SOME REMARKS ON QUIGGIN'S ANTICIPATED UTILITY*

by

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Abstract

This remark proves that Quiggin's anticipated utility function may solve the Allais paradox and the common ratio effect. For some generalizations of these it is needed to assume that the decision-weight function is concave.

1 Introduction

Quiggin (1982) presented anticipated utility as a generalization of expected utility theory. Let A be a random variable on $[\alpha,\beta]$, and let $F_A(x) = \Pr(A \le x)$. Quiggin suggested that the value of A is

$$\int_\alpha^\beta u(x)df(F_A(x)) \tag{1}$$

where $u(0) = 0$, $f(0) = 0$, and $f(1) = 1$.

The support of A may be finite. Let $x_1 < \dots < x_n$, $p_1, \dots, p_n > 0$, $\Sigma p_i = 1$. The vector $(x_1,p_1;\dots;x_n,p_n)$ denotes a lottery yielding x_i dollars with probability p_i, $i = 1,\dots,n$. By (1), the value of this lottery is

$$u(x_1)f(p_1) + \sum_{i=2}^{n} [f(\sum_{j=1}^{i} p_j) - f(\sum_{j=1}^{i-1} p_j)]u(x_i) \tag{2}$$

Quiggin's assumptions imply that $f(\frac{1}{2}) = \frac{1}{2}$. No further restrictions on f are imposed by his assumptions, but for the sake of simplicity, Quiggin assumed that either

(A*) f is concave on $[0,\frac{1}{2}]$ and convex on $[\frac{1}{2},1]$

or

(B*) f is convex on $[0,\frac{1}{2}]$ and concave on $[\frac{1}{2},1]$.

This paper discusses these assumptions, and particularly the assumption that $f(\frac{1}{2}) = \frac{1}{2}$. Empirical evidence, as well as results obtained by other theories, suggest that f is concave on $[0,1]$, but in this case, $f(\frac{1}{2}) > \frac{1}{2}$.

Let $g(p) = 1 - f(1-p)$. Obviously, $g(0) = 0$, $g(1) = 1$, $g(\frac{1}{2}) = \frac{1}{2}$ iff $f(\frac{1}{2}) = \frac{1}{2}$, and g is convex iff f is concave. Rewrite (1) and (2) with g instead of f and obtain

$$\int_\alpha^\beta u(x)dg(1-F_A(x)) \tag{3}$$

$$u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})]g(\sum_{j=i}^{n} p_j) \tag{4}$$

As will become apparent later, g is sometimes more useful than f (see the remark at the end of Section 2).

Section 2 shows that (4) is compatible with the Allais paradox when g is convex. If, in addition, the elasticity of g is increasing, (4) is compatible with the common ratio effect. Section 3 discusses some generalizations of these phenomena, and Section 4 compares Quiggin's theory with some other recent theories.

2. Unravelling of "Paradoxes"

In this section I show that some behavioral patterns, although inconsistent with expected utility theory (EU), may agree with anticipated utility theory (AU). In each case I will present the behavioral pattern, explain why it contradicts EU, and show that it may be consistent with AU. (MacCrimmon and Larsson (1979) discussed these patterns and their relationship to EU hypothesis in detail.) Recall that in (1)-(4), $u(0) = 0$. $A \succsim B$ means A is (weakly) preferred to B, $A \succ B$ iff $A \succsim B$ but not $B \succsim A$, and $A \sim B$ iff $A \succsim B$ and $B \succsim A$.

2.1 The Paradox of Allais

Problem 1: Choose between

$$A_1 = (0,0.9;\ 5000000,0.1) \quad \text{and} \quad B_1 = (0,0.89;\ 1000000,0.11)$$

Problem 2: Choose between

$$A_2 = (0,0.01;\ 1000000,0.89;\ 5000000,0.1) \quad \text{and} \quad B_2 = (1000000,1).$$

Let 1M denote 1000000. According to EU, $A_1 \succsim B_1$ iff $0.1u(5M) + 0.9u(0)$ $> 0.11u(1M) + 0.89u(0)$ iff $0.1u(5M) + 0.89u(1M) + 0.01u(0) > u(1M)$ iff $A_2 \succsim B_2$. However, most people prefer A_1 to B_1, but B_2 to A_2 (see Allais (1953)).

