The Jensen's Inequality "Paradox": Its Economic Meaning in the Term Structure, The Fisher Equation, and Foreign Exchange

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Susan Woodward

University of California, Los Angeles

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In models valuing risky assets, a difference between a forward and expected future price is usually taken to indicate a risk premium paid to the holder of a risky asset. When we realize that the numeraire commodity in such a model is also a risky asset, a paradox arises. Take as an example the exchange of dollars for pounds. Suppose that because holding pounds is risky for those operating in the dollar arena, the forward pound/dollar price ratio, is lower than the expected future pound/dollar price ratio, so those who buy pounds now (to be delivered at a future date), with the plan of changing back into dollars, earn a premium, for holding pounds, i.e., they expect the future price of pounds in terms of dollars to be higher than the current forward price of pounds in terms of dollars.

Then what of the forward dollar/pound price ratio? Should it be higher or lower than the expected future dollar/pound price ratio? The symmetric risk-aversion argument suggests that because holding dollars is risky for those operating in the pound arena, the forward dollar/pound price ratio ought to be below the expected future dollar/pound price ratio. But can a premium be earned by both Americans with pound balances and Britons with dollar balances?

One might argue that because both assets are risky the fair resolution is for "expectations" to prevail. That is, the forward pound/dollar price ratio should equal the expected future pound/dollar price ratio and likewise for the dollar/pound price ratio. But this is logically impossible. If $E(x) = y$, then by Jensen's Inequality, $E(1/x) > \frac{1}{E(x)}$, and of course $1/y = \frac{1}{E(x)}$. Therefore, expectations cannot prevail in a price and its reciprocal (which is also a price) at the same time.

The issue of how a price and its reciprocal are related to their respective expected future prices and the economic meaning of these relations is not germane only to foreign exchange, but arises in at least three other familiar contexts: 1) in the exchange of ordinary commodities, such as wheat and corn; 2) in the term structure of interest rates; 3) and in the Fisher Equation relating the nominal interest rate and expected rate of inflation.

Previous discussions of the Jensen's Inequality paradox yielded altogether unsatisfactory outcomes. In an early discussion of the phenomenon, Roper (1972) acknowledged the problem and offered the harmonic mean in place of the arithmetic mean as the relevant concept of expectations. Siegel (1975) responded by defending the arithmetic mean and arguing that one numeraire was logically the "right" numeraire. McCulloch's (1975) response was of a practical bent: examining some actual foreign exchange data, he concluded that even with large data sets, the issue was empirically trivial. None satisfactorily resolved the basic economic meaning of the paradox.

Besides the above papers, which discuss this issue directly, there are numerous papers, both theoretical and empirical, in which this issue naturally crops up; it is always finessed. For example, Benninga and Protopapadakis (1983), in analyzing the Fisher Equation conclude "...when the Fisher theorem is restated in terms of expected depreciation of money, the Jensen inequality term disappears. The Jensen inequality term seems to be a result of definitions, and as Fama concludes, it appears to be of no economic consequence."

The term does, in fact, have economic consequence, as I will show in the following sections. A thorough understanding of the term yields a more satisfactory interpretation of prices and risk premiums not only in foreign exchange, but also in the term structure and real and nominal interest rates.

Section I addresses the case easiest to interpret, that of two risky non-monetary commodities, wheat and corn, where the principles are laid out. Section II addresses the problem as it arises in the real term structure, still two risky non-monetary commodities. Section III addresses the issue in the context of the Fisher Equation, and Section IV in foreign exchange.

I. Corn and Wheat: Two Risky Non-Monetary Commodities

Take as a simplest model a representative individual economy with one future date. At the future date, both corn and wheat will be harvested. Either or both may have risky outcomes. There will be two rounds of trading, one prior to the revelation of the state of the world, and one posterior to it. No actual trading will take place, this economy being composed of representative individuals. The prices prevailing before knowledge of the harvest and after knowledge of the harvest are thus sustaining prices.

All individuals have the same state-contingent endowments of corn and wheat, the vector (C_s, W_s), $s = 1, \ldots, S$, and the same beliefs about the likelihood of each state, the vector π_s, $s = 1, \ldots, S$. The economy has complete markets for contingent claims over all possible corn and wheat harvest outcomes. Individuals have risk-averse, additively separable, continuously differentiable tastes for both corn and wheat: $U = E(V(c) + V(w))$. Again, for simplicity, assume that the preference scaling functions

V are the same for both corn and wheat.

