PROBABILISTIC INSURANCE

AND ANTICIPATED UTILITY

by

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Abstract

Kahneman and Tversky found that decision makers' reactions toward probabilistic insurance imply risk loving over the range of negative outcomes. This paper proves that the probabilistic insurance phenomenon is consistent with risk aversion and a concave utility function, provided the decision maker satisfies the reduction of compound lotteries axiom <u>or</u> the independence axiom, but not both.

1. Introduction

One of the most puzzling findings of Kahneman and Tversky in their 1979 paper is the probabilistic insurance phenomenon, where "intuitive" risk aversion seems to contradict the Arrow-Pratt formal definition of risk aversion. Consider the following problem [2, p. 269]:

> Suppose you consider the possibility of insuring some property against damages, e.g., fire or theft. After examining the risks and the premium you find that you have no clear preference between the option of purchasing insurance or leaving the property uninsured.
>
> It is then called to your attention that the insurance company offered a new program called probabilistic insurance. In this program you pay half of the regular premium. In the case of damage, there is a 50 percent chance that you pay the other half of the premium and the insurance company covers all the losses; and there is a 50 percent chance that you get back your insurance payment and suffer all the losses.

Kahneman and Tversky found that most subjects preferred not to buy the probabilistic insurance. They proved that if a decision maker is an expected utility maximizer, then this behavior is consistent with convex utility functions, i.e., risk loving (see also Section 3 below). In Kahneman and Tversky's words, [2, p. 270] "This is a rather puzzling consequence of the risk aversion hypothesis of utility theory, because probabilistic insurance appears intuitively riskier than regular insurance, which entirely eliminates the element of risk." Kahneman and Tversky suggested instead that the utility function for losses is convex.

This paper proves that the probabilistic insurance phenomenon does not imply risk loving and is consistent with concave utility functions provided the decision maker satisfies the reduction of compound lotteries axiom <u>or</u> the independence axiom, but not both. I demonstrate this by using the anticipated utility function (Quiggin [5], Segal [6], Yaari [7]), which is described, together with the reduction of compound lotteries axiom and the independence axiom, in Section 2.

2. Definitions

A lottery is a vector of the form $X = (x_1,p_1;\dots;x_n,p_n)$ where $x_1,\dots,x_n \in R$, $x_1 < \dots < x_n$, $p_1,\dots,p_n > 0$, and $\Sigma p_i = 1$. This lottery yields x_i dollars with probability p_i, $i = 1,\dots,n$. Denote the set of all these lotteries by L_1. On L_1 there exists a complete and transitive binary preference relation $\succsim$. $X \sim Y$ iff $X \succsim Y$ and $Y \succsim X$, and $X \succ Y$ iff $X \succsim Y$ but not $Y \succsim X$.

$V: L_1 \to R$ represents the order $\succsim$ if for every $X,Y \in L_1$, $X \succsim Y$ iff $V(X) \geq V(Y)$. By expected utility theory,

$$V(x_1,p_1;\dots;x_n,p_n) = \Sigma p_i u(x_i). \tag{2.1}$$

Kahneman and Tversky suggested prospect theory as an alternative to expected utility theory and claimed that

$$V(x_1,p_1;\dots;x_n,p_n) = \Sigma \pi(p_i)u(x_i).$$

This function is unacceptable from a normative standpoint. Unless $\pi(p) = p$, it necessarily contradicts first order stochastic dominance[1] (see Machina [4]).

An alternative generalization of expected utility theory, called anticipated utility, was suggested by Quiggins [5] (see also Yaari [7] and Segal [6]). By this theory,

$$V(x_1,p_1;\dots;x_n,p_n) = u(x_n)f(p_n) + \sum_{i=1}^{n-1} u(x_i)\Big[f\Big(\sum_{j=i}^{n} p_j\Big) - f\Big(\sum_{j=i+1}^{n} p_j\Big)\Big] = \tag{2.2}$$

$$u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})]f\Big(\sum_{j=i}^{n} p_j\Big)$$

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[1] Let $X = (x_1,p_1;\dots;x_n,p_n)$ and $Y = (y_1,p_1;\dots;y_n,p_n)$. $\forall i$ $x_i \geq y_i \Rightarrow X \succsim Y$.

where $f(0) = 0$, $f(1) = 1$, and $u(0) = 0$. When $f(p) = p$ this function is reduced to (2.1). Anticipated utility satisfies first order stochastic dominance (for nondecreasing u and f) and aids in unravelling some well known paradoxes in expected utility theory. Chew, Karni, and Safra [1] recently proved that (2.2) satisfies Machina's [4] conditions for risk aversion if u is concave and f convex. Yaari [7] too proved that whenever u is linear, (2.2) represents risk aversion behavior if and only if f is convex (see also Segal [6]). Indeed, let $X_p = (0,p;1,1-p)$. If the decision maker is risk averse, then his appreciation of an ε-reduction in p is probably decreasing with p. The convexity of f implies diminishing marginal value of p.

