ERRORS IN VARIABLES IN LINEAR SYSTEMS

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This paper extends the simple errors-in-variable bound to the setting of systems of equations. Both diagonal and nondiagonal measurement error covariance matrices are considered. In the nondiagonal case, the analogue of the simple errors-in-variable interval of estimates is an ellipsoid with diagonal equal to the line segment connecting the direct least squares with a two-stage least squares estimate. For the diagonal case, the set of estimates under some conditions must lie within the convex hull of 2^k points.

1. INTRODUCTION

If a single explanatory variable in a linear regression model is subject to a form of stochastic measurement error, the model is not identified, but a set of estimates that asymptotically contains the true value is the interval between the ordinary regression and the reverse regression, the latter computed by minimizing the sum-of-squares in the direction of the mismeasured explanatory variable. One generalization of this two-dimensional result of Gini (1921) has been conjectured by Frisch (1934) and a partial proof can be found in Koopmans (1937). The Frisch-Koopmans result can be stated in the following way: If all the $k+1$ regressions formed by minimizing the sum of squared errors in the $k+1$ different orthogonal directions are in the same orthant, then the convex hull of these $k+1$ regressions form a consistent set that asymptotically contains the true parameter. Clearer proofs may be found in Patefield (1981), Kalman (1982) and Klepper and Leamer (1984), the latter two references dealing with the converse as well. However, few data sets that economists analyze will satisfy the condition that all $k+1$ regressions are in the same orthant, and in that event Klepper and Leamer (1984) demonstrate that the minimal consistent set is unbounded. In the absence of other information most data sets are thereby rendered useless for inference about structural coefficients, though parameters defining the distribution of observables of course remain identifiable.

Several kinds of additional information that can narrow the bounds have been considered in the recent literature. Klepper and Leamer (1984) explore the usefulness of bounds on the "true" R^2 and on the measurement error variances. Bekker, Kapteyn and Wansbeek (1985) consider limits on the measurement error covariance matrix, and Klepper and Stapleton (1985) consider restrictions on the regression coefficients.

In this paper, I revert to the earlier tradition of using only the information contained in the first two moments of the data. The additional information that is used to narrow the bounds is a set of observations of other dependent variables that are influenced by the same set of mismeasured explanatory variables. In general, the errors-in-variables bounds applicable to systems of equations are narrower than the bounds for single equations, and the multiple-equations bounds may be limited even when the single-equation bounds are unlimited. In particular, the bounds will necessarily be limited if the number of equations is no less than the number of variables measured with error. Then it is mathematically convenient not to make the assumption that errors in different variables are uncorrelated, and the consequent set of estimates for a vector of coefficients for one equation is an ellipsoid which, loosely speaking, lies between the ordinary least squares estimate and a two-stage least squares estimate in which the dependent variables are used as predictor variables to form predicted explanatory variables in the first stage of estimation.

The familiar bivariate errors-in-variable interval between the direct and reverse regression can be described this way since the reverse regression can be found by two stage least squares with a "predicted x" formed by regressing the explanatory variable x on the dependent variable y. This upper bound (the reverse regression) can also be found by dividing the direct regression by the squared correlation between the dependent variable and the mismeasured explanatory variable. If there is more than one dependent variable but still one explanatory variable, then the two-stage least squares estimate can be found by dividing the direct regression by the multiple correlation coefficient formed when the mismeasured variable is regressed on all the dependent variables. This interval is necessarily shorter than the interval applicable in the bivariate case, just as an R^2 must increase when a variable is added to an equation.

If, as in the classic work of Koopmans (1937) and Frisch (1934), the errors in variables are assumed to be distributed independently, a convenient complete characterization of the analogue of the bivariate errors-in-variables interval is not yet available for the multivariate model considered here. Permissible estimates of the diagonal measurement error covariance matrix can be shown to be smaller (in a matrix sense) than the partial covariance matrix of the explanatory variables given the dependent variables, but this set of measurement error covariance matrices maps into a rather complicated set of regression estimates. The exception to this statement is the single equation bound discussed in Klepper and Leamer (1984) which is applicable only when the k+1 regressions are in the same orthant, although, when they are not, a precise characterization of the set of regression estimates is rendered unnecessary because the set is unbounded in all relevant directions. Although the set of regression estimates resists a complete convenient characterization, it is possible to identify a set of 2^k estimates which under certain circumstances contains the minimal consistent set. This is discussed in Section 4.

In Section 2 of this paper, the errors-in-variables problem is shown to require the removal from the observed covariability of the explanatory variables that part which is due to measurement errors. A measurement error matrix that is compatible with the first two moments of the data must be less than the observed partial covariance matrix of the explanatory variables given the dependent variables. In Section 3, this set of measurement error covariance matrices is shown to map into an ellipsoid of estimates for a regression vector. In Section 4, the measurement error covariance matrix is assumed to be diagonal. Finally, an example is presented in Section 5.

2. LINEAR REGRESSION SYSTEMS WITH MEASUREMENT ERRORS

The multivariate linear regression system with all variables measured with error is written as

$$y_t = \alpha + B'\chi_t + u_t$$

$$x_t = \chi_t + e_t$$

$$\chi_t = \bar{\chi} + \varepsilon_t$$

where y_t is a (px1) observable vector

α is a px1 vector of unobservable "intercepts"

B is a kxp matrix of unobservable "slopes"

χ_t is a kx1 vector of "true" explanatory variables

u_t is a px1 vector of unobservable "disturbances"

x_t is a kx1 vector of observable measurements of χ_t

e_t is a kx1 vector of unobservable measurement errors

$\bar{\chi}$ is a kx1 vector of unobservable means of the "signal" χ_t

ε_t is a kx1 vector of unobservable signal departures from the mean.