The forward corn-wheat price ratio is a price agreed upon at date-0, in terms of wheat, for delivery of a given amount of corn at date-1. It is a certainty price, the price at which one unit of corn for sure is exchanged for a given amount of wheat for sure, prior to knowledge of the outcome. The price of a certainty bundle of a given commodity is simply the sum of the contingent claim prices for that commodity. Thus

$$\frac{{}_oP_c}{{}_oP_w} = \frac{\sum_s {}_oP_{cs}}{\sum_s {}_oP_{ws}} = \frac{E(V_c)}{E(V_w)} \tag{1}$$

where ${}_oP_c$ is the prior (date-0) round price of corn, ${}_oP_{cs}$ is the prior round price of corn contingent on state-s, and V_c is the marginal utility of corn, etc.

The future price of corn will be given simply by the ratio of the marginal utilities of the corn and wheat harvest outcomes. Thus, the expected future price ratio is

$$E\left(\frac{{}_1P_c}{{}_1P_w}\right) = E\left(\frac{V_c}{V_w}\right) \tag{2}$$

The difference between the forward and expected future prices can be expressed (simply by rearranging) as a multiple of the covariance between the marginal utility of the corn outcome and the future corn/wheat price ratio. We shall call this difference the Corn Risk Premium (CRP):

$$\begin{aligned} CRP &= \frac{{}_oP_c}{{}_oP_w} - E\left(\frac{{}_1P_c}{{}_1P_w}\right) \\ &= \frac{E(V_c)}{E(V_w)} - E\left(\frac{V_c}{V_w}\right) \\ &= \frac{1}{E(V_w)} \text{Cov } (V_c/V_w, V_w) \end{aligned} \tag{3}$$

$$= \frac{1}{E(V_w)} \text{Cov} \left(\frac{{}_1P_c}{{}_1P_w}, V_w\right)$$

(Proof in Appendix I.)

We can interpret this as a risk premium earned by holders of forward certainty corn bundles, and the risk premium is positively related to the covariance of the corn/wheat price ratio with the marginal utility of wheat. If the corn/wheat price is low when the wheat crop is poor (making the marginal utility of wheat high) and high when the wheat crop is good, then $\text{Cov}({}_1P_c/{}_1P_w, V_w) < 0$, so holding corn with the plan to trade for wheat increases risk with respect to one's wheat consumption; a premium earned by that position in the sense that the forward price is lower than the expected future price. (Note CRP < 0 when the forward price ratio is lower than the expected future price ratio.)

A negative covariance arises when the corn/wheat price is low if the wheat crop is good and high if the wheat crop is bad. When this prevails, holding corn is a form of insurance against endowed wheat risk, and the risk premium reverses. (Note that buying wheat certainties neither increases nor reduces uncertainty with respect to future wheat consumption.) Those who buy corn for future delivery must pay more for it than its expected furutre price because it is a form of insurance against endowed wheat risk.

If the "numeraire" crop, wheat, is nonrandom, the covariance between the corn/wheat price ratio and the marginal utility of wheat is zero. Thus, no risk premium is earned by holders of corn, even if it is a risky crop and the corn/wheat price ratio is consequently uncertain. A risk premium arises only when the price of the risky commodity has a nonzero covariance with the marginal utility of the numeraire.

Now of course we could just as easily have written all these prices as wheat/corn prices. Then we would have arrived at an expression that says that the Wheat Risk Premium (WRP), the difference between the wheat/corn forward price ratio and the wheat/corn expected future price ratio, goes as the covariance between the future wheat/corn price and the marginal utility of corn:

$$WRP = \frac{{}_oP_w}{{}_oP_w} - E\left(\frac{{}_1P_w}{{}_1P_c}\right) \tag{4}$$

$$= \frac{1}{E(V_c)} \text{Cov}\left(\frac{{}_1P_w}{{}_1P_c}, V_c\right)$$

The "paradox" from Jensen's Inequality arises when one realizes that these two risk premiums need not be of opposite sign. In fact, if both the corn and wheat harvests are nondegenerate random variables, by Jensen's Inequality, we can have

$CRP > 0$, and $WRP < 0$

or $WRP > 0$, and $CRP < 0$

or $CRP = 0$, and $WRP = 0$

or $CRP < 0$, and $WRP < 0$

but not $CRP > 0$, and $WRP > 0$, again by Jensen's Inequality.

The economic meaning of this paradox is revealed by expressing the covariance in terms of the correlation between the corn and wheat crops and their respective coefficients of variation:

$$CRP = E(V_c)\, \sigma_{1/w} \left[\frac{\sigma_w}{E(V_w)} \rho_{w,1/w} - \frac{\sigma_c}{E(V_c)} \rho_{c,1/w}\right] \tag{5}$$

$$WRP = E(V_w)\, \sigma_{1/c} \left[\frac{\sigma_c}{E(V_c)} \rho_{c,1/c} - \frac{\sigma_w}{E(V_w)} \rho_{w,1/c}\right]$$

(Proof in Appendix II.)

