EX POST INFORMATION IN AUCTIONS

by

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ABSTRACT

EX POST INFORMATION IN AUCTIONS

When buyers' private information about the value of an item for sale is correlated, the seller can increase expected revenue in a sealed bid auction by making the winner's payment a function of information available after the end of the auction. Specifically, revenue can be increased by making the payment a function of <u>all</u> the losing bids. In addition there are gains to making the payment contingent upon some signal of the object's value which becomes public at a later date. That is, there are gains to introducing positive royalty rates.

The foundation stone of the recent literature on auction choice is the revenue equivalence theorem. This states that, under the assumptions of risk neutrality, independence of valuations and symmetry of the underlying distribution of valuations, (i) sealed high bid and open ascending bid auctions generate the same expected revenue, and (ii) under a mild additional restriction on the underlying distribution of valuations, these auctions maximize expected seller revenue.[1]

Much of the ensuing literature has focused on relaxing one or more of these assumptions. Maskin and Riley (1984a) show that when the amount a buyer is willing to pay for the auctioned object is not independent of his wealth, expected revenue tends to be lower under open bidding. In contrast, Milgrom and Weber (1982) relax the assumption that valuations are independent. Maintaining the other assumptions they then show that expected revenue is higher in the open ascending bid auction. The intuition behind this result is that, under open bidding, more information is revealed in the auction and this helps to reduce the difference in bidders' willingness to pay. This, in turn results in buyers bidding away more of the surplus. Milgrom and Weber also show that if the seller purchases and then publicly reveals information that affects bidders' valuations, this also raises expected (gross) revenue from an auction. Once again the intuition is that this helps to reduce the difference in bidders' willingness to pay.

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[1]Part (i) was first derived for the special uniform case by Vickrey (1961). The optimality of the common auction forms was analyzed independently by Myerson (1981) and Riley and Samuelson (1981). As Maskin and Riley (1984b) have since shown, even when the additional restriction is not satisfied, the common auctions are optimal if modified to exclude bidding over certain intervals.

That such a reduction should raise expected revenue is easily understood for the limiting case in which the seller's information includes all the buyers' private information. For then all buyer asymmetry is eliminated and so expected buyer profit is bid away to zero. The seller therefore reaps the entire surplus.

In the following pages the implications of correlated signals are further explored. While Milgrom and Weber emphasized the potential gains to a seller who reveals information **ex ante**, the focus here is on **ex post** information. In Section 2 it is shown that, under weak assumptions, the sealed high bid auction is dominated by any auction in which the winners payment is a weighted average of all bids. Surprisingly it appears that for auctions in which payment is made by the winner, only a weak statement is possible about the optimality of placing no weight on the high bid.

In Section 3 the focus switches to auctions in which public information about the value of the object becomes available after the auction. It is shown that if this **ex post** information is anticipated by the seller, expected seller revenue can always be increased by making the final buyer payment contingent upon the nature of the information. As a practical application, suppose that the seller of an oilfield observes the quantity of oil extracted and that this quantity is a noisy signal of oilfield profitability. Then expected revenue is raised by announcing that the winning bidder must pay a royalty on each unit. Alternatively, expected revenue can be increased by having buyers bid on the royalty <u>rate</u> that they are willing to pay rather than a fixed fee.

Again the intuition behind these results is that <u>differences</u> in buyers' valuations are reduced through the introduction of a royalty and so the expected surplus of the winning bidder is reduced.

1. The Model

To simplify the exposition we begin by considering a two buyer auction. Buyer 1 observes a signal s and buyer 2 a signal t, each of which yields private information about the value of the item for sale. These signals are realizations from an underlying distribution with joint density function $f(s,t)$.

Assumption 1. Symmetry of Signals:

The joint density function $f(s,t)$ is symmetric and strictly positive if and only if s and t belong to the unit interval.

Assumption 2. Linked Signals:[2]

For any permutation (x,y) of (s,t) and $x < x'$

$$\text{Prob}\{\tilde{x} < x \mid \tilde{x} < x', y\} = F(x|y)/F(x'|y)$$

is a nonincreasing function of y.

If the probability is strictly increasing for all x, x', y on the unit interval we shall say that beliefs are strictly linked.

As an immediate implication of Assumption 2, if s and t are linked $s < s'$ implies that $\dfrac{F(s'|t) - F(s|t)}{F(s'|t)}$ is nondecreasing in t. Taking the limit as $s \to s'$ we obtain:

Lemma 1: If s and t are linked $f(s|t)/F(s|t)$ is a nondecreasing function of t.

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[2] In the case of an n-dimensional vector of signals $(s_1,s_2,\ldots,x_n)$, we shall say that signals are linked if for all i and $s_i' < s_i$

$$\text{Prob}\{\tilde{s}_i < s_i' \mid \tilde{s}_i < s_i, s_{-i}\}$$

is a nonincreasing function of $s_{-i} \equiv (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$.

Assumption 2 says that for any $t' > t$, the cumulative distribution function for s, given that s is no greater than s', $F(s|s < s',t')$ exhibits first order stochastic dominance over $F(s|s < s',t)$. Then if utility is increasing in s, conditional expected utility is nondecreasing in t. To summarize:

Lemma 2: If s and t are linked and $u(s)$ is a nondecreasing function

$$\underset{s}{E}\left\{u(s) \mid s < s',t\right\} \text{ is nondecreasing in } t.$$

Milgrom and Weber make frequent use of this result and we shall do so here as well. Actually Milgrom and Weber start with the assumption that signals are affiliated. In the two signal case this is the requirement that, for any $s < s'$ and $t < t'$, the joint density function satisfies

$$f(s,t)\ f(s',t') \geq f(s,t')\ f(s',t).$$

It is readily shown that affiliatedness is a strongly sufficient condition for signals to be linked.[3]

Finally, we assume that buyers are risk neutral and the value to buyer 1 of the item for sale $V(s,t)$ is a strictly increasing function of buyer 1's private signal and a nondecreasing function of buyer 2's private signal. Maintaining the symmetry of the model, the value to buyer 2 is assumed to be $V(t,s)$. These assumptions can be summarized as follows.