Two crucial assumptions in this analysis are that the random vector $(u_t, e_t, \varepsilon_t)$ is serially uncorrelated and has the block diagonal covariance matrix

$$\phi = \text{Var}(u,e,\varepsilon) = \begin{vmatrix} \Omega & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & \Sigma \end{vmatrix}$$

where Ω, D, and Σ are symmetric positive semi-definite matrices. These assumptions will not always be appropriate and require careful scrutiny in serious applications. Obviously, the assumption of serial independence is inappropriate for time series data. In addition, the assumption that the covariance matrix is block diagonal is often doubtful. In this normalized form of the errors-in-variable model, the "equation error" can be thought to be composed of two parts: $u = e_y + \varepsilon_y$ where e_y is the measurement error in the "dependent" variable y, and ε_y is the combined effect of unmeasured "explanatory" variables. The other two random vectors in the model are e, the measurement error of the observed explanatory variables and ε, the difference between the true observed explanatory variable and its mean. The usual assumption underlying the selection of a dependent variable in a regression analysis is that the observed and unobserved "explanatory" variables are distributed independently: $\text{cov}(\varepsilon_y, \varepsilon) = 0$. It has furthermore been the tradition in the errors-in-variable literature to assume that measurement errors and true variables are distributed independently: $0 = \text{cov}(e,\varepsilon) = \text{cov}(e,\varepsilon_y) = \text{cov}(e_y,\varepsilon)$. Although covariances of this type are traditionally assumed to be zero, it is not difficult to conjure up reasons why they might be either negative or positive. A completely convincing analysis therefore would probably have to allow for these covariances to be different from zero.

For block diagonality of ϕ, we need also the assumption of independence between the measurement errors of y, ε_y, and the measurement errors of x, ε. This too is an assumption that needs careful scrutiny in actual applications. Incidentally, there is an apparent inconsistency between the assumption of independence in the measurement errors in x and y and the assumption that the covariance matrix of the errors in x is nondiagonal. But there are settings in which this asymmetrical treatment of the measurement errors seems justified, an example of which is offered in Section 3. Moreover, the "symmetric" treatment with diagonal D leads to the complicated sets of estimates discussed in Section 3, and for ease of computation the wider bounds associated with a free D may be preferred, particularly when these wider bounds are narrow "enough".

Given these assumptions, the observables y and x have moments

$$E \begin{vmatrix} y \\ x \end{vmatrix} = \begin{vmatrix} \alpha + \bar{\chi}'B \\ \bar{\chi} \end{vmatrix}, \tag{1}$$

$$\text{Var} \begin{vmatrix} y \\ x \end{vmatrix} = \begin{vmatrix} B'\Sigma B + \Omega & B'\Sigma \\ \Sigma B & \Sigma + D \end{vmatrix}. \tag{2}$$

Method-of-moments estimators for the unknown parameters may be found by setting these hypothetical moments equal to their observable counterparts. The equations formed by setting the theoretical means (1) equal to the observed means can be solved uniquely for estimates of α and $\bar{\chi}$, given an estimate of B. This leaves estimates of B unrestricted. Restrictions on the estimate of B must therefore come entirely from the second moments. Setting the observed covariances S_{ij} equal to the theoretical moments, we obtain the estimating equations

$$\begin{vmatrix} S_{yy} & S_{yx} \\ S_{xy} & S_{xx}-\hat{D} \end{vmatrix} = \begin{vmatrix} \hat{B}'\hat{\Sigma}\hat{B} + \hat{\Omega} & \hat{B}'\hat{\Sigma} \\ \hat{\Sigma}\hat{B} & \hat{\Sigma} \end{vmatrix}, \tag{3}$$

where $\hat{B}$ is an estimate of B. In this expression, the estimated measurement-error covariance matrix, $\hat{D}$, has been moved to the left of the equals sign to produce a matrix on the right that takes the familiar form of the covariance matrix of the multivariate regression model. On the left is the matrix of "true" covariances which are the observed covariances purged of the measurement error covariances $\hat{D}$. It seems clear from the analysis of multivariate regression that these equations have solutions with $\hat{\Omega}$ and $\hat{\Sigma}$ positive semi-definite provided the matrix on the left is positive semi-definite. (This is verified in the Appendix.) In particular, $\hat{B}$ is a solution to the normal equations

$$(S_{xx}-\hat{D})\hat{B} = S_{xy} \qquad (4)$$

with suitably selected $\hat{D}$.

The requirement that the left-hand matrix in (3) be positive semi-definite of course is not enough to determine $\hat{D}$ precisely but it does restrict its domain. If S_{yy} is positive definite, $S_{yy} > 0$, the matrix (3) is positive semi-definite if and only if $S_{xx} - \hat{D} - S_{xy}S_{yy}^{-1} S_{yx} \geq 0$ (see Appendix). This together with $\hat{D} \geq 0$ can be written as:

$$S_{xx.y} \geq \hat{D} \geq 0 \qquad (5)$$

where $S_{xx.y}$ is the partial covariance matrix of x given y, $S_{xx.y} = S_{xx} - S_{xy} S_{yy}^{-1}S_{yx}$.

In summary, the set of estimates $\hat{B}$ compatible with the first two moments of the data is the set of solutions to the normal equations (4) with the error covariance matrix $\hat{D}$ selected from the set (5). This set of estimates is bounded unless a singular $S_{xx} - \hat{D}$ is compatible with $S_{xx.y} \geq \hat{D}$, since $\hat{B} = (S_{xx}-\hat{D})^{-1}S_{xy}$ is a continuous function of $\hat{D}$ over a compact domain. This set of estimates is a consistent set, and, if the random vectors are all normally distributed, it is a minimal consistent set since higher order moments contain no further information.

3. NON-DIAGONAL ERROR COVARIANCE MATRIX

The set of estimates $\hat{B}$ which are solutions to (4) with $\hat{D}$ satisfying (5) is most easily described if the measurement error covariance matrix $\hat{D}$ is not restricted to be diagonal. Then, in order for the set of estimates to be bounded, it is necessary for the number of explanatory variables measured with error to be no greater than the number of dependent variables. More precisely, the matrix $S_{xy}S_{yy}^{-1}S_{yx}$ must be positive definite, and, if it is, the set of estimates compatible with the first two moments of the data is an ellipsoid which is easily computed and easily described.

The first result establishes conditions under which the set is finite. The second result indicates the ellipsoid of possible estimates.

<u>Theorem 1</u>: $S_{xx.y} \geq \hat{D}$ and $S_{xy}S_{yy}^{-1}S_{yx} > 0$ imply $S_{xx} - \hat{D} > 0$. Conversely, if $S_{xy}S_{yy}^{-1}S_{yx}$ is singular, there exists a matrix $\hat{D} \geq 0$, such that $S_{xx} - \hat{D}$ is singular and $S_{xx.y} \geq \hat{D}$.

