STOCHASTIC DOMINANCE FOR TWO-STAGE LOTTERIES*

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June 1985

Revised July 1986

UCLA Working Paper #416

*I am grateful to Larry Epstein, Giora Hanoch, Joseph Ostroy, Joel Sobel, Shlomo Weber, and William Zame for their comments.

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Abstract

In this paper I discuss several possible extensions of the concept of stochastic dominance for two-stage lotteries. I show that the independence axiom, together with one such possible extension, which is strictly weaker than the reduction of compound lotteries axiom, yield expected utility theory. The main result of this paper is that the independence axiom, together with an irrelevance axiom and another possible extension of stochastic dominance for two-stage lotteries may serve as an axiomatic basis for anticipated utility theory.

1. Introduction

First order stochastic dominance is probably the most acceptable axiom ruling decision makers' behavior under uncertainty. Roughly speaking, this axiom states that if whatever happens, lottery X gives a higher prize than Y, where X and Y are two real random variables, then X is preferred to Y. It is well known that the extension of this axiom to more general random variables is not straightforward, especially when there is no natural complete order on the set of prizes. Levhari, Paroush, and Peleg (1975) discussed random variables over $\mathbb{R}^n$, while Hansen, Holt, and Peled (1978), and Fishburn and Vickson (1978) discussed more general spaces.

One of the most important spaces, though, neglected in this context, is the space of the lotteries themselves. However, as simple, one-stage lotteries are partially ordered by first order stochastic dominance, the concept of stochastic dominance is meaningful for the space of two-stage lotteries. This paper investigates possible extensions of the stochastic dominance definition for such lotteries.

As the concept of stochastic dominance for simple lotteries is independent of the decision maker's actual preferences, the suggested extensions of stochastic dominance for two-stage lotteries are objective in the sense that they do not depend on the decision maker's preferences on simple lotteries, nor do they depend on the way he evalutes two-stage lotteries. Green (1984) suggested a different approach, in which he justifies the independence axiom by using stochastic dominance arguments. According to his understanding of the nature of this concept, it may happen that one decision maker believes that the two-stage lottery A stochastically dominates B, while other people believe that B dominates A. This may happen when the possible outcomes in A and B cannot be compared through first order stochastic dominance. The stochastic dominance concepts I develop in this paper are different in the sense that they do not depend on the decision maker's preferences and they compare only those compound lotteries whose outcomes can be compared through first order stochastic dominance.

Expected utility analysis of two stage lotteries is usually based on the independence axiom, namely, the assumption that for every three lotteries X, Y, and Z, X is preferred to Y iff for every $0 < P < 1$, the two stage lottery yielding X with probability p and Z with probability 1-p is preferred to the same lottery with Y instead of X (Samuelson (1952)). To this it is necessary to add the reduction of compound lotteries axiom, that a two-stage lottery is equivalent to the simple lottery yielding the same prizes with reduced probabilities. This axiom, although self-evident when no real time is involved, is not at all natural in the context of lotteries over time especially if the decision maker makes new decisions at each stage. (For such models see Kreps and Porteus (1978) and Epstein (1985.)).

In recent years, several authors suggested to depart from expected utility theory either by weakening the independence axiom, or by totally omitting it. (See especially Machina (1982)). However, it is evident that decision makers are more willing to accept the independence axiom than the reduction of compound lotteries axiom (see, for example, Kahneman and Tversky (1979)). Moreover, the independence axioms has a clear normative appeal, while the reduction of compound lotteries axiom can hardly be justified in the context of lotteries over time. (See Sections 2 and 8 below.)

In this paper I assume only one decision period. In Section 3 I show that in the presence of either the independence axiom or the reduction of compound lotteries axiom, first order stochastic dominance and stochastic dominance for two stage lotteries are equivalent. Section 4 presents stronger versions of stochastic dominance. In Section 5 I prove that we can replace the reduction of compound lotteries axiom by a strong version of stochastic dominance, which is strictly weaker than the reduction axiom and which, together with continuity and the independence axiom, imply expected utility theory.

One of the new alternatives to expected utility is anticipated utility theory (or expected utility with rank-dependent probabilities). It was first developed by Quiggin (1982), and later on, independently, by Yaari (1985a). This theory suggests that decision makers are interested not only in the winning probabilities of every given prize, but also in the probability of receiving more than each possible outcome. Formally, it suggests that the value of the lottery $(x_1,p_1;\ldots;x_n,p_n)$, yielding x_i dollars with probability p_i, $i = 1,\ldots,n$, $x_1 < \ldots < x_n$, is

$$u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{j=i}^{n} p_j).$$

This theory is efficient in unravelling some well known paradoxes in expected utility theory, like the Allais paradox (Quiggin (1982), Segal (1985a)), the common ratio effect (Yaari (1985a), Segal (1985a)), the preference reversal phenomenon (Karni and Safra (1984)), and the Ellsberg paradox (Segal (1985b)). This paper shows that the concept of stochastic dominance for two-stage lotteries may be useful in establishing an axiomatic basis for this new theory (Section 6). The shape of the decision-weight function f is discussed in Section 7. Some concluding remarks appear in Section 9.

2. Definitions

Let $L_1 = \{(x_1,p_1;\ldots;x_n,p_n)\colon 0 < x_1 < \ldots < x_n,\ p_1 > 0,\ldots,p_n > 0,$ $\Sigma p_i = 1\}$. Elements of L_1, denoted by X, Y etc., represent simple lotteries, yielding x_i dollars with probability p_i, $i = 1,\ldots,n$. Let $X = (x_1,p_1,\ldots,x_n,p_n) \in L_1$, and define the cumulative distribution function F_X

by $F_X(x) = Pr(X \leq x)$.

On L_1 there exists a complete and transitive binary preference relation $\succsim$. $X \sim Y$ iff $X \succsim Y$ and $Y \succsim X$, and $X \succ Y$ iff $X \succsim Y$ but not $Y \succsim X$. Assume that the relation $\succsim$ satisfies the two following axioms:

Continuity: $\succsim$ is continuous in the topology of weak convergence. That is, if $X, Y, Y_1, Y_2, \ldots \varepsilon L_1$ such that at each continuity point x of F_Y, $F_{Y_i}(x) \to F_Y(x)$, and if for all i, $X \succsim Y_i$, then $X \succsim Y$. Similarly, if for all i $Y_i \succsim X$, then $Y \succsim X$.

