 \leq q(y_1,\theta)\}.$$

In this case the net consumption schedule $\langle x_1(\theta),x_2(\theta)\rangle$ must satisfy

(8) $$\frac{d}{d\theta} u(x_1(\theta),x_2(\theta)) \geq 0.$$[2]

Tax revenue from an θ-agent, $T(\theta)$, must satisfy an aggregate feasibility constraint. When tax revenue is measured in good 2 units, this constraint reads

(9) $$\int T(\theta)\, dF(\theta) \leq \int(q(-x_1(\theta),\theta) - x_2(\theta))\, dF(\theta).$$

When tax revenue is collected in good 1 units, it becomes

(10) $$\int T(\theta)\, dF(\theta) \leq \int(-x_1(\theta) - c(x_2(\theta),\theta)\, dF(\theta).$$

---

[2]Here we assume that the input and output schedules are piecewise differentiable functions of θ. We shall maintain this assumption throughout. We shall also often argue as though $x_1(\theta)$ and $x_2(\theta)$ were continuous. All our conclusions can be established rigorously for finite distributions of θ's.

It is easy to see that, when (1) is maximized subject to (3) and either (9) or (10), the solution satisfies

$$\frac{d}{d\theta} u(x_1(\theta),x_2(\theta)) = 0.$$

Thus constraint (8) holds automatically. We conclude that the first-best is attainable when the welfare constraint is maximin. Moreover, for a utilitarian welfare constraint, we have the following straightforward generalization of a result due to Arrow [1971], Sadka [1976] and Dasgupta-Hammond [1980].

Proposition 1: If good 1 is everywhere not inferior and $q_{12} > 0$, then when revenue (collected in either good 1 or good 2) is maximized subject to a utilitarian constraint, all individuals have the same utility.

Proof: We must show that in the solution to the maximization of (1) subject to (2), (8) and either (9) or (10), (8) is satisfied with equality everywhere.

Suppose that (8) holds strictly for some interval and that taxes are measured in good 2 units. Then, differentiating the relevant Lagrangian with respect to $x_2(\theta)$ and $x_1(\theta)$ and setting these derivatives equal to zero, we obtain

(11) $$u_2(x_1(\theta),x_2(\theta)) = \alpha$$

and

(12) $$u_1(x_1(\theta),x_2(\theta)) = \alpha q_1(-x_1(\theta),\theta),$$

where α is a constant. Differentiating (11) and (12) with respect to θ, we have

$$u_{21}x_1' + u_{22}x_2' = 0$$

and

$$(u_{11}+\alpha q_{11})x_1' + u_{12}x_2' = \alpha q_{12},$$

Solving for x_1' and x_2', we obtain

$$x_1' = k(\theta)\; u_{22}q_{11}$$

and

$$x_2' = -k(\theta)\; u_{12}q_{12}$$

where $k(\theta) > 0$ because u and $q(.,\theta)$ are concave. Now,

$$\frac{d}{d\theta}\, u(x_1(\theta),x_2(\theta)) = u_1x_1' + u_2x_2' \tag{13}$$

$$= k(\theta)(u_1u_{22} - u_2u_{12})q_{12}.$$

From our hypotheses, the right hand side of (13) is nonpositive, which contradicts our assumption that (8) holds strictly. Hence, (8) is satisfied with equality everywhere. Q.E.D.

Remark 1: We provide a formal derivation of this and a number of the following results for the case in which the tax is paid in units of good 2, so that the relevant constraint is (9). Completely analogous arguments apply to the case where (10) pertains.[3]

Remark 2: In the monopoly aplication above, good 2 is neither inferior nor normal (i.e., $u_2u_{11} - u_1u_{12} = 0$). Hence (8) is satisfied with equality regardless of the sign of q_{12}.

[3]Actually, the hypotheses can be weakened somewhat when the tax is collected in good 1 units. In that case, our results follow if

$$c_{12} = \frac{-q_{12}}{q_1^2} - c_{11}q_2 < 0.$$

3. Input Monitoring or Output Monitoring

If the authority can observe only the input, $y_1 = -x_1$, of good 1, it must set a tax schedule $T = T_1(y_1)$. Given schedule $T = T_1(y_1)$ let $y_1(\theta)$ be the utility maximizing choice of a θ-agent, and take $T(\theta) = T_1(y_1(\theta_1))$. Then, for each θ, $\hat{\theta}$

$$\langle y_1(\theta), T(\theta)\rangle \succsim_{\theta} \langle y_1(\hat{\theta}), T(\hat{\theta})\rangle. \tag{14}$$

Facing only an input tax, an agent of type θ will always operate on his cost function. Thus, for (14) to hold $\hat{\theta} = \theta$ must maximize

$$U(\hat{\theta},\theta) \equiv u(-y_1(\hat{\theta}), q(y_1(\hat{\theta}),\theta) - T(\hat{\theta})) \tag{15}$$

when taxes are paid in good 2.

The first-order condition of the maximization is

$$\frac{\partial}{\partial\hat{\theta}} U(\hat{\theta},\theta) = u_1[-y_1'(\hat{\theta})] + u_2[q_1(y_1(\hat{\theta}),\theta)y_1'(\hat{\theta}) - T'(\hat{\theta})] = 0 \text{ at } \hat{\theta} = \theta. \tag{16}$$

From (15) the rate at which utility varies under an input tax scheme is

$$\frac{d}{d\theta} U(\theta,\theta) = \frac{\partial}{\partial\hat{\theta}} U(\hat{\theta},\theta)\Big|_{\hat{\theta}=\theta} + \frac{\partial}{\partial\theta} U(\hat{\theta},\theta)\Big|_{\theta=\hat{\theta}}.$$

From (16) the first term is zero. From (15) the second term is $u_2 q_2(y_1(\theta),\theta)$. Hence, since $q_2(y_1(\theta),\theta) > 0$ if $y_1(\theta) > 0$, utility increases with θ. However, from Section 2, utility is constant in θ for an optimal tax on both input and output with either a maximin constraint or, under the hypotheses of Proposition 1, a utilitarian constraint. We there have

Proposition 2: If the optimal input tax does not induce agents to choose $y_1 = 0$ for all θ, then the authority cannot raise as high a level of revenue with an input tax alone as with both an input and output tax for a maximin welfare constraint or, under the hypotheses of Proposition 1, for a utilitarian welfare constraint.

