EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

IN

SEALED HIGH BID AUCTIONS*

by

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A central assumption underlying almost all of the recent theoretical work on auctions has been that any two buyers with the same private estimate of the value of object for sale should have the same beliefs about the estimates of the other buyers. Given this strong symmetry assumption it is natural that the literature has focused almost exclusively on equilibria in which strategies are also symmetric. However, it is not unreasonable to suppose that a particular buyer might establish a reputation as an aggressive bidder if it is in his interest to do so. Riley [1980] provides an example of the "war of attrition" in which this is indeed the case.[1] Each of two buyers have identical beliefs about the other's valuation of the object for sale. Each must submit a sealed bid b_i, $i = 1,2$. The high bidder is the winner. The special feature of the war of attrition is that both buyers must pay the lower of the two bids. In addition to a unique symmetric equilibrium, Riley shows that there exists a continuum of asymmetric equilibria in which one buyer bids "aggressively" and the other "passively." Furthermore, the expected gain of the aggressive buyer is larger in each of the asymmetric equilibria than in the symmetric equilibrium.

In this paper we consider the possibility of asymmetric equilibria in the sealed high bid auction.

In Section I we consider the high bid auction under the assumption that private signals are independent. Then in Section II consider the more difficult case in which signals are positively correlated. The general conclusion is that, under rather mild assumptions, there exists a unique equilibrium.

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[1] For a fascinating discussion of the art of "brinksmanship" using a generalization of the simple war of attrition see B. J. Nalebuff [1982]. Similar problems arise in certain formulations of R&D races. See also Nalebuff and Riley [1985].

I. The High Bid Auction With Independent and Symmetric Beliefs

Differences among buyers are parameterized by s. Without loss of generality we normalize so that the utility of a buyer of type s who does not make a successful bid is zero. If a bid of b is successful the buyer's gain in utility is $U(-b;s)$. We impose the following assumptions on U and on the distribution of the different types of buyer.

Assumption A: Characterization of Preferences

A buyer with parameter value s who pays b for the object has an increase in utility of $U(-b;s)$ satisfying

A1: $U(-b;s)$ is twice continuously differentiable

A2: $U_1 > 0$

A3: $U_{11} < 0$

A4: $U_2 > 0$

Assumption B: Characterization of Beliefs

Each buyer's beliefs about the "signal" s of any other buyer are described by the c.d.f. $F(.)$ with $F(\underline{s}) = 0$, $F(\bar{s}) = 1$, and $F(.)$ is strictly increasing and continuously differentiable on $(\underline{s},\bar{s})$.

Elsewhere (Maskin and Riley [1984]) we have argued that these assumptions are sufficiently general to incorporate a wide range of interesting cases.

In the sealed high bid auction each buyer can submit a bid b not less than the announced floor price b^o. The buyer submitting the highest bid pays his bid and is awarded the object. In the case of ties the winner is selected at random. The central result of this section is that the only noncooperative equilibrium set of bidding strategies for the n buyers is the unique symmetric equilibrium. More precisely we establish

Theorem 1: Uniqueness of Equilibrium in the High Bid Auction

In a high bid auction with $n \geq 2$ buyers suppose the seller announces a minimum price b^o such that some buyer type $s^o > \underline{s}$ is indifferent between buying and not buying at this price, that is

$$U(-b^o;s^o) = 0 \qquad s^o \in (\underline{s},\bar{s})$$

Then if Assumptions A and B are satisfied and $\frac{\partial}{\partial s}(\frac{U_1}{U}) < 0$, there exists a unique equilibrium.

To prove Theorem 1 we proceed with a sequence of preliminary Lemmas, each of which is stated under the assumption that the hypotheses of Theorem 1 are satisfied. First, however, some remarks are in order on the hypothesis that U_1/U is a decreasing function of s.

Since

$$\frac{\partial}{\partial s}(\frac{U_1}{U}) = \frac{U_{12}U - U_1U_2}{U^2}$$

this condition necessarily holds as long as U_{12} is nonpositive. Moreover, suppose we introduce a small probability p that the object will be withdrawn and define $\delta(s)$ to be the insurance that a buyer of type s would be willing to pay to eliminate the resulting risk. Since we have normalized by setting $U(0,s) = 0$, $\delta(s)$ must satisfy

$$(1 - p)U(-b;s) = U(-b - \delta(s);s)$$

Taking a Taylor expansion of the right hand side and collecting terms we obtain

$$\delta(s) \simeq p\,\frac{U(-b;s)}{U_1(-b,s)}$$

Thus our hypothesis is the innocuous requirement that an individual who values the object more is willing to pay more to insure against the risk of the object's withdrawal.

To prove our theorem we begin by deriving a sequence of necessary conditions for equilibrium (Lemma 1-7). We then show that there can be no asymmetric equilibrium satisfying these conditions and that any symmetric equilibrium must satisfy some strong regularity restrictions. The latter imply uniqueness.

Lemma 1: Suppose $\hat{b}$ is optimal for buyer 1 with parameter $\hat{s}$. Suppose furthermore that the probability of buyers $3,\ldots,n$ bidding in $[\hat{b} - \varepsilon, \hat{b}]$ is zero. Then for all $\delta \in (0,\varepsilon)$ the probability of buyers 1 and 2 bidding in $[\hat{b} - \delta, \hat{b})$ is strictly positive.

Proof: Let $G_i(b)$ be the probability that buyer i wins with a bid of b. If for some δ the probability of buyers $2,\ldots,n$ bidding in $[\hat{b} - \delta, \hat{b})$ is zero $G_1(b) = G_1(\hat{b}-\delta)$, $b \in [\hat{b} - \delta, \hat{b}]$. Then the expected utility of buyer 1 is

$$G_1(\hat{b}-\delta)U(-b,\hat{s}), \quad b \in [\hat{b}-\delta,\hat{b})$$

Since this is strictly decreasing in b, buyer 1 is better off bidding $\hat{b} - \delta$ than $\hat{b}$, contradicting the definition of $\hat{b}$. Then the Lemma holds for buyer 2.

Now suppose that for some δ the probability of buyer 1 bidding in $[\hat{b} - \delta, \hat{b})$ is zero then $G_2(b) = G_2(\hat{b}-\delta)$ for $b \in [\hat{b} - \delta, \hat{b})$. Applying the same argument as before, buyer 2 is strictly better off bidding $\hat{b} - \delta$ than in the interval $[\hat{b} - \delta, \hat{b})$. But then the probability of buyer 2 bidding in $[\hat{b} - \delta/2, \hat{b})$ is zero, a contradiction. Q.E.D.