Let $L_2 = \{(X_1,p_1;\ldots;X_m,p_m)\colon\ \Sigma p_i = 1,\ \ p_1,\ldots,p_m > 0,\ \ X_1,\ldots,X_m \varepsilon L_1\}$. Elements of L_2, called two-stage lotteries, are denoted by A, B, etc. A lottery $A \varepsilon L_2$ yields a ticket to lottery X_i with probability p_i, $i = 1,\ldots,m$. More specifically, at time t_1 the decision maker faces the lottery $(1,p_1;\ldots;m,p_m)$. Upon winning the number i, he participates at time $t_2 > t_1$ in the lottery X_i. It is assumed that the decision maker's discount rate for future income is 1. Thus, once he knows that he won a certain amount of money, the actual time at which he receives this prize does not make any different to him. Let $\succsim_2$ be a complete and transitive preference relation on L_2. The decision maker is time neutral, thus L_1 naturally becomes isomorphic to a subspace of L_2, where $(x_1,p_1;\ldots;x_n,p_n)$ and $((x_1,1),p_1;\ldots;(x_n,1),p_n)$ are equally attractive. The subscript 2 is therefore omitted and the preference relation over one- and two-stage lotteries is denoted by $\succsim$. A similar discussion holds for mixed lotteries, where the set of prizes is $\mathbb{R} \cup L_1$.

This last discussion is relevant for lotteries of the form $((x_1,1),p_1;\ldots;(x_n,1),p_n)$ only. So far nothing restricts the decision maker in comparing other lotteries in L_2 with lotteries in L_1. The following two axioms deal with such comparisons.

1. Reduction of Compound Lotteries Axiom (RCLA): If the decision maker is indifferent to the resolution timing of the uncertainty, then he may assume both stages to be conducted at time t_1. Thus, a two-stage lottery is reduced to a simple one-stage lottery (see, for example, Kreps and Porteus [3]). Formally, let $X_i = (x_1^i,p_1^i;\ldots;x_{n_i}^i,p_{n_i}^i)$, $i = 1,\ldots,m$.

$$(2.3) \qquad (X_1,p_1;\ldots;X_m,p_m) \sim (x_1^1,p_1p_1^1;\ldots;x_{n_1}^1,p_1p_{n_1}^1;\ldots;x_1^m,p_mp_1^m;\ldots;x_{n_m}^m,p_mp_{n_m}^m)$$

2. Independence Axiom (IA): The relation $\succsim$ on L_2 induces several relations on L_1. The independence axiom assumes that these relations coincide and are equal to $\succsim$ on L_1. Formally,

$$(2.4) \qquad (X_1,p_1;\ldots;Y,p_i;\ldots;X_m,p_m) \succsim (X_1,p_1;\ldots;Z,p_i;\ldots;X_m,p_m) \Leftrightarrow Y \succsim Z.$$

Let $CE(X)$ be the certainty equivalence of X, given implicitly by $(CE(X),1) \sim X$. If $\succsim$ satisfies IA, then

$$(2.5) \qquad (X_1,p_1;\ldots;X_m,p_m) \sim (CE(X_1),p_1;\ldots;CE(X_m),p_m)$$

(2.1) is the only continuous function satisfying both (2.3) and (2.5). Anticipated utility is compatible with RCLA or IA.

3. Probabilistic Insurance

Denote the possible loss in the probabilistic insurance problem by x, its probability by p, and the insurance premium by k. Let $X = (-x,p;0,1-p)$ (no insurance), $Y = (-k,1)$ (full insurance), and $A =$

$((-x,\frac{1}{2};-k,\frac{1}{2}),p;-\frac{k}{2},1-p)$ (probabilistic insurance). By expected utility theory, $X \sim Y \succ A$ iff

$$pu(-x) = u(-k) > \frac{p}{2}u(-x) + \frac{p}{2}u(-k) + [1-p]u(-\frac{k}{2}) \Rightarrow$$

$$1 < \frac{1}{2} + \frac{p}{2} + [1-p]\frac{u(-k/2)}{u(-k)} \Rightarrow$$

$$\frac{u(-k/2)}{u(-k)} > \frac{1}{2} \tag{3.1}$$

It thus follows that u is not a concave function, and the rejection of the probabilistic insurance is compatible with risk loving.