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[3]Except for the special case in which the signals can take on only two values the affiliatedness assumption is stronger. Consider for example the matrix

$$[f_{ij}] \equiv \begin{bmatrix} .09 & .02 & .06 \\ .09 & .12 & .12 \\ .12 & .16 & .12 \end{bmatrix}$$

If $f(s_i,t_j) = f_{ij}$ signals are linked but $f_{21}f_{22} < f_{23}f_{13}$ so signals are not affiliated.

Assumption 3: Symmetry of Valuations:

If s and t are the private signals of buyers 1 and 2, their valuations of the object for sale are, respectively $V(s,t)$ and $V(t,s)$ where V is strictly increasing in its first argument and is nondecreasing in its second argument.

2. Utilizing all the Information in a Sealed Bid Auction

Consider now a sealed bid auction in which each buyer is asked to submit an "estimate" m of the object's value. The buyer submitting the high bid will be awarded the object and will pay some weighted average $A(m_{(1)},m_{(2)})$ of the two bids, where $m_{(1)}$ is the high and $m_{(2)}$ is the second bid.

Assumption 4. Form of the Weighting Function:

The weighting function $A(m_{(1)},m_{(2)})$ is continuously differentiable, strictly increasing in $m_{(1)}$, and nondecreasing in $m_{(2)}$. Moreover,

$$A(m,m) = m. \tag{1}$$

The seller also announces that he will only accept estimates above some minimum m_0 and that, in the absence of a second bid, replace $m_{(2)}$ by m_0 in the weighting function. Given Assumption 4, the minimum price paid or "reserve price" is therefore m_0.

Let s_0 be the lowest private signal for which, in equilibrium, a buyer is willing to submit a bid. Let $M(s)$ be the symmetric equilibrium bid function for all $s > s_0$ and let $M(s) = m_0$ for all $s < s_0$. Then if buyer 2 bids according to the equilibrium bid function and buyer 1 bids $m_1 = M(x)$, and if, as we shall later confirm, $M(.)$ is strictly increasing, buyer 1 wins if and only if $t < x$. Buyer 1's expected return is therefore

$$\Pi(x|s) = \int_0^x [V(s,t) - A(M(x),M(t))]f(t|s)dt \tag{2}$$

Differentiating with respect to x and appealing to (1) we obtain

$$\frac{\partial \Pi}{\partial x}(x|s) = [V(s,x) - M(x)]f(x|s) - M'(x)\int_0^x A_1(M(x),M(t))f(t|s)dt \tag{3}$$

But, for $M(\cdot)$ to be the equilibrium bid function, buyer 1's best reply must be to bid $m_1 = M(s)$. That is $\Pi(x|s)$ must take on its maximum at $x = s$. Therefore, from (3)

$$[V(s,s) - M(s)]f(s|s) - M'(s)\int_0^s A_1(M(s),M(t))f(t|s)dt = 0. \tag{4}$$

Given Assumption 1, $\Pi(s|t)$ is a continuous function. Therefore, for s_o to be signal level below which it is not worthwhile bidding,

$$\Pi(s_o|s_o) = E\{V(s_o,t) \mid t<s_o\} - m_o = 0. \tag{5}$$

Conditions (4) and (5) are necessary for $M(\cdot)$ to be the equilibrium bid function. The following theorem establishes conditions under which they are also sufficient.

<u>Proposition 1</u>: Existence of an Equilibrium Bid Function:
If Assumptions 1-4 hold there exists a solution $M(\cdot)$ to (4) and (5). If, in addition, $A_{12}(m_{(1)},m_{(2)})$ is nonpositive, $M(\cdot)$ is an equilibrium bid function.[4]

In deriving this result, and some of the results to follow we appeal to the following simple Lemma. (For a proof see Riley and Samuelson (1981).)

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[4]This additional restriction is strongly sufficient. Clearly it holds if A is a convex combination of the two bids.

Lemma 3: Suppose that $g(s)$ and $h(s)$ are continuous and differentiable on $[0,1]$ and that

$$g(s) > h(s) \Rightarrow g'(s) < h'(s)$$

Then

$$g(x^*) \leq h(x^*) \Rightarrow g(x) < h(x) \quad \text{for all} \quad x \in (x^*,1]$$

Proof of Proposition 1: Let Ω be the family of differentiable, nondecreasing functions bounded above in the sup norm by $V(1,1)$ on the interval $[s_o,1]$, where s_o is defined implicitly by (5). By Assumption 4, $A_1(m_{(1)},m_{(2)}) > 0$ for all $m_{(1)}, m_{(2)} \in [0,V(1,1)]$. Therefore for any $H(t) \in \Omega$ there is a unique solution $M(s)$ to the ordinary differential equation

$$M'(s) \int_0^s A_1(H(s),H(t))\, f(t|s)ds = [V(s,s) - M(s)]f(s|s) \tag{6}$$

with boundary condition $M(s_o) = m_o$.

We first establish that $m_o \leq M(s) < V(s,s)$. For any s such that $M(s) > V(s,s)$, it follows from (6) that $M'(s) \leq 0$. Since $V(s,s)$ is strictly increasing in s it follows that

$$M(s) > V(s,s) \Rightarrow M'(s) < \frac{dV}{ds}$$

Also, from (5) $M(s_o) = E\{V(s_o,t)|t \leq s_o\} \leq V(s_o,s_o)$. Therefore, by Lemma 3,

$$m_o \leq M(s) < V(s,s), \quad s \in [s_o,1]$$

From (6) it follows that $M(s)$ is an increasing function, hence $M(s) \in \Omega$.

Since $M(s)$ is bounded it follows also from (6) that $M'(s)$ is bounded. Indeed there exists some k such that, for all $H \in \Omega$ and all $s \in [s_o,1]$

$$0 < M'(s) \leq k \tag{7}$$

We can now use a fixed point argument to establish the existence of a solution to the differential equation (4). Let T be the mapping from H to M and define Γ to be the image of Ω. Define

$$\Omega^* = \{H \in \Omega \mid H \text{ is differentiable and } H'(s) \leq k, \quad s \in [s_0,1]\}$$

By Arzela's Theorem Ω^* is compact. By construction Ω^* is also convex. From (7) $\Gamma \subset \Omega^*$. Then we can appeal to the fixed point theorem of Schauder (1930) to establish the existence of some $H \in \Omega^*$ such that $H = TH$.