Lemma 2: If $\tilde{b}_i(s_i)$, $i = 1,\ldots,n$ is an equilibrium bidding strategy (possibly a mixed strategy) then $\tilde{b}_i(.)$ is strictly increasing when it exceeds b^o.

<u>Proof</u>: We show that if $\bar{b}_i(.)$ is a deterministic selection from $\tilde{b}_i(.)$ (i.e., for all s, $\bar{b}_i(s)$ is in the support of $\tilde{b}_i(s)$), then $\bar{b}_i$ must be strictly increasing for $\bar{b}_i(s) > b^o$.

Suppose that for some i $\bar{b}_i(.)$ is decreasing over an interval. Then there exists x^1 and x^2 with $x^1 < x^2$ such that $\bar{b}_i(x^1) > \bar{b}_i(x^2)$. Hence $\bar{G}^1 > \bar{G}^2$, where $\bar{G}^j$, j = 1,2 is the probability that buyer i wins if he bids $\bar{b}_i(x^j)$; otherwise if he has parameter x^1 he is better off bidding $\bar{b}_i(x^2)$. By definition of equilibrium

(1) $$\bar{G}^2 U(-\bar{b}(x^2);x^2) > \bar{G}^1 U(-\bar{b}(x^1);x^2)$$

and

(2) $$\bar{G}^2 U(-\bar{b}(x^2);x^1) < \bar{G}^1 U(-\bar{b}(x^1);x^1) \quad .$$

Combining (1) and (2), we obtain

(3) $$\frac{U(-\bar{b}(x^2);x^2))}{U(-\bar{b}(x^2);x^1))} > \frac{U(-\bar{b}(x^1);x^2))}{U(-\bar{b}(x^1);x^1))}$$

Also, for any b^1, b_2

(4) $$0 < \frac{U(-b^2.x^2)}{U(-b^2,x^1)} - \frac{U(-b^1,x^2)}{U(-b^1,x^1)}$$

=>

$$0 < \log \frac{U(-b^2,x^2)}{U(-b^2,x^1)} - \log \frac{U(-b^1,x^2)}{U(-b^1,x^1)}$$

$$= \int_{b^1}^{b^2} \int_{x^1}^{x^2} - \frac{\partial}{\partial x} \left(\frac{U_1}{U}\right) dx\, db$$

Then, since $\frac{\partial}{\partial x}\left(\frac{U_1}{U}\right) < 0$, (4) holds if and only if $b^2 > b^1$. Hence (3) holds only if $\bar{b}(x^2) > \bar{b}(x^1)$ contradicting our hypothesis. Thus $\bar{b}_i(\ )$ and hence $\tilde{b}_i(\ )$ is nondecreasing.

Now suppose that $\tilde{b}_i(s) = \hat{b}$ on the interval $[x^1,x^2]$ where $\hat{b} > b^o$. If there exists $j \neq i$ and a sequence $\{s^t\}$ such that $\bar{b}_j(s^t)$ converges to $\hat{b}$ from below, then

$$\text{(5)} \qquad \limsup_{t\to\infty} G_j(\bar{b}_j(s^t)) < G_j(\hat{b})$$

where $G_j(b)$ is buyer j's probability of winning from bidding b. For t sufficiently large $\bar{b}_j(s^t) > b^o$ and $G_j(b_j(s^t)) > 0$. Therefore,

$$\text{(6)} \qquad G_j(\bar{b}_j(s^t))U(-\bar{b}_j(s^t);s^t) > 0,$$

since buyer j, with parameter s^t can get positive utility by bidding $\hat{b}$. From (5) and (6) we conclude that for t large enough

$$G_j(\hat{b})U(-\hat{b};s^t) > G_j(-\bar{b}_j(s^t))U(-\bar{b}_j(s^t);s^t),$$

a contradiction of the assumption that $\bar{b}_j$ is a selection from an equilibrium strategy. Hence, the hypothesized sequences $\{s^t\}$ does not exist. But this means that there exists $\varepsilon > 0$ such that for all $j \neq i$ and $s \in [\underline{s},\bar{s}]$, $\bar{b}_j(s) \notin [\hat{b}-\varepsilon,\hat{b}]$. Thus if buyer i bids $b_i \in [\hat{b}-\varepsilon,\hat{b})$, $G_i(b_i) = G_i(\hat{b})$ and so $G_i(b_i)U(-b_i,x^1) > G_i(\hat{b})U(-\hat{b},x^1)$, a contradiction. Hence, $\bar{b}_i$ must be strictly increasing for $\bar{b}_i(\theta) > b^o$. Q.E.D.

Since Lemma 2 holds for all selections from $\tilde{b}_i$,

$$\text{(7)} \qquad y_i(\cdot) \equiv \tilde{b}_i^{-1}(\cdot)$$

is an increasing function that is well defined at all b for which there exists s with $b \in \text{supp}\, \tilde{b}_i(s)$. Then for all $b > b^o$ define

$$\text{(8)} \qquad \phi_i(b) = \sup\{y_i(\hat{b}) \mid \hat{b} < b,\ y_i(\hat{b}) \text{ defined}\}$$

Because y_i is increasing ϕ_i is nondecreasing and continuous for all $b > b^o$. Note furthermore that the probability of winning can be written as

$$\text{(9)} \qquad G_i(b) = \prod_{j\neq i} F(\phi_i(b)),\ b > b^o.$$

Since $\phi_j(b)$ is continuous for all j so is $G_j(b)$.

Lemma 3: If $\phi_i(b)$ is strictly increasing to the right at $b = \beta$, then $y_i(b)$ is defined at β.

Proof: If $\phi_i(b)$ is strictly increasing from the left the Lemma follows immediately. Then suppose that $\phi_i(b) = \hat{s}$ if and only if $b \in [\alpha,\beta]$. That is, for some $\hat{b} \in [\alpha,\beta]$, $y_i(\hat{b}) = \hat{s}$. Since $\phi_i(b)$ is strictly increasing to the right at β, there exists a decreasing sequence $\{b^1,\ldots,b^t,\ldots\}$ approaching β and a corresponding nonincreasing sequence $\{y_i(b^1),\ldots, y_i(b^t),\ldots\}$ approaching $\hat{s}$. Since b^t is optimal for parameter $y_i(b^t)$ we have

(10) $$G_i(b^t)\, U(-b^t;y_i(b^t)) - G_i(\hat{b})\, U(-\hat{b};y_i(b^t)) \geq 0, \quad \text{for all } t.$$

Since $G_i(.)$ and U are continuous, we have in the limit,

(11) $$G_i(\beta)\, U(-\beta;\hat{s}) - G_i(\hat{b})\, U(-\hat{b};\hat{s}) \geq 0.$$

From (11) it follows that buyer i, with parameter $\hat{s}$, is at least as well off choosing β as $\hat{b}$. Since $\hat{b}$ is optimal he must be equally well off choosing β. Q.E.D.