<u>Proof</u>: $S_{xx.y} - \hat{D} \geq 0$ implies $S_{xx} - \hat{D} \geq S_{xx}S_{yy}^{-1}S_{yx}$, and $S_{xy}S_{yy}^{-1}S_{yx} > 0$ thus implies $S_{xx} - \hat{D} > 0$. Conversely, if $S_{xy}S_{yy}^{-1}S_{yx}$ is singular let $\hat{D} = S_{xx} - S_{xy}S_{yy}^{-1}S_{yx}$. Then $S_{xx.y} - \hat{D} = 0 \geq 0$ and $S_{xx} - \hat{D} = S_{xy}S_{yy}^{-1}S_{yx}$, which is singular.

<u>Theorem 2</u>: If $S_{yy} > 0$, $S_{xy}S_{yy}^{-1}S_{yx} > 0$ and $S_{xx.y} \geq \hat{D} \geq 0$, the vector of estimates for one equation

$$\hat{\beta} = (S_{xx} - \hat{D})^{-1} r,$$

lies in the ellipsoid

$$(\hat{\beta}-f)'\, H(\hat{\beta}-f) \leq c \qquad (6)$$

where r is the vector of covariances between the y for the selected equation and the k x-variables, and

$$f = S_{xx}^{-1} r/2 + (S_{xy}S_{yy}^{-1}S_{yx})^{-1}\, r/2$$

$$H = A\, S_{xx \cdot y}^{-1}\, A + A$$

$$A = S_{xy}S_{yy}^{-1}S_{yx}$$

$$c = r'A^{-1}r/4 - r'S_{xx}^{-1}\, r/4.$$

Conversely, for any $\hat{\beta}$ in this ellipsoid, there exists a $\hat{D}$ such that $(S_{xx}-\hat{D})\hat{\beta} = r$ and $S_{xx \cdot y} \geq \hat{D} \geq 0$.

Proof: This is a straightforward corollary of Theorem 2 in Leamer (1982, p. 727) which deals with matrix weighted averages with bounded weights. Here we may obtain the matrix weighted form by writing $\hat{\beta} = (S_{xy}S_{yy}^{-1}S_{yx} + V^{-1})^{-1}\, r$ where $V^{-1} = S_{xx \cdot y} - \hat{D}$ and satisfies $S_{xx \cdot y} \geq V^{-1} \geq 0$, or equivalently, $V \geq (S_{xx \cdot y})^{-1}$.

The extreme estimates of a linear combination $\psi'\hat{\beta}$ over the ellipsoid (8) are implied by Lemma 3 in Leamer (1982):

$$\left|\psi'\hat{\beta} - \psi'f\right| \leq (\psi'H^{-1}\psi c)^{1/2}$$

where

$$H^{-1} = (A+AS_{xx \cdot y}^{-1}A)^{-1} = A^{-1} - A^{-1}A(AA^{-1}A+S_{xx \cdot y})^{-1}\, AA^{-1}$$

$$= A^{-1} - S_{xx}^{-1}$$

The special choice of $\psi = r$ implies the inequalities

$$r'S_{xx}^{-1}\, r \leq r'\hat{\beta} \leq r'A^{-1}r$$

or equivalently

$$r'b \leq r'\hat{\beta} \leq r'\ \hat{\beta}_{2SLS} \tag{7}$$

where $b = S_{xx}^{-1}\ r$ is the least-squares vector and $\hat{\beta}_{2SLS} = (S_{xy}S_{yy}^{-1}S_{yx})^{-1}r$ is the two-stage least-squares vector formed with the dependent variables as explanatory variables in the first stage. This two-stage least squares interpretation follows from the fact that the "predicted" X matrix is $\hat{X} = Y\hat{B} = YS_{yy}^{-1}S_{yx}$, and $(\hat{X}'\hat{X})^{-1}\ \hat{X}'Y = [S_{xy}S_{yy}^{-1}Y'YS_{yy}^{-1}S_{yx}]^{-1}\ S_{xy}S_{yy}^{-1}Y'Y = (S_{xy}S_{yy}^{-1}S_{yx})^{-1}\ S_{xy}$ where Y and X are matrices of data with means removed. In words, the line segment connecting b to $\hat{\beta}_{2SLS}$ is a diagonal of the ellipsoid of estimates (6), where a diagonal is defined as the line segment connecting the tangency points of a pair of parallel supporting hyperplanes.

The simple errors-in-variable bound with $p = k = 1$ is a corollary in which the ellipsoid (6) becomes an interval between the "direct regression" $S_{xx}^{-1}S_{xy}$ and the "reverse regression" $(S_{xy}S_{yy}^{-1}S_{yx})^{-1}S_{xy} = S_{yy}S_{yx}^{-1}$. This reverse regression can be written as

$$\begin{aligned} S_{yy}S_{yx}^{-1} &= S_{xx}^{-1}S_{xy}\ /\ (S_{yx}^{2}/S_{xx}S_{yy}) \\ &= S_{xx}^{-1}S_{xy}\ /\ R^2 \end{aligned}$$

where R^2 is the squared correlation between the two variables. Another simple corollary results when $k = 1$ and $p \geq 1$. Then the ellipsoid (6) is again an interval, this time extending from the direct regression to the 2SLS regression which can be written as

$$\begin{aligned} (S_{xy}S_{yy}^{-1}S_{yx})^{-1}S_{xy} &= S_{xx}^{-1}S_{xy}\ /\ (S_{xy}S_{yy}^{-1}S_{yx}/S_{xx}) \\ &= S_{xx}^{-1}S_{xy}\ /\ R^2 \end{aligned}$$

where R^2 is the squared multiple correlation coefficient between the one explanatory variable and the set of dependent variables. The extra dependent variables in this case can easily be seen to narrow the errors-in-variables bound since the upper limit is found by dividing the usual least-squares estimate by the squared multiple correlation, not the simple correlation.

For expository purposes, Theorem 2 has made reference to the inverse of $S_{xy}S_{yy}^{-1}S_{yx}$ for defining the location and size of the ellipsoid of estimates (6). In fact, the quadratic inequality (6) makes no use of this inverse, nor does the proof of the inequality. Thus even if there are more "x-variables" than "y-variables", the set of estimates is constrained, not to an ellipsoid, but to a cylinder with an elliptical base. This implies that there are some linear combinations of parameters that are bounded, even though most are unbounded. For emphasis, I will state this as a theorem.