Here σ_c and σ_w are the standard deviations of the marginal utilities of corn and wheat; $\sigma_{1/c}$ and $\sigma_{1/w}$ are the standard deviations of their reciprocals. ρ is the coefficient of correlation. The sign of $\rho_{w,c}$ (correlation between V_w and V_c) does not conclusively determine the sign of $\rho_{w,1/c}$, but the two will fail to take on different signs only where one or both variables are highly skewed. For purposes of exposition, I shall carry on as though $\rho_{c,w}$ and $\rho_{c,1/w}$ are always of opposite sign. (As for $\rho_{c,1/c}$, the correlation of any variable with its own reciprocal is always negative.)

Subject to this imprecision, the interpretation is then straightforward. When the corn and wheat crops are positively correlated (more precisely, when the marginal utilities of wheat and corn are positively correlated) then, generally speaking, one risk premium will be positive and the other negative, depending on which crop has the higher coefficient of variation. If corn has the higher coefficient of variation, for example, the corn/wheat price will be high in the bad states (bad for both crops, as they are correlated, but worse for corn than for wheat) and low in the good states. So holding corn will be insurance against endowed wheat risk, but holding wheat will be risky with respect to endowed corn risk. The opposite occurs if wheat has the higher variance.

If the crops are perfectly correlated and their coefficients of variation are the same, the price ratio will always be the same and consequently the covariance term is degenerately zero and both risk premiums are zero. If one or the other of the crops is nonrandom, one risk premium will be zero, i.e., one of the inequalities in the first two expressions will be strict. Which will be zero again depends on whether the risk is coming from the corn side of things or from the wheat side of things. If it is corn that is risky, the premium for holding corn to trade back into wheat will be zero because the covariance of everything with the marginal utility of wheat is zero.

If the crops are uncorrelated or are negatively correlated, then the "paradoxical" case arises, where each crop is a risky asset with respect to the endowed social risk in the other. The corn/wheat forward price is less than the expected future corn/wheat price, and as well the wheat/corn forward price is less than the expected future wheat/corn price (CRP < 0, WRP < 0)). It is in fact possible for holders of both wheat and corn to earn a risk premium, but only when the crops are either uncorrelated or negatively correlated.

Only when the question "Which is more risky, corn or wheat?" has an unambiguous answer do we get an unambiguous interpretation for our risk premiums. This is, of course, the case in which the wheat and corn crops are positively correlated, and consequently futures contracts on one of them is a form of insurance with respect to endowed risk in the other. This potential for ambiguity arises simply because when we address the pricing of risky assets in the context of multiple goods, we generally cannot summarize the riskiness of an asset with a single risk measure. To fully describe how risky an asset is, we must state its risk characteristics with respect to <u>each</u> consumption good (see Khilstrom and Mirman, 1974). Wheat certainty bundles are of course riskless with respect to wheat consumption, but can be either risky or insuring with respect to corn.

It will be useful for the additional assets we shall consider to note how in this simple case an appeal to "risk-neutrality" gets us nowhere. With two goods and endogenously varying price ratios, the only sense that can be made of risk-neutrality is for U to be linear in both arguments. If this is the case, both marginal utilities are fixed and the price ratio for two certainty bundles never varies.

This should give the reader the flavor of the further results. If we take seriously the notion of one or more forms of money as assets willingly held and yielding utility either directly or indirectly, then money cannot be treated simply as a unit of account. We will have to be concerned not only about how risky money is for obtaining goods, but how risky goods are for obtaining money.

NUMERICAL EXAMPLES

The reader may find helpful numerical examples for two interesting cases.

EXAMPLE 1

Let tastes be described by the simple utility function $U = \ln c + \ln w$, where c is corn and w is wheat. Suppose there are only two states of the world, the two states are equally probable, and the corn and wheat outcomes in the two states are as follows:

	Corn Crop	Wheat Crop
State 1	125	250
State 2	80	200

Then the marginal utilities are:

	Corn	Wheat
State 1	.008	.004
State 2	.0125	.005

The forward corn/wheat price, ${}_0P_c/{}_0P_w$, is given by $E(V_c)/E(V_w) = .5(.008 + .0125)/.5(.004 + .005) = 2.27$.

If State-1, the good state, obtains, the corn/wheat price will be $V_{c,good}/V_{w,good} = .008/.004 = 2$. If State-2, the bad state, obtains, the price will be $.0125/.005 = 2.5$. Thus, the expected future corn/wheat price $E({}_1P_c/{}_1P_w)$, is given by $E(V_c/V_w) = .5(2 + 2.5) = 2.25$.