First Order Stochastic Dominance (FOSD): If, for every x, $F_X(x) \leq F_Y(y)$, then $X \succsim Y$.

Let $X = (x_1, p_1; \ldots; x_n, p_n)$, $Y = (y_1, q_1; \ldots; y_m, q_m)$. It is well known that X dominates Y by FOSD iff for every nondecreasing real function u, $\Sigma p_i u(x_i) \geq \Sigma q_i u(y_i)$ (see Fishburn and Vickson (1978)).

Later on I use the following stronger version of stochastic dominance:

Strict First Order Stochastic Dominance: If, for every x, $F_X(x) \leq F_Y(x)$, and there is x for which $F_X(x) < F_Y(x)$, then $X \succ Y$.

$U: L_1 \to R$ represents the order $\succsim$ if for every $X, Y \varepsilon L_1$, $X \succsim Y$ iff $U(X) \geq U(Y)$. The most celebrated representation is the expected utility function.

(2.1) $$U(X) = \Sigma p_i u(x_i)$$

Preference relations represented by this function satisfy FOSD (provided u is an increasing function). If u is continuous, they satisfy the continuity axiom as well. Of course, these two axioms do not imply the expected utility functional. Further assumptions are required, either on $\succsim$

itself, or on its extension to two-stage lotteries.

One possible extension of (2.1) is the anticipated utility functional (Quiggin (1982), Yaari (1985a)):

$$V(X) = u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{j=i}^{n} p_j) \tag{2.2}$$

where $f(0) = 0$, $f(1) = 1$ and $u(0) = 0$. If f is linear, then this function reduces to the expected utility representation (2.1). It is easy to verify that this functional satisfies the continuity axiom (for continuous u and f) and FOSD (for increasing u and f). I discuss this theory in detail in Sections 6 and 7 below.

Let $L_2 = \{(X_1,q_1;\ldots;X_m,q_m)\colon \Sigma q_i = 1,\ \ q_1,\ldots,q_m > 0,\ \ X_1,\ldots,X_m \in L_1\}$. Elements of L_2, called two-stage lotteries, are denoted by A,B, etc. A lottery $A \in L_2$ yields a ticket to lottery X_i with probability q_i, $i = 1,\ldots,m$. More specifically, at time t_1 the decision maker faces the lottery $(1,q_1;\ldots;m,q_m)$. Upon winning the number i, he participates at time $t_2 > t_1$ in the lottery X_i. Assume that the decision maker's discount rate for future income is 1. Thus, once he knows that he won a certain amount of money, the actual time at which he receives this prize does not make any difference to him.

Let $\succsim_2$ be a complete and transitive preference relation on L_2. $\succsim_2$ is the decision maker's preference relation at time t_1 over two-stage lotteries, the only time at which he faces a choice problem. In particular, he is not allowed to make any new decision at time t_2.[1] The decision maker is time neutral, thus L_1 naturally becomes isomorphic to a subspace of L_2, where

[1] Other models, like those of Kreps and Porteus (1978) and Epstein (1985) assume several decision periods.

$(x_1,p_1;\ldots;x_n,p_n)$ and $((x_1,1),p_1;\ldots;(x_n,1),p_n)$ are equally attractive. The subscript 2 is therefore omitted and the preference relation over one- and two-stage lotteries is denoted by $\succsim$. A similar discussion holds for mixed lotteries, where the set of prizes is $\mathbb{R} \cup L_1$.

This last discussion is relevant only for lotteries of the form $((x_1,1),p_1;\ldots;(x_n,1),p_n)$. So far, nothing restricts the decision maker in comparing other lotteries in L_2 with lotteries in L_1. The following two axioms deal with such comparisons:[2]

1. Reduction of Compound Lotteries Axiom (RCLA): If the decision maker is indifferent to the resolution timing of the uncertainty, then he may assume both stages to be conducted at time t_1. Thus, a two-stage lottery is reduced to a simple one-stage lottery. Formally, let $X_i = (x^i_1,p^i_1;\ldots;x^i_{n_i},p^i_{n_i})$, $i = 1,\ldots,m$.

$$(3.1) \qquad (X_1,q_1;\ldots;X_m,q_m) \sim (x^1_1,q_1p^1_1;\ldots;x^1_{n_1},q_1p^1_{n_1};\ldots;x^m_1,q_mp^m_1;\ldots;x^m_{n_m},q_mp^m_{n_m})$$

2. Independence Axiom (IA): Let $X \succ Y$ and consider the lotteries $A = (X,q;Z,1-q)$ and $B = (Y,q;Z,1-q)$. With probability $1-q$ these two lotteries yield the same outcome Z. Therefore, the decision maker's attitude toward these lotteries depends on the alternative possible outcomes. As $X \quad Y$, the IA claims that he should also prefer A to B. Formally,

$$(X_1,q_1;\ldots;Y,q_i;\ldots;X_m,q_m) \succsim (X_1,q_1;\ldots;Z,q_i;\ldots;X_m,q_m) \iff Y \succsim Z$$

[2]In this section I restrict myself to a formal presentation of these axioms. For a more detailed discussion, including some remarks on the history of these axioms, see Section 8 below.

Let CE(X) be the certainty equivalent of X, given implicitly by $(CE(X),1) \sim X$. If $\succsim$ satisfies IA, then

(3.2) $$(X_1,q_1;\ldots;X_m,q_m) \sim (CE(X_1),q_1;\ldots;CE(X_m),q_m)$$

Finally, note that both IA and RCLA are compatible with continuity and FOSD. Formally,

<u>Theorem 1</u>: Let $\succsim$ be a continuous and transitive preference relation on L_1 and assume that $\succsim$ satisfies FOSD. $\succsim$ can be uniquely extended to L_2 to satisfy RCLA, and it can be uniquely extended to L_2 to satisfy IA.