Using (4) yields $A_1 \succ B_1$ iff

$$u(5M)g(0.1) > u(1M)g(0.11) \tag{5}$$

and $B_2 \succ A_2$ iff

$$u(1M)g(0.99) + [u(5M) - u(1M)]g(0.1) < u(1M)g(1). \qquad (6)$$

(5) and (6) together imply

$$u(1M)[g(1) - g(0.99)] > u(5M)g(0.1) - u(1M)g(0.1) > u(1M)[g(0.11) - g(0.1)].$$

Thus, if g is convex and if

$$\frac{g(1) - g(0.99) + g(0.1)}{g(0.1)} > \frac{u(5M)}{u(1M)} > \frac{g(0.11)}{g(0.1)}$$

then the choices $A_1 \succ B_1$ and $B_2 \succ A_2$ are compatible with (4).

The following data come from Kahneman and Tversky (1979):

Problem 3: Choose between

$$A_3 = (0,0.67;\ 2500,0.33) \quad \text{and} \quad B_3 = (0,0.66;\ 2400,0.34)$$

Problem 4: Choose between

$$A_4 = (0,0.01;\ 2400,0.66;\ 2500,0.33) \quad \text{and} \quad B_4 = (2400,1).$$

According to EU, $A_3 \succsim B_3$ iff $A_4 \succsim B_4$. However, most people prefer A_3 to B_3, but B_4 to A_4. Thse results are compatible with AU theory if g is convex and if

$$\frac{g(1) - g(0.99) + g(0.33)}{g(0.33)} > \frac{u(2500)}{u(2400)} > \frac{g(0.34)}{g(0.33)}.$$

2.2 The Common Ratio Effect

Problem 5: Choose between

$$A_5 = (1000000,1) \quad \text{and} \quad B_5 = (0,0.2;\ 5000000,0.8)$$

Problem 6: Choose between

$$A_6 = (0,0.95;\ 1000000,0.05) \quad \text{and} \quad B_6 = (0,0.96;\ 5000000,0.04).$$

According to EU, $A_5 \succsim B_5$ iff $u(1M) > 0.8u(5M) + 0.2u(0)$ iff $0.05u(1M) > 0.04u(5M) + 0.01u(0)$ iff $0.05u(1M) + 0.95u(0) > 0.04u(5M) + 0.96u(0)$ iff

$A_6 \succsim B_6$. Most people prefer A_5 to B_5, but B_6 to A_6.

By (4), $A_5 \succ B_5$ and $B_6 \succ A_6$ iff

$$\left.\begin{array}{l} u(1M)g(1) > u(5M)g(0.8) \\ \\ u(5M)g(0.04) > u(1M)g(0.05) \end{array}\right\} \Rightarrow \frac{g(1)}{g(0.8)} > \frac{u(5M)}{u(1M)} > \frac{g(0.05)}{g(0.04)}.$$

A sufficient condition for $g(1)/g(0.8) > g(0.05)/g(0.04)$ is that for every $\alpha > 1$, $g(\alpha p)/g(p)$ is increasing with p. (In this example $\alpha = 5/4$). This occurs iff

$$\alpha g'(\alpha p)g(p) > g(\alpha p)g'(p) \iff \frac{\alpha p g'(\alpha p)}{g(\alpha p)} > \frac{p g'(p)}{g(p)}.$$

The elasticity of a function g is defined as $xg'(x)/g(x)$. Thus, if the elasticity of g is increasing, then choosing A_5 and B_6 is compatible with (4).

MacCrimmon and Larsson (1979) investigated a more general form of this decision problem:

Problem 5*: Choose between

$$A^*_5 = (0,1-p;\ x,p) \quad \text{and} \quad B^*_5 = (0,1-0.8p;\ 5x,0.8p)$$

By (4), $A^*_5 \quad B^*_5$ iff $u(x)g(p) > u(5x)g(0.8p)$ iff

$$\frac{g(p)}{g(0.8p)} > \frac{u(5x)}{u(x)}$$

MacCrimmon and Larsson found that the preference for A^*_5 is increasing with x and with p. One obtains these results if the elasticity of g is increasing and the elasticity of u, decreasing.

Kahneman and Tversky (1979) observed similar patterns. For example:

Problem 7: Choose between

$$A_7 = (3000,1) \quad \text{and} \quad B_7 = (0,0.2;\ 4000,0.8)$$

Problem 8: Choose between

$A_8 = (0,0.75;\ 3000,0.25)$ and $B_8 = (0,0.8;\ 4000,0.2)$.