Expected utility theory employs both axioms, IA and RCLA. This section proves that the rejection of the probabilistic insurance is compatible with concave utility functions (hence, risk aversion behavior) provided the decision maker satisfies IA or RCLA, but not both. For this I assume that the decision maker is an anticipated utility maximizer, that is, his value function is given by (2.2). In this model, risk aversion implies concave u and convex f (see Chew et al. [1], Quiggin [5], and Segal [6]). For the general case, let

$$A^* = ((-x,1-q;-k,q),p;-qk,1-p).$$

Note that for $q = 0$, $A^* = X$; for $q = 1$, $A^* = Y$; and for $q = \frac{1}{2}$, $A^* = A$. For the sake of simplicity I assume the existence of $a > -\infty$ such that for every $X \in L_1$, $x_1 > a$. In particular, $-x,-k > a$.

3.1 Probabilistic Insurance and RCLA

If $\succsim$ satisfies RCLA, then by (2.3), $A^* \sim (-x,(1-q)p;-k,qp;-qk,1-p)$. By (2.2), $X \sim Y \succ A^*$ iff

$$u(-x)[1-f(1-p)] = u(-k) >$$

$$u(-x)[1-f(1-p + qp)] + u(-k)[f(1-p + qp) - f(1-p)] + u(-qk)f(1-p) \iff$$

$$1 < \frac{1-f(1-p+qp)}{1-f(1-p)} + f(1-p + qp) - f(1-p) + \frac{u(-qk)}{u(-k)} f(1-p) \iff$$

$$\frac{f(1-p+qp) - f(1-p)}{1-f(1-p)} < \frac{u(-qk)}{u(-k)} \tag{3.2}$$

Inequality (3.2) holds for every convex f and $u(x) = x$. There are therefore strictly concave u and convex f satisfying (3.2).

In Figure 1, $X = \alpha \cup \beta \cup \gamma$, $Y = \alpha \cup \delta \cup \varepsilon$, and, by RCLA, $A^* \sim \alpha \cup \beta \cup \delta$. $\mu(\beta) = \mu(\gamma)$ and $\mu(\delta) = \mu(\varepsilon)$ (μ denotes Lebesgues' measure). By

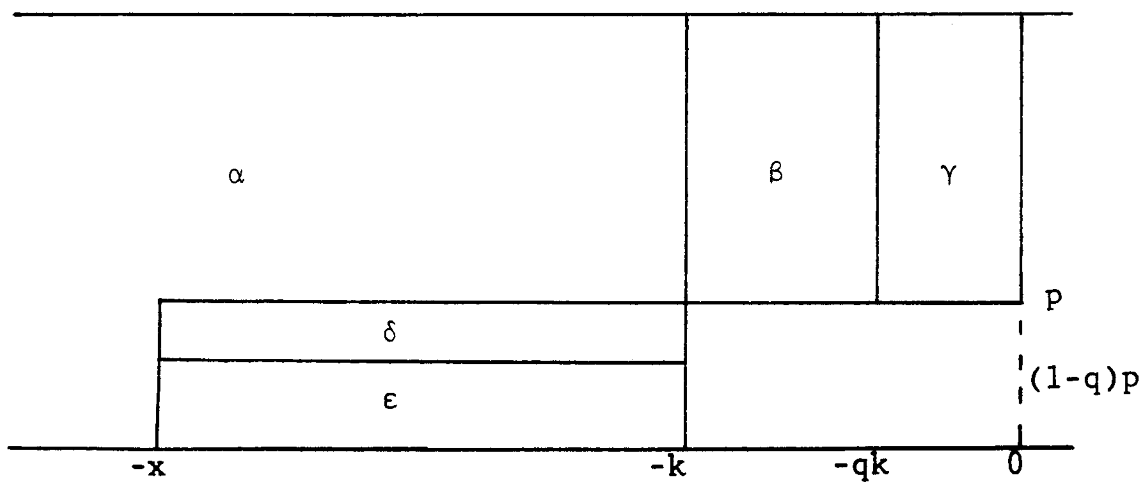

Figure 1

adapting the risk aversion concept to anticipated utility theory, it follows that the additional contribution of β to the value of a lottery is greater than that of γ, and the additional contribution of ε to the value of a lottery is greater than that of δ. Since $X \sim Y$, the contribution of $\beta \cup \gamma$ equals that of $\delta \cup \varepsilon$. $Y \succ A^*$ suggests that the decision maker is less risk averse in money than in the probabilities, at least for small probabilities. Note, however, that one can be risk averse in money and still prefer A^* to Y, as is the case with expected utility maximizers.