<u>Proof</u>: Let $X_i = (x^i_1,p^i_1;\ldots;x^i_{n_i},p^i_{n_i})$, $i = 1,\ldots,\ell$, let $Y_i = (y^i_1,q^i_1;\ldots;y^i_{m_i},q^i_{m_i})$, $i = 1,\ldots,k$, and let $A = (X_1,p_1;\ldots;X_\ell,p_\ell)$, $B = (Y_1,q_1;\ldots;Y_k,q_k)$. $A \succsim B \Leftrightarrow (x^1_1,p_1p^1_1;\ldots;x^\ell_{n_\ell},p_\ell p^\ell_{n_\ell}) \succsim (y^1_1,q_1q^1_1;\ldots;y^k_{n_k},q_kq^k_{n_k})$ is the only extension satisfying RCLA, while $A \succsim B \Leftrightarrow (CE(X_1),p_1;\ldots;CE(X_\ell),p_\ell) \succsim (CE(Y_1),q_1;\ldots;CE(Y_k),q_k)$ is the only extension satisfying IA. Q.E.D.

To illustrate Theorem 1, consider extensions of the anticipated utility functional (2.2) to two-stage lotteries. Let $X_i = (x_1,p^i_1;\ldots;x_n,p^i_n)$, $i = 1,\ldots,m$, and let $A = (X_1,q_1;\ldots;X_m,q_m)$. Suppose that $\succsim$ can be represented by the anticipated utility functional (2.2). If $\succsim$ satisfies RCLA, then

(2.3) $$V(A) = u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{k=1}^{m}\sum_{j=i}^{n} q_kp^k_j).$$

On the other hand, if $\succsim$ satisfies the independence axioms, then by (2.2) $CE(X_k) = u^{-1}[u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{j=i}^{n} p^k_j)$ and if $CE(X_1) < \ldots < CE(X_m)$, then

$$(2.4)\qquad V(A) = V(A_1) + \sum_{k=2}^{m} [V(X_k)-V(X_{k-1})]f(\sum_{\ell=k}^{m} q_\ell) =$$

$$u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]\{f(\sum_{j=i}^{n} p_j^1) + \sum_{k=2}^{m} [f(\sum_{j=i}^{n} p_j^k) - f(\sum_{j=i}^{n} p_j^{k-1})]f(\sum_{\ell=k}^{m} q_\ell)\}.$$

It is easy to verify that (2.3) and (2.4) are equal if and only if f is linear, i.e., when anticipated utility reduces to expected utility.

3. Stochastic Dominance

This section discusses conditions under which the two stage lottery A stochastically dominates the two stage lottery B.

$U: L_1 \to \mathbf{R}$ is called an increasing function if whenever X dominates Y by FOSD, $U(X) \geq U(Y)$. Let U^* be the set of all the continuous and increasing functions $U: L_1 \to \mathbf{R}$.

Definition: Let $A = (X_1,p_1;\ldots;X_m,p_m)$, $B = (Y_1,q_1;\ldots;Y_\ell,q_\ell)$ be two two-stage lotteries. A dominates B by VWSD (very weak stochastic dominance) if for every $U \in U^*$, $\Sigma p_i U(X_i) \geq \Sigma q_i U(Y_i)$. $\succsim$ is said to satisfy the VWSD axiom if $A \succsim B$ whenever A dominates B by VWSD.

Remark: This should be the definition of stochastic dominance for two-stage lotteries according to Levhari, Paroush, and Peleg (1975), Hansen, Holt, and Peled (1978), and Fishburn and Vickson (1978). I call it very weak stochastic dominance because it is the weakest definition I discuss.

Lemma 2: Let $A = (X_1,\frac{1}{m};\ldots;X_m,\frac{1}{m})$, $B = (Y_1,\frac{1}{m};\ldots;Y_m,\frac{1}{m})$ be two two-stage lotteries. A dominates B by VWSD if and only if there is a permutation σ

on $\{1,\ldots,m\}$ such that $X_{\sigma(i)}$ dominates Y_i by FOSD, $i = 1,\ldots,m$.

<u>Proof:</u> The "if" part of the Lemma is obvious. Assume now that for all $U \varepsilon U^*$,

(3.1) $$\Sigma U(X_i) > \Sigma U(Y_i)$$

Let $X^* = \{Y: Y \succsim X \text{ by FOSD}\}$. Define $\phi(Z_1,\ldots,Z_\ell;\cdot)$: $L_1 \to R$ by

$$\phi(Z_1,\ldots,Z_\ell;X) = \begin{cases} 1 & X \varepsilon \quad Z_i^* \\ 0 & X \not\varepsilon \quad Z_i^* \end{cases}$$

and let

$$\psi(Y_{i_1},\ldots,Y_{i_\ell}) = \#\{(\bigcup_j Y^*_{i_j}) \cap \{X_1,\ldots,X_m\}\}$$

$\phi(Z_1,\ldots,Z_\ell;\cdot)$ is increasing (with respect to FOSD), hence by (3.1)

(3.2) $$\psi(Y_{i_1},\ldots,Y_{i_\ell}) > \ell$$

I now prove by induction on m that if (3.2) holds for all subsets of $\{Y_1,\ldots,Y_m)$, then there exists a permutation σ on $\{1,\ldots,m\}$ such that $X_{\sigma(i)}$ dominates Y_i by FOSD. For $m = 1$ the claim is obvious. Suppose that it holds true for $m - 1$, and prove it for m. If there exists $\emptyset \neq I \subseteq \{1,\ldots,m\}$ such that $\psi(\{Y_i\}_{i\varepsilon I}) = \#I$, then by the induction hypothesis we can define σ on I. For each $J \subseteq \{1,\ldots,m\} \setminus I$, it follows that

$$\#\{(\bigcup_{j\varepsilon J} Y^*_j) \cap \{X_i\}_{i\not\varepsilon I}\} > \#J$$

Otherwise, $\psi(\{Y_i\}_{i\varepsilon I \cup J}) < \#I + \#J = \#(I \cup J)$. By the induction hypothesis we can extend σ to $\{1,\ldots,m\} \setminus I$.