The forward corn/wheat price is 2.27, and the expected future price is 2.25. It is on average more expensive to buy corn now than to wait until the size of the harvest is known and buy then. Note that when the state is good, corn is cheap and wheat is dear, and when the state is bad, corn is dear and wheat is cheap. This helps us interpret these results: holding corn futures contracts is a form of insurance against endowed wheat risk, and hence a premium is paid for these contracts. Corn futures contracts are, with respect to corn endowed risk, merely riskless, of course.

Looking at the prices from the wheat side of things, the forward price is ${}_0P_w/{}_0P_c = 1/({}_0P_c/{}_0P_w) = 1/2.27 = .439$.

The expected future wheat/corn price is $.5(.004/.008)/.5(.005/.0125) = .45$. Thus, from the viewpoint of corn as numeraire, the forward price, .439, is below the expected future price, .45, so it is on average cheaper to buy wheat futures than to wait and buy wheat once the harvest outcomes are known. This result arises because wheat futures contracts are risk-increasing with respect to the endowed corn risk -- note that in the good state of the world corn is relatively cheap, and in the bad state it is relatively dear. Consequently, holding wheat futures contracts, rather than reducing the risk of corn consumption outcomes, increases it.

In this example that the corn and wheat outcomes are positively correlated, and it is unambiguous that wheat futures are a risky asset (more precisely, riskless with respect to wheat and risky with respect to corn), and that corn futures are a form of insurance with respect to wheat (and riskless with respect to corn).

EXAMPLE #2

Now we try a different risk configuration, one in which the harvest outcomes of corn and wheat are negatively correlated. The two states are equally probable and tastes are the same as above.

Harvests:	Corn	Wheat
State 1	125	200
State 2	80	250
Marginal Utilities:		
State 1	.008	.005
State 2	.0125	.004

Then the corn/wheat price ratio will be .008/.005 = 1.6 in State 1, and .0125/.004 = 3.125 in State 2, for an expected price of 2.36. The forward corn/wheat price ratio is (.008 + .0125)/(.005 + .004) = 2.27.

The wheat/corn price ratio will be 1/1.6 = .625 in State 1, and 1/3.125 = .32 in State 2, and the expected price .439. The forward wheat/corn price ratio is 1/2.27 = .4725.

In this case we have both of the forward prices (corn/wheat and wheat/corn) below the expected future prices. If we inspect the relative prices in each state, we find that wheat futures contracts are risky with respect to corn consumption (corn is cheap in the good corn state, corn is expensive in the bad corn state) and the corn futures contracts are risky with respect to wheat consumption (wheat is cheap in the good wheat state, and expensive in the bad wheat state). Thus, from <u>both</u> sides of the market, a risk premium is earned.

II. The Term Structure

[The material in this section is a pared-down version of my June 1983, AER piece.]

We start again with a representative individual economy, a single good at each of three different dates. The first date, date-0, has a certain endowment, and dates-1 and -2 have risky endowments. Everyone has a non-contingent endowment at date-0, and a state-contingent endowment at both dates-1 and -2. Tastes are additively separable, continuously differentiable, and risk-averse, $U = E[V(c_0) + V(c_1) + V(c_2)]$. Individuals hold identical beliefs about the possible states of the world, e, at date-1 and about the possible states of the world, s, at date-2, given by π_e and π_s. A message may be received at date-1 about the endowment at date-2, revealing the posterior belief, $\pi_{s.e}$ (read s given e). Prior and posterior beliefs are related by the usual $\sum_e \pi_e \pi_{s.e} = \pi_s$.

We can express term premiums as either the difference between the forward rate of interest and the expected future rate of interest, the traditional liquidity premium, L, or in terms of the difference between the forward discount and the expected future discount, S. Though L and S have the same arguments, neither is a sufficient statistic for the other. L measures the difference in the return from holding a long-term bond (in this paradigm a two-period pure discount bond) to maturity versus holding a short-term bond (one period) and rolling over to date-2. S measures the difference in the returns from holding a long-term bond and liquidating it at date-1 versus simply holding a short-term bond. Again, forward and expected future prices can be expressed as ratios of expected marginal utilities and expected marginal rates of substitution. As above, certainty prices are equal to ratios of expected marginal utilities. (For more details, see my 1983 paper). The expressions for L and for S in terms of interest rates, price ratios, and marginal utilities are:

$$L = (1 + {}_0r_2) - E(1 + {}_1r_2) \tag{6}$$

$$= \frac{{}_oP_2}{{}_oP_1} - E(\frac{{}_1P_2}{{}_1P_1})$$

$$= \frac{E(V_1)}{E(V_2)} - E(\frac{V_1}{\underset{e}{E}(V_2)})$$

$$S = 1/(1 + {}_0r_2) - E\{1/(1 + {}_1r_2)\}$$

$$= \frac{{}_oP_1}{{}_oP_2} - E(\frac{{}_1P_1}{{}_1P_2})$$

$$= \frac{E(V_2)}{E(V_1)} - E(\frac{\underset{e}{E}(V_2)}{V_1})$$

Note here an important difference in the expected marginal utility expressions from those in Section I above. In the expression for the expected future price a <u>conditional</u> expected marginal utility appears. This is because as of date-1, the uncertainty at date-2 is as yet unresolved. In terms of covariances L and S are