Lemma 4: If $\phi_i(b)$ is strictly increasing to the right at $\hat{b}$ then $G_i(b)$ is right differentiable at $\hat{b}$. Furthermore the right derivative satisfies

(12) $$G_i'(\hat{b})\, U(-\hat{b};\phi_i(\hat{b})) - G_i(\hat{b})\, U_1(-\hat{b};\phi_i(\hat{b})) = 0.$$

Proof: By Lemma 2 $\phi_i(b)$ is continuous. Then, since it is strictly increasing to the right at $\hat{b}$ there exists a decreasing sequence $\{b^1,b^2,\ldots\}$ approaching $\hat{b}$ such that $y_i(b^t)$ is defined for all t and approaches $\hat{s} = y_i(\hat{b})$ monotonically from above.

Since b^t is optimal for $s^t = y_i(b^t)$ we require

$$G_i(\hat{b})\ U(-\hat{b};y_i(b^t)) < G_i(b^t)\ U(-b^t;y_i(b^t)).$$

Subtracting $G_i(b^t)\ U(-\hat{b};y_i(b^t))$ from both sides, we obtain

$$[G_i(\hat{b}) - G_i(b^t)]\ U(-\hat{b};y_i(b^t)) < G_i(b^t)\ [U(-b^t;y_i(b^t)) - U(-\hat{b};y_i(b^t))]$$

Dividing through by $(b^t-\hat{b})\ U(-\hat{b},y_i(b^t))$ we then obtain

$$\text{(13)} \qquad \frac{G_i(b^t) - G_i(\hat{b})}{b^t - \hat{b}} > \frac{-G_i(b^t)}{U(-\hat{b};y_i(b^t))} \left| \frac{U(-b^t;y_i(b^t)) - U(-\hat{b};y_i(b^t))}{b^t - \hat{b}} \right|$$

By Lemma 3 $y_i(b)$ is defined at $\hat{b}$ and is equal to $\hat{s}$. Since $\hat{b}$ is optimal for $\hat{s}$ we require

$$G_i(\hat{b})\ U(-\hat{b};\hat{s}) > G_i(b^t)\ U(-b^t;\hat{s}) \quad \text{for all} \quad t.$$

Subtracting $G_i(b^t)\ U(-\hat{b};\hat{s})$ from both sides and then dividing by $(b^t-\hat{b})\ U(-\hat{b};\hat{s})$ we then obtain

$$\text{(14)} \qquad \frac{G_i(b^t) - G_i(\hat{b})}{b^t - \hat{b}} < -\frac{G_i(b^t)}{U(-\hat{b};\hat{s})} \left| \frac{U(-b^t;\hat{s}) - U(-\hat{b};\hat{s})}{b^t - \hat{b}} \right|$$

In the limit as $b^t \to \hat{b}$ the right hand sides of (13) and (14) coincide. Then $G_i(b)$ is right differentiable at $\hat{b}$. Moreover the right derivative satisfies (12). Q.E.D.

<u>Lemma 5</u>: $\phi_i(b)$ is right differentiable for all $b > b^o$ and all i.

<u>Proof</u>: Suppose $\phi_1(b),\ldots,\phi_k(b)$ are strictly increasing at $\hat{b}$ and that $\phi_{k+1}(b),\ldots,\phi_n(b)$ are constant at $\hat{b}$. By Lemma 1 $k \geq 2$. By Lemma 3 $G_i(b)$ is differentiable at $\hat{b}$, $i = 1,\ldots,k$. Also, from the definition of b^o in the statement of Theorem 1 $\phi_i(b^o) > \underline{s}$. Then $F(\phi_i(\hat{b})) > 0$ and we may take the logarithm of both sides of (9) to obtain

(15) $$\log G_i = \sum_{\substack{j=1 \\ j \neq i}}^{k} \log F(\phi_j(b)) + c_i$$

where

$$c_i \equiv \sum_{j=k+1}^{n} \log F(\phi_j(b))$$

Subtracting c_i from both sides we can express (15) in matrix form as follows:

$$\begin{bmatrix} \log G_1 - c_1 \\ \vdots \\ \log G_k - c_k \end{bmatrix} = B \begin{bmatrix} \log F(\phi_1(b)) \\ \vdots \\ \log F(\phi_k(b)) \end{bmatrix}$$

where $b_{pp} = 0$, $b_{pq} = 1$, $p \neq q$.

By inspection

$$B^{-1} = \frac{1}{k-1} \begin{bmatrix} -(k-2) & & 1 \\ 1 & -(k-2) \ \cdots & 1 \\ \vdots & & \\ 1 & 1 \ \cdots & -(k-2) \end{bmatrix} ; \quad k \neq 1$$

Then since $k \geq 2$ $\log F(\phi_i(b))$ and hence $\phi_i(b)$ is differentiable.

Q.E.D.

Lemma 6: For any pair of buyers r, v, if $\phi_r(b) > \phi_v(b)$ at $\hat{b}$ and $\phi_r(b)$ is strictly increasing then

$$\log (F(\phi_r(b))/F(\phi_v(b)))$$

is strictly increasing at $\hat{b}$.

Proof: By Lemma 2 $\phi_i(b)$ is a nondecreasing continuous function for all i. If $\phi_v(b)$ is constant at $\hat{b}$ the Lemma obviously holds. If $\phi_v(b)$ is strictly increasing at $\hat{b}$ we know from Lemma 4 that $G_r(b)$ and $G_v(b)$ are right differentiable at $\hat{b}$ and satisfy (12). Since $G_r(\hat{b})$ and $G_v(\hat{b})$ are strictly positive we may rewrite (12) as

$$\frac{d}{db} \log G_i(b) = \frac{U_1(-\hat{b};\phi_i(\hat{b}))}{U(-\hat{b};\phi_i(\hat{b}))} \tag{16}$$

Substituting (16) with $i = r$ from (16) with $i = v$ we then obtain

$$\frac{d}{db} \log (G_v(b)/G_r(b)) = -\left[\frac{U_1(-\hat{b};\phi_r(\hat{b}))}{U(-\hat{b};\phi_r(\hat{b}))} - \frac{U_1(-\hat{b};\phi_v(\hat{b}))}{U(-\hat{b};\phi_v(\hat{b}))}\right] \tag{17}$$

But, $\frac{\partial}{\partial s}(\frac{U_1}{U}) < 0$ and by hypothesis $\phi_r(\hat{b}) > \phi_v(\hat{b})$. Then the right hand side of (17) is positive. Moreover, from (9) the left hand side is

$$\frac{d}{db} \log(F(\phi_r(b))/F(\phi_v(b)))$$ Q.E.D.