Suppose now that for every $\emptyset \neq I \subseteq \{1,\ldots,m\}$,

(3.3) $$\psi(\{Y_i\}_{i\varepsilon I}) > \#I$$

Let $X_i \in Y_1^*$ and define $\sigma(1) = i$. The sets $X' = \{X_j\}_{j \neq i}$ and $Y' = \{Y_j\}_{j \neq 1}$ satisfy the induction hypothesis. To see this, as with ψ, define ψ' for X' and Y'. If there exists $I \subseteq \{2,\ldots,m\}$ such that $\psi'(\{Y_i\}_{i \in I}) < \# I$, then $\psi(\{Y_i\}_{i \in I}) = \# I$, in contradiction to (3.3). Q.E.D.

Consider now two general two stage lotteries A and B. By the continuity assumption it follows that A dominates B by VWSD if and only if A and B can be rewritten as $A = (X_1,r_1;\ldots;X_\ell,r_\ell)$, $B = (Y_1,r_1;\ldots;Y_\ell,r_\ell)$ with X_i dominating Y_i by FOSD, $i = 1,\ldots,\ell$. This last observation leads to the following conclusion:

Theorem 3: If $\succsim$ satisfies RCLA, or if it satisfies IA, then it satisfies VWSD if and only if it satisfies FOSD.

4. Other Concepts of Stochastic Dominance

This section discusses some alternative definitions of stochastic dominance for two stage lotteries. FOSD for simple lotteries states that if for every x the probability of winning more than x under X is at least as large as the corresponding probability under Y, then X should be preferred to Y. The major problem in adapting this idea to two-stage lotteries is the lack of objective complete order on L_1. Instead, we can try to use an objective partial order on this space, namely, the partial FOSD order. Formally, let $X^* = \{Y\colon Y$ dominates X by FOSD$\}$. For each $A = (X_1,p_1;\ldots;X_m,p_m)$ and $S \subseteq L_1$, let $P_A(S) = \Sigma_{i:X_i \in S}\, p_i$. The above discussion suggests that if for every X, $P_A(X^*) \geq P_B(X^*)$, then $A \succsim B$. Denote this axiom by SD*. However, this is not the only possible extension. FOSD for simple lotteries also says that if for every x the probability of winning less than x under X is less than the corresponding probability under Y, then X is preferred to Y. Let $X_* = \{Y\colon X$ dominates Y by FOSD$\}$. This last observation leads to the assumption that if for all X, $P_A(X_*) \leq P_B(X_*)$, then $A \succsim B$. Denote this axioms by SD_*. I use these two axioms in Section 6 to construct an axiomatic basis for anticipated utility theory.

These two interpretations of stochastic dominance coincide on $\mathbf{R}$, but not on L_1. (Counterexamples can easily be constructed.) The following weak stochastic dominance axiom therefore seems a possible extension of FOSD:

<u>WSD</u>: If for every X, $P_A(X^*) \geq P_B(X^*)$, and if for every X, $P_A(X_*) \leq P_B(X_*)$, then $A \succsim B$.

Alternatively, one may suggest the following stochastic dominance axiom:

<u>SD</u>: If for every X, $P_A(X^*) \geq P_B(X^*)$, or if for every X, $P_A(X_*) \leq P_B(X_*)$, then $A \succsim B$.

The following theorem discusses the connection between these axioms and RCLA and VWSD:

<u>Theorem 4</u>: If $\succsim$ satisfies FOSD, then RCLA => SD => SD^*, SD_* => WSD => VWSD, but VWSD $\not\Rightarrow$ WSD; WSD $\not\Rightarrow$ SD^*, SD_*; SD^*, SD_* $\not\Rightarrow$ SD, and SD $\not\Rightarrow$ RCLA.

<u>Proof</u>:

RCLA => SD: I will prove that if $\succsim$ satisfies FOSD and RCLA, then $[\forall X\ P_A(X^*) \geq P_B(X^*)] \Rightarrow A \succsim B$. Let $A = (X_1,p_1;\ldots;X_m,p_m)$ and $B = (Y_1,p_1;\ldots;Y_m,p_m)$ (there is no loss of generality by assuming the same probabilities vectors), such that for all $Z \in L_1$, $P_A(Z^*) \geq P_B(Z^*)$. $\succsim$ satisfies FOSD and RCLA, hence it is sufficient to prove that for all x, $\Sigma p_i F_{X_i}(x) \leq \Sigma p_i F_{Y_i}(x)$. Let $Z_p = (0,1-p;x,p)$. For every p, $P_A(Z_p^*) \geq P_B(Z_p^*)$, hence

$$\Sigma_{i:F_{X_i}(x) \leq 1-p}\ p_i \geq \Sigma_{i:F_{Y_i}(x) \leq 1-p}\ p_i$$

It thus follows that

$$\Sigma_{i:F_{X_i}(x) > 1-p}\, p_i < \Sigma_{i:F_{Y_i}(x) > 1-p}\, p_i.$$

As stated in Section 2, X dominates Y by FOSD iff for every monotonic utility function u, $E_X[u] \geq E_Y[u]$. It thus follows that $\Sigma p_i F_{X_i}(x) \leq \Sigma p_i F_{Y_i}(x)$.

SD => SD^*, SD_*; SD^*, SD_* => WSD; WSD => VWSD: Obvious.