Above we noted that any tax function $T_1(y_1)$ induces an optimal choice function $\langle y_1(\theta),T(\theta)\rangle$. We then noted that such a choice function must satisfy (14). As a prelude to the discussion in Section 4 we now ask what restrictions on a pair of functions $\langle y_1(\theta),T(\theta)\rangle$ are necessary and sufficient for these functions to represent a choice function. That is, we seek necessary and sufficient conditions for $\langle y_1(\theta),T(\theta)\rangle$ to satisfy (14).

Lemma 4: If Assumption 2 holds then, for any function $y_1(\theta)$, $y_1'(\theta) > 0$ is a necessary and sufficient condition for the existence of a tax in good 1 or good 2 units, $T(\theta)$, such that $\hat{\theta} = \theta$ maximizes either $u(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta}))$ or $u(-y_1(\hat{\theta}) - T(\hat{\theta}),q(y_1(\hat{\theta}),\theta))$ as appropriate.

Conversely if Assumption 2 fails and instead, with lump sum taxation an agent with higher θ supplies a lower input, the necessary and sufficient condition is $y_1'(\theta) < 0$.

Proof: See Appendix.

Since it will be useful below, we note that if, for all θ, $\hat{\theta} = \theta$ solves

$$\max_{\hat{\theta}} u(-u_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta})),$$

we must have

$$-u_1 y_1'(\theta) + u_2 q_1 y_1'(\theta) - u_2 T'(\theta) = 0.$$

Rearranging we obtain

$$T'(\theta) = \left[q_1(y_1(\theta),\theta) - \frac{u_1(-y_1(\theta),q(y_1(\theta),\theta) - T(\theta))}{u_2(-y_1(\theta),q(y_1(\theta),\theta) - T(\theta))}\right] y_1'(\theta). \tag{17}$$

We conclude this section by summarizing the parallel arguments which apply when only output is observable. By analogy with Proposition 2 we have

Proposition 3: If the optimal output tax does not induce agents to choose $y_2 = 0$ for all θ, then the authority cannot raise as high a level of revenue with an output tax alone as with both an input and output tax for a maximin welfare constraint or, under the hypotheses of Proposition 1, for a utilitarian welfare constraint.

Similarly, by analogy with Lemma 4, we have

Lemma 5: If Assumption 3 holds then, for any function $y_2(\theta)$, $y_2'(\theta) > 0$ is a necessary and sufficient condition for the existence of a tax (in good 1 or good 2 units), $T(\theta)$, such that $\hat{\theta} = \theta$ maximizes either $u(-c(y_2(\hat{\theta})\theta), y_2(\hat{\theta}) - T(\hat{\theta}))$ or $u(-c(y_2(\hat{\theta}),\theta) - T(\hat{\theta}), y_2(\hat{\theta}))$ as appropriate.

Conversely, if Assumption 3 fails and instead, with lump sum taxation an agent with a higher θ supplies a lower output, the necessary and sufficient condition is $y_2'(\theta) < 0$.

Finally, with only output observable, the analogue of (17) is

$$T'(\theta) = [1 - c_1(y_2(\theta),\theta) \frac{u_1(-c_1(y_2(\theta),\theta), y_2(\theta) - T(\theta))}{u_2(-c_1(y_2(\theta),\theta), y_2(\theta) - T(\theta)}] y_2'(\theta). \tag{18}$$

4. Input Versus Output Monitoring

So far we have examined input and output taxes separately. We now compare their effectiveness. We begin with preferences and cost functions satisfying Assumption 2, that is, the supply function with lump sum taxation is increasing. Of course if input, $z_1(\theta)$, increases with θ then output, $q(z_1(\theta),\theta)$, also increases. Thus Assumption 2 implies Assumption 3.

We first observe that under Assumption 2, the input function induced by any input tax schedule can be duplicated by an output tax.

Proposition 4: Under Assumption 2, if $y_1(\theta)$ is the optimal input choice for a θ-agent when faced with an input tax schedule $T_1 = T_1(y_1)$, then there exists an output tax schedule $T_2(y_2)$ that induces the same input choices $y_1(\theta)$.

Proof: Suppose that the hypotheses of the proposition are satisfied. Then from Lemma 4, $y_1'(\theta) > 0$. Define $y_2(\theta) = q(y_1(\theta),\theta)$. Then $y_2'(\theta) > 0$, and so, from Lemma 5 there exists an output tax schedule, $T_2 = T_2(y_2)$, that induces the output function $y_2(\theta)$ and, hence, the input function $y_1(\theta)$.

Q.E.D.

We next show that, under the hypotheses of Proposition 4, the marginal tax rate in an optimal input tax schedule is nonnegative when the welfare constraint is maximin. We also establish that, under the optimal inut tax schedule work incentives are blunted for (almost) all θ. These results are crucial preliminaries for the comparison of input and output taxes.

Proposition 5: Under Assumption 2, if $\langle y_1^*(\theta), T^*(\theta)\rangle$ is the revenue-maximizing input tax for a maximin welfare constraint, then $dT^*/d\theta > 0$ for all θ. Moreover,

$$(19) \qquad q_1(y_1^*(\theta),\theta) - \frac{u_1(-y_1^*(\theta), q_1(y_1^*(\theta),\theta) - T^*(\theta))}{u_2(-y_1^*(\theta), q_1(y_1^*(\theta),\theta) - T^*(\theta))} > 0 \quad \text{a.e.}$$

that is, almost all agents choose a smaller input supply than under perfect information about θ.