It remains to confirm that, for such a function, the necessary condition (4) is buyer 1's best reply. But, from (3)

$$\frac{\frac{\partial \Pi}{\partial x}(x|s)}{\int_0^x A_1(m(s),M(t))f(t|s)dt} = \left[\frac{[V(s,x) - M(x)]\frac{f(x|s)}{F(x|s)}}{\int_0^x A_1(M(s),M(t))\,\frac{f(t|s)}{F(x|s)}\,dt} - M'(s)\right].$$

By Lemma 1 the numerator of the first expression inside the large bracket is increasing in s. By Lemma 2, if $A_{12} \leq 0$ the denominator of this expression is nonincreasing in s. Therefore

$$s \underset{(<)}{>} x \Rightarrow \frac{\partial \Pi}{\partial x}(x,s) \underset{(<)}{>} \frac{\partial \Pi}{\partial x}(x,x) = 0$$

It follows that, for each $x > s_0$, $\Pi(x,s)$ takes on its global maximum at $x = s$. That is, bidding $m_1 = M(s)$ is buyer 1's best reply. Q.E.D.

We now turn to an examination of the equilibrium expected revenue received by the seller. If buyer 1 wins, buyer 2's signal must lie on the interval $[0,s]$. The expected payment by buyer 1, if he is the winner, is therefore

$$P(s) = \int_0^s A(M(s),M(t))\,\frac{f(t|s)}{F(s|s)}\,dt. \tag{8}$$

Differentiating (8) we obtain

$$(9) \qquad P'(s) = A(M(s),M(s)) \frac{f(s|s)}{F(s|s)} + \int_0^s M'(s)A_1(M(s),M(t)) \frac{f(t|s)}{F(s|s)}\, dt$$

$$- \left(\int_0^s A(M(s),M(t) \frac{f(t|s)}{F(s|s)}\, dt\right) \frac{f(s|s)}{F(s|s)}$$

$$+ \frac{\partial}{\partial x} \int_0^s A(M(s),M(t) \frac{f(t|x)}{F(s|x)}\, dt \Big|_{x=s}.$$

Substituting for the second term from (3) and for the third term from (8) we obtain

$$(10) \qquad P'(s) = [V(s,s) - P(s)] \frac{f(s|s)}{F(s|s)} + \frac{\partial}{\partial x} \int_o^s A(M(s),M(t)) \frac{f(t|x)}{F(s|x)}\, dt \Big|_{x=s}.$$

From Lemma 2, since A is nondecreasing in its second argument, the last term on the right hand side is nonnegative. Moreover, if A is strictly increasing in its second argument and signals are strictly linked this term is strictly greater than zero. We now compare revenue from this auction with that derived from the common sealed high bid auction. The latter is just the special case in which $A(m_{(1)},m_{(2)}) = m_{(1)}$. Writing the expected payment by the winner of this auction as $P_1(s)$ we therefore obtain

$$(11) \qquad P_1'(s) = [V(s,s) - P_1(s)] \frac{f(s|s)}{F(s|s)}.$$

Since $P_1(s_o) = m_o = P(s_o)$, (10) and (11) together lead to the following result.

<u>Proposition 2</u>: If signals are strictly linked, expected revenue from a sealed bid auction in which the winner's payment is a strictly increasing function of all bids strictly exceeds expected revenue from the common sealed high bid auction.

We now consider the "Vickrey auction" in which the winner's payment is equal to the second highest bid. From (1)

$$\Pi(x|s) = \int_0^x [V(s,t) - M(t)]\ f(t|s)dt.$$

Differentiating with respect to x we obtain

$$\frac{\partial\Pi}{\partial x}(x|s) = [V(s,s) - M(s)]f(x|s).$$

Arguing as before, for $M(.)$ to be the symmetric equilibrium bid function, $\partial\Pi/\partial x$ must be zero at $x = s$, that is

$$V(s,s) - M(s) = 0.$$

Thus the bid by buyer 1, $M(s)$ is equal to buyer 1's valuation if he and the second buyer both have the same signal s.[5]

Given such a bidding strategy, the expected payment by buyer 1, if he is the winner is

$$P_2(s) = \int_0^s V(t,t)\ \frac{f(t|s)}{F(s|s)}\ dt. \tag{12}$$

Differentiating by s and substituting from (12) we obtain

$$P_2'(s) = [V(s,s) - P_2(s)]\ \frac{f(s|s)}{F(s|s)} + \frac{\partial}{\partial x}\int_0^s V(t,t)\ \frac{f(t|x)}{F(s|x)}\ dt\Big|_{x=s}. \tag{13}$$

Comparing (11) and (13) we have the Milgrom and Weber result, viz. expected revenue from the Vickrey or second bid auction exceeds that from the high bid auction. Note furthermore that this holds even if valuations are independent, that is $V(s,t) \equiv \hat{V}(s)$. Conversely, if beliefs are independent so that $f(t|x)$ is independent of x, (13) reduces to

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[5]To understand this result suppose buyer 1 were instead to bid $M(s-\varepsilon) = V(s-\varepsilon,s-\varepsilon)$ and find that he had lost out to a bid of $M(s-\delta)$ where $\delta < \varepsilon$. Since $M(s-\delta) = V(s-\delta,s-\delta)$ is less than buyer 1's ex post valuation, $V(s,s-\delta)$, this strategy is dominated by the bid $M(s) = V(s,s)$.

$$(14) \qquad P_2'(s) = [V(s,s) - P_2(s)]\, \frac{f(s|s)}{F(s|s)}.$$

Comparing (11) and (14) it follows that expected revenue from the high and second bid auctions is the same. Since, with independent beliefs, the second bid auction is equivalent to the open ascending bid auction we therefore have the following generalization of the original revenue equivalence theorem.

Proposition 3: General Revenue Equivalence Theorem

Suppose buyers' signals are independent draws from the same distribution. Then expected revenue from the sealed high bid and open ascending bid auctions is the same, even if valuations are not independent.