SD ≠> RCLA: Let $Z = (0,\frac{1}{2};1,\frac{1}{2})$. Define $V: L_1 \to \mathbf{R}$ by

$$V(X) = \begin{cases} 1 & X \in Z^* \\ 0 & X \notin Z^* \end{cases}$$

$\succsim$ on L_1 is represented by V, and $A = (X_1,p_1;\ldots;X_m,p_m) \succsim (Y_1,q_1;\ldots;Y_\ell,q_\ell)$ iff $V(V(X_1),p_1;\ldots;V(X_m),p_m) \geq V(V(Y_1),q_1;\ldots;V(Y_\ell),q_\ell)$. Obviously, $\succsim$ satisfies IA and FOSD. It also satisfies SD. Indeed, if $\forall X\ P_A(X^*) \geq P_B(X^*)$, then in particular $P_A(Z^*) \geq P_B(Z^*)$, and by FOSD $A \succsim B$. Suppose that $\forall X\ P_A(X_*) \leq P_B(X_*)$ and let

$$F_W(x) = \min\{\min\{F_{X_i}(x): X_i \notin Z^*\}, \min\{F_{Y_j}(x): Y_j \notin Z^*\}\}$$

It follows that

$$\Sigma_{i:X_i \in Z^*}\, p_i = 1 - \Sigma_{i:X_i \in W_*}\, p_i \geq 1 - \Sigma_{j:Y_j \in W_*}\, q_j = \Sigma_{j:Y_j \in Z^*}\, q_j$$

hence $A \succsim B$.

$\succsim$ does not satisfy RCLA. For example, $V(V(0,1),\frac{1}{3};V(0,\frac{1}{3};1,\frac{2}{3}),\frac{2}{3}) = V(0,\frac{1}{3};1,\frac{2}{3}) = 1$, but $V(V(0,\frac{5}{9};1,\frac{4}{9}),1) = V(0,1) = 0$, although these two lotteries are equivalent by RCLA.

WSD ≠> SD^*, SD_*; SD^*, SD_* ≠> SD: Construct counter-examples based on the observation that by SD_*, $A = ((0,\frac{1}{3};1,\frac{2}{3}),\frac{1}{2};(0,\frac{2}{3};2,\frac{1}{3}),\frac{1}{2}) \succ B = ((0,\frac{2}{3};1,\frac{1}{3}),\frac{1}{2};(0,\frac{1}{3};1,\frac{1}{3};2,\frac{1}{3}),\frac{1}{2})$, by SD^*, $B \succ A$, and by SD, $A \sim B$, while

WSD does not compare these two lotteries.

VWSD $\not\Rightarrow$ WSD: Contruct a counter-example based on the observation that by WSD,

$$A = ((0,\tfrac{1}{3};2,\tfrac{2}{3}),\tfrac{1}{3};(0,\tfrac{1}{3};1,\tfrac{1}{3};3,\tfrac{1}{3}),\tfrac{1}{3};(1,\tfrac{1}{3};2,\tfrac{1}{3};3,\tfrac{1}{3}),\tfrac{1}{3}) \succsim$$

$$B = ((0,\tfrac{1}{3};1,\tfrac{2}{3}),\tfrac{1}{3};(1,1),\tfrac{1}{3};(0,\tfrac{1}{3};2,\tfrac{1}{3};3,\tfrac{1}{3}),\tfrac{1}{3})$$

while VWSD does not compare these two lotteries. Q.E.D.

5. Expected Utility

Let $\succsim$ be a transitive and continuous preference relation. We know that if $\succsim$ satisfies RCLA and IA, then it can be represented by the expected utility functional (2.1). On the other hand, it follows from Theorem 3 that if $\succsim$ satisfies IA, then it also satisfies VWSD, hence all continuous and transitive preference relations on L_1 can be extended to L_2 through IA satisfying VWSD. As SD, SD*, SD$_*$, and WSD are successive weakenings of RCLA, the question naturally arises as to which preference relations are consistent with IA and these axioms. A partial answer to this question is given in this and in the next section.

Theorem 5: If $\succsim$ is continuous and satisfies IA and SD, then it can be represented by the expected utility function (2.1).

Proof: Let $X \in L_1$ and define $X^o = \{(x,p): p > F_X(x)\}$. Let $L_1^o = \{X^o: X \in L_1\}$, let $\Delta = \{[x,y] \times [p,q] \subset [0,\infty) \times [0,1]: x < y,\ p < q\}$, and let $\Psi = \{(X^o,\delta) \in L_1^o \times \Delta: \text{Int } X^o \cap \text{Int } \delta = \emptyset,\ X^o \cup \delta \in L_1^o\}$. Finally, for $S \in L_1^o$, S^+ is the lottery in L_1 such that $S = (S^+)^o$.

Proposition 5.1: Let $X,Y \in L_1$ such that $X^o \subseteq Y^o$, and let $\delta \in \Delta$ such that $(X^o,\delta),(Y^o,\delta) \in \Psi$. $A = ((X^o \cup \delta)^+,\tfrac{1}{2};Y,\tfrac{1}{2}) \sim (X,\tfrac{1}{2};(Y^o \cup \delta)^+,\tfrac{1}{2}) = B$.

Proof: Let $Z \in L_1$. If $P_A(Z^*) = 0$, then obviously $P_B(Z^*) > P_A(Z^*)$. If $P_A(Z^*) = \frac{1}{2}$, then either $(X^o \cup \delta)^+$ dominates Z by FOSD, but Y does not, or Y dominates Z by FOSD, but $(X^o \cup \delta)^+$ does not. In both cases $(Y^o \cup \delta)^+ \in Z^*$, hence $P_B(Z^*) > P_A(Z^*)$. If $P_A(Z^*) = 1$, then $X = ((X^o \cup \delta) \cap \dot{Y})^+ \in Z^*$, and $P_B(Z^*) = P_A(Z^*)$. By SD, $B \succsim A$. Similarly, for each $Z \in L_1$, $P_A(Z_*) < P_B(Z_*)$, hence $A \succsim B$. It thus follows that $A \sim B$.

Proposition 5.2: Let $(X^o, \delta), (Y^o, \delta) \in \Psi$. $(X^o \cup \delta)^+ \succsim (Y^o \cup \delta)^+ \iff X \succsim Y$.

Proof: By IA and Proposition 5.1, $(X^o \cup \delta)^+ \succsim (Y^o \cup \delta)^+ \iff$
$((X^o \cup Y^o)^+, \frac{1}{2}; (X^o \cup \delta)^+, \frac{1}{2}) \succsim ((X^o \cup Y^o)^+, \frac{1}{2}; (Y^o \cup \delta)^+, \frac{1}{2}) \iff$
$(((X^o \cup Y^o) \cup \delta)^+, \frac{1}{2}; X, \frac{1}{2}) \succsim (((X^o \cup Y^o) \cup \delta)^+, \frac{1}{2}; Y, \frac{1}{2}) \iff X \succsim Y$.