Proof: Suppose that $T^*(\theta) \equiv T(y_1^*(\theta)) < T^*(\theta_1)$ for $\theta \in (\theta_1, \theta_2)$, where either $T(\theta_2) = T(\theta_1)$ or $\theta_2 = 1$. Define $\langle \bar{y}_1(.), \bar{T}(.)\rangle$ such that

$$\langle \bar{y}_1(\hat{\theta}),\bar{T}(\hat{\theta})\rangle = \begin{cases} \langle y_1^*(\hat{\theta}),T^*(\hat{\theta})\rangle, & \hat{\theta} \notin [\theta_1,\theta_2] \\ \langle y_1^*(\theta_1),T^*(\theta_1)\rangle, & \hat{\theta} \in [\theta_1,\theta_2] \end{cases}$$

Let $\hat{\theta}(\theta)$ be the optimal choice of a θ-agent faced with the schedule $\langle \bar{y}_1(\hat{\theta}),\bar{T}(\hat{\theta})\rangle$. For $\theta \notin [\theta_1,\theta_2]$, it is clear that $\hat{\theta}(\theta) = \theta$. For $\theta \in [\theta_1,\theta_2]$, $u(-y_1^*(\theta_1),q(y_1^*(\theta_1),\theta) - T^*(\theta_1)) > u(-y_1^*(\hat{\theta}_1),q(y_1^*(\hat{\theta}),\theta) - T^*(\hat{\theta}))$ for $\hat{\theta} < \theta_1$ since $\bar{U}(\hat{\theta},\theta) \equiv u(-\bar{y}_1(\hat{\theta}),q(\bar{y}_1(\hat{\theta}),\theta) - \bar{T}(\hat{\theta}))$ is pseudoconcave in $\hat{\theta}$. (Here we assume that the tax is paid in units of good 2. The argument is completely analogous for a tax paid in good 1 units.) Similarly, $\bar{U}(\theta_2,\theta) > \bar{U}(\hat{\theta},\theta)$ for $\hat{\theta} > \theta_2$. Therefore, $\hat{\theta}(\theta)$ is either θ_1 or θ_2 for all $\theta \in (\theta_1,\theta_2)$, and we conclude that $\bar{T}(\hat{\theta}(\theta)) > T^*(\theta)$ for all $\theta \in (\theta_1,\theta_2)$, a contradiction of the optimality of
$\langle y_1^*(\theta),T^*(\theta)\rangle$. Hence $T^*(\theta)$ is nondecreasing.

To prove (19) we show that there can be no open interval (θ_1,θ_2) over which it is violated. Suppose first that for a tax paid in good 2 units there exists (θ_1,θ_2) such that

(20) $$q_1(y_1^*(\theta),\theta) < \frac{u_1(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T^*(\theta))}{u_2(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T^*(\theta))},$$

for $\theta \in (\theta_1,\theta_2)$. From continuity, we may suppose that either

(21) $$q_1(y_1^*(\theta_1),\theta_1) = \frac{u_1(-y_1^*(\theta_1),q(y_1^*(\theta_1),\theta_1) - T^*(\theta_1))}{u_2(-y_1^*(\theta_1),q(y_1^*(\theta_1),\theta_1) - T^*(\theta_1))}$$

or

(22) $$\theta_1 = 0.$$

Because, $T^*(\theta)$ is nondecreasing, (17) and (20) imply that $y_1^*(\theta)$ and $T^*(\theta)$ are constant on $[\theta_1,\theta_2]$. Therefore, if (21) holds, then because

$(u_2u_{12} - u_1u_{22})q_2 - q_{12}u_2^2 < 0$, the sign of (20) is reversed for all $\theta \in (\theta_1,\theta_2)$, a contradiction. Thus assume that (22) holds. Choose $\theta^* \in (\theta_1,\theta_2)$. Consider varying y_1 and adjusting T so as to keep a θ^*-agent's utility constant. That is, define $\hat{T}(y)$ so that

$$(23) \qquad u(-y_1,q(y_1,\theta^*) - \hat{T}(y_1)) = u(-y_1^*(\theta^*),q(y_1(\theta^*),\theta^*) - T^*(\theta^*)).$$

Implicitly differentiating (23) with respect to y_1, we obtain

$$(24) \qquad \frac{d\hat{T}}{dy_1} = q_1 - \frac{u_1}{u_2}.$$

Notice the right hand side of (24) is negative at $y_1 = y_1^*(\theta^*)$. Consider the tax schedule $\langle y_1^{**}(\theta),T^{**}(\theta)\rangle$ such that

$$\langle y_1^{**}(\theta),T^{**}(\theta)\rangle = \begin{cases} \langle y_1^*(\theta),T^*(\theta)\rangle, & \theta > \theta^* \\ \langle \hat{y}_1,\hat{T}\rangle, & \theta < \theta^* \end{cases},$$

where $\hat{y}_1 = y_1^*(\theta^*) - \varepsilon$ and $\hat{T} = \hat{T}(\hat{y}_1)$, for some small $\varepsilon > 0$. Because the marginal cost (in terms of a reduction in good 2 taxes) of inducing greater labor supply

$$\left.\frac{-dT}{dy}\right|_{u(-y_1,q(y_1,\theta)-T)} = \frac{u_1}{u_2} - q_1$$

is, by Assumption 2, decreasing in θ, all those types with $\theta < \theta^*$ will strictly prefer $\langle\hat{y}_1,\hat{T}(\hat{y}_1)\rangle$ to $\langle y_1^*(\theta^*),T^*(\theta^*)\rangle$, while those types with $\theta > \theta^*$ will continue to choose $\langle y_1^*(\theta),T^*(\theta)\rangle$. That is, for all θ, a θ-agent sets $\hat{\theta} = \theta$ when faced with $\langle y_1^{**}(\theta),T^{**}(\theta)\rangle$. But $\langle y_1^{**}(\theta),T^{**}(\theta)\rangle$ generates more revenue than $\langle y_1^*(\theta),T^*(\theta)\rangle$ (since $\hat{T} > T(\theta^*)$), contradicting the latter's optimality. This establishes that (20) cannot hold on any interval. Assume, finally, that

$$(25) \qquad q_1(y_1^*(\theta),\theta) = \frac{u_1(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T^*(\theta))}{u_2(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T^*(\theta))}$$

on the interval $[\theta_1, \theta_2]$. Then from (17) $T^*(\theta)$ is constant on that interval. Moreover, by Assumption 2, since $T^*(\theta)$ is constant, $y_1^*(\theta)$ is strictly increasing on $[\theta_1, \theta_2]$. Choose $\theta^* \varepsilon (\theta_1, \theta_2)$ and $\hat{T}(\theta^*)$ such that

$$(26) \qquad u(-y_1^*(\theta^*), q(y_1^*(\theta^*), \theta^*) - \hat{T}(\theta^*)) = u(-y_1^*(\theta_1), q(y_1^*(\theta_1), \theta^*) - T^*(\theta_1)).$$