<u>Theorem 3</u>. Given $\hat{D}$ satisfying $S_{xx.y} > \hat{D} > 0$, and $\hat{\beta}$ satisfying the "normal equations" $(S_{xx}-\hat{D})\hat{\beta} = r$, then $\hat{\theta} = S_{yx}\hat{\beta}$ lies in the ellipsoid

$$(\hat{\theta}-g)'S_{yy.x}^{-1}(\hat{\theta}-g) < \delta'S_{yy.x}\delta/4 \tag{8}$$

where $g = (S_{yy} + S_{yx}S_{xx}^{-1}S_{xy})\ \delta/2$, and δ is the coordinate vector with one element equal to one and others equal to zero such that $S_{xy}\delta = r$. Conversely, for any $\hat{\theta}$ in this ellipsoid, there exists a $\hat{D}$ and $\hat{\beta}$ such that $(S_{xx}-\hat{D})\hat{\beta} = r$, $\hat{\theta} = S_{yx}\hat{\beta}$ and $S_{xx.y} > \hat{D} > 0$.

The verification of this result requires only that we rewrite (6) in the form of (8). This requires some effort that is better relegated to the Appendix.

The supporting hyperplanes of this ellipsoid (8)

$$|\psi'\hat{\theta} - \psi'g| < (\psi'S_{yy.x}\psi\ \delta'S_{yy.x}\delta)^{1/2}/2 \tag{9}$$

identify a set of linear inequalities which necessarily bound the set of estimates. If $\psi' = (1,0,0,\ldots,0) = \delta'$, then $\psi'\hat{\theta} = r'\hat{\beta}$, and then the bound analogous to (7) is

$$r'b = \delta' S_{yx} S_{xx}^{-1} S_{xy} \delta < r'\hat{\beta} < \delta' S_{yy} \delta \tag{10}$$

It may be noted that if the number of x's equals the number of y's, then $r'\hat{\beta}_{2SLS} = \delta' S_{yy}\delta$, and consequently inequalities (7) and (10) conform. They conform in general if you interpret $\hat{\beta}_{2SLS}$ to be a solution to the equations $S_{xy}S_{yy}^{-1}S_{yx}\hat{\beta}_{2SLS} = S_{yx}\delta$, since this implies $S_{xy}S_{yy}^{-1}(S_{yx}\hat{\beta}_{2SLS} - S_{yy}\delta) = 0$, which, since $S_{xy}S_{yy}^{-1}$ has rank equal to p, the number of y's, implies $S_{xy}\hat{\beta}_{2SLS} - S_{yy}\delta = 0$. Thus $\delta' S_{yx}\hat{\beta}_{2SLS} = \delta' S_{yy}\delta$.

4. DIAGONAL MEASUREMENT ERROR COVARIANCE MATRIX

When the set of solutions to the moment equations (3) is too large to be useful, it is necessary either to discard the given data set as useless, or to use additional information that might narrow the set of estimates. The additional information that has been traditionally employed is that the off-diagonal elements of the measurement error covariance matrix are zero. If $\hat{D}$ is restricted to be a diagonal matrix, $\hat{D} = \text{diag}\{\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_k\}$, then the normal equations can be solved for $\hat{D}$ as a function of one of the columns of $\hat{B}$. A column of $\hat{B}$ is the vector of estimates for one equation, $\hat{\beta} = \hat{B}\delta$, where δ is a vector with a single element equal to one and zeroes elsewhere. The normal equations that define the least-squares vector for this equation are $S_{xx}b = r$, where $r = S_{xy}\delta$. Then postmultiplying (4) by δ we obtain $S_{xx}(\hat{\beta} - b) = \hat{D}\hat{\beta}$. This set of equations identifies a mapping from values of $\hat{\beta}$ into values of $\hat{D}$ which is one-to-one provided $\hat{\beta}_j \neq 0$, $j = 1,\ldots,k$. If $\hat{\beta}_j = 0$, any value of $\hat{d}_j$ is compatible with these equations, though $\hat{\beta}$ must satisfy the normal equation $(S_{xx})_i'(\hat{\beta} - b) = 0$, where $(S_{xx})_i'$ is the i^{th} row of S_{xx}. Otherwise, the elements of $\hat{D}$ are $\hat{d}_i = (S_{xx})_i'(\hat{\beta} - b)/\hat{\beta}_i$. Within an orthant, $\hat{\beta}_i \neq 0$ $(i = 1,\ldots,k)$, the condition $S_{xx.y} > \hat{D} > 0$, can then be written as:

$$(S_{xx})_i' \, (\hat{\beta}-b)/\hat{\beta}_i > 0 \quad (i = 1,\ldots,k), \tag{11}$$

$$S_{xx.y} - \mathrm{diag}\left\{(S_{xx})_i' \, (\hat{\beta}-b)/\hat{\beta}_i\right\} > 0. \tag{12}$$

On the boundary of the orthants, with $\hat{\beta}_j = 0$ for one or more values of j, the conditions analogous to (11) are $d_j = (S_{xx})_j' \, (\hat{\beta}-b) = 0$, $i = j$, $d_i = (S_{xx})_i' \, (\hat{\beta}-b)/\hat{\beta}_i > 0$, $i \neq j$.

The set of estimates $\hat{\beta}$ satisfying these conditions is highly complex and there is no clear algorithm for finding the exact extremes of linear combinations $\psi'\hat{\beta}$. However, it is sometimes possible to encompass this set with a larger set that is much more tractable. One possibility is the set formed from the k linear inequalities (11) and the k linear inequalities implied by the restriction that $\hat{d}_i$ must be less than the partial variance of the i^{th} explanatory variable conditional on all the other variables (both dependent and explanatory):

$$d_i^* \equiv [\delta'(S_{xx.y})^{-1}\delta]^{-1} > (S_{xx})_i' \, (\hat{\beta}-b)/\hat{\beta}_i = \hat{d}_i \tag{13}$$

where δ is the coordinate vector selecting the i^{th} variable.

This inequality is implied by the partitioned determinant rule as follows. Let A stand for the partial covariance matrix $S_{xx.y}$ and let $D_2 = \mathrm{diag}(d_2, d_3, \ldots, d_k)$. Then the partitioned determinant rule implies $A_{11} - d_1 - A_{12}(A_{22}-D_2)^{-1}A_{21} > 0$, which can be written as $A_{11} - A_{12}(A_{22}-D_2)^{-1}A_{21} > d_1$. It is straightforward to show that $A_{11} - A_{12}(A_{22})^{-1}A_{21} > A_{11} - A_{12}(A_{22}-D_2)^{-1}A_{21}$, which implies that $A_{11} - A_{12}(A_{22})^{-1}A_{21} > d_1$.