3.2 Probabilistic Insurance and IA

If $\succsim$ satisfies IA, then by (2.4),

(3.3) $$A^* \sim (CE(-x,1-q;-k,q),p;-qk,1-p).$$

Obviously, $CE(-x,1-q;-k,q) = u^{-1}(u(-x)[1-f(q)] + u(-k)f(q))$. Hence, by (3.3) and (2.2), $X \sim Y \succ A^*$ iff

$$u(-x)[1-f(1-p)] = u(-k) >$$

$$[u(-x)[1-f(q)] + u(-k)f(q)][1-f(1-p)] + u(-qk)f(1-p) \iff$$

$$1 < 1-f(q) + f(q)[1-f(1-p)] + \frac{u(-qk)}{u(-k)} f(1-p) \iff$$

(3.4) $$f(q) < \frac{u(-qk)}{u(-k)}.$$

Assume that $u(a) > -\infty$. Let $g(q) = \min \{\frac{u(-qk)}{u(-k)}: 0 < k < -a\}$. By L'Hospital's rule,

$$\lim_{k \to 0} \frac{u(-qk)}{u(-k)} = q.$$

It thus follows that $g(0) = 0$, $g(1) = 1$, and g is strictly increasing. For every function u (concave or convex), there thus exists a convex function f satisfying (3.4).

Let $q = \frac{1}{2}$. The lottery $(-x,\frac{1}{2};-k,\frac{1}{2})$ is represented by the area $\alpha \cup \beta \cup \gamma \cup \delta \cup \varepsilon$ (Figure 2a). By expected utility theory, the value of $\xi \cup \eta \cup \theta$ equals that of $\beta \cup \gamma \cup \delta$. If the decision maker is risk averse, i.e., if u is concave, then those values are greater than that of ε. It thus follows that the expected utility certainty equivalence (EUCE) of $(-x,\frac{1}{2};-k,\frac{1}{2})$ is less than $\frac{-k-x}{2}$. By (2.1), it also follows that the value of $\xi \cup \eta$ equals that of $\theta \cup \lambda$ and that the value of $\beta \cup \gamma$ equals that of $\delta \cup \varepsilon$.

Assume now that the decision maker is an anticipated utility maximizer. If he is risk averse, i.e., if u is concave and f is convex, then the value of $\xi \cup \eta$ exceeds that of $\beta \cup \gamma$. It thus follows that for the same utility function, the anticipated utility certainty equivalence (AUCE) of $(-x,\frac{1}{2};-k,\frac{1}{2})$ is less than EUCE.

By IA and (2.1), A* is equivalent to the lottery $\alpha' \cup \beta' \cup \delta' \cup \varepsilon'$ (Figure 2b) and by IA and (2.2), A* is equivalent to $\alpha' \cup \beta' \cup \delta'$. By (2.1), the value of $\delta' \cup \varepsilon'$ equals that of ξ', while risk aversion implies that the value of β' exceeds that of γ'. It thus follows that given $X \sim Y$, (i.e., given that the value of $\beta' \cup \gamma'$ equals that of $\delta' \cup \varepsilon' \cup \xi'$), a risk-averse expected utility maximizer will prefer A* to both X and Y. However, if ε' is sufficiently large (i.e., the decision maker is sufficiently risk averse in the probabilities), then an anticipated utility maximizer will prefer X and Y to A*.