Define on Δ partial orders R_X by $\delta_1 R_X \delta_2$ iff $(X^o \cup \delta_1)^+ \succsim (X^o \cup \delta_2)^+$.

Proposition 5.3: For every X and Y, R_X and R_Y do not contradict each other. In other words, if δ_1 and δ_2 can be compared by both R_X and R_Y, then $\delta_1 R_X \delta_2$ iff $\delta_1 R_Y \delta_2$.

Proof: Let $X, Y \in L_1$ such that $(X^o, \delta_i), (Y^o, \delta_i) \in \Psi$, $i = 1,2$, and let $Z^o = X^o \cap Y^o$. Obviously, $Z^o \in L_1^o$, and $(Z^o, \delta_i) \in \Psi$, $i = 1,2$. There exist $\delta_1^1, \ldots, \delta_s^1$, and $\delta_1^2, \ldots, \delta_t^2$ such that $\forall j (Z^o \cup \bigcup_{k=1}^{j-1} \delta_k^i, \delta_j^i) \in \Psi$, $i = 1,2$, $X^o = Z^o \cup \bigcup_{k=1}^{s} \delta_k^1$, and $Y^o = Z^o \cup \bigcup_{k=1}^{t} \delta_k^2$. By Proposition 2, $\delta_1 R_X \delta_2 \iff (Z^o \cup \delta_1^1 \cup \ldots \cup \delta_s^1 \cup \delta_1)^+ \succsim (Z^o \cup \delta_1^1 \cup \ldots \cup \delta_s^1 \cup \delta_2)^+ \iff \ldots \iff (Z^o \cup \delta_1)^+ \succsim (Z^o \cup \delta_2)^+ \iff \ldots \iff (Z^o \cup \delta_1^2 \cup \ldots \cup \delta_t^2 \cup \delta_1)^+ \succsim (Z^o \cup \delta_1^2 \cup \ldots \cup \delta_t^2 \cup \delta_2)^+ \iff \delta_1 R_Y \delta_2$.

For the proof of the next proposition, I assume strict first-order stochastic dominance. Although not essential, this assumption makes the proof of Proposition 5.4 less tiresome.

Let $R = \bigcup_{X \in L_1} R_X$. That is, $\delta_1 R \delta_2$ iff there exists $X \in L_1$ such that $\delta_1 R_X \delta_2$. We can prove that R is acyclic. That is, $\delta_1 R \delta_2 R \ldots R \delta_t R \delta_1$ imply $\delta_1 R \delta_t R \ldots R \delta_2 R \delta_1$. Let $\succsim^*$ be the transitive closure of R: $\delta_1 \succsim^* \delta_2$ iff there are $\delta_3, \ldots, \delta_t$ such that $\delta_1 R \delta_3 R \ldots R \delta_t R \delta_2$.

<u>Proposition 5.4</u>: There exist $V: L_1 \to R$ and $W: \Delta \to R$ such that

a. V represents the relation $\succsim$

b. W is finitely additive, i.e., if δ_1 $\delta_2 \in \Delta$, then $W(\delta_1 \cup \delta_2) = W(\delta_1) + W(\delta_2) - W(\delta_1 \cap \delta_2)$

c. If $X^o = \bigcup_{k=1}^{t} \delta_k$ where $\forall j (\bigcup_{k=1}^{j-1} \delta_k, \delta_j) \in \Psi$, then $V(X) = \sum_{k=1}^{t} W(\delta_k)$.

<u>Proof</u>: Let $[0,x] \times [0,p] \sim^* [x,y_1] \times [p,1]$ (see Figure 1)

[Insert Figure 1 here.]

and let $W([0,x] \times [0,p]) = W([x,y_1] \times [p,1]) = 1$. By the continuity assumption there exist z and w such that $[x,w] \times [p,1] \sim^* [0,z] \times [0,p] \sim^* [w,y_1] \times [p,1]$. Define $W([x,w] \times [p,1]) = W([w,y_1] \times [p,1]) = \frac{1}{2}$. This can be repeated again and again for the x as well as for the p axes. By the monotonicity assumption, the areas of all these rectangles will become smaller and smaller. W can thus be defined as an atomless, continuous, finitely additive measure on $[0,x] \times [0,p]$ and $[x,y_1] \times [p,1]$. Similarly, it can be defined for the rectangles $[y_i, y_{i+1}] \times [p,1] \sim^* [0,x] \times [0,p]$, $i = 1, \ldots$. By the continuity assumption, $\{y_i\}$ is not bounded. Indeed, let $\lim y_i = y < \infty$. For all i, $([0,y] \times [p,1])^+ \succ ([0,y_i] \times [p,1])^+ \sim (([0,y_{i-1}] \times [p,1]) \cup ([0,x] \times [0,p]))^+$, in contradiction of the continuity and monotonicity assumptions. This process defines a finitely additive measure W on $[x,\infty) \times [p,1]$, which can be extended to $[0,\infty) \times [0,p]$ and to $[0,x] \times [p,1]$, and thus to $[0,\infty) \times [0,1]$. Define V as in part c of Proposition 5.4. Because W is finitely additive, V does not depend on the choice of $\delta_1,\ldots,\delta_t$.

Let $X^o = \bigcup_{k=1}^{t} \delta_k$ and $Y^o = \bigcup_{\ell=1}^{s} \zeta_\ell$ where $\forall j \; (\bigcup_{k=1}^{j-1} \delta_k, \delta_j), (\bigcup_{\ell=1}^{j-1} \zeta_\ell, \zeta_j) \in \Psi$, such that $X \succsim Y$. Let $k_o < s$ be the first index for which $\delta_{k_o} \not\sim^* \zeta_{k_o}$. If $\delta_{k_o} \succ^* \zeta_{k_o}$, construct $\zeta'_{k_o} \in \Delta$ such that $\zeta'_{k_o} \sim^* \delta_{k_o}$ and $\zeta_{k_1} = C\ell(\zeta_{k_o} \setminus \zeta'_{k_o}) \in \Delta$. We can therefore assume, without loss of generality, that for $k < t$, $\delta_k \sim^* \zeta_k$. Obviously, $V((\bigcup_{k=1}^{t} \delta_k)^+) = V((\bigcup_{\ell=1}^{t} \zeta_\ell)^+)$, and as $s > t$, it follows that $V(X) > V(Y)$.