These $2k$ inequalities (11) and (13) define a region that can be difficult to characterize in general since the directions of the inequalities change as the sign of $\hat{\beta}_i$ changes. The set may in fact be unbounded. The set will be bounded if $X'X - D > 0$ for all D satisfying (13), the worst case being $D = D^* = \mathrm{diag}\left\{d_1^*, d_2^*, \ldots, d_k^*\right\}$. Then the following applies.

Theorem 4: If $X'X - D^* > 0$, D is diagonal, and $D^* \geq D$, then $\hat{\beta} = (S_{xx}-D)^{-1} r$ lies in the convex hull of 2^k points, each point defined by k linear equalities selected from the k pairs of inequalities (11) and (13). Equivalently, these 2^k points can be found by setting the diagonal elements of the matrix D to one of the extreme values, 0 or d_i^*.

Proof: $D^* > D > 0$ is equivalent to $D^* > D^* - D > 0$ which in turn is equivalent to $D^{*-1} < (D^*-D)^{-1}$. The last condition can be written as $D^{*-1} + B = (D^*-D)^{-1}$ where B is an arbitrary non-negative diagonal matrix $B > 0$. Then we can write $\hat{\beta} = (X'X-D)^{-1}r = ([X'X-D^*] + [D^*-D])^{-1}r = (X'X - D^* + (D^{*-1}+B)^{-1})^{-1}r$. This can be written using Lemma 1 in Leamer (1982) as $\hat{\beta} = (X'X)^{-1}r + (X'X)^{-1} D^*(D^*(X'X)^{-1} [(X'X)-D^*] + B^{-1})^{-1}(X'X)^{-1}r$. The second term in this expression is a matrix weighted average with one weight matrix being an arbitrary diagonal matrix. The theorem then follows from a result of Leamer and Chamberlain (1976) which expresses the matrix weighted average as a weighted average of the 2^k points implied by the 2^k extreme weight matrices.

5.0 AN EXAMPLE: A LINEAR SYSTEM OF NET EXPORT EQUATIONS

The multivariate system of net export equations that is analyzed in Leamer(1985) will serve as an example. This system of linear equations applies at a point in time when commodity prices can be taken as fixed. The system explains the levels of net exports of a vector of commodities in terms of the availability of various resources such as capital and labor. One function of the data analysis is to identify the sources of comparative advantage in each of the commodities, a goal which may be hindered by the presence of errors in measurement in the resource variables. The bounds that have been formulated in this paper may help to clarify the consequences of measurement errors.

An important question concerning the applicability of these errors-in-variables bounds is whether the covariance matrix can credibly be assumed to be block diagonal. The assumption of independence between the measurement errors in the dependent variables and the measurement errors of the explanatory variables will usually be open to question, but in this case the measurement processes for the net export data and for the resource data are so different that it is hard to imagine reasons why the two kinds of measurement errors would be substantially correlated.

Another doubtful assumption that underlies the bounds is the assumption of homoscedasticity. This seems doubtful for two reasons. First, the errors in measurement can be expected to be relatively large when the true variable is relatively large (e.g. the U.S. capital stock is surely measured with an error that exceeds even the true capital stock of several small countries). Second, during the review of the data, we may discover one or more extreme outliers that suggest gross errors of measurement. The treatment of gross errors requires a model that allows ex post heteroscedasticity. The errors-in-variable model that is used here allows for chronic measurement errors by adding a normal random variable with a fixed variance to each observation. A model that could generate gross errors selects the measurement error variance from a distribution of variances. This kind of model implies that, before the data are observed, the process is homoscedastic. However, after the data are observed, the process is estimated to be heteroscedastic, since extreme data points are signals that the corresponding error variances are relatively large.

The linear system that is analyzed in Leamer(1984) explains the net export of ten commodity aggregates in terms of the availability of eleven resources. The results in this paper cannot be used for this full system, first because the number of explanatory variables exceeds the number of dependent variables, and second because the upper diagonal matrix D^* defined by (13) leaves $X'X - D^*$ singular. For illustrative purposes, several smaller systems are discussed instead of this full system. Alternatively, the net export variables could have been disaggregated. This raises the real research question concerning the appropriate level of aggregation of the variables. By concentrating on the problem of finding point estimates, this paper has made it appear that there is a clear benefit from increasing the number of equations. But a proper treatment of the uncertainty associated with a higher dimensional parameter space is likely to lead to a more ambiguous conclusion. Expressed in terms of the shape of the likelihood function, the problem is that an increase in the number of equations though reducing the size of the region over which the likelihood function attains its maximum also makes the likelihood function around the maximum more and more flat. Expressed in terms of properties of estimators, the border of the elliptical region which asymptotically contains the true parameter is estimated with greater inaccuracy as the number of equations is increased.

The variables are defined in Table 1. The data set, which is published in Leamer(1984), consists of observations of these variables for fifty eight countries in 1975. The first set of results in Tables 2 - 6 use models with only three explanatory variables: capital, labor and land. A system with only two dependent variables (MACH and CHEM) is reported in Table 3. The ordinary least squares estimates (OLS) are reported, as are the t-values and the extreme estimates allowing for independent measurement errors. These estimates have to be interpreted one at a time, each being the solution to a different maximization problem. In this case the set of estimates would be unbounded if the measurement error covariance matrix were allowed to be free. A diagonal measurement error covariance matrix does imply useful bounds, in particular selecting unique signs for the coefficients of capital(+) and land(-), but nonunique signs for the coefficients on labor. This finding is consistent with the view that labor embodies in some countries a substantial amount of human capital and in other countries much less, making it difficult to discern the effect of labor on comparative advantage.

A number analogous to a t-statistic that measures the distance of the errors-in-variable interval from the origin is

$$eov = (max + min) / (max - min).$$

This statistic exceeds one in absolute value if the interval of possible estimates does not include the origin. An analogous relationship exists between a one-standard deviation confidence interval and the corresponding t-value. Generally there is a fairly good correspondence between the t-values and the eov insensitivity indicators. The other numbers reported in Table 2 are nse, the maximal noise variances (13) relative to the observed variances of the variable. Equivalently, these ratios are $1-R^2$, where R^2 is the squared multiple correlation between one explanatory variable and all the other variables (both dependent and explanatory). These data indicate that no more than 14.5 per cent of the variance in capital could be attributed to measurement error, but land could be contaminate with as much as 44 percent error, and labor 73 percent error.

Tables 3 and 4 contain results if the full set of ten dependent variables is used, Table 3 dealing with the nondiagonal case(Theorem 2) and Table 4 with the diagonal case(Theorem 4). Note the substantial reduction in the errors-in-variables bounds afforded by the extra information contained in the other eight dependent variables. In Table 3 only the coefficient of labor on the machinery equation remains indeterminant in sign.