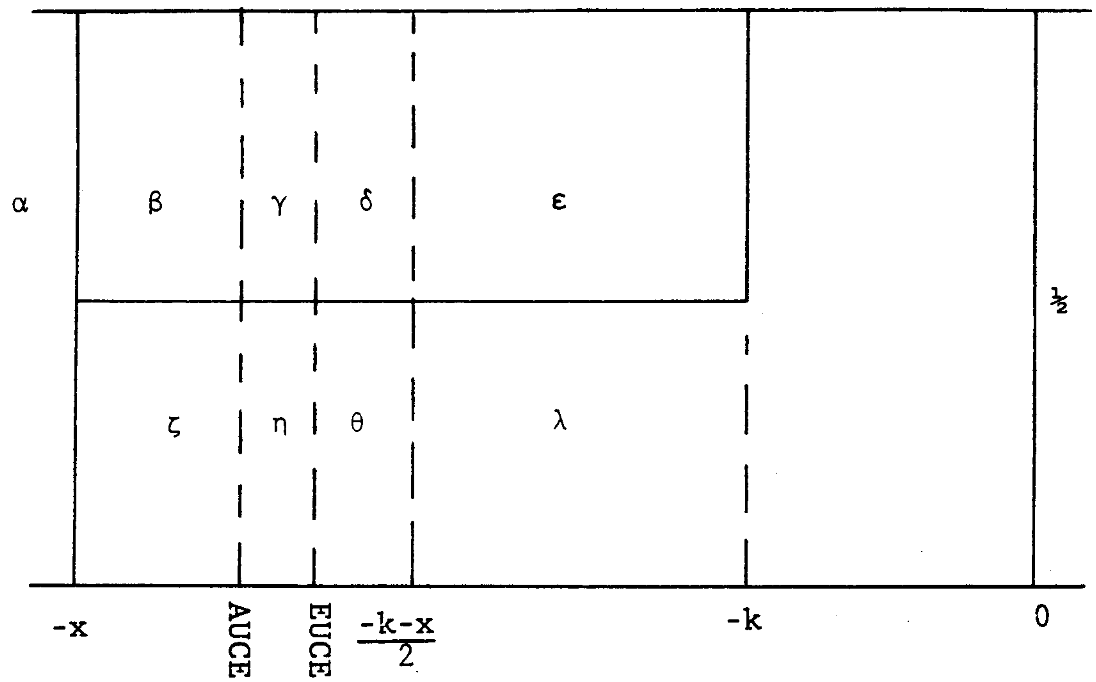

Figure 2a

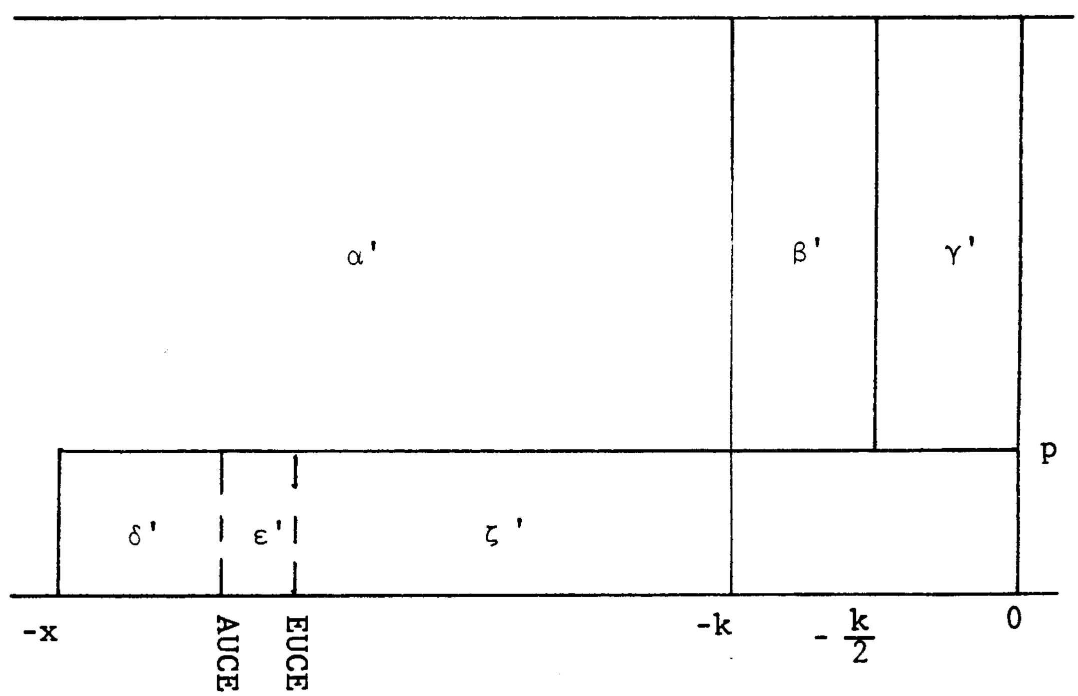

Figure 2b

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