Again by Assumption 2, $\hat{T}(\theta^*) > T^*(\theta^*)$. Define $\langle y_1^{**}(\theta), T^{**}(\theta)\rangle$ so that

$$(27) \qquad \langle y_1^{**}(\theta), T^{**}(\theta)\rangle = \begin{cases} \langle y_1^*(\theta), T^*(\theta)\rangle, & \theta < \theta_1 \\ \langle y_1^*(\theta_1), T^*(\theta_1)\rangle, & \theta_1 < \theta < \theta^* \\ \langle y_1^*(\theta), \hat{T}(\theta)\rangle, & \theta^* < \theta, \end{cases}$$

where $\hat{T}(\theta)$ satisfies (17) for $y_1(\theta) = y_1^*(\theta)$. Because $y_1^{**}(\theta)$ is nondecreasing, each agent chooses $\hat{\theta} = \theta$ when facing (27). Because $\hat{T}(\theta^*) > T^*(\theta^*)$, we conclude that $T^{**}(\theta) > T^*(\theta)$ for $\theta > \theta^*$, and $T^{**}(\theta) = T^*(\theta)$ for $\theta < \theta^*$. Thus $\langle y_1^{**}(\theta), T^{**}(\theta)\rangle$ generates strictly more revenue than $\langle y_1^*(\theta), T^*(\theta)\rangle$, contradicting the latter's optimality. Q.E.D.

Using Propositons 4 and 5 we can establish that, when Assumption 2 holds, the optimal output tax is superior to the optimal input tax.

<u>Proposition 6</u>: Under Assumption 2, the optimal output tax generates more revenue than the optimal input tax when the welfare constraint is maximin.

<u>Proof</u>: Suppose that $\langle y_1^*(\theta), T_1^*(\theta)\rangle$ is the optimal input tax. From Proposition 4 there exists an output tax schedule $\langle y_2^*(\theta), T_2^*(\theta)\rangle$, where

$$(28) \qquad y_2^*(\theta) = q(y_1^*(\theta), \theta).$$

Suppose, for convenience, that taxes are paid in good 2 units. Then, from (17), $T_1^*(\theta)$ satisfies

$$(29) \qquad \frac{dT_1^*(\theta)}{d\theta} = [q_1(y_1^*(\theta),\theta) - \frac{u_1(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T_1^*(\theta))}{u_2(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T_1^*(\theta))}] \frac{dy_1^*}{d\theta}(\theta).$$

From (18), $T_2^*(\theta)$ satisfies

$$(30) \qquad \frac{dT_2^*}{d\theta} = [1 - \frac{u_1(-c(y_2^*(\theta),\theta),y_2^*(\theta) - T_2^*(\theta))}{u_2(-c(y_2^*(\theta),\theta),y_2^*(\theta) - T_2^*(\theta))} c_1(y_2^*(\theta),\theta)] \frac{dy_2^*}{d\theta}(\theta),$$

where we choose the boundary condition

$$T_2^*(0) = T_1^*(0).$$

From (28), we can rewrite (30) as

$$(31) \qquad \frac{dT_2^*}{d\theta} = [q_1(y_1^*(\theta),\theta) - \frac{u_1(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T_2^*(\theta))}{u_2(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T_2^*(\theta))}] (\frac{dy_1^*}{d\theta}(\theta) + \frac{q_2(y_1^*(\theta),\theta)}{q_1(y_1^*(\theta),\theta)}).$$

Because the bracketed expression in (31) is positive almost everywhere (Proposition 5),

$$\frac{dT_2^*}{d\theta} > \frac{dT_1^*}{d\theta} \quad \text{wherever} \quad T_2^*(\theta) = T_1^*(\theta).$$

Therefore, $T_2^*(\theta) > T_1^*(\theta)$ for all $\theta > 0$. We conclude that the optimal output tax generates greater revenue than the optimal input tax. Q.E.D.

It may be helpful to give a heuristic explanation of Proposition 6. Suppose that θ assumes the values θ^a and θ^b, where $\theta^a < \theta^b$, with probability 1/2 each. Then the optimal input tax is the solution to the program

$$\max\ T_1^a + T_1^b$$

such that

$$u(-y_1^b, q(y_1^b, \theta^b) - T_1^b) \geq u(-y_1^a, q(y^a, \theta^b) - T_1^a) \tag{32}$$

$$u(-y_1^a, q(y_1^a, \theta^a) - T_1^a) \geq u(-y_1^b, q(y^b, \theta^a) - T_1^b) \tag{33}$$

and

$$u(-y_1^a, q(y_1^a, \theta^a) - T_1^a) \geq \bar{u}. \tag{34}$$

Given our assumptions on preferences and technology, incentive constraint (32), not (33), is the binding one, and the solution satisfies

$$q_1(y_1^a, \theta^a) > \frac{u_1(-y_1^a, q(y_1^a, \theta^a) - T_1^a)}{u_2(-y_1^a, q(y_1^a, \theta^a) - T_1^a)}\ . \tag{35}$$