Remark 1: When signals are linked and not independent it is not necessarily the case that expected revenue from the open aucton is strictly greater. One density function which generates linked signals and equal expected revenue is that depicted in Figure 1. For this example, the c.d.f. $F(t|t < x,s)$ is not strictly decreasing in s at $s = x$, for all $s \neq a$, and so the second term in equation (13) is zero.

Remark 2: For the Milgrom and Weber ranking theorem it is critical that signals be linked. Figure 2 depicts a density function similar to that in Figure 1 except that cuts have been made along the planes ABD and ACD. It is clear from the figure that signals are "pairwise positively correlated", that is, for all s and t,

$$f(s,s)f(t,t) > f(s,t)f(t,s).$$

Moreover, a minor modification of the proof of Proposition 1 establishes the existence of a monotonic equilibrium bid function as long as the vertical

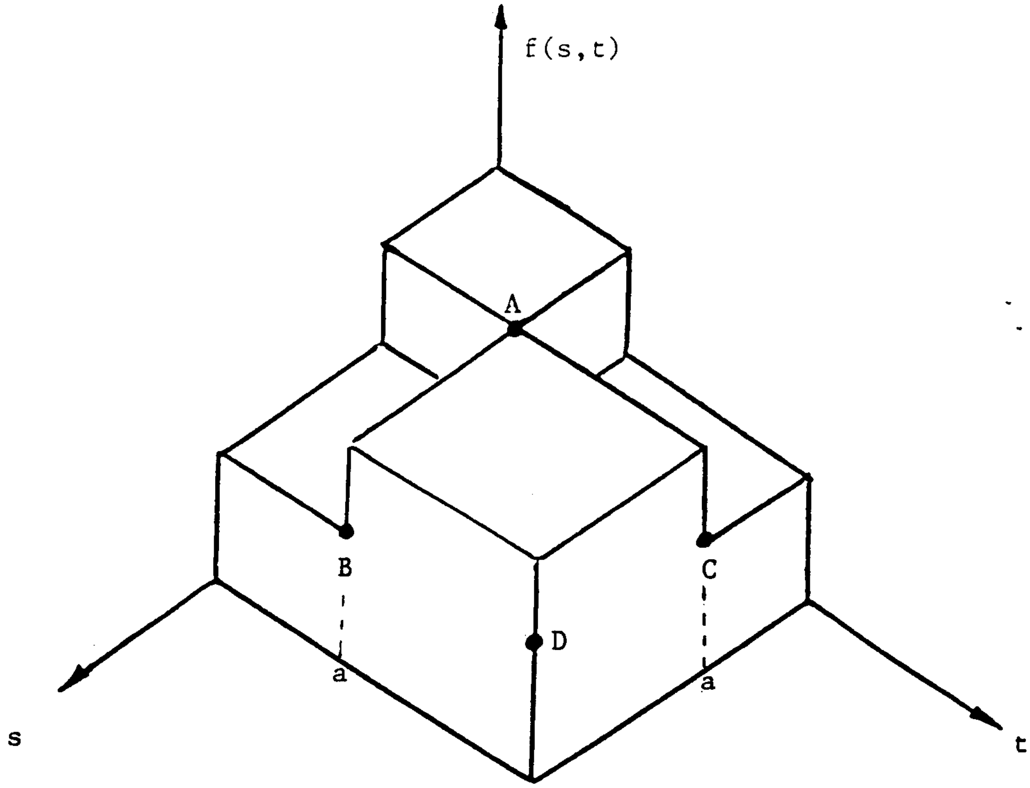

Figure 1: First example

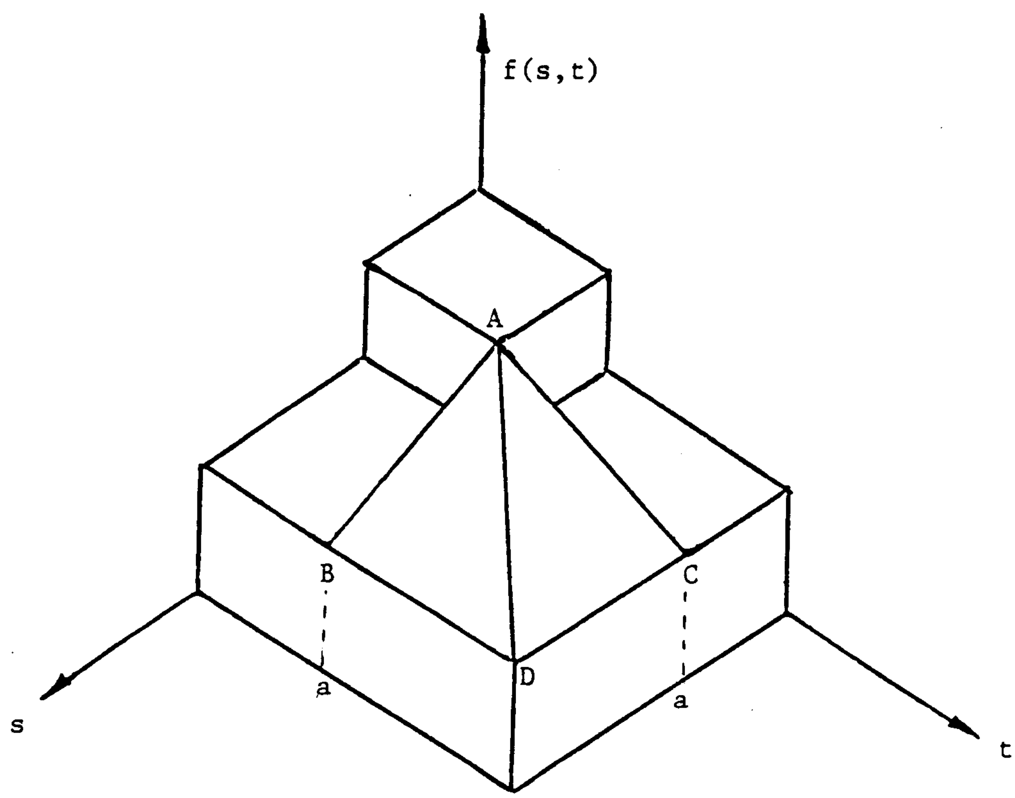

Figure 2: Second Example

distance between A and D is sufficiently small.[6] It is also straightforward to confirm that, for this example, the c.d.f. $F(t|t < x,s)$ is everywhere nondecreasing in s and is strictly increasing for $s > a$. Therefore the second term in (13) is nonpositive for all s and negative for $s > a$. It follows that expected revenue is strictly higher in the sealed high bid auction.