Most people prefer A_7 to B_7, but B_8 to A_8. Increasing elasticity of g may explain this phenomenon.

Remark: Increasing elasticity of g does not imply decreasing elasticity of f. Let $g(p) = 1 - \sqrt{1-p}$ and let $f(p) = \sqrt{p}$. $g(p) = 1 - f(1-p)$, the elasticity of g is increasing and the elasticity of f is constant.

3. The Convexity of g

This section discusses some properties of the preference relation $\succsim$ resulting from the assumption that g is a convex function.

Definition: F_B is said to differ from F_A by a simple compensated spread if $A \sim B$ and if there exists a point x^* such that for every $x < x^*$ $F_B(x) > F_A(x)$ and for every $x > x^*$ $F_B(x) < F_A(x)$ (Machina (1982, p. 281)).

Definition: A stochastically dominates B if for every x, $F_A(x) < f_B(x)$.

Generalized Common Ratio Effect (GCRE): Let A, B, C, and D be lotteries such that C and D stochastically dominate A and B respectively, and $F_D - F_C \equiv \xi(F_B - F_A)$ for some $\xi > 0$. If F_B differs from F_A by a simple compensated spread, then $C \succsim D$, and if F_D differs from F_C by a simple compensated spread, then $B \succsim A$ (Machina (1982, p. 305)).

The common ratio effect (2.2 above) constitutes a special case of the GCRE. Let $A = (0,1-p;\ x,p)$ and $B = (0,1-q;\ y,q)$ such that $0 < x < y$, $1 > p > q$ and $A \sim B$. By definition, B differs from A by a simple compensated spread. Let $1 < \lambda < \frac{1}{p}$ and let $C = (0,1-\lambda p;\ x,\lambda p)$, $D = (0,1-\lambda q;\ y,\lambda q)$. C and D stochastically dominate A and B respectively, and $F_D - F_C \equiv \lambda(F_B - F_A)$. The GCRE requires that $C \succsim D$, as MacCrimmon and Larsson (1979) found.

If $\succsim$ can be represented by an EU function, then it satisfies the GCRE assumption because $B \sim A$ iff $D \sim C$. I now prove that the EU function is the only AU function satisfying the GCRE. This functional form cannot resolve the Allais paradox and the common ratio effect. Hence, if a decision maker behaves in accordance with AU and the Allais paradox or the common ratio effect, then he cannot satisfy the GCRE.

Theorem 1: If $\succsim$ can be represented by (1)-(4), and satisfies the GCRE, then it can be represented by an EU function. In other words, AU reduces to EU.[1]

Proof: Let $0 < x < y$, $p > q$ be such that $(0,1-p;\ x,p) \sim (0,1-q;\ y,q)$. By the GCRE, it follows that for every $p < p' < 1$ and for every $0 < r < p'$, $(0,1-p';\ x,p'-r;\ y,r) \succsim (0,1 - [q(p'-r) + rp]/p;\ y,[q(p'-r) + rp]/p)$. $\succsim$ can be represented by (4), hence

$$u(x)g(p) = u(y)g(q)$$

$$u(x)g(p') + [u(y) - u(x)]g(r) > u(y)g\left(\frac{q(p'-r) + rp}{p}\right)$$

and it follows that

$$\frac{g(p)}{g(q)} = \frac{u(y)}{u(x)} < \frac{g(p') - g(r)}{g\left(\frac{q(p'-r) + rp}{p}\right) - g(r)}$$

Because g is increasing, it is almost everywhere differentiable. If g is differentiable at p', then by the L'Hospital's Rule

$$\frac{g(p)}{g(q)} < \lim_{r \to p'} \frac{g(p') - g(r)}{g\left(\frac{q(p'-r) + rp}{p}\right) - g(r)} =$$

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[1]This was pointed out to me by Mark Machina.

$$= \lim_{r \to p'} \frac{-g'(r)}{(\frac{p-q}{p})g'(\frac{q(p'-r) + rp}{p}) - g'(r)} = \frac{p}{q}$$