Similarly, the optimal income tax solves

$$\max\ T_2^a + T_2^b$$

such that

$$u(-c(y_2^b, \theta^b), y_2^b - T_2^b) \geq u(-c(y_2^a, \theta^b), y_2^a - T_2^a) \tag{36}$$

and

$$u(-c(y_2^a, \theta^a), y_2^a - T_2^a) \geq \bar{u}. \tag{37}$$

The incentive constraints (32) and (36) represent the requirement that a θ^b-agent should not find it advantageous to "pretend" to be a θ^a-agent. With an input tax, a θ^b-agent who did so pretend would supply the same quantity, y_1^a, of input as a θ^a-agent. However, he would produce $q(y_1^a, \theta^b)$ rather than $q(y_1^a, \theta^a)$ units of output. We denote

$$q(y_1^a, \theta^b) - q(y_1^a, \theta^a) \tag{38}$$

as a θ^b's "advantage over θ^a." Similarly,

$$c(y_2^a,\theta^a) - c(y_2^a,\theta^b) \tag{39}$$

is θ^b's advantage over θ^a in the case of an output tax. Now in view of (35), a θ^a-agent supplies too little input and produces too little output relative to efficiency. Thus if a θ^b-agent pretends to be θ^a-agent under an output tax, he induces a further deviation from efficiency by supplying even less input than a θ^a-agent. This further deviation from efficiency means that the contribution of (39) to a θ^b-agent's utility is relatively small. By contrast, a θ^b-agent who pretends to be a θ^a-agent under an input tax enhances efficiency relative to a θ^a-agent and so (38) makes a correspondingly higher contribution to utility than (39). Thus an output tax is more effective than an input tax in deterring θ^b-agents from pretending to be θ^a-agents, since the gain to be had through dissembling is smaller. An output tax therefore enables the tax authority to extract more revenue from the θ^b-agents.

This rough argument applies as well to a utilitarian welfare constraint, which we turn to next. As an intermediate step, we first generalize and strengthen a result due to Mirrlees [1971].

Proposition 7: Under Assumption 2, for a utilitarian welfare constraint, if $\langle y_1^*(\theta),T^*(\theta)\rangle$ is the revenue-maximizing input tax paid in good 2 and good 1 is not inferior, then $dT^*(\theta)/d\theta > 0$ for all θ. Furthermore

$$q_1(y_1^*(\theta),\theta) > \frac{u_1(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T^*(\theta))}{u_2(-y_1^*(\theta),q(y_1^*(\theta),\theta) - T^*(\theta))} \quad \text{a.e.} \tag{40}$$

Moreover, if $\langle y_1^*(\theta),T^*(\theta)\rangle$ is the revenue-maximizing input tax paid in good 1 and good 2 is everywhere not inferior

(41) $$q_1(y_1^*(\theta),\theta) > \frac{u_1(-y_1^*(\theta) - T^*(\theta),q(y_1^*(\theta),\theta))}{u_2(-y_1^*(\theta) - T^*(\theta),q(y_1^*(\theta),\theta))} \qquad \text{a.e.}$$

Proof: Because the argument is easier to follow, we establish the proposition for the case where θ assumes only finitely many values $\theta^1 < \ldots < \theta^n$. We consider only taxes paid in good 2. Consider the problem of maximizing

(41) $$\sum_{i=1}^{n} T^i f^i,$$

where f^i corresponds to $F'(\theta)$ in the continuous setting, subject to

(42) $$\sum_{i=1}^{n} u(-y_1^i,q(y_1^i,\theta^i) - T^i) > \bar{u}$$

(43) $$u(-y_1^i,q(y_1^i,\theta^i) - T^i) > u(-y_1^{i-1},q(y_1^{i-1},\theta^i) - T^{i-1}), \quad i = 2,\ldots,n$$

(44) $$y_1^i > y_1^{i-1}, \quad i = 2,\ldots,n.$$

Given Assumption 2 it follows from Lemma 4 that the conditions (44) must necessarily be satisfied by an input tax. The condition (43) are the adjacent downward incentive constraints. Suppose (43) holds with equality for all i, that is

(45) $$y_1^{i-1} \sim_i y_1^i; \quad i = 1,\ldots,n$$

Since, by (44), $y_1^{i-1} < y_1^i$, (45) implies that no type with a lower marginal rate of substitution of good 1 for good 2 will strictly prefer y_1^{i-1} over y_1^i. But, by Assumption 2,

$$\left.\frac{dy_2}{dy_1}\right|_{u(-y_1,q(y_1,\theta)-T)} = \frac{u_1}{u_2} - q_1$$

is decreasing in θ.[4] Therefore

$$(46) \qquad y_1^{i-1} \,\hat{j}\, y_1^i, \quad j > i$$

Moreover, using essentially this same argument we also have

$$(47) \qquad y_1^{i-1} \,\hat{j}\, y_1^i, \quad j < i$$

Combining (45)-(47) it follows that <u>all</u> incentive constraints will be satisfied if (43) holds with equality for all i and (44) holds, in which case the solution to the program (41)-(44) is the optimal input tax schedule. To see that (43) indeed <u>does</u> hold with equality consider the function $h(y_1)$ such that for all y_1

$$(48) \qquad u(-y_1,h(y_1)) = u(-y_1^{i-1},q(y_1^{i-1},\theta^i) - T^{i-1}).$$

Implicitly differentiating (48) with respect to y_1, we obtain

$$(49) \qquad h'(y_1) = \frac{u_1(-y_1,h(y_1))}{u_2(-y_1,h(y_1))}.$$

From (49),

$$(50) \qquad \frac{d}{dy_1} u_2(-y_1,h(y_1))$$

$$= -u_{12}(-y_1,h(y_1)) + u_{22}(-y_1,h(y_1))h'(y_1)$$

$$= -u_{12} + u_{22}\frac{u_1}{u_2},$$

which is nonpositive from the noninferiority of good 1. From (44) and the nonpositivity of (50),

$$(51) \qquad u_2(-y_1^{i-1},q(y_1^{i-1},\theta^i) - T^{i-1}) > u_2(-y_1^i,h(y_1^i)).$$

---

[4]This is discrete equivalent of the pseudo-concavity argument used in the proof of Lemma 4.