Proposition 3 is important because it provides a strong rationalization of the common practice in art auctions of accepting a mixture of sealed and open bids. For even if a buyer is considering a possible later resale so that his valuation is a function of other buyers' private signals, the two auctions generate the same expected revenue as long as buyers' signals (use valuations) are independent.

Returning to the case when private signals are strictly linked, the remaining question is whether or not there are weighting functions $A(m_{(1)},m_{(2)})$ which generate higher expected revenue than the second bid auction. While I do not have a general answer to this question, my conjecture is that, under mild additional restrictions, the second bid auction is optimal among auctions in which only the winner makes a payment to the seller.[7] This conjecture is suggested by the following result:

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[6] In general, even with pairwise positive correlation, it is possible to construct examples in which the equilibrium bid function for the high bid auction is not monotonic.

[7] More generally an auction could have payments by losers as well as the winner. In this case, if valuations are strictly affiliated it is possible to design a selling procedure which has a Bayesian equilibrium with all surplus going to the seller (Myerson (1981), Cremer and McLean (1985)).

Proposition 4: Without loss of generality we can choose a scale so that the signal s is equal to a buyer's reservation price in a symmetric open auction, that is $V(s,s) = s$. Suppose that for such a scale, $f(s|s)/F(s|s)$ is a strictly decreasing function and that signals are strictly linked. Then for sealed bid auctions in which the payment is a convex combination of the high and second bids, expected revenue is a strictly increasing function of the weight on the second bid.

Proof: We must show that if

$$A(m_{(1)},m_{(2)}) = (1-\lambda)m_{(1)} + \lambda m_{(2)}, \quad 0 < \lambda < 1 \tag{15}$$

expected revenue is a strictly increasing function of λ. Let $M(s,\lambda)$ be the equilibrium bid function for each value of the parameter λ. From (4)

$$[V(s,s) - M(s,\lambda)]\ f(s|s) = \frac{\partial M}{\partial s}(1-\lambda)F(s|s)$$

Dividing both sides by $F(s|s)$ and then differentiating logarithmically we obtain

$$\frac{\dfrac{\partial^2 M}{\partial s^2}}{\dfrac{\partial M}{\partial s}} = \frac{\dfrac{d}{ds}V(s,s) - \dfrac{\partial M}{\partial s}}{V(s,s) - M(s)} + \frac{\dfrac{d}{ds}\dfrac{f(s|s)}{F(s|s)}}{\dfrac{f(s|s)}{F(s|s)}}.$$

By hypothesis the last term on the right hand side is negative. Therefore

$$\frac{\partial M}{\partial s} > \frac{d}{ds}V(s|s) \Rightarrow \frac{\partial^2 M}{\partial s^2} < 0 = \frac{d^2}{ds^2}V(s,s).$$

Since $\frac{\partial M}{\partial s} < \frac{dV}{ds}$ at $s = s_o$ it follows from Lemma 3 that for all s

$$\frac{\partial}{\partial s}M(s,\lambda) < \frac{d}{ds}V(s|s), \quad 0 < \lambda < 1 \tag{17}$$

Next let $P(s,\lambda)$ be the equilibrium expected payment by a buyer if he wins with a bid of $M(s,\lambda)$. From (10)

$$\frac{\partial P}{\partial s}(s,\lambda) = [V(s,s) - P(s,\lambda)] \frac{f(s|s)}{F(s|s)} +$$

$$\frac{\partial}{\partial x} \int_0^s [(1-\lambda)M(s,\lambda) + \lambda M(t,\lambda)] \frac{f(t|x)}{F(t|x)} dt \Big|_{x=s}$$

$$= [V(s,s) - P(s,\lambda)] \frac{f(s|s)}{F(s|s)} + \lambda \frac{\partial}{\partial x} \int_0^s M(t,\lambda) \frac{f(t|x)}{F(t|x)} dt \Big|_{x=s}$$

Also, from (13)

$$\frac{\partial P}{\partial s}(s,1) = [V(s,s) - P(s,1)] \frac{f(s|s)}{F(s|s)} + \frac{\partial}{\partial x} \int_0^s V(t,t) \frac{f(t|x)}{F(t|x)} dt \Big|_{x=s}$$

Hence, for all $\lambda < 1$

$$\frac{\partial P}{\partial s}(s,1) - \frac{\partial P}{\partial s}(s,\lambda) = [P(s,\lambda) - P(s,1)] \frac{f(s|s)}{F(s|s)} + \frac{\partial}{\partial x} \int_0^s [V(t,t) - \lambda M(t,\lambda)] \frac{f(t|x)}{F(t|x)} dt \Big|_{x=s} . \tag{18}$$

From (17) $V(t,t) - \lambda M(t,\lambda)$ is a strictly increasing function of t. Therefore, by Lemma 2, the second expression on the right hand side of (18) is positive. Hence,

$$\frac{\partial P}{\partial s}(s,1) - \frac{\partial P}{\partial s}(s,\lambda) > [P(s,\lambda) - P(s,1)] \frac{f(s|s)}{F(s|s)} \tag{19}$$

Hence $P(s,\lambda) - P(s,1) > 0 \Rightarrow \frac{\partial P}{\partial s}(s,\lambda) < \frac{\partial P}{\partial s}(s,1)$. Since $P(s_o,\lambda) = P(s_o,1) = m_o$ it follows from Lemma 3 that $P(s,1) > P(s,\lambda)$. Q.E.D.

Remark 3: The assumption that $f(s|s)/F(s|s)$ is decreasing is satisfied for broad classes of distribution functions. For example s and t are strictly linked if

$$F(s|t) = s^{a+t}, \quad a > 0$$

In this case

$$\frac{f(s|s)}{F(s|s)} = \frac{a+s}{s} .$$

I conclude this section with some observations on open auctions with more than two buyers. Consider an "open exit" auction in which the auctioneer raises the price continuously and a buyer is committed to purchase unless he signals his decision to exit. Suppose that buyer 1 has signal s and when the price reaches P there is only one remaining buyer.