Since g is a continuous function, it follows that

$$p > q \Rightarrow \frac{g(p)}{p} < \frac{g(q)}{q}. \tag{7}$$

Let $0 < x < y$, $p > q$ be such that $(0,1-p;\ x,p) \sim (0,1-q;\ y,q)$. By the GCRE, $(x,1) \succsim (0,p-q;\ x,1-p;\ y,q)$. By (4),

$$u(x)g(p) = u(y)g(q)$$

$$u(x) \geq u(x)g(1-p+q) + [u(y) - u(x)]g(q)$$

hence $1 \geq f(1-p+q) + g(p) - g(q)$. For $q = 0$ we obtain

$$g(p) + g(1-p) \leq 1. \tag{8}$$

Since $g(1) = 1$, it follows from (7) and (8) that $g(1/2) = 1/2$. Let $1/2 < p < 1$. By (7),

$$1 - \frac{g(1/2)}{1/2} \geq \frac{g(p)}{p} \geq \frac{g(1)}{1} = 1$$

hence $g(p) = p$.

Let $0 < p < 1/2$. By (8) $g(p) + 1 - p \leq 1$, thus $g(p) \leq p$. By (7)

$$\frac{g(p)}{p} \geq \frac{g(1/2)}{1/2} = 1$$

hence $g(p) = p$. Q.E.D.

Remark: This proof does not depend on the assumption that $f(\frac{1}{2}) = \frac{1}{2}$.

Although AU cannot satisfy GCRE (unless $g(p) = p$), it satisfies some modifications of this assumption.

Generalized Allais Paradox (GAP): Let A, B, C, and D be lotteries such that C and D stochastically dominate A and B respectively, and F_D -

$F_C \equiv F_B - F_A$. Assume, moreover, that B differs from A by a simple compensated spread, and let x^* be such that for $x < x^*$ $F_B(x) > F_A(x)$ and for $x > x^*$ $F_B(x) < F_A(x)$. If for $x > x^*$ $F_C(x) = F_A(x)$ (and $F_D(x) = F_B(x)$), then $C \succsim D$.

To obtain the Allais paradox, let A = (0,0.89; 1000000,0.11), B = (0,0.9; 5000000,0.1), C = (1000000,1), D = (0,0.01; 1000000,0.89; 5000000,0.1), and $x^* = 1000000$.

It is reasonable to assume that decision makers obey the GAP. Let A = (0,1-p; x,p), B = (0,1-q; y,q), C = (0,1-p-r; x,p+r) and D = (0,1-q-r; x,r; y,q) such that $0 < x < y$, $p > q$ and $A \sim B$. C may be understood as A plus an r chance of receiving x, while D equals B plus an r chance of receiving x. Note, however, that with the shift from A to C the probability of 0 is reduced relatively more than with the shift from B to D. Since $A \sim B$ and $F_C - F_A \equiv F_D - F_B$, C should be preferred to D, as predicted by the GAP.

<u>Theorem 2</u>: Assume that $\succsim$ can be represented by (4). $\succsim$ satisfies the GAP iff g is convex.

<u>Proof</u>: Let A, B, C, and D be as in the definition of the GAP, and assume, without loss of generality, that they are all bounded by α and β. Assume that $\succsim$ can be represented by (3) with a convex function g. $A \sim B$ implies

$$\int_\alpha^\beta u(x)dg(1-F_A(x)) = \int_\alpha^\beta u(x)dg(1-F_B(x)) \tag{9}$$

Similarly, $C \succsim D$ iff

$$\int_\alpha^\beta u(x)dg(1-F_C(x)) > \int_\alpha^\beta u(x)dg(1-F_D(x)) \tag{10}$$

Subtract (9) from (10), and obtain that $C \succsim D$ iff

$$\int_{\alpha}^{x^*} u(x)d[g(1-F_C(x)) - g(1-F_A(x))] \geq \int_{\alpha}^{x^*} u(x)d[g(1-F_D(x)) - g(1-F_B(x))] \quad (11)$$

According to the definition of the GAP, $F_D - F_B \equiv F_C - F_A$ and on $[\alpha,x^*]$, $F_D \geq F_C$. Inequality (11) thus holds by the convexity of g.

Assume now that $\succsim$ satisfies the GAP. Let $0 < x < y$ and $p > q$ such that $(0,1-p;\ x,p) \sim (0,1-q;\ y,q)$. Hence

$$u(x)g(p) = u(y)g(q). \quad (12)$$

By the GAP, for every $0 < r \leq 1 - p$, $(0,1-p-r;\ x,p+r) \succsim (0,1-q-r;\ x,r;\ y,q)$. By (4), this preference holds iff $u(x)g(p+r) \geq u(x)g(q+r) + [u(y) - u(x)]g(q)$ iff (by (12)) $g(p+r) - g(p) \geq g(q+r) - g(q)$. Hence $g'(p) \geq f'(q)$ and g is convex. Q.E.D.