The bounds in Table 4 might be expected to be narrower than the bounds in Table 3 since they make use of a restricted class of covariance matrices, but the set defined by Theorem 4 is not a minimal set of estimates, consequently the bounds can and in several cases are wider than the bounds applicable if the covariance matrix is free. In fact, only the minima for the land coefficients are improved.

Bounds for the complete 10x3 system are reported in Tables 5 and 6. The discussion of the last two equations could more or less be repeated for the whole system: the data admit very sturdy conclusions about the effect of capital on comparative advantage, less sturdy inferences about land, and somewhat fragile inferences about labor.

A larger system with nine explanatory variables is reported in Tables 7, 8 and 9, which contrast the t-values with the errors-in-variables sensitivity indicators. Very few of the inferences about the signs of the coefficients can be said to be resistant to concerns about measurement errors. The two most resistant variables are CAPITAL and OIL, with four and five resistant coefficients, respectively. In addition, COAL has three resistant coefficients, and LAND3 has one. The signs of these resistant effects seem quite predictable, with some exceptions for the OIL variable.

The t-values in Table 9 generally tell quite a different story. These large t-values suggest that in the absence of measurement errors the data would allow very sharp inferences about the signs of the coefficients. The errors-in-variables possibilities of course weaken these inferences, but one might have hoped not to the point indicated in Table 8.

Notice also that both large t-values and small noise ratios are indicators, though imperfect ones, of the resistance to errors-in-variables issues. The largest t-values in order are (-16.8, -16.5, 15.9, 13.2, 7.8, -7.4, 7.1) with corresponding insensitivity indicators (-1.8, -2.5, 2.0, 1.0, 1.8, -.9, 1.4). Thus a t of -7.4 does not assure that the coefficient is resistant. Similarly, COAL which has a maximal noise ratio of only .006, has only four insensitivity indicators less than one in absolute value.

In conclusion, though it is obvious at the start that concerns about errors-in-variables will weaken the inferences that may be obtained from a data set, the extent of the deterioration is not so obvious. Klepper and Leamer(1984) leave the impression that the situation may be very dire indeed. The basic conclusion from this paper is that things are not so bad, if, as is likely, there is a system of equations to be estimated. Nonetheless, data analyses that do not consider the deteriorating effect of errors-in-variables may greatly overstate the precision of the inferences.

Footnotes

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APPENDIX

The first two results in this Appendix make use of some standard theorems concerning positive semi-definite matrices. Provided the operations are well defined:

(A1) $H \geq 0 \Rightarrow G'HG \geq 0$

(A2) $H \geq 0, \quad G \geq 0 \Rightarrow H + G \geq 0.$

First it is shown that the matrix of equations (3) has a solution with $\hat{\Omega} \geq 0$ and $\hat{\Sigma} \geq 0$, if and only if the matrix on the left is positive semi-definite. The system admits a solution: $\hat{\Sigma} = S_{xx} - \hat{D}$, $\hat{B}$ satisfying $\hat{\Sigma}\hat{B} = S_{xy}$, and $\hat{\Omega} = S_{yy} - S_{yx}\hat{B}$. Then the following result establishes the property of the solution:

Result 1:

$$S = \begin{vmatrix} \hat{B}' \\ I \end{vmatrix} \hat{\Sigma}[\hat{B},I] + \begin{vmatrix} \hat{\Omega} & 0 \\ 0 & 0 \end{vmatrix} \geq 0$$
$$\Leftrightarrow \hat{\Sigma} \geq 0, \quad \hat{\Omega} \geq 0.$$

Proof: (A1) and (A2) imply that $(\hat{\Sigma} \geq 0, \hat{\Omega} \geq 0) \Rightarrow (S \geq 0)$. (A1) with $G' = (0,I)$ implies $(S \geq 0 \Rightarrow \hat{\Sigma} \geq 0)$ and with $G' = (I,-\hat{B}')$ implies $(S \geq 0 \Rightarrow \hat{\Omega} \geq 0)$.

The second result is:

Result 2

$$S \geq 0,\ S_{yy} > 0 \Rightarrow S_{xx.y} \geq 0$$

$$S_{xx.y} \geq 0,\ S_{yy} > 0 \Rightarrow S \geq 0.$$

Proof: The first part is established using A1 and premultiplying S by $G' = (-S_{xy}S_{yy}^{-1}, I)$ and postmultiplying by G. The second part again using A1 and writing $S = G'HG$ where

$$G = \begin{vmatrix} I & S_{yy}^{-1}S_{yx} \\ 0 & I \end{vmatrix}$$

$$H = \begin{vmatrix} S_{yy} & 0 \\ 0 & S_{xx}-S_{xy}S_{yy}^{-1}S_{yx} \end{vmatrix}$$

The last result reports the tedious algebra establishing the equivalence of (6) and (8).

Result 3: Inequalities (6) and (8) are equivalent.

Use: $A = S_{xy}S_{yy}^{-1}S_{yx}$, $B = S_{yx}S_{xx}^{-1}S_{xy}$, $r = S_{xy}\delta$,

$$(S_{yy}^{-1}S_{yx}S_{xx.y}^{-1}S_{xy}S_{yy}^{-1} + S_{yy}^{-1}) = S_{yy.x}^{-1}$$

$$(AS_{xx.y}^{-1}+I) = (A+S_{xx.y})S_{xx.y}^{-1} = S_{xx}S_{xx.y}^{-1}$$

$$\begin{aligned}(AS_{xx.y}^{-1}+I)r &= S_{xy}(S_{yy}^{-1}S_{yx}S_{xx.y}^{-1}S_{xy}S_{yy}^{-1} + S_{yy}^{-1})S_{yy}\delta \\ &= S_{xy}S_{yy.x}^{-1}S_{yy}\delta\end{aligned}$$

$$\begin{aligned}H &= S_{xy}(S_{yy}^{-1}S_{yx}S_{xx.y}^{-1}S_{xy}S_{yy}^{-1} + S_{yy}^{-1})S_{yx} \\ &= S_{xy}S_{yy.x}^{-1}S_{yx}\end{aligned}$$