But $u(-y_1^i, h(y_1^i)) < u(-y_1^i, q(y_1^i, \theta^i) - T^i)$. Hence (51) implies

$$u_2(-y_1^{i-1}, q(y_1^{i-1}, \theta^i) - T^{i-1}) > u_2(-y_1^i, q(y_1^i, \theta^i) - T^i). \tag{52}$$

Now if (43) held strictly in the solution to (41)-(44), so would (52). Thus we could increase T^i by ΔT^i and lower T^{i-1} by $-\Delta T^{i-1}$, where $\Delta T^{i-1} < \Delta T^i \frac{f^i}{f^{i-1}}$, and continue to satisfy (42). But this adjustment increases total revenue, a contradiction of the assumed optimality of the tax schedule. Therefore, (43) holds with equality after all. This also shows that the Lagrange multiplier for (43) must be positive. Now if $T^j < T^{j-1}$ for some j, define $\langle \hat{y}_1^i, \hat{T}^i \rangle_{i=1}^n$ so that

$$\langle \hat{y}_1^i, \hat{T}^i \rangle = \begin{cases} \langle y_1^i, T^i \rangle, & i \neq j \\ \langle y_1^{j-1}, T^{j-1} \rangle, & i = j \end{cases}.$$

Now $\langle \hat{y}_1^i, \hat{T}^i \rangle_{i=1}^n$ satisfies all the downward adjacent incentive constraints. Furthermore

$$\Sigma\, \hat{T}^i f^i > \Sigma\, T^i f^i$$

Therefore, $\langle y_1^i, T^i \rangle_{i=1}^n$ is not optimal, a contradiction. We conclude that y_1^i is nondecreasing in i after all.

To see that (40) holds for all i, suppose to the contrary that there exists j for which

$$q_1(y_1^j, \theta^j) < \frac{u_1(y_1^j, q(y_1^j, \theta^j) - T^j)}{u_2(-y_1^j, q(y_1^j, \theta^j) - T^j)}. \tag{53}$$

Let j be the smallest such superscript. Consider $\hat{T}(y_1)$ as defined by (23) with θ^* replaced by θ^j. By hypothesis, (24), and (53) we can find $\bar{y}_1 < y_1^j$, and $\bar{T} > T^j$ such that

$$u(-\bar{y}_1, q(\bar{y}_1, \theta^j) - \bar{T}) = u(-y_1^j, q(y_1^j, \theta^j) - T^j) \tag{54}$$

and

$$q_1(\bar{y}_1, \theta^j) = \frac{u_1(-\bar{y}_1, q(\bar{y}_1, \theta^j) - \bar{T})}{u_2(-\bar{y}_1, q(\bar{y}_1, \theta^j) - T)}. \tag{55}$$

If we replace (y_1^j, T^j) by $(\bar{y}_1, \bar{T})$ in the incentive scheme, we generate more revenue, leave the utilitarian welfare constraint unaffected, and satisfy all the incentive constraints. To complete the contradiction, the only remaining point to check is that $\bar{y} > y_1^{j-1}$. But (54) implies that

$$u(-\bar{y}_1, q(\bar{y}_1, \theta^j) - \bar{T}) = u(-y_1^{j-1}, q(y_1^{j-1}, \theta^j) - T^{j-1}),$$

and so if $y_1^{j-1} > \bar{y}$, (55) implies that

$$q_1(y_1^{j-1}, \theta^j) < \frac{u_1(-y_1^{j-1}, q(y_1^{j-1}, \theta^j) - T^{j-1})}{u_2(-y_1^{j-1}, q(y_1^{j-1}, \theta^j) - T^{j-1})},$$

which, because the noninterventionist input function is increasing, contradicts the fact that

$$q_1(y_1^{j-1}, \theta^{j-1}) > \frac{u_1(-y_1^{j-1}, q(y_1^{j-1}, \theta^{j-1}) - T^{j-1})}{u_2(-y_1^{j-1}, q(y_1^{j-1}, \theta^{j-1}) - T^{j-1})}.$$

We conclude that

$$q_1(y_1^i, \theta^i) > \frac{u_1(-y_1^i, q(y_1^i, \theta^i) - T^i)}{u_2(-y_1^i, q(y_1^i, \theta^i) - T^i)} \tag{56}$$

for all i, as required.

It remains only to show that the inequality (56) is strict for $1 < i < n$. Consider the Lagrangian for the program (41)-(44). The first order condition with respect to y_1^i is

$$(\alpha+\beta_i)\,(-u_1(-y_1^i,q(y_1^i,\theta^i) - T^i) + q_1(y_1^i,\theta^i)u_2(-y_1^i,q(y_1^i,\theta^i) - T^i))$$

$$-\beta_{i+1}(-u_1(-y_1^i,q(y_1^i,\theta^{i+1}) - T^i) + q_1(y_1^i,\theta^i)u_2(-y_1^i,q(y_1^i,\theta^{i+1}) - T^i)) \tag{57}$$

$$+ \gamma_i - \gamma_{i+1} = 0,$$

where α, β_i, and γ_i are the Lagrange multipliers for (42), (43), and (44), respectively, and all multipliers are nonnegative. Now $\alpha > 0$ because if $\bar{u}$ in (42) were lowered, the authority could raise more revenue by increasing T^1. We demonstrated above that $\beta_i > 0$ for all i. Suppose that for some j with $1 < j < n$

$$q_1(y_1^j,\theta^j) = \frac{u_1(-y_1^j,q(y_1^j,\theta^j) - T^j)}{u_2(-y_1^j,q(y_1^j,\theta^j) - T^j)}. \tag{58}$$

Let j be the greatest such index. In this case (57) reduces to

$$-\beta_{j+1}(-u_1(-y_1^j,q(y_1^j,\theta^{j+1}) - T^j) + q_1(y_1^j,\theta^{j+1})u_2(-y_1^j,q(y_1^j,\theta^{j+1}) - T^j)) \tag{59}$$

$$+ \gamma_j - \gamma_{j+1} = 0.$$

The first term on the left hand side of (59) is negative because the noninterventionist input function is increasing. Therefore, γ_j must be positive. This implies that $y_1^j = y_1^{j-1}$. Thus, since (43) holds with equality for all i, $T^j = T^{j-1}$. But then from (58) and the fact that the input function is increasing, we conclude that

$$q_1(y_1^{j-1},\theta^{j-1}) < \frac{u_1(-y_1^{j-1},q(y_1^{j-1},\theta^{j-1}) - T^{j-1})}{u_2(-y_1^{j-1},q(y_1^{j-1},\theta^{j-1}) - T^{j-1})},$$

a contraction of (56). Therefore (40) must hold strictly for all θ^i with $1 < i < n$. Q.E.D.