<u>Lemma 7</u>: Suppose that $\phi_n(\hat{b}) = y_n(\hat{b})$ and that $\phi_n(\hat{b}) > \phi_i(\hat{b})$, $i \neq n$. Then $\phi_n(b)$ is strictly increasing to the right at $\hat{b}$.

<u>Proof</u>: If $\phi_i(b)$ is constant to the right at $\hat{b}$ for all $i < r$ the result follows immediately from Lemma 1. Then suppose that for $i = 1,\ldots,k$ $\phi_i(b)$ is strictly increasing to the right of $\hat{b}$ and for $i > k$ $\phi_i(b)$ is constant to the right of $\hat{b}$.

From the proof of Lemma 5, if all the right derivatives are zero, $G_i'(\hat{b}) = 0$, $i = 1,\ldots,k$. But by Lemma 3 $G_i'(\hat{b}) > 0$. Then the right derivative of $\phi_i(b)$ is strictly positive for some $i \leq k$. Without loss of generality suppose this to be true for $i = 1$. Then, from Lemma 4 the following condition must hold at $b = \hat{b}$ (where all derivatives are right derivatives).

$$-\left[\prod_{i=2}^{n} F(\phi_i(\hat{b}))\right] U_1(-b,\phi_1(\hat{b})) + \frac{d}{db}\left[\prod_{i=2}^{n} F(\phi_i(\hat{b}))\right] U(-b;\phi_1(\hat{b})) = 0$$

Multiplying this by $F(\phi_1(\hat{b}))/F(\phi_n(\hat{b}))$ and adding

(18) $$\left[\prod_{i=2}^{n-1} F(\phi_i(\hat{b}))\right] F'(\phi_1(\hat{b}))\ \phi_1'(\hat{b})\ U(-\hat{b};\phi_1(\hat{b}))$$

we have

$$- G_n(\hat{b})\ U_1(-\hat{b};\phi_1(\hat{b})) + G_n'(\hat{b})\ U(-\hat{b};\phi_1(\hat{b})) > 0$$

Note that the inequality is strict since, by hypothesis, $\phi_1(b)$ has a strictly positive right derivative at $\hat{b}$, so that (18) is strictly positive. We have therefore established that

$$G_n(x)\ U(-x;\phi_1(\hat{b}))$$

is right differentiable at $x = \hat{b}$ and that the right derivative is strictly positive. Define

$$\xi(b,\hat{b},s) = \frac{G_n(b)U(-b;s)}{G_n(\hat{b})U(-\hat{b};s)}$$

By inspection ξ is equal to 1 at $b = \hat{b}$. Moreover, for $s = \phi_1(\hat{b})$ we have established that ξ is strictly increasing in b at $b = \hat{b}$. Then, for some $\beta > \hat{b}$, $\xi(\beta,\hat{b},s) > 1$, $s = \phi_1(\hat{b})$. Furthermore, from (4), for any $s > \phi_1(\hat{b})$,

$$\frac{U(-\beta,s)}{U(-\beta,\phi_1(\hat{b}))} > \frac{U(-\hat{b},s)}{U(-\hat{b},\phi_1(\hat{b}))}$$

Hence

$$\frac{U(-\beta,s)}{U(-\hat{b},s)} > \frac{U(-\beta,\phi_1(\hat{b}))}{U(-\hat{b},\phi_1(\hat{b}))}$$

Then $\xi(\beta,\hat{b},s) > \xi(\beta,\hat{b},\phi_1(\hat{b})) > 1$, $s > \phi_1(\hat{b})$. But, by hypothesis $\phi_n(\hat{b}) > \phi_1(\hat{b})$. Then, from the definition of ξ,

$$G_n(\beta)\ U(-\beta;\phi_n(\hat{b})) - G_n(\hat{b})\ U(-\hat{b};\phi_n(\hat{b})) > 0.$$

But this contradicts the definition of $\phi_n(\hat{b})$. Q.E.D.

We are now in a position to prove Theorem 1. Define

$$\alpha = \inf\{\hat{b} \mid \log(F(\phi_i(b))/F(\phi_j(b))) < \varepsilon \quad \forall\, i,j \text{ and all } b > \hat{b}\}$$

We shall show that for any $\varepsilon > 0$ $\alpha = b^o$. Without loss of generality we may assume that the maximum bids by each buyer satisfy

$$b_1^* \geq b_2^* \geq \ldots \geq b_n^*$$

Suppose that $\alpha \geq b_i^*$ if and only if $i \leq k$. Certainly it cannot be the case that $b_1^* > b_2^*$ for if no other buyer bids more than b_2^* buyer 1 is strictly better off bidding $\frac{1}{2}(b_1^* + b_2^*)$ rather than b_1^*. Then $k \geq 2$.

From Lemma 2 $\phi_i(b)$ is continuous and nondecreasing. By Lemma 6, if $\phi_i(b)$ is strictly increasing at $b = \alpha$ for all $i = 1,\ldots,k$, then

$$\log(F(\phi_i(b))/F(\phi_j(b)))$$

is strictly increasing whenever $\phi_i(b) > \phi_j(b)$, $i,j < k$. But this contradicts the definition of α. Then for at least some $i < k$, $\phi_i(b)$ is constant to the right at $b = \alpha$. Suppose this to be true for $i \in I$ where $I \subset I^c = \{1,2,\ldots,k\}$. By Lemma 7

$$\max_{i \in I^c} \phi_i(\alpha) > \phi_i(\alpha) \quad i \in I$$

If $\min_i \phi_i(b)$ is constant at α we have a contradiction since it then follows that

$$F(\max \phi_i(b))/F(\min \phi_i(b))$$

is increasing at $b = \alpha$. But if $\min_i \phi_i(b)$ is strictly increasing at α we again have a contradiction by our earlier argument. Then $\alpha = b^o$. We have therefore etablished that for all $\varepsilon > 0$ and all i,j

$$\log(F(\phi_i(b))/F(\phi_j(b))) < \varepsilon, \quad b > b^o.$$

We have therefore proved that, under the hypotheses of the theorem, there exists no asymmetric equilibrium.