$$f = (S_{xx}^{-1} + A^{-1})r/2$$

$$\begin{aligned}Hf &= (HS_{xx}^{-1} + AS_{xx.y}^{-1} + I)r/2 \\ &= S_{xy}S_{yy.x}^{-1}(S_{yx}S_{xx}^{-1}S_{xy} + S_{yy})\delta/2 \\ &= S_{xy}S_{yy.x}^{-1}g\end{aligned}$$

where

$$g = (S_{yx}S_{xx}^{-1}S_{xy} + S_{yy})\delta/2$$

$$g = (S_{yy.x} + 2B)\delta/2$$

$$g'S_{yy.x}^{-1}g = \delta'(S_{yy.x} + 2B)S_{yy.x}^{-1}(S_{yy.x} + 2B)\delta/4$$

$$= \delta'S_{yy.x}\delta/4 + 4\delta'B\delta/4 + 4\delta'BS_{yy.x}^{-1}B\delta/4.$$

$$H = AS_{xx.y}^{-1}A + A$$

$$f = (S_{xx}^{-1} + A^{-1})S_{xy}\delta/2$$

$$f'Hf = \delta'S_{yx}(S_{xx}^{-1}+A^{-1})(AS_{xx.y}^{-1}A+A)(S_{xx}^{-1}+A^{-1})S_{xy}\delta/4$$

$$= \delta'S_{xy}S_{xx}^{-1}S_{xy}[S_{yy}^{-1}S_{yx}S_{xx.y}^{-1}S_{xy}S_{yy}^{-1}+S_{yy}^{-1}]S_{yx}S_{xx}^{-1}S_{xy}\delta/4$$

$$+ 2\delta'S_{yx}(S_{xx.y}^{-1}A+I)S_{xx}^{-1}S_{xy}\delta/4.$$

$$+ \delta'S_{yx}(S_{xx.y}^{-1}+A^{-1})S_{xy}\delta/4$$

$$= \delta'BS_{yy.x}^{-1}B\delta/4$$

$$+ 2\delta'S_{yx}S_{xx.y}^{-1}S_{xy}\delta/4$$

$$+ \delta'S_{yx}S_{xx.y}^{-1}S_{xy}\delta/4 + \delta'S_{yx}A^{-1}S_{xy}\delta/4$$

Using $S_{xx.y}^{-1} = S_{xx}^{-1} + S_{xx}^{-1}S_{xy}S_{yy.x}^{-1}S_{yx}S_{xx}^{-1}$, this can be written as

$$f'Hf = \delta'BS_{yy.x}^{-1}B\delta/4 + \frac{3}{4}\,\delta'B\delta + \frac{3}{4}\,\delta'BS_{yy.x}^{-1}B\delta/4$$

$$+ \delta'S_{yx}A^{-1}S_{xy}\delta/4$$

$$= g'S_{yy.x}^{-1}g - \delta'S_{yy.x}\delta/4$$

$$- \delta'B\delta/4 + r'A^{-1}r/4$$

Thus, after substitution, the inequality

$$\hat{\beta}'H\beta - 2\hat{\beta}'Hf + f'Hf < r'A^{-1}r/4 - r'S_{xx}^{-1}r/4$$

can be written as

$$\hat{\theta}'S_{yy.x}^{-1}\hat{\theta} - 2\hat{\beta}'S_{yy.x}^{-1}g + g'S_{yy.x}^{-1}g < \delta'S_{yy.x}\delta/4.$$

TABLE 1: Definitions of Variables

Net Export Data ($ thousands)

PETRO	Petroleum and Petroleum Products
MAT	Raw Materials
FOR	Forest Products
TROP	Tropical Agricultural Products
ANL	Animal Products
CER	Cereals and Fibers
LAB	Labor-Intensive Manufactures
CAP	Capital-Intensive Manufactures
MACH	Machinery
CHEM	Chemicals

Resources

CAPITAL	Millions of U.S. dollars of accumulated and discounted gross domestic investment flows since 1948, assuming an average life of 15 years.
LABOR 1	Thousands of workers classified as professional or technical.
LABOR 2	Thousands of literate nonprofessional workers.
LABOR 3	Thousands of illiterate workers
LAND 1	Thousands of hectares of land area in tropical rainy climate zone (comprises 30% of total area).
LAND 2	Thousands of hectares of land area in dry climate zone (comprises 30% of total area).
LAND 3	Thousands of hectares of land area in humid mesothermal climate zone (for example, California; comprises 15% of total area).
LAND 4	Thousands of hectares of land area in humid microthermal climate (for example, Michigan; comprises 17% of total area).
COAL	Thousands of dollars of production of primary solid fuels (coal, lignite, and brown coal).
MINERALS	Thousands of dollars of production of minerals: bauxite, copper, flourspar, ironore, lead, manganese, nickel, potash, pyrite, salt, tin, zinc. (Copper and iron ore make up about 50% of the value of minerals.)
OIL	Thousands of U.S. dollars of oil and gas production.

TABLE 2 Estimates of a 3 x 2 Model
(Diagonal Measurement Error Covariance Matrix)

	CAPITAL			LABOR			LAND		
	OLS	min	max	OLS	min	max	OLS	min	max
MACH	22.5	6.6	35.3	9.3	-461	595.	-13.	-24.	-3.9
CHEM	3.9	1.4	5.9	-3.9	-78.	87.2	-1.8	-3.5	-.4
	t	eov	nse	t	eov	nse	t	eov	nse
MACH	14.2	1.46	.145	.75	.127	.731	-6	-1.4	.441
CHEM	10.4	1.62	.145	-1.3	.055	.731	-3.6	-1.3	.441

Note: eov = (max+min)/(max-min)
nse = maximal error variance divided by measured variance

TABLE 3 Estimates of a 3 x 10 Model
(Nondiagonal Measurement Error Covariance Matrix)

	CAPITAL			LABOR			LAND		
	min	max	eov	min	max	eov	min	max	eov
MACH	19.6	23.7	10.6	-7.0	135	.901	-21.	-12.	-3.9
CHEM	3.87	4.41	15.3	-19.	-.47	-1.1	-2.3	-1.2	-3.1

TABLE 4 Estimates of a 3 x 10 Model
(Diagonal Measurement Error Covariance Matrix)

	CAPITAL				LABOR				LAND			
	min	max	eov	nse	min	max	eov	nse	min	max	eov	nse
MACH	18.7	27.9	5.07	.022	-188	151.	-.11	.63	-16.	-9.1	-3.8	.16
CHEM	3.56	4.89	6.35	.022	-40.	8.7	-.65	.63	-2.0	-1.1	-3.4	.16