Modifying the proof of Proposition 6 only slightly and making use of Proposition 7, we now establish

Proposition 8: If Assumption 2 holds and either good 1 (for a tax paid in good 2) or good 2 (for a tax paid in good 1) is noninferior, the optimal output tax generates more revenue than the optimal input tax for a utilitarian welfare constraint.

Proof: We argue for the case of taxes paid in good 2. Let $\langle y_1^*(\theta), T_1^*(\theta)\rangle$ be the optimal input tax. Under our hypotheses there exists an output tax $\langle y_2^*(\theta), T_2^*(\theta)\rangle$ where $y_2^*(\theta) = q(y_1^*(\theta),\theta)$ for all θ. From (31) and because (40) holds strictly for (almost) all $\theta \in (0,1)$,

$$T_2^*(\theta) = T_1^*(\theta) \quad \text{at} \quad \theta \quad \text{implies} \quad \frac{dT_2^*(\theta)}{d\theta} > \frac{dT_1^*(\theta)}{d\theta}. \tag{60}$$

It follows that if T_2^* intersects T_1^*, it does so from below. Define

$$v^1(\theta) = u(-y_1^*(\theta), q_1(y_1^*(\theta),\theta) - T_1^*(\theta))$$

and

$$v^2(\theta) = u(-c(y_2^*(\theta),\theta), y_2^*(\theta) - T_2^*(\theta)).$$

From (60) it follows that if T_1^* intersects T_2^* at θ^* then

$$v^1(\theta) > v^2(\theta) \quad \text{if and only if} \quad \theta > \theta^*. \tag{61}$$

Define $v_2^2(\theta) = u_2(-c(y_2^*(\theta),\theta), y_2^*(\theta) - T_2^*(\theta))$ and $v_2^1(\theta) = u_2(-y_1^*(\theta), q(y_1^*(\theta),\theta) - T_1^*(\theta))$. Then, from the concavity of u, it follows that for all θ

$$v^1(\theta) - v^2(\theta) < v_2^2(\theta)(T_2^*(\theta) - T_1^*(\theta)), \tag{62}$$

with strict inequality for $\theta \neq \theta^*$. Because good 1 is noninferior and q_{12}

> 0, $v_2^1(\theta)$ and $v_2^2(\theta)$ are nonincreasing. Hence

(63) $$v_2^2(\theta) < v_2^2(\theta^*) \text{ if and only if } \theta > \theta^*.$$

Then, combining (61)-(63), we have

$$v^1(\theta) - v^2(\theta) < v_2^2(\theta^*)(T_2^*(\theta) - T_1^*(\theta))$$

for all $\theta \neq \theta^*$. Hence

$$\int_0^1 v^1(\theta)dF(\theta) - \int_0^1 v^2(\theta)dF(\theta) < v_2^2(\theta^*) \int_0^1 (T_2^*(\theta) - T_1^*(\theta))dF(\theta).$$

Therefore, if the boundary value $T_2^*(0)$ is chosen to equate $\int_0^1 v^1(\theta)dF(\theta)$ and $\int_0^1 v^2(\theta)dF(\theta)$ -- the levels of social welfare under the input and output taxes respectively -- we conclude that $\langle y_2^*(\theta),T_2^*(\theta)\rangle$ generates higher revenue than $\langle y_1^*(\theta),T_1^*(\theta)\rangle$. Q.E.D.

Propositions 6 and 8 compare input and output taxes for maximin and utilitarian welfare constraints, respectively. In some circumstances, however, both constraints arise naturally. For instance, recall the monopsonist-workers example of Section 1. Suppose that a worker signs a contract with the monopsonist before he learns his input costs, which are therefore regarded as a random variable. Presumably the contract assures the worker of some expected utility level. Such a constraint corresponds to the utilitarian constraint. Imagine, however, that the worker cannot be bound to monopsonist, so that in no state can his utility be set at less than the reservation level $\bar{u}$. This corresponds to a maximin constraint (see Foster and Wan [1983] for a greater elaboration of this model). It is quite possible for both constraints to bind simultaneously. In that case, the arguments of Propositions 6 and 8 easily extend to give us the following.

<u>Proposition 9</u>: Under the hypotheses of Proposition 8, the optimal output tax generates more revenue than the optimal input tax for the case where both the maximin and utilitarian constraints pertain.

By reversing our hypotheses, we can obtain results that are diametrically opposied to Propositions 6 and 8.

<u>Proposition 10</u>: If Assumption 2 fails then the optimal input tax generates more revenue than the optimal output tax when the welfare constraint is maximin. If, furthermore, good 1 (good 2) is not normal, then the same result holds for a utilitarian welfare constraint when the tax is paid in good 2 (good 1).

References

Arrow, K.J. [1971], "A Utilitarian Approach to the Concept of Equity in Public Expenditure," Quarterly Journal of Economics, 85, 408-15.

Baron, D. and R. Myerson [1982], "Regulating a Monopolist With Unknown Costs," Econometrica, 50, 911-30.

Dasgupta, P. and P. Hammond [1980], "Fully Progressive Taxation," Journal of Public Economics, 13, 141-54.