Extending the earlier definition, let $V(s_1,s_2,\ldots,s_n)$ be the value to buyer 1 if the private signals to the n buyers are $s_1,s_2,\ldots,s_n$. Suppose that buyer 1 has signal s_1 and all buyers but the second have exited. Since the private valuations of all the other buyers can be inferred from the exit points, the problem reduces to a two buyer problem. Indeed it becomes equivalent to the Vickrey auction analyzed above. Thus buyer 1's equilibrium strategy is to remain in the auction as long as

$$V(s_1,s_1,s_3,\ldots,s_n) > P. \tag{20}$$

With three more bidders this auction is not equivalent, however, to the Vickrey "second price" sealed bid auction, since in the latter the signals of the other buyers remain unknown. As Milgrom and Weber show, the additional information available in the "open exit" auction increases bidding competition and hence raises expected revenue, whenever signals are strictly linked. Nonetheless, it is wrong to infer that open bidding dominates sealed bidding. It is not really the availability of the information which is critical to the extraction of greater revenue, but the fact that the final payment is contingent on all available information. We now show that the seller can replicate the open auction by using a sealed bid auction in which the final payment is a weighted average of all the losing bids.

As a preliminary, the symmetry assumption of Section 1 is generalized as follows:

<u>Assumption 3'</u>: Symmetry of Valuations

The value of the object to buyer i is

$$v_i = V(s_i,s_{-i}),$$

where $s_{-i} = (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$, $V(s_i,s_{-i})$ is a strictly increasing function of s_i and is a symmetric nondecreasing function of s_{-i}.

Proposition 5 Replication of the "Open Exit" Auction:

If Assumption 3' holds the symmetric Bayesian equilibrium of an open exit auction is also the symmetric Bayesian equilibrium of a sealed bid auction in which the winner's payment is a weighted average

$$A(m_{(2)},\ldots,m_{(n)}) = V(m_{(2)},m_{(2)},m_{(3)},\ldots,m_{(n)})$$

of the losing bids, where $m_{(j)}$ is the j^{th} ranked bid.

Proof: Suppose all buyers other than the i^{th} bid their true signals while buyer i submits a bid of x. Let $s_{(j)}$ be the j^{th} ranked bid among buyers other than i. Then buyer i's expected gain is

$$\Pi(x|s_i) = \int_0^x E_{s_{(2)},\ldots,s_{(n-1)}}\{V(s_i,s_{(1)},\ldots,s_{(n-1)})$$

$$- V(s_{(1)},s_{(1)},s_{(2)},\ldots,s_{(n-1)})\}\, dF_{(1)}(s_{(1)}|s_i).$$

Differentiating by x

$$\frac{\partial\Pi}{\partial x}(x,s_o) = E_{s_{(2)},\ldots,s_{(n-1)}}\{V(s_i,x,s_{(2)},\ldots,s_{(n-1)})$$

$$- V(x,x,s_{(2)},\ldots,s_{(n-1)})\}\, f_{(1)}(x|s_i) \gtreqless \text{ as } x \lesseqgtr s_i$$

Thus buyer i's best reply is to bid his true signal also. Q.E.D.

Milgrom and Weber have also argued that the common English ascending bid auction is, to a first approximation, equivalent to what I have called the "open exit" auction. However, in the English auction buyers take pains to conceal their identities from each other. With strictly affiliated signals there is a clear incentive to do so. Suppose there are just two active bidders in the early going and both conclude that a third potential buyer has a valuation less than the opening price P_0. Suppose buyer 2 drops out at a price $\hat{P}$, and buyer 3 continues the bidding in his place. If buyer 1 does not observe the switch he will exit at the price

$$\underset{t}{E}\{V(s,s,t) \mid t < P_0)\}$$

However, buyer 3, even if he has the identical signal will have a higher valuation and will therefore gain from the deception. Whenever such deception is successful it follows also that expected revenue will fall short of that achievable in the "open exit" auction.

3. Exploiting Public Information Available After the Auction

Returning again to the 2 buyer case, we now assume that, at some date after the auction, the winning buyer and seller observe a further verifiable signal q which is affiliated with each buyer's private signal.

Let $f(s,t,q)$ be the joint density function of the three signals. If for any permutation (x,y,z) of (s,t,q) and any x',x''

$$\text{Prob}\{x \leq x' \mid x \leq x'',y,z\}, \quad x' < x''$$

is nonincreasing in y and z then we shall say that the three signals are linked. If the probability is strictly decreasing in y we shall say that x and y are strictly linked signals.

Ex post signals are almost always present when the winning bidder must undertake some production decision. For example in oil lease bidding, while profit is extremely difficult to verify, the output of the oil field is readily monitored. Moreover in competitive bidding for contracts, while actual costs are often hard to verify, certain types of inputs into the production process can be monitored.

With buyers bidding for some valuable right, and with q observable, the seller can make the winner's payment a function of q instead of a pure "fee". We begin by examining an extreme alternative in which the total payment is a proportion r of the observable q. Instead of submitting a fixed fee, each buyer submits the "royalty rate" r that he is willing to pay.[8]

Let $V(s,t,q)$ be the value of the item for sale to buyer 1. As in Section 1 we continue to assume symmetry across buyers so that the value to buyer 2 is $V(t,s,q)$. It will also be convenient to define the expected value

$$\bar{V}(s,t) = \underset{q}{E}\{V(s,t,q)\}. \tag{21}$$

We seek to characterize the equilibrium royalty bid function

$$r = R(\cdot). \tag{22}$$

Suppose, as we shall later verify, $R(\cdot)$ is strictly increasing. Then if buyer 2 adopts the equilibrium bidding strategy and buyer 1 bids $r_1 = R(x)$, buyer 1's expected profit is

$$\Pi_*(x|s) = \int_0^x [\bar{V}(s,t) - R(x)E\{q|s,t\}]\, f(t|s)dt. \tag{23}$$

Arguing almost exactly as above, $\Pi_*(x|s)$ must take on its maximum at $x = s$. Then differentiating (23) with respect to x and setting $x = s$ we obtain

$$R'(s) = \frac{[\bar{V}(s,s) - R(s)E\{q|s,s\}]f(s|s)}{\int_0^s E\{q|s,t\}f(t|s)ds}.$$

Rearranging we have

$$R'(s) = \frac{\left[\frac{\bar{V}(s,s)}{E\{q|s,s\}} - R(s)\right]f(s|s)}{E\{q|s,t<s\}/E\{q|s,s\}} \tag{24}$$

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[8]Reece (1978) also examines royalty rate bidding for the special case in which the observable q is equal to the actual value V.