$$L = \frac{1}{E(V_2)} \text{Cov}(1 + {}_1r_2, \underset{e}{E}(V_2)) \tag{7}$$

$$S = \frac{1}{E(V_1)} \text{Cov}\{1/(1 + {}_1r_2), V_1\}$$

Here again, L and S need not be of opposite sign. We can have

$L > 0, \quad S < 0$

or $\quad S > 0, \quad L < 0$

or $\quad L = 0, \quad S = 0$

or $\quad L < 0, \quad S < 0,$ but not $L > 0$ and $S > 0$, by Jensen's Inequality.

Interpretation is similar to the previous case. Only if the question "which bond is more risky?" has an unambiguous answer do we have an unambiguous interpretation for the term premiums. L is positive if the strategy of going short-term and rolling over covaries negatively with date-2 endowed consumption, so that with respect to date-2 consumption risk, it is an insuring strategy. Then with respect to the short-term, the short-term bond strategy is less risky, and it is as well less risky with respect to the long-term. S is positive when the long-term bond is a form of insurance for date-1 endowed consumption risk. The ambiguous case occurs, ironically, in the case we usually think of most naturally, that where each bond is riskless with respect to consumption at its own maturity date but risky with respect to that of the other maturity date.

The variation on the results of the first section is that the relevant correlation now becomes the serial correlation in consumption. The coefficients of variation appear again as well:

$$L = E(V_1)\ \sigma\left(\frac{1}{\underset{e}{E}(V_2)}\right) \left[\frac{\sigma(\underset{e}{E}(V_2))}{E(V_2)}\ \rho\left(\underset{e}{E}(V_2), \frac{1}{\underset{e}{E}(V_2)}\right) - \frac{\sigma(V_1)}{E(V_1)}\ \rho\left(V_1, \frac{1}{\underset{e}{E}(V_2)}\right)\right] \qquad (8)$$

$$S = E(V_2)\ \sigma\left(\frac{1}{V_1}\right) \left[\frac{\sigma V_1}{E(V_1)}\ \rho\left(V_1, \frac{1}{V_1}\right) - \frac{\sigma(\underset{e}{E}(V_2))}{E(V_2)}\ \rho\left(\underset{e}{E}(V_2), \frac{1}{V_1}\right)\right]$$

Parallel to the corn and wheat case, unambiguous term premiums (one bond or the other having the higher expected rate of return over both the short and long hauls) arise only when the serial correlation in consumption is positive. Given positive serial correlation, the direction of the premium again depends on the coefficients of variation. In the case of the term structure, even the coefficients of variation have an extra subtlety of meaning. If the coefficient of variation of date-1 marginal utility is greater than that of conditional expected date-2 marginal utility, then we can say that the news at date-1 is more belief-revising with respect to date-1 than to date-2, and $L > 0$. Date-2 anticipations are revised, to be sure, but with "regression toward the mean." If the reverse relation holds for the coefficients of variation, the news revises date-2 beliefs more, (in a sense the news is "explosive"), and $S > 0$.

In the simple corn and wheat case we could interpret the coefficients of variation simply in terms of which commodity was more risky. In the context of the term structure, the modifier "conditional, expected" on date-2 marginal utility precludes this easy interpretation. A positive term premium arises when the correlation between consumption at the two date's is positive and the coefficient of variation of conditional expected date-2 marginal utility is smaller than that of date-1 marginal utility. This need not imply that the date-2 endowment is less risky. It could be less or it could be more. What is important is that at date-1, beliefs regarding date-2 be revised less than those for date-1.

Simply defining the term premiums via one particular version of the expectations hypothesis, i.e., with L or with S, does not banish the inherent ambiguity. The ambiguity returns as soon as more than one horizon is considered. Only when nature is cooperative, and gives us the positive serial correlation in consumption which makes either one bond or the other command an obvious premium, is the ambiguity resolved.

III. The Fisher Equation

The first step to a more profound understanding of the Fisher equation is to realize the sense in which it is an identity, even in the context of risk. Taking the traditional notation

$$(1+i) = (1+r)(1+a) \tag{9}$$

where i is the nominal rate of interest and r is the real rate of interest, a is usually taken to be the expected or anticipated rate of inflation. It can be given another meaning: the forward price level, which we will call $(1+a_f)$.