Foster, J. and H. Wan [1983], "'Involuntary' Unemployment as a Principal-Agent Equilibrium," mimeo.

Maskin, E. and J. Riley [1984], "Monopoly With Incomplete Information," Rand Journal of Economics, 15, 171-96.

Mirrlees, J. [1971], "An Exploration in the theory of Optimum Income Taxation," Review of Economic Studies, 38, 175-208.

Sadka, E. [1976], "Social Welfare and Incomd Distribution," Econometrica, 44, 1238-52.

Appendix

Lemma 4 If Assumption 2 holds then, for any function $y_1(\theta)$, $y_1'(\theta) > 0$ is a necessary and sufficient condition for the existence of a tax in good 1 or good 2 units, $T(\theta)$, such that $\hat{\theta} = \theta$ maximizes either $u(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta}))$ or $u(-y_1(\hat{\theta}) - T(\hat{\theta}),q(y_1(\hat{\theta}),\theta))$ as appropriate.

Conversely if Assumption 2 fails and instead, with lump sum taxation an agent with higher θ supplies a lower input, the necessary and sufficient condition is $y_1'(\theta) < 0$.

Proof: Condition (16) is the first order condition for an agent's maximum. Solving for $T'(\theta)$ from (16), we obtain

$$\text{(a-1)} \qquad T'(\theta) = [q_1(y_1(\theta),\theta) - \frac{u_1(-y_1(\theta),q(y_1(\theta),\theta) - T(\theta))}{u_2(-y_1(\theta),q(y_1(\theta),\theta) - T(\theta))}]\, y_1'(\theta).$$

Then define $T(\theta)$ to be the solution to (a-1), with $T(0) = T_0$.

We first show that $y'(\theta) > 0$ is a necessary condition for $U(\hat{\theta},\theta) = u(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta}))$ to take in its maximum at $\hat{\theta} = \theta$. We then show that if $y'(\theta) > 0$, U is a pseudo-concave function of $\hat{\theta}$ so that the first order condition (16), yields the global maximum.

Making use of (a-1) we can rewrite $\frac{\partial}{\partial\hat{\theta}} U(\hat{\theta},\theta)$ as

$$\text{(a-2)} \quad \frac{\partial}{\partial\hat{\theta}} U(\hat{\theta},\theta) =$$

$$= u_2(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta}))[q_1(y_1(\hat{\theta}),\theta) - q_1(y_1(\hat{\theta}),\hat{\theta})$$

$$+ \frac{u_1(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\hat{\theta}) - T(\hat{\theta}))}{u_2(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\hat{\theta}) - T(\hat{\theta}))} - \frac{u_1(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta}))}{u_2(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta})}]\, y_1'(\hat{\theta})$$

$$= u_2(\hat{\theta},\theta)\ [q_1(\hat{\theta},\theta) - q_1(\hat{\theta},\hat{\theta}) + \frac{u_1(\hat{\theta},\hat{\theta})}{u_2(\hat{\theta},\hat{\theta})} - \frac{u_1(\hat{\theta},\theta)}{u_2(\hat{\theta},\theta)}]\ y_1'(\hat{\theta}),$$

where $u_i(\hat{\theta},\theta) \equiv u_i(-y_1(\hat{\theta}),q(y_1(\hat{\theta}),\theta) - T(\hat{\theta}))$ and $q_1(\hat{\theta},\theta) \equiv q_1(y_1(\hat{\theta}),\theta)$.

Consider the cross partial derivative $\frac{\partial^2 U}{\partial\hat{\theta}\partial\theta}$ where at a point where (a-2) vanishes. We have

$$\text{(a-3)} \qquad \frac{\partial^2}{\partial\hat{\theta}\partial\theta} U(\hat{\theta},\theta) = u_2(\hat{\theta},\theta)[q_{12}(\hat{\theta},\theta) - \frac{u_2(\hat{\theta},\theta)u_{12}(\hat{\theta},\theta)q_2(\hat{\theta},\theta)}{(u_2(\hat{\theta},\theta))^2} + \frac{u_1(\hat{\theta},\theta)u_{22}(\hat{\theta},\theta)q_2(\hat{\theta},\theta)}{(u_2(\hat{\theta},\theta))^2}]\, y_1'(\theta).$$

By Assumption 2 and Lemma 2 the bracketed expression is positive. Therefore the sign of $\frac{\partial^2}{\partial\hat{\theta}\partial\theta} U$ is the same as that of $y_1'(\hat{\theta})$.

Now, for $\hat{\theta} = \theta$ to maximize $U(\hat{\theta},\theta)$, it is neessary that $\frac{\partial U}{\partial\hat{\theta}}$ should be nonnegative for $\hat{\theta}$ in a left neighborhood of θ and nonpositive in a right neighborhood of θ. But, for each $\hat{\theta}$, $\frac{\partial U}{\partial\hat{\theta}} U(\hat{\theta},\theta)$ is zero at $\theta = \hat{\theta}$. Hence, it is necessary that

$$\text{(a-4)} \qquad \frac{\partial^2}{\partial\hat{\theta}\partial\theta} U(\hat{\theta},\theta) > 0, \quad \text{at} \quad \hat{\theta} = \theta.$$

But from the above argument, (a-4) implies that

$$\text{(a-5)} \qquad y_1'(\theta) > 0.$$

In turn, (a-5) implies that $\frac{\partial^2}{\partial\hat{\theta}\partial\theta} U(\hat{\theta},\theta)$ is nonnegative at <u>any</u> critical point (any point where $\frac{\partial}{\partial\hat{\theta}} U(\hat{\theta},\theta)$ vanishes). Therefore, (a-5) implies that $U(\hat{\theta},\theta)$ is a pseudoconcave function of $\hat{\theta}$. If (a-5) is satisfied, $\hat{\theta} = \theta$ maximizes $U(\hat{\theta},\theta)$.

Similarly, if Assumption 2 fails, that is, with lump sum taxation a higher θ implies a lower input supply, $y_1'(\theta) < 0$ is a necessary and sufficient condition. Q.E.D.