Table 5: Extreme Estimates, Three Variable Model

Maximum

	PETRO	MAT	FOR	TROP	ANL	CER	LAB	CAP	MACH	CHEM
CAPITAL	-15.0	-2.6	-.8	-3.8	-1.4	5.7	-1.5	5.4	23.7	4.4
LABOR	6.9	.9	-2.1	22.8	-1.0	2.1	35.9	74.3	135.8	-.5
LAND	8.5	10.1	4.4	1.9	2.2	12.4	-2.1	-5.7	-12.1	-1.2

Minimum

	PETRO	MAT	FOR	TROP	ANL	CER	LAB	CAP	MACH	CHEM
CAPITAL	-17.7	-4.6	-1.8	-4.4	-1.9	2.9	-2.6	3.3	19.6	3.9
LABOR	-83.4	-67.5	-35.9	2.2	-18.7	-93.9	.9	2.3	-7.0	-19.2
LAND	3.2	6.1	2.4	.6	1.1	6.7	-4.2	-10.0	-20.5	-2.3

Table 6: Insensitivity Indicators and Noise Ratios, Three Variable Model

	PETRO	MAT	FOR	TROP	ANL	CER	LAB	CAP	MACH	CHEM	Noise
CAPITAL	-12.5	-3.6	-2.6	-13.8	-6.4	3.1	-4.0	4.2	10.5	15.3	.022
LABOR	-.8	-1.0	-1.1	1.2	-1.1	-1.0	1.0	1.1	.9	-1.1	.628
LAND	2.2	4.0	3.4	2.0	3.2	3.4	-3.0	-3.7	-3.9	-3.1	.158

TABLE 7: Extreme Estimates

<u>Maximum Estimates</u>

	PETRO	MAT	FOR	TROP	ANL	CER	LAB	CAP	MACH	CHEM
CAPITAL	-9.2	-6.4	.9	1.5	3.2	.5	3.7	24.4	41.6	9.5
LABOR	178.6	52.6	118.6	46.7	22.7	81.9	110.6	125.7	450.0	48.5
LAND1	42.8	14.4	13.0	18.6	17.0	22.2	5.9	28.9	45.0	20.2
LAND2	75.6	14.0	10.3	11.4	18.7	14.8	3.6	54.7	96.8	29.5
LAND3	41.4	18.6	33.3	45.7	50.1	69.4	49.1	27.0	-.5	6.9
LAND4	78.4	19.4	24.7	14.5	23.5	24.0	6.7	52.2	90.5	29.6
COAL	.8	1.0	.0	.0	.2	.7	-.3	.1	3.0	1.1
MIN	7.8	6.5	8.7	5.6	3.6	9.5	8.9	6.0	21.9	4.5
OIL	.7	.1	.4	.2	.1	.5	.2	.0	.2	.0

<u>Minimum Estimates</u>

	PETRO	MAT	FOR	TROP	ANL	CER	LAB	CAP	MACH	CHEM
CAPITAL	-31.7	-15.1	-11.3	-7.0	-4.6	-13.3	-7.3	8.2	-.4	-.8
LABOR	-125.8	-65.8	-46.7	-69.4	-83.1	-104.6	-39.1	-93.2	-119.9	-91.1
LAND1	-38.4	-17.2	-31.1	-12.4	-11.2	-27.6	-34.0	-29.5	-107.1	-17.0
LAND2	-24.0	-24.7	-43.7	-26.6	-15.9	-46.2	-45.4	-16.9	-89.6	-16.1
LAND3	-127.4	-47.1	-58.4	-18.8	-8.6	-34.2	-33.9	-94.4	-316.7	-70.6
LAND4	-52.3	-31.5	-46.3	-35.4	-21.9	-56.1	-57.6	-41.8	-154.3	-30.4
COAL	-1.4	.2	-1.1	-.9	-.6	-.7	-1.4	-1.5	-1.2	.0
MIN	-13.1	-1.7	-2.7	-2.4	-3.7	-3.4	-1.5	-9.1	-17.4	-5.1
OIL	.1	.0	.0	.0	-.1	.2	-.1	-.4	-.9	-.3

Table 8: Insensitivity Indicators and Noise Ratios
Indicator = (Max + Min)/ (Max - Min)

	PETRO	MAT	FOR	TROP	ANL	CER	LAB	CAP	MACH	CHEM	Noise
CAPITAL	-1.8	-2.5	-.8	-.6	-.2	-.9	-.3	2.0	1.0	.8	.006
LABOR	.2	-.1	.4	-.2	-.6	-.1	.5	.1	.6	-.3	.013
LAND1	.0	.0	-.4	.2	.2	-.1	-.7	.0	-.4	.0	.435
LAND2	.5	-.3	-.6	-.4	.0	-.5	-.9	.5	.0	.3	.090
LAND3	-.5	-.4	-.3	.4	.7	.3	.2	-.6	-1.0	-.8	.193
LAND4	.2	-.2	-.3	-.4	.0	-.4	-.8	.1	-.3	.0	.056
COAL	-.3	1.4	-.8	-1.1	-.5	.0	-1.5	-.9	.4	1.2	.006
MIN	-.3	.6	.5	.4	.0	.5	.7	-.2	.1	.0	.021
OIL	1.4	.2	1.3	.5	.0	1.8	.3	-1.0	-.6	-1.0	.012

Table 9: t-values

	PETRO	MAT	FOR	TROP	ANL	CER	LAB	CAP	MACH	CHEM
CAPITAL	-16.8	-16.5	-4.5	-6.9	-2.4	-7.4	1.5	15.9	13.2	5.5
LABOR	1.2	-3.0	.4	3.2	.1	-4.0	1.7	3.1	-.8	-2.9
LAND1	.6	.7	-1.1	2.1	-.4	.2	-.6	-1.0	.1	-.5
LAND2	2.8	.0	-1.6	-1.7	.0	-2.5	-.6	2.4	1.3	1.2
LAND3	-1.0	.5	.1	2.3	.7	3.7	.5	-3.2	-2.6	-2.4
LAND4	1.6	1.7	.8	-4.3	-.6	-2.5	-1.4	.3	.8	1.3
COAL	-2.4	5.1	-1.5	-5.6	-1.2	1.6	-2.7	3.2	3.4	4.7
MIN	-1.8	2.8	2.1	3.3	.7	3.4	.4	-1.3	-2.6	-2.0
OIL	7.1	.9	3.0	4.2	1.3	7.8	-.1	-5.4	-5.5	-3.7