The forward price level is a price agreed upon now at which the exchange of future money and future goods will be executed at a future date. (Anyone who doubts the value of studying such an abstract price should consider a) mere abstractness of a price never stopped economists before, and b) such an item, CPI futures, is now being traded. Moreover, the forward price level is analogous to any other forward commodity price, such as the forward price of corn or gold.) The term $(1+i)$ is the exchange of money now for money later, $-dm(1)/dm(0)$ along the budget constraint, and in terms of the amount of nominal money to be delivered, it is not a risky trade. The term $(1+r)$, the trade of consumption now for consumption later, $-dc(1)/dc(0)$ along the budget constraint, is also not risky. The current price level we assign without loss of generality the value one (1), so the term $(1+i)/(1+r)$ defines the forward price level, $-dm(1)/dc(1)$ along the budget constraint. This, as well, is not a risky trade.

Then the question as to whether

$$(1+i) = (1+r)\, E(1+a) \tag{10}$$

is a matter of whether the forward price level, $(1+a_f)$ equals the expected future price level, $E(1+a)$. An alternative version of the Fisher relation is

$$1/(1+i) = 1/(1+r)\ E\{1/(1+a)\} \tag{11}$$

where the analogous question is whether the forward deflator (the reciprocal of the forward price level, $1/(1+a_f)$,) equals the expected future deflator. As the reader may now suspect, whether either expression holds depends on whether the variation in the price level arises as a result of real phenomena or monetary phenomena, and on the correlation between these variables and on their coefficients of variation.

If we take the current date's nominal money and assign it a marginal utility of one (1), then the relation between the forward and expected future price level, parallel to the results in the previous sections, is

$$(1+a_f) - E(1+a) = \left(\frac{1}{E(V_{m(1)})}\right) Cov(V_{m(1)},(1+a)) \tag{12}$$

That is, the risk premium is a multiple of the covariance of the marginal utility of a unit of nominal money, $V_{m(1)}$, with the future price level. This equation says that if the covariance between the marginal utility of nominal money and the rate of inflation is negative (real money holdings are low when the price level is high), then holding claims to future goods with the plan to trade for money is risky and the forward price level will be lower than the expected future price level.

The term $1/E(V_{m(1)})$ is of course just $(1+i)$ since we already assigned a marginal utility of one to date-0 nominal money. Then putting this result back in the context of the Fisher equation, we have

$$(1+a_f) - E(1+a) = (1+i)\ Cov(V_{m(1)},1+a) \tag{13}$$

For the expression showing how risky it is to hold money with the plan to exchange for goods we have:

$$\frac{1}{(1+a_f)} - E\left(\frac{1}{(1+a)}\right) = \left\{\frac{1}{E(V_{c(1)})}\right\} \mathrm{Cov}\left(V_{c(1)}, \frac{1}{(1+a)}\right) \quad (14)$$

Since we have already assigned a value of unity to both the current price level and the current marginal utility of nominal money, we have implicitly also assigned the value unity to the marginal utility of date-0 consumption. So we can write the above equation as

$$\frac{1}{(1+a_f)} - E\left(\frac{1}{(1+a)}\right) = (1+r)\ \mathrm{Cov}\left\{V_{c(1)}, \frac{1}{(1+a)}\right\} \quad (15)$$

which is the same as equation (5) in Beninga and Protopapadakis (JPE 83), except that they use, instead of the marginal utility of consumption, the real state price, q_s, (note that B and P use m to denote the state of the world) divided by the probability of the state, π_s, inside the covariance operator. Since each contingent claim price q_s is just $\pi_s V_{cs}$, these are equivalent.

From equation (15) we see that if the covariance between the marginal utility of consumption and the deflator is positive, the forward deflator exceeds the expected future deflator. If the covariance is negative the forward deflator is below the expected future deflator. If $V_{c(1)}$ is degenerate, the covariance is zero and the "inverse" Fisher relation, equation (11) holds exactly. It also holds exactly, as aruged by Beninga and Protopapadakis, when there is no correlation between consumption outcomes and the future price level.

On the other hand, if nominal money is constant, so that all of the variation in the price level arises from the real side of things, then the original version of the Fisher equation holds exactly.

The ambiguous cases occur when both consumption and nominal money are uncertain and monetary policy is countercyclical, as pointed out by B and P. Here we can see this clearly from the two expressions: money is risky for obtaining goods and goods are risky for obtaining money.