This has exactly the same general form as the differential equation (4). Therefore, from the proof of Proposition 1, as long as $\bar{V}(s,s)/E\{q|s,s\}$ is strictly increasing, there exists a royalty bid function $R(s)$, satisfying (24), which is strictly increasing and satisfies the boundary condition

$$\Pi_*(s_o|s_o) = E\{\bar{V}(s_o,t)|t < s_o\} - R(s_o)\, E\{q|s_o,t<s_o\} = 0. \tag{25}$$

If buyer 1 is the winner, his expected payment $P_*(s)$, can be expressed as

$$P_*(s) = R(s) \int_0^s E\{q|s,t\} \frac{f(t|s)}{F(s|s)}\, dt \tag{26}$$

Differentiating (26) by s and then substituting from (24) and (26) we obtain

$$P_*'(s) = [\bar{V}(s,s) - P_*(s)] \frac{f(s|s)}{F(s|s)} + R(s) \frac{\partial}{\partial x} \int_0^s E\{q|x,t\} \frac{f(t|x)}{F(s|x)}\, dt \Big|_{x=s}. \tag{27}$$

But if q and t are strictly linked $E\{q|s,t\}$ is a strictly increasing function of t. Then by Lemma 2, if s and t are strictly linked, the second term on the right hand side of (27) is positive and so

$$P_*'(s) > [\bar{V}(s,s) - P_*(s)] \frac{f(s|s)}{F(s|s)} \tag{28}$$

From (25)

$$P_*(0) = R(s_o)\, E\{q|s_o,t < s_o\} = E\{\bar{V}(s_o,t)|t < s_o\} = m_o$$

where m_o is the reserve price in a common sealed high bid auction. Then comparing (28) and (11) we have the following result.

<u>Proposition 6</u>: Royalty versus Fee Bidding

If Assumptions 1-4 hold, private and public signals are strictly linked and $\bar{V}(s,s)/E\{q|s,s\}$ is strictly increasing, then the expected payment by the winning bidder is strictly greater under royalty bidding than under pure fee bidding, for each value of the signal $s > s_o$.

Remark: While the assumption that $\bar{V}(s,s)/E\{q|s,s\}$ increases with s is not innocuous, it is easy to write down examples for which it holds. Indeed, for any ex post signal q, it is always possible to define a concave transformation $\hat{q} = g(q)$ such that $\bar{V}(s,s)/E\{\hat{q}|s,s\}$ is increasing.

Of course royalty bidding is just one possible way of exploiting the additional information contained in the ex post signal. In the auctioning of oil field leases the winning bidder must pay a pre-set royalty rate in addition to his bid. As Robinson (1985a) has shown, for some simple examples, it is possible to raise expected revenue by introducing such a royalty rate. We now show that this result is a very general one.

Proposition 7: The gains to pre-set royalty rates

Suppose Assumptions 1-4 hold, the public signal is strictly linked with the private signal of the winning bidder and

$$V_N(s,t) = \bar{V}(s,t) - RE\{q|s,t\} \tag{29}$$

is positive, strictly increasing in s and nondecreasing in t. Then if the minimum bid is set so that, in equilibrium, only those with signals of at least s_0 have an incentive to bid, expected revenue from a high bid auction is a strictly icreasing function of the royalty rate.

Proof: Let $b = B_R(\cdot)$ be the equilibrium bid function and suppose that $B_R(\cdot)$ is strictly increasing. (Since the arguments parallel those in the proof of Proposition 1 we omit a formal derivation of existence.) Then buyer 1, bidding $b_1 = B_R(x)$, has an expected profit of

$$\Pi_R(x|s) = \int_0^x [\bar{V}(s,t) - RE\{q|s,t\} - B_R(x)]\ f(t|s)dt$$

But $\Pi_R(x|s)$ must take on the maximum at $x = s$. Setting $\partial\Pi_R/\partial x = 0$ at

$x = s$ we obtain

$$(30) \qquad B_R'(s) = [\bar{V}(s,s) - RE\{q|s,s\} - B_R(s)] \frac{f(s|s)}{F(s|s)}$$

By hypothesis, s_o is the value of the private signal below which, in equilibrium it is not worthwhile bidding. Therefore the minimum fee B_R^o must satisfy

$$(31) \qquad \Pi_R(s_o|s_o) = \underset{t}{E} \{\bar{V}(s_o,b) - R\underset{q}{E} \{q|s_o,t\} \mid t < s_o\} - B_R^o = 0$$

If buyer 1 wins with a bid of $B_R(s)$ his equilibrium expected payment is

$$(32) \qquad P(s,R) = R \int_0^s E\{q|s,t\} \frac{f(t|s)}{F(s|s)} dt + B_R(s)$$

Differentiating by s and substituting for B_R' and B_R from (30) and (32) we obtain

$$(33) \qquad \frac{\partial P}{\partial s}(s,R) = [\bar{V}(s,s) - P(s,R)] \frac{f(s|s)}{F(s|s)} + R\frac{\partial}{\partial x} \int_0^s E\{q|x,t\} \frac{f(t|x)}{F(s|x)} dt \Big|_{x=s}$$

Also, from (31), the minimum total payment $P(s_o,R)$ is given by

$$(34) \qquad P(s_o,R) = B_R^o + R\underset{q}{E} \{q|s_o,t < s_o\} = \underset{t}{E} \{\bar{V}(s_o,t) \mid t < s_o\}$$

As long as q and s are strictly linked $E(q|s,t)$ is strictly increasing in s. Therefore the second term on the right hand side of (33) is strictly positive. It follows that even if private signals are independent

$$(35) \qquad \frac{\partial P}{\partial s}(s,R) > [\bar{V}(s,s) - P(s,R)] \frac{f(s|s)}{F(s|s)}, \quad \text{for all } s > 0$$

Comparing (35) and (11) we can conclude that, for each $s > 0$ the expected payment by the winner is higher with a positive royalty rate than under pure fee bidding.