From Lemma 1, if there exists a symmetric equilibrium, $\phi_o(b)$, it is strictly increasing over the entire range of bids. Then, from Lemma 4, $G_o(b) \equiv F^{n-1}(\phi_o(b))$ is differentiable. Then $\phi_o(b)$ is also differentiable and, from (12), the derivative satisfies

$$(n-1)\ F^{n-2}(\phi_o)F'(\phi_o)\phi_o'(b)\ U(-b,\phi_o) = F^{n-1}(\phi_o)U_1(-b,\phi_o)$$

Rearranging we obtain

(19) $$\phi_o'(b) = \frac{F(\phi_o)}{(n-1)F'(\phi_o)}\,\frac{U_1(-b,\phi_o)}{U(-b,\phi_o)}$$

The ordinary differential equation (19) along with the boundary condition

(20) $$\phi_o(b_o) = s_o$$

uniquely defines a strictly increasing differentiable function $\phi_o(b)$.

Suppose all buyer except the first adopt the bidding strategy

$$b(v) = \phi_o^{-1}(s)$$

Then, if buyer 1, with parameter $s_1 = \phi_o(b_1)$ bids $b \neq b_1$ his expected utility is

$$u(b;b_1) = F^{n-1}(\phi_o(b))U(-b,\phi_o(b_1))$$

Differentiating by b and rearranging we obtain

$$\frac{\partial u}{\partial b}(b;b_1) = F^{n-2}(\phi_o(b_1))\ \left[\frac{(n-1)F'(\phi_o(b))\phi_o'(b)}{F(\phi_o(b))} - \frac{U_1(-b,\phi_o(b_1))}{U(-b,\phi_o(b_1))}\right]$$

Substituting for $\phi_o'(b)$ from (19) we obtain

$$\frac{\partial u}{\partial b}(b;b_1) = F^{n-2}(\phi_o(b_1))\ \left[\frac{U_1(-b,\phi_o(b))}{U(-b,\phi_o(b))} - \frac{U_1(-b,\phi_o(b_1))}{U(-b,\phi_o(b_1))}\right]$$

By hypothesis U_1/U is decreasing in s and by construction $\phi_o(b)$ is an increasing function. Hence

$$\frac{\partial u}{\partial b}(b;b_1) \gtreqless 0 \quad \text{as} \quad b \lesseqgtr b_1$$

Thus buyer 1's optimal response is to bid $b_1 = \phi_o^{-1}(s_1)$. Hence $\phi_o^{-1}(s)$ is the unique symmetric equilibrium. Q.E.D.

II. Existence and Uniqueness With Affiliated Signals

We now relax a key simplifying assumption of the above analysis, namely that buyers' signals are independently distributed. At the same time we allow the value one buyer places on an object to depend on other buyers' signals. Paralleling Assumptions A and B of Section I we impose the following restrictions on the preferences and beliefs of the different buyers.

Assumption C: Characterization of Preferences

If buyer i, $i = 1,\ldots,n$ pays b for the object his increase in utility is $U(-b;s_i,s_{-i})$, where

$$s_{-i} = (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$$

satisfying

C1: $U(-b,s_i,s_{-i})$ is twice continuously differentiable and symmetric in s_{-i}.

C2: $U(-b,s_i,s_{-i})$ is nondecreasing and strictly increasing in its first two arguments.

C3: $s_i > s_j \Rightarrow U(-b,s_i,s_j,s_{-i-j}) > U(-b,s_j,s_i,s_{-i-j})$

C4: $U_o(w,s) \equiv \frac{\partial}{\partial w} U(w,s)$ is a nonincreasing function.

C5: $s_i > s_j \Rightarrow U_o(-b,s_i,s_j,s_{-i-j}) < U_o(-b,s_i,s_j,s_{-i-j})$

Assumption D: Characterization of Beliefs

Buyers' signals are affiliated.[2]

Assumption C1 maintains the symmetry of the model. That is, if buyer i and j have the same private signal they will have the same beliefs about the signals of the other buyers. Assumption C3 requires that if the vector of private signals is held constant but the rank of buyer i's signal rises, the utility of the object to buyer i does not decline. Assumption C4 and C5 place related restrictions on marginal utility. C4 requires that if any signal becomes more favorable, marginal utility should not increase. C5 requires that if the vector of private signals is held constant but the rank of buyer i's signal rises, the marginal utility of the object to buyer i does not increase.

Taken together these assumptions should be interpreted as imposing the restrictions that (i) a buyer's own signal is not less influential than that of another buyer in determining value and (ii) buyers are risk averse. Assumptions C and D not only include the model of Section I as a special case but also that of Milgrom and Weber [1982]. In considering risk aversion, MW assumed preferences of the form

$$U(-b,s) = u(v(s_1,\ldots,s_n)-b), \quad u' > 0 \quad u'' < 0.$$

Clearly this satisfies Assumption C if $v(s)$ satisfies C3 and C5.

While Assumptions C and D are sufficient for a proof of the existence of an equilibrium, we shall make the following further assumption in the proof of uniqueness.

Assumption E: The conditional density $g(s_i|s_{-i})/G(s_i|s_{-i})$ is strictly decreasing in s_i.

---

[2]Actually all we need is the weaker assumption that signals are "linked". See Riley [1986].

Assumption E is a fairly mild restriction which requires that the density not rise too rapidly with s.[3]

We now provide conditions sufficient to ensure the existence of a symmetric equilibrium.

Theorem 2: Existence of a Symmetric Equilibrium[4]

If Assumptions C1-C4 and D hold there exists a symmetric equilibrium bid function

$$b_i = b(s_i), \quad i = 1,\ldots,n$$

for any minimum bid b^o between the pre-auction reservation prices of buyers with the lowest and highest possible signals.

Proof: Let $y = \text{Max}\{s_2,\ldots,s_n\}$. Then if agent 1 bids b and all the other signals are no greater than y, his expected utility is

$$\bar{U}(-b,s_1,y) = \underset{s_2,\ldots,s_n}{E} \{U(-b,s_1,\ldots,s_n) \mid s_1,y\} \tag{21}$$

By Assumption C2 U and hence $\bar{U}$ is a nondecreasing function. Moreover U is strictly increasing in its first two arguments $(U_o,U_1 > 0)$. By Assumption C4 $\bar{U}_o(-b,s_1,y)$ is a nonincreasing function.