I now turn to the proof of the theorem. Let $(x_1,\frac{1}{n};\ldots;x_n,\frac{1}{n}) \in L_1$, $x_1 < \ldots < x_n$. By the continuity assumption, there exist $0 < p < 1$ and $y_1,\ldots,y_n$ such that $X_i = (0,p;y_i,1-p) \sim (x_i,1)$. By IA, $(X_1,\frac{1}{n};\ldots;X_n,\frac{1}{n}) \sim (x_1,\frac{1}{n};\ldots;x_n,\frac{1}{n})$. Let k be the first index for which $x_i > 0$, and let $0 < \varepsilon < p$. Define $Y_i = (0,p-\varepsilon;y_k,\varepsilon;y_i,1-p)$, $i = k,\ldots,n$. Let $i,j > k$. By SD,

$$(X_1,\tfrac{1}{n};\ldots;Y_i,\tfrac{1}{n};\ldots;X_n,\tfrac{1}{n}) \sim (X_1,\tfrac{1}{n};\ldots;Y_j,\tfrac{1}{n};\ldots;X_n,\tfrac{1}{n})$$

IA implies that on L_2, $\succsim$ can be represented by $\phi(V(X_1),\frac{1}{n};\ldots;V(X_n),\frac{1}{n})$. By Proposition 5.4 there exists $\beta > 0$ such that $V(Y_i) = V(X_i) + \beta$, $i = 1,\ldots,n$. Moreover, for every sufficiently small β, there exists an appropriate ε. It thus follows that for every $(y_1^*,\ldots,y_n^*)$ there exists β^* such that for every $(y_1,\ldots,y_n)$[3] satisfying $y_i < y_i^*$, $i = 1,\ldots,n$, for every $0 < \beta < \beta^*$, and for every i and j,

$$\phi(y_1,\ldots,y_i + \beta,\ldots,y_n) = \phi(y_1,\ldots,y_j + \beta,\ldots,y_n) \tag{5.1}$$

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[3]The assumption that $y_1 < \ldots < y_n$ is not essential, because the value of a lottery depends on its prizes and not on their order.

Let $(y_1,\ldots,y_n) \in \mathbb{R}_+^n$. Let $y^* = \Sigma y_i$, and let β^* be appropriate for $(y^*,\ldots,y^*)$. Let $(z_1,\ldots,z_n) \in \mathbb{R}_+^n$ such that $\Sigma z_i = y^*$ and $\max |y_i - z_i| < \frac{\beta^*}{n}$. By (5.1).

$$\phi(y_1,\ldots,y_n) = \phi(y_1,\ldots,y_{n-1} + y_n - z_n, z_n) =$$

$$\phi(y_1,\ldots,y_{n-2} + y_{n-1} - z_{n-1} + y_n - z_n, z_{n-1}, z_n) = \ldots = \phi(z_1,\ldots,z_n)$$

hence $\phi(V(x_1),\frac{1}{n};\ldots;V(X_n),\frac{1}{n}) = f(\Sigma V(X_i))$. Let $x > 0$. $V(x,1) = \phi(V(x,1),\frac{1}{n};\ldots;V(x,1),\frac{1}{n}) = f(nV(x,1))$, hence $f(\alpha) = \frac{\alpha}{n}$. It thus follows that $\phi(V(X_1),\frac{1}{n};\ldots;V(X_n),\frac{1}{n}) = \frac{1}{n}\,\Sigma V(X_i)$. By the continuity assumption it follows that

$$\phi(V(X_1),p_1;\ldots;V(X_n),p_n) = \Sigma p_i V(X_i)$$

Let $u(x) = V(x,1)$. It follows that on L_1, $\succsim$ can be represented by $\Sigma p_i u(x_i)$. Q.E.D.

6. Anticipated Utility[4]

In the last years, several authors suggested alternatives to expected utility theory. One of the most promising of these new theories is the anticipated utility theory (also known as "expected utility with rank dependent probabilities"), first suggested by Quiggin (1982). The anticipated utility functional is a special case of an extension of Machina's functional (Machina (1982), Chew, Karni, and Safra (1985)), and it is helpful in solving several paradoxes, including the Allais paradox (Quiggin (1982), Segal (1985a)), the common ratio effect (Yaari (1985a), Segal (1985a)), the preference reversal phenomenon (Karni and Safra (1984)) and the Ellsberg paradox

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[4] I am especially thankful to Bill Zame for extensive discussions of this section.

(Segal (1985b)). This theory is a natural extension of expected utility theory, and it suggests that the value of the lottery X is

(6.1) $$V(X) = u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{j=i}^{n} p_j)$$

where f(0) = 0 and f(1) = 1. Note that when f is linear, this functional reduces to the expected utility form (2.1). In the general case, the anticipated utility functional is defined as

(6.2) $$\int_0^\infty u(x)df(1-F(x)).$$

Although its descriptive force is evident, this new theory did not receive so far a clear normative justification. Quiggin himself suggested to weaken the independence axiom, but an essential part of his axiomatic basis leads to the conclusion that $f(\frac{1}{2}) = \frac{1}{2}$. However, we now know that in this theory risk aversion is associated with convex f (see Chew, Karni, and Safra (1985), Yaari (1985b), and Segal (1985a)). Assuming that $f(\frac{1}{2}) = \frac{1}{2}$ thus takes a lot of power out of this theory.