However we can prove a stronger result. Differentiating (33) by R we obtain

(36) $$\frac{\partial}{\partial R}\left(\frac{\partial P}{\partial s}\right) = \frac{f(s|s)}{F(s|s)}\left[\frac{F(s|s)}{f(s|s)}\frac{\partial}{\partial x}\int_0^s E\{q|x,t\}\frac{f(t|x)}{F(s|x)}\,dt\Big|_{x=s} - \frac{\partial P}{\partial R}\right]$$

But we have just argued that the first term in the bracket is positive if q and s are strictly affiliated. Hence

$$\frac{\partial P}{\partial R}(s,R) = 0 \Rightarrow \frac{\partial}{\partial s}\left(\frac{\partial P}{\partial R}\right) > 0$$

But from (34), $P(s_o,R)$ is independent of R. Therefore $\partial P/\partial R > 0$ for all $R > 0$. Q.E.D.

This result has been termed the "bid intensification" effect of a higher royalty rate by McAfee and McMillan (1984), who analyze the special case in which private signals are independent and the public signal is an unbiased estimate of the winner's private signal. The intuition is that by introducing a higher royalty rate, the seller reduces the remaining asymmetry in buyers' valuations net of the royalty payments. This induces the buyers to bid more aggressively and thus increases the expected revenue of the seller.

As McAfee and McMillan also observe, in many practical applications the **ex post** signal q is not exogenous but instead is influenced by decisions made after the auction. For example, in the oil field case the amount of oil extracted is a choice variable for the firm. Taking this simple case, with a royalty rate R, the **ex post** choice of the winning bidder is to choose $q^*(R)$ to solve

$$\max_q \{V(s,t,q) - Rq\}$$

It is readily confirmed that the value of the right to drill

$$\bar{V}(s,t,R) \equiv V(s,t,q^*(R))$$

is a decreasing function of R. That is, the bid competition effect of a higher royalty rate is offset by the moral hazard effect on production.

However, it can also be readly confirmed that

$$\frac{\partial \bar{V}}{\partial R}(s,t,R) = 0 \quad \text{at} \quad R = 0. \tag{37}$$

Since the bid competition effect of increasing R is strictly positive, and since the moral hazard effect is zero at $R = 0$ we can conclude that the expected revenue maximizing R is strictly positive.

Finally, it should be noted that, with a royalty rate R and a minimum fee B_R^o, the winning bidder has an expected minimum payment of B_R^o + $RE\{\tilde{q}|s,t < s\}$. Thus we can interpret the latter as a minimum bid which varies with the **ex post** signal. A natural question, therefore, is whether it is necessarily optimal for the seller to choose a strictly positive minimum fee.

Recently, Robinson (1985a) has extended the earlier literature on minimum prices to show that, with affiliated beliefs, the seller can always increase expected revenue from the sealed high bid auction by announcing a positive minimum price. It is a straightforward matter to apply his arguments to show that this result continues to hold even with positive royalty rates.[9]

4. Concluding Remarks

The primary conclusion of this paper is that, whenever a seller has an opportunity to collect and utilize ex post information, he has an incentive to do so. In Section 2 it was argued that expected revenue can always be increased by making the winner's payment a function of all the losers bids as well as his own. Then, in Section 3 it was shown that if information will emerge after the auction the seller can exploit this to his advantage by making

[9] More precisely, Robinson's results generalize as long as the seller has no use value, and for sufficiently low s, $\bar{V}(s,s) = E\{\tilde{q}|s,s\} = 0$.

the winner's payment depend on this information. More specifically, expected revenue is increased with the introduction of a positive royalty rate.

There remain, however, a number of important open issues. First of all, it is assumed throughout the paper that buyers are risk neutral. Analysis of the risk averse case is significantly more complicated. Under strong simplifying assumptions McAfee and McMillan (1984) and Samuelson (1984) have shown that risk aversion also tends to create an incentive for the seller to charge a positive royalty rate. The intuition behind this conclusion parallels that for the risk neutral case. By charging a royalty the seller reduces the residual risk facing the buyer and hence increases the certainty equivalent value of the item for sale. This tends to raise bids. At the same time the positive royalty rate reduces the size of the asymmetry between buyers valuations and this also intensifies buyers' bidding. The two effects are therefore reinforcing. However, it remains an open question whether this result continues to hold under less stringent assumptions than those of constant absolute risk aversion and normally distributed returns.

Second, there has been no attempt in this paper to derive profit maximizing rules for the seller. In Maskin and Riley (1980) a very simple example is analyzed in which buyers' signals can take on one of 2 values. If these signals are affiliated it is shown that there exists a selling procedure, involving payments by both winners and losers, which extracts the entire surplus when buyers adopt their unique Bayesian equilibrium strategies. Whether or not this result can be generalized remains to be resolved.

In order to extract the entire surplus, it is necessary to exploit the assumption that private signals are strictly affiliated. Another interesting case arises if private signals are independent but an **ex post** signal is observed which is positively correlated with the winner's private signal. In such an environment, extraction of all buyer surplus is no longer possible. However, the simple linear royalty schemes analyzed in this paper are certainly not optimal from the seller's viewpoint. A natural next step would be to attempt to characterize the royalty function which maximizes expected revenue.

Finally, throughout the paper it has been assumed that buyers behave noncooperatively. Recently Robinson (1985b) has argued that the open ascending bid auction (and sealed second bid auction) is particularly susceptible to manipulation by a group of bidders behaving as a cartel (or "ring"). Here we have argued that, when buyers do behave noncooperatively, and the payment is a convex combination of the first and second bid, expected revenue is a strictly increasing function of the weight on the second bid. Together, these arguments suggest that there is, on average, an advantage to employing a payment scheme that depends on both the high and second bids (and, possibly, lower bids as well).

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