Since the minimum price is between the reservation price of buyers with the correct and highest possible signals, there exists s^o such that

$$\int_0^{s^o} \bar{U}(-b^o,s^o,y)\, g(y|s^o)dy = 0. \tag{22}$$

---

[3]Assumption E is satisfied, for example, if the joint distribution of signals is normal or if $F(x|t) = x^{\alpha(t)}\beta(t)$.

[4]The proof of this result is based, in part, on the proof for the risk neutral case by Milgrom and Weber.

Define $b(s)$ to satisfy

$$\bar{U}(-b(s),s,s)\ g(s|s) - b'(s) \int_0^s \bar{U}_o(-b(s),s,y)\ g(y|s)dy = 0 \tag{23}$$

with endpoint condition $b(s^o) = b^o$.

Arguing almost exactly as in Riley (1986), it can be shown that Assumption C is sufficient to ensure the existence of such a function. To complete the proof we must show that $b(s)$ is indeed a best reply when all other buyers adopt it as their bidding strategy.

Since s^o satisfies (22) and $\bar{U}$ is nondecreasing

$$\bar{U}(-b(s^o),s^o,s^o) > 0.$$

Then $\qquad b'(s^o) > 0.$

Next define

$$u(x,s_1) = \int_0^x \bar{U}(-b(x),s_1,y)\ g(y|s_1)dy. \tag{24}$$

Then

$$\frac{\partial u}{\partial x}(x,s_1) = \bar{U}(-b(x),s_1,x)\ g(x|s_1) - b'(x) \int_0^x \bar{U}_o(-b(x),s_1,y)\ g(y|s_1)dy \tag{25}$$

From (23)

$$\frac{\partial u}{\partial x}(x,s_1) = 0 \quad \text{at} \quad x = s_1.$$

From Milgrom and Weber [1982] if the n private signals are affiliated then s_1 and $y \equiv \text{Max}\{s_2,\ldots,s_n\}$ are affiliated. Since $\bar{U}$ is strictly increasing in s_1 and nondecreasing in y it follows that

$$\frac{\partial u}{\partial s_1}(x,s_1) > 0.$$

Therefore $u(s,s)$ is a strictly increasing function. Also, from (22) $u(s^o,s^o) = 0$. Then $u(s_1,s_1) > 0$ for all $s_1 > s^o$. From (24)

$\bar{U}(-b(s_1),s_1,s_1) > u(s_1,s_1)$ and so

$$\bar{U}(-b(s_1)s_1,s_1) > 0, \quad s_1 > s^o.$$

Then, from (23)

$$b'(s) > 0, \quad s > s^o.$$

Thus (23) defines a strictly increasing function. But if all other buyers bid according to $b(s)$, and buyer 1 bids $b_1 = b(x)$ his expected utility is given by (24). Thus $b(s_1)$ is his best reply if $u(x,s_1)$ takes on its maximum at $x = s_1$. We have already seen that

$$\frac{\partial u}{\partial x}(x,s_1) = 0 \quad \text{at} \quad x = s_1.$$

Then, from (25)

$$(26) \qquad \frac{\frac{\partial u}{\partial x}(x,s_1)}{g(x|s_1)} = \frac{\frac{\partial u}{\partial x}(x,s_1)}{g(x|s_1)} - \frac{\frac{\partial u}{\partial x}(x,x)}{g(x|x)} = \bar{U}(-b(x),s_1,x) - \bar{U}(-b(x),x,x)$$

$$- b'(x) \int_0^x [\bar{U}_o(-b(x),s_1,y) \frac{g(y|s_1)}{g(x|s_1)} - \bar{U}_o(-b(x),x,y) \frac{g(t|x)}{g(x|x)}] dy$$

We can rewrite the integrand as

$$\bar{U}_o(-b(x),x,y) \frac{g(y|x)}{g(x|s_1)} \{\frac{\bar{U}_o(-b(x),s_1,y)g(y|s_1)}{\bar{U}_o(-b(x),x,y)g(y|x)} - \frac{g(x|s_1)}{g(x|x)}\}.$$

But, since y and s_1 are affiliated,

$$y < x \text{ and } x < s_1 \Rightarrow g(y|s_1)g(x|x) < g(y|x)g(x|s_1).$$

Rearranging we obtain

$$y < x \text{ and } x < s_1 \Rightarrow \frac{g(y|s_1)}{g(y|x)} < \frac{g(x|s_1)}{g(x|x)}.$$

Since $\bar{U}_o$ is a nonincreasing function it follows that the integrand in (26) is nonpositive for $x < s_1$. Then, since $\bar{U}$ is a nondecreasing function,

$$x < s_1 \Rightarrow \frac{\partial u}{\partial x}(x,s_1) > 0.$$

By a symmetrical arrangement we also have

$$x > s_1 \Rightarrow \frac{\partial u}{\partial x}(x,s_1) < 0.$$

Thus $b(s_1)$ is indeed buyer 1's best reply. Q.E.D.

Having established existence we now show that the equilibrium bid function characterized in Theorem 2 is the unique equilibrium. While a complete proof is provided only for the 2 buyer case we conjecture that our method of proof could be generalized further to cover the n buyer case.

As in Section 1 we begin by showing that under our primary assumptions, an equilibrium bid function must be nondecreasing.

Lemma 8: Under Assumptions C1-C2 and D, suppose $\hat{b}$ is optimal for buyer 1 with parameter $\hat{s}$. Suppose furthermore that the probability of buyers $3,\ldots,n$ bidding in $[\hat{b}-\varepsilon,\hat{b}]$ is zero. Then for all $\delta \in (0,\varepsilon)$ the probability of buyers 1 and 2 bidding in $[\hat{b}-\delta,\hat{b}]$ is strictly positive.

The proof is omitted since the argument is almost exactly the same as for Lemma 1.

Lemma 9: If Assumptions C1-C4 and D hold then buyer i's equilibrium bidding strategy $\tilde{b}_i(s_i)$ (possibly a mixed strategy) is nondecreasing in s_i.