Yaari (1985a) suggested another axiomatic basis, necessarily leading to the conclusion that the utility function u is linear. An attempt to obtain the general function (6.1) was done in Segal (1984), but this approach lacks in normative appeal. In this section I suggest what I believe to be a normatively appealing set of axioms implying (6.1) with general utility function u (thus avoiding the linearity of Yaari's functional) and which allows f to be either concave or convex (thus letting in the concept of risk aversion). This axiomatic basis includes the independence axiom and extended concepts of the first order stochastic dominance (or monotonicity) axiom. One advantage of this set of axioms is that it makes anticipated utility a natural extension of expected utility theory.

Consider again the first order stochastic dominance axiom. One possible interpretation of it is that if for every x, $Pr(X > x) > Pr(Y > x)$, then $X \succsim Y$. According to this interpretation, the decision maker is interested in the probability of receiving more (or less) than every possible outcome x. It is therefore a natural extension of this axiom to assume that whenever he compares X and Y, the decision maker examines those prizes x for which these probabilities do not equal. Formally, I suggest the following "irrelevance" axiom.

<u>Irrelevance</u>: Let $X,Y,X',Y' \in L_1$, and let $S \subset \mathbb{R}$. If on S, $F_X = F_Y$, $F_{X'} = F_{Y'}$, and on $\mathbb{R} \setminus S$, $F_X = F_{X'}$, $F_Y = F_{Y'}$, then $X \succsim Y$ if and only if $X' \succsim Y'$. (See Figure 2).[5]

[Insert Figure 2 here.]

This axiom suggests that decision makers first eliminate all the points x for which the probability of receiving more than x is the same at X and Y. Then, they compare X and Y by their corresponding cumulative distribution functions at those points where these probabilities do not equal. Furthermore, they do it independently on the probabilities that are equal. This axioms resembles Savage's sure thing principle (Savage (1954)), but it is much weaker. Savage suggested the following axiom as a basic rule to be used in uncertain situations.

<u>Sure Thing Principle</u>: Let S be an event and let X, Y, X', and Y' be lotteries. If on S, $X = Y$, $X' = Y'$, and on "not S" $X = X'$, $Y = Y'$, then $X \succsim Y$ if and only if $X' \succsim Y'$.

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[5]This axiom is equivalent to the cancellation axiom in Segal (1984).

Among other things, this axiom assumes that the evaluation of the prizes available if S happens does not depend on the prizes available if "not S" happens. In particular, it does not depend on whether the prizes at "not S" are larger or smaller than those of S. It is well known that this axiom is unacceptable from a descriptive point of view, apparently because of this last objection. The irrelevance axiom, on the other hand, restricts the sure thing principle to those cases where the order of the prizes is not reversed. For example, the irrelevance axiom agrees with the sure thing principle that $(5,0.1;0,0.01;0,0.89) \succsim (1,0.1;1,0.01;0,0.89) \Leftrightarrow (5,0.1;0,0.01;-1,0.89) \succsim (1,0.1;1,0.01;-1,0.89)$, but it does not agree that $(5,0.1;0,0.01;0,0.89) \succsim (1,0.1;1,0.01;0,0.89) \Leftrightarrow (5,0.1;0,0.01;1,0.89) \succsim (1,0.1;1,0.01;1,0.89)$. Therefore, the irrelevance axiom does not rule out Allais-type behavior patterns, as does the sure thing principle. Indeed, the reason decision makers violate the sure thing principle through the Allais paradox is that replacing (0,0.89) by (1,0.89) makes (0,0.01) strictly worse than all other prices. (See Figure 3.)

[Insert Figure 3 here.]

<u>Lemma 6</u>: If $\succsim$ on L_1 is continuous and satisfies strict first-order stochastic dominance and the irrelevance axiom, then there is a finitely additive measure ν on $\mathbb{R}_+ \times [0,1]$ such that $X \succsim Y$ if and only if $\nu(X^o) \geq \nu(Y^o)$.

<u>Proof</u>: Follows from Propositions 5.3 and 5.4 in the proof of Theorem 5. See also the proof of the first part of Theorem 2.2 in Segal (1984).

In anticipated utility theory (and expected utility theory), the value of the lottery X is a measure of the set X^o. These theories require in addition that the measure is a product measure. The measure of $[x,y]$ is $u(y)-u(x)$, and the measure of $[p,q] \subseteq [0,1]$ is $f(1-p) - f(1-q)$, reducing to $q-p$ in the expected utility model. I now show that given a measure, SD^* and SD_* guarantee that it is a product measure.

As is clear from the discussion in Section 5, SD^* implies that $A = (X_1,p_1;\ldots;(X_i^o \; X_j^o)^+,p;\ldots;(X_i^o \; X_j^o)^+,p;\ldots;X_n,p_n) \succsim B = (X_1,p_1;\ldots;X_i,p;\ldots;X_j,p;\ldots;X_n,p_n)$ while by SD_*, $B \succsim A$. I do not know whether these conditions are equivalent to SD^* and SD_*, but they are certainly not stronger. I will therefore replace SD^* and SD_* by these weaker axioms.

<u>WSD*</u>: $(X_1,p_1;\ldots;(X_i^o \; X_j^o)^+,p;\ldots;(X_i^o \; X_j^o)^+,p;\ldots;X_n,p_n) \succsim (X_1,p_1;\ldots;X_i,p;\ldots;X_j,p;\ldots;X_n,p_n)$.

<u>WSD*</u>: $(X_1,p_1;\ldots;X_i,p;\ldots;X_j,p;\ldots;X_n,p_n) \succsim (X_1,p_1;\ldots;(X_i^o \cap X_j^o)^+,p;\ldots;(X_i^o \cup X_j^o)^+,p;\ldots;X_n,p_n)$.

The main result of this section is presented in the following theorem:

<u>Theorem 7</u>: Let $\succsim$ satisfy the continuity, strict first-order stochastic dominance, irrelevance and independence axioms. It can be represented by the anticipated utility functional (6.1) with concave (convex) f if and only if it satisfies the WSD^* (WSD_*) axiom.

<u>Proof</u>: By Lemma 6, $\succsim$ can be represented by a measure ν. I first prove that if $\succsim$ satisfies