4. Some Remarks on the Literature

Machina (1982) defined the local utility function $U(x,F)$ and proved that $A \succsim B$ whenever A stochastically dominates B iff $U(x,F)$ is nondecreasing in x for every cumulative distribution function F. Also, $A \succsim B$ whenever B differs from A by a mean preserving increase in risk iff $U(x,F)$ is a concave function of x for every F.[2] Machina assumed that the representation function is Frechet differentiable (as a function of F_A). Chew, Karni, and Safra (1985) proved that (3) is not Frechet differentiable, but they showed that Machina's results hold if the representation function is Gateaux differentiable, and that (3) is Gateaux differentiable. One can prove that if $\succsim$ is represented by (3), then the local utility function $U(\cdot,F)$ is given by

[2] The local utility function is defined in Machina (1982), Section 3.1. For the definition of mean preserving increase in risk see Rothschild and Stiglitz (1970).

$$U(x,F) = \int^{x} u'(s)g'(1-F(s))ds.$$

Differentiating twice with respect to x implies

$$U_1(x,F) = u'(x)g'(1-F(x)) \tag{13}$$

$$U_{11}(x,F) = u''(x)g'(1-F(x)) - u'(x)g''(1-F(x))F'(x) \tag{14}$$

Because u and g are increasing functions, by (13), U is nondecreasing in x. $\succsim$ indeed satisfies the first order stochastic dominance axiom (Machina (1982)). If u is concave and g is convex, then by (14), U is a concave function of x. According to Machina's theorem, if B differs from A by a mean preserving increase in risk, then $A \succsim B$.

Yaari (1984, 1985) discussed AU and risk aversion. In his approach the utility function u is linear, and a decision maker is called risk averse if g is convex. Similar results were obtained by Chew, Karni, and Safra (1985). Segal (1985) suggests a solution to the Ellsberg paradox, based on the assumption that g is convex.

5. Conclusions

Quiggin's axioms, especially his weak independence axiom, imply that $f(\frac{1}{2}) = \frac{1}{2}$. However, other axiomatic bases (Yaari (1985), Segal (1984)) yield a similar representation function without this restriction. Indeed, this restriction on f is not an essential part of anticipated utility theory, moreover, some experimental data, as well as results obtained by other theories, suggest that f is concave.

References

Allais, M., 1953: "Le Compartment de l'Homme Rationnel devante le Risque, Critique des Postulates et Axiomes de l'Ecole Americaine," Econometrica 21, 503-546.

Chew, S.H., E. Karni, and Z. Safra, 1985: "Risk Aversion in the Theory of Expected Utility with Rank-Dependent Probabilities," Tel Aviv University, Israel.

Kahneman, D. and A. Tversky, 1979: "Prospect Theory: An Analysis of Decision Under Risk," Econometrica 47, 263-291.

MacCrimmon, K.R. and S. Larsson, 1979: "Utility Theory: Axioms versus 'Paradoxes'," in Expected Utility Hypotheses and the Allais Paradox, M. Allais and O. Hagen (eds). Holland: D. Reidel Publishing Company.

Machina, M.J., 1982: "'Expected Utility' Analysis Without the Independence Axiom," Econometrica 51, 277-323.

Quiggin, J., 1982: "A Theory of Anticipated Utility," Journal of Economic Behavior and Organization 3, 323-343.

Rothschild, M. and J. Stiglitz, 1970: "Increasing Risk: I. A Definition," Journal of Economic Theory 2, 225-243.

Segal, U., 1984: "Nonlinear Decision Weights With the Independence Axiom," UCLA Working Paper #353.

__________, 1985: "The Ellsberg Paradox, Anticipated Utility, and the Independence Axiom," UCLA Working Paper #362.

Yaari, M.E., 1984: "Univariate and Multivariate Comparisons of Risk Aversion: A New Approach," CRMEGT Research Memorandum #64, Hebrew University, Jerusalem, Israel.

__________, 1985: "Risk Aversion Without Diminishing Marginal Utility and the Dual Theory of Choice Under Risk," CRMEGT Research Memorandum #65, Hebrew University, Jerusalem, Israel.