Proof: While we consider only the two buyer case, the generalization to n buyers is only expositionally more complicated. Let s be the signal of buyer 1 and t be the signal of buyer 2. Suppose $s' < s''$ and that buyer 1 bids b' with positive probability when his signal is s' and $b'' < b'$ with positive probability when his signal is s''. Define

$$p(b|t) = \text{Prob}\,\{\text{buyer 1 wins with a bid of } b \text{ when buyer 2's signal is } t\}.$$

Then buyer 1's expected utility if he bids b is

$$\bar{u}(b,s) = \int_0^1 p(b|t)\ U(-b,s,t)\ g(t|s)dt.$$

It follows that

$$(27) \qquad \frac{\bar{u}(b',s) - \bar{u}(b'',s)}{g(0|s)} = \int_0^1 p(b'|t)\ [U(-b',s,t) - U(b'',s,t)]\ \frac{g(t|s)}{g(0|s)}\ dt$$

$$+ \int_0^1 U(-b'',s,t)\ [p(b'|t) - p(b''|t)]\ \frac{g(t|s)}{g(0|s)}\ dt.$$

Since $\bar{u}(b,s')$ takes on its maximum at $b = b'$

$$(28) \qquad \bar{u}(b',s') - \bar{u}(b'',s') \geq 0.$$

Also, since s and t are affiliated,

$$\hat{s} > s \Rightarrow g(0|s)g(t|\hat{s}) \geq g(0|\hat{s})g(t|s).$$

Rearranging we obtain

$$\hat{s} > s \Rightarrow \frac{g(t|\hat{s})}{g(0|\hat{s})} \geq \frac{g(t|s)}{g(0|s)}.$$

But $U(-b,s,t)$ is strictly increasing in s and, since $b' > b''$, $p(b'|t) \geq p(b''|t)$. Then the second integral on the right hand side of expression (27) is nondecreasing in s.

Moreover, the first integral on the right hand side of (27) can be rewritten as

$$\int_0^1 p(b'|t) \int_{b''}^{b'} - U_0(b,s,t)db\ \frac{g(t|s)}{g(0|s)}\ dt.$$

By Assumption C5 U_0 is strictly decreasing in s, therefore this expression is strictly increasing in s. It follows from (27) that

$$\frac{\bar{u}(b',s'') - \bar{u}(b'',s'')}{g(0|s'')} > \frac{\bar{u}(b',s') - \bar{u}(b'',s')}{g(0|s')} \geq 0,$$

by condition (28). But this contradicts the assumption that b" is optimal for buyer 1 when his signal is s". Q.E.D.

For the two buyer case Lemmas 8 and 9 imply that the equilibrium bidding strategies are strictly increasing continuous functions from $[s^o,1] \to [b^o,\bar{b}]$. Therefore the equilibrium inverse bid functions

$$y_i(\cdot) = b^{-1}(\cdot) \quad i = 1,2$$

are also strictly increasing. We now show that the equilibrium inverse bid functions are also differentiable.

Lemma 10: Under Assumptions C1-C2 and D the equilibrium bid inverse bid functions $y_i(\cdot) \equiv b^{-1}(\cdot) \quad i = 1,2$ are defined implicitly by the following system of differential equations

$$
(29) \qquad
\begin{aligned}
y_2'(b)U(-b,y_1,y_2)\,\frac{g(y_2|y_1)}{G(y_2|y_1)} &= \underset{s_2}{E}\{U_o(-b,y_1,s_2)\,|\,s_2 < y_2\} \\
y_1'(b)U(-b,y_2,y_1)\,\frac{g(y_1|y_2)}{G(y_1|y_2)} &= \underset{s_1}{E}\{U_o(-b,y_2,s_1)\,|\,s_1 < y_1\}
\end{aligned}
$$

Proof: Let $\{s^1,s^2,\ldots\}$ be an increasing sequence approaching $\hat{s}$. Then if $b^r = b(s^r)$, $\{b^1,b^2,\ldots\}$ is an increasing sequence approaching $\hat{b}$. Since b^r is optimal for s^r we require

$$\int_0^{y_2(b^r)} U(-b^r,s^r,t)g(t|s^r)dt > \int_0^{y_2(\hat{b})} U(-\hat{b},s^r,t)g(t|s^r)dt$$

Subtracting $\int_0^{y_2(b^r)} U(-\hat{b},s^r,t)g(t|s^r)dt$ from both sides we obtain

$$\int_0^{y_2(b^r)} [U(-b^r,s^r,t) - U(-\hat{b},s^r,t)]g(t|s^r)dt > \int_{y_2(b^r)}^{y_2(\hat{b})} U(-\hat{b},s^r,t)g(t|s^r)dt$$

Dividing both sides by $(\hat{b}-b^r)$ and taking limits we obtain

$$\int_0^{y_2(\hat{b})} U_o(-\hat{b},\hat{s},t)g(t|\hat{s})dt > U(-\hat{b},\hat{s},y_2(\hat{b}))g(y_2(\hat{b})|\hat{s}) \lim_{b^r\uparrow\hat{b}} \left[\frac{y_2(\hat{b})-y_2(b^r)}{\hat{b}-b^r}\right]$$

Similarly, since $\hat{b}$ is optimal for $\hat{s}$ we require

$$\int_0^{y_2(\hat{b})} U(-\hat{b},\hat{s},t)g(t|\hat{s})dt \geq \int_0^{y_2(b^r)} U(-b^r,\hat{s},t)g(t|\hat{s})dt$$

Subtracting $\int_0^{y_2(b^r)} U(-\hat{b},\hat{s},t)g(t|\hat{s})dt$ from both sides we obtain

$$\int_{y_2(b^r)}^{y_2(\hat{b})} U(-\hat{b},\hat{s},t)g(t|\hat{s})dt \geq \int_0^{y_2(b^r)} [U(-\hat{b},\hat{s},t) - U(-b^r,\hat{s},t)]g(t|\hat{s})dt$$

Again dividing by $\hat{b} - b^r$ and taking limits we obtain

$$U(-\hat{b},\hat{s},y_2(\hat{b}))g(y_2(\hat{b})|\hat{s}) \lim_{b^r \uparrow \hat{b}} \left[\frac{y_2(\hat{b})-y_2(b^r)}{\hat{b}-b^r}\right] \geq \int_0^{y_2(\hat{b})} U_0(-\hat{b},\hat{s},t)g(t|\hat{s})dt$$

Combining inequalities yields the first of the two differential equations defined by (29). Arguing symmetrically yields the second of the two differential equations. Q.E.D.

As a final preliminary to the uniqueness proof we also derive the following result.