The cases of a clear premium, that is, the forward price level above the expected future price level (or forward deflator above the expected deflator) occur when the price level is procyclical. B and P argue that "real bills" monetary policy (increase nominal money when the consumption state is good, decrease it when consumption is poor) always results in the lowest nominal interest rates given an expected purchasing power loss on money. The caveat "given an expected purchasing power loss on money" is necessary because both the forward and all possible future deflators are endogenous. If the real bills monetary policy is slightly under target (so we get a little deflation in good consumption states and a little inflation in bad consumption states), then the money price of consumption is low in the good states and high in the bad states. Thus, holding money is rather risky with respect to consumption of goods, and hence the expected real rate of return on money (the goods rate on money) will be high, higher than the goods (real) rate of interest, i.e.,

$$E(\frac{1+i}{1+a}) > 1+r = (\frac{1+i}{1+a_f})$$

the expected deflator is higher than the forward deflator.

On the other hand, holding goods is a hedge with respect to money. The expected money rate of return on goods will be lower than the nominal rate of interest. This is the same thing as saying the forward price level is higher than the expected future price level:

$$E((1+r)(1+a)) < (1+i)$$

$$(1+r)E(1+a) < (1+i)$$

$$E(1+a) < (1+i)/(1+r) = (1+a_f).$$

If the real bills policy is slightly over target, so we get a little inflation in the good state and a little deflation in the bad, then the money price of goods is high in the good state and low in the bad state, and holding money is a hedge against the real risk in goods. The expected real return on an asset with a fixed and certain nominal return will be lower than the real rate of interest on goods, because holding money is a form of insurance against real consumption risk. Another way of saying this is that the forward deflator is higher than the expected future deflator:

$$(1+r) > E((1+i)(1/(1+a)))$$

$$(1+r)/(1+i) = 1/(1+a_f) > E(1/(1+a)).$$

If the "real bills" policy is right on target, so that the price level does not vary as real consumption varies, i.e., the price level is degenerate, then the Fisher equation in both its original and reciprocal versions will hold exactly.

The meaning of the mysterious "Jensen Inequality" term in B and P's Equation (3) is now clear. It arises simply because they try to force the expected rate of inflation into an equation where the expected deflator is more natural. Like expressing the demand for corn in terms of the wheat/corn price instead of the corn/wheat price, it can be done, but the results are neither as easy to interpret not as elegant.

The above interpretation of money and real risk also help us understand the results of LeRoy (1984a,b). LeRoy models the effects of both money shocks and real shocks on the price level and interest rates. Due to the intractability of these problems in infinite time models, he addresses these risks one-at-a-time.

In the "real shocks" paper (1984b) LeRoy addresses the issue of the sign of the "Fisher Premium", the difference between the expected real rate of return on a nominal bond and the real interest rate. LeRoy's equation (14) is, in our notation here:

$$F = \frac{1+i}{E(1+a)} - 1+r = \frac{1+i}{E(1+a)} - \frac{1+i}{(1+a_f)} = 1+i\ [\frac{1}{E(1+a)} - \frac{1}{(1+a_f)}] \qquad (16)$$

Our equation (15) shows (16) will be positive, negative, or zero depending on the covariance between the marginal utility of date-1 consumption and the rate of inflation. LeRoy, in contrast, proves F is always positive so long as consumption is risky. There is no paradox: in his "Real Shocks" paper, LeRoy was assuming that nominal money was fixed. If this is so, then all changes in the price level must come from the real side of things, a fortiori, the covariance of consumption and the price level must be negative (hence the covariance between the marginal utility of consumption and the deflator is also negative) and his result follows.

But when risks arise both on the real and on the money side, the premium can be positive or negative. Recall that when there is no real consumption risk, $V_{c(1)}$ is degenerate and F is zero. But of course, we could have as easily defined "the" Fisher Premium as

$$F' = \frac{E(1+a)}{1+i} - \frac{1}{1+r} = \frac{E(1+a)}{(1+i)} - \frac{(1+a_f)}{(1+c)} = \frac{1}{1+i}\ (E(1+a) - (1+a_f)]$$

which will not, as seen in (12) be invariant to money risk.

Neither will the demand for real balances be invariant to money or real risk. LeRoy's equation (6) in (1984a), he interprets as revealing that the demand for real balances depends only on the nominal interest rate and not on either inflation or risk, and presents a persuasive arbitrage argument to this effect. Let us say rather that the nominal interest rate captures the effect of both inflation and risk on the demand for real balances. Since the nominal interest rate is endogenous and clearly is driven by both real and money risk, these risks do influence the demand for real balances, as expressed in their relation to the nominal interest rate.

The results of this section are completely parallel to those of the first two sections. With two "goods", consumption and money, the risk premium in forward prices can be analyzed with either good as numeraire. The risk premium will be a positive linear function of the covariance of the realized future price with the marginal utility of the numeraire. The two risk premiums may be of opposite sign or both be negative. They are both negative when each is risky for obtaining the other. One is positive when one good is a hedge against the real, endowed risk in the other.