Lemma 11: Suppose that s and t are affiliated and that $g(x|y)/G(x|y)$ is strictly decreasing in x. Suppose also that $\Omega(x,y)$ is a strictly increasing continuously differentiable function such that $\Omega_1(x,y) \geq \Omega_2(x,y)$. Then there exists $\delta > 0$ such that if $0 < t - s < \delta$

$$E_x \{\Omega(s,x) \mid x < t\} < E_x \{\Omega(t,s) \mid x < s\}.$$

Proof: Define

$$\omega(s,t) = E_y \{\Omega(s,y) \mid y < t\} = \int_0^t \Omega(s,x) \frac{g(x|s)}{g(t|s)} dx.$$

Differentiating by t

$$\omega_2(s,t) = \Omega(s,t)\,\frac{g(t|s)}{G(t|s)} - \frac{g(t|s)}{G(t|s)}\int_0^t \Omega(s,x)\,\frac{g(x|s)}{G(t|s)}\,dx.$$

Therefore

$$w_2(s,s) = \left\{\int_0^s [\Omega(s,s) - \Omega(s,x)]\,\frac{g(x|s)}{G(s|s)}\,dx\right\}\frac{g(s|s)}{G(s|s)}$$

$$= \left\{\int_0^s \Omega_2(s,x)\,\frac{G(x|s)}{G(s|s)}\,dx\right\}\frac{g(s|s)}{G(s|s)}.$$

Similarly define

$$\bar{\omega}(s,t) = E\{\Omega(t,x) \mid x < s\} = \int_0^s \Omega(t,x)\,\frac{g(x|t)}{G(s|t)}\,dx$$

Differentiating by t

$$\bar{\omega}_2(s,t) = \int_0^s \Omega_1(t,x)\,\frac{g(x|t)}{G(s|t)}\,dx + \frac{\partial}{\partial y}\,\underset{x}{E}\{\Omega(t,x) \mid x < y\}\Big|_{y=t}.$$

Since Ω is increasing and s and t are affiliated the second term is nonnegative. Also, by assumption, $\Omega_1(t,x) \geqslant \Omega_2(t,x)$. Therefore

$$\bar{\omega}_2(s,t) > \int_0^s \Omega_2(t,x)\,\frac{g(x|t)}{G(s|t)}\,dx$$

and so

$$\bar{\omega}_2(s,s) - \omega_2(s,s) > \int_0^s \Omega_2(s,s)\left[\frac{g(x|s)}{G(x|s)} - \frac{g(s|s)}{G(s|s)}\right]dx > 0 \quad \text{by hypothesis.}$$

Appealing to Taylor's Expansion it follows immediately that for all $\delta > 0$ and sufficiently small

$$0 < t - s < \delta \Rightarrow \bar{\omega}(s,t) - \omega(s,t) > 0. \qquad \text{Q.E.D.}$$

We can now prove

<u>Theorem 3</u>: Uniqueness

If Assumptions C, D and E hold the symmetric equilibrium is the only equilibrium in the 2 buyer auction.

Proof: We must show that there is a unique solution to the pair of differential equations defined by (29). Dividing the second equation by the first and rearranging we obtain

$$\frac{y_2'(b)}{y_1'(b)} = \left[\frac{\dfrac{g(y_1|y_2)}{G(y_1|y_2)}}{\dfrac{g(y_2|y_1)}{G(y_2|y_1)}}\right] \left[\frac{U(-b,y_2,y_1)}{U(-b,y_1,y_2)}\right] \left[\frac{\underset{s}{E}\{U_o(-b,y_1,s) \mid s < y_2\}}{\underset{s}{E}\{U_o(-b,y_2,s) \mid s < y_1\}}\right]$$

Suppose $y_2 > y_1$. Since s and t are affiliated $g(s|t)/G(s|t)$ is nondecreasing in t. Then the first bracketed expression is greater than or equal to

$$\frac{g(y_1|y_1)}{G(y_1|y_2)} \bigg/ \frac{g(y_2|y_2)}{G(y_2|y_2)}.$$

Furthermore, given Assumption C3 it follows immediately that the second bracketed expression exceeds unity. Finally, by C4 and C5 $U_o(-b,x,y)$ is nondecreasing in x and y and $U_{o1} < U_{o2}$. It follows from Lemma 11 that, for δ sufficiently small and $0 < y_2 - y_1 < \delta$, the third bracketed expression is greater than or equal to unity. Therefore,

$$\frac{y_2'(b)}{y_1'(b)} > \frac{g(y_1|y_1)}{G(y_1|y_2)} \bigg/ \frac{g(y_2|y_2)}{G(y_2|y_2)}. \tag{30}$$

Define

$$H(y) = \int_0^y \frac{g(x|x)}{G(x|x)}\,dx.$$

For $y \in [0,1]$, $H(y)$ is strictly increasing. Suppose that for some b, the equilibrium inverse bid functions $y_1(b)$, $y_2(b)$ satisfy

$$0 < y_2(b) - y_1(b) < \delta.$$

From what we have just shown

$$H(y_2(b)) > H(y_1(b))$$

and

$$H'(y_2)y_2'(b) - H'(y_1(y_1'(b)) = \frac{g(y_2|y_2)}{G(y_2|y_2)} y_2'(b) - \frac{g(y_1|y_1)}{G(y_1|y_1)} y_1'(b) > 0$$

Then $$\frac{\partial}{\partial b} [H(y_2(b)) - H(y_1(b))] > 0.$$

It follows that if $y_2(\hat{b}) > y_1(\hat{b})$, then $y_2(b) > y_1(b)$ for all $b > \hat{b}$. But, by Lemma 8, both buyers must have the same maximum bid $\bar{b}$. Thus there can be no such $\hat{b}$ and so $y_1(b) = y_2(b)$. Q.E.D.

III. Concluding Remarks

In this paper we have established existence and uniqueness of equilibrium in sealed high bid auctions under quite weak assumptions. While the final uniqueness theorem was only derived for the two buyer case we believe that an extension of our method of proof will yield a uniqueness theorem for the n buyer auction as well.

However, the analysis remains special in one important respect. That is, we follow most of the literature in assuming an underlying symmetry among buyers. To be precise, if buyers i and j happen to receive the same signal they will have identical beliefs about the signals of the other buyers. While this assumption is a natural first approximation there are many environments in which one or more buyers are known to be different. For example, in art auctions all other buyers have to give special consideration to the possibility of a bid from the Getty Museum.

While the existence theorem can be readily generalized to allow for asymmetry in buyers' beliefs, our proof of existence hinges critically on the symmetry assumption. Whether or not uniqueness hinges on this assumption therefore remains an open question.

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