IV. Foreign Exchange

The presence of international trade depends on asymmetries in tastes, endowments, or productive opportunities. In this section I am not going to try to build a general equilibrium model of two risky goods and two risky currencies in a market with asymmetric agents, but instead simply to use the earlier results to give the reader a flavor of the nature of the pattern of risk premiums in currency markets. Parallel to equations (3) and (4), we have

$$\$RP = \frac{{}_oP_\$}{{}_oP_£} - E\left(\frac{{}_1P_\$}{{}_1P_£}\right) = \frac{1}{E(V_{£(1)})}\,\mathrm{Cov}\left(V_{£(1)}, \frac{{}_1P_\$}{{}_1P_£}\right)$$

$$£RP = \frac{{}_oP_£}{{}_oP_\$} - E\left(\frac{{}_1P_£}{{}_1P_\$}\right) = \frac{1}{E(V_{\$(1)})}\,\mathrm{Cov}\left(V_{\$(1)}, \frac{{}_1P_£}{{}_1P_\$}\right)$$

As before, the two premiums may be either both negative or of opposite sign. I conjecture the following:

If the rates of inflation in the two currencies are negatively correlated, both premiums will be negative because holding Pounds will be risky for Dollar holdings and vice versa.

If the rates of inflation are positively correlated one currency is risky with respect to the other and the other is a hedging instrument for the one. For example, suppose the rates of inflation in Pounds and Dollars are positively correlated but the variability of the inflation rate is higher in the Pound. If inflation is high relative to expectations, (the bad state), the Dollar rises against the Pound. If it is lower, (the good) the Dollar falls. Thus, holding Pounds is risk-increasing with respect to trading back into dollars and consuming, but holding dollars is a hedge with respect to trading back into pounds and consuming. I expect the forward Dollar price of Pounds to be lower than the expected future price to compensate for the risk, and the forward Pound price of Dollars to be higher than the expected price to charge for the insurance.

The complete analysis will likely take account of how "monetary" the two economies are (i.e., the degree to which money is used internally), the covariance in the activities of the two central banks, and the covariances and exportability of the risks on the real side. I leave the more detailed analysis for another effort.

V. Conclusion

This paper is about the risk presmiums earned on assets and how they are related to the numeraire commodities in which the returns to holding the asset are measured. I show that the average rate of return to holding corn, measured in wheat, will, in general, be different from the average rate of return to holding corn measured in terms of some other numeraire, like eggs or money. This is simply because the forces determining these relative prices are different, and result in different covariances between the price ratio and the marginal utility of the numeraire, the essential variable in explaining risk premiums.

Given this dependence of risk presmiums on the choice of numeraire, a puzzle regarding the interpretation of risk premiums for a price and its reciprocal arises. I also explain this puzzle. For two risky commodities (like future corn and future wheat), if both are risky with respect to obtaining the other, each will earn a risk premium measured in terms of the other. But if one performs an insurance role with respect to the endowed risk in the other, then one risk premium will be positive and the other negative. The essential force underlying the premiums is the endowed social risk, and the critical features of this risk turn out to be the correlation in the crop outcomes and their respective coefficients of variation.

This is a difficult result to explain in the conclusion. If the reader wishes the most brief exposition that conveys the idea, go back and read Section I.

The results allow me to interpret what turns out to be the same puzzle arising in three other different contexts: 1) the term structure of interest rates and the short-term bonds vs. long term bonds risk choice, 2) the "Fisher Equation" and the relation of nominal interest rates to inflation risk, and finally, the context in which I first encountered the problem 3) currency exchange.

Appendix I

$$
\begin{aligned}
CRP &= \frac{{}_{o}P_{c}}{{}_{o}P_{w}} - E\left(\frac{{}_{1}P_{c}}{{}_{1}P_{w}}\right) \\
&= \frac{E(V_c)}{E(V_w)} - E\left(\frac{V_c}{V_w}\right) \\
&= \frac{1}{E(V_w)} \left[E(V_c) - \left(\frac{V_c}{V_w}\right) E(V_w)\right] \\
&= \frac{1}{E(V_w)} \left[E\left(\frac{V_c}{V_w} \cdot V_w\right) - E\left(\frac{V_c}{V_w}\right) E(V_w)\right] \\
&= \frac{1}{E(V_w)} \operatorname{Cov}\left(\frac{V_c}{V_w}, V_w\right) \qquad \text{Q.E.D.}
\end{aligned}
$$

Appendix II

$$
\operatorname{Cov}(y, x/y) = E(x)\sigma_y \, \sigma_{1/y} \, \rho_{y,1/y} - E(y) \, \sigma_x \sigma_{1/y} \, \rho_x, y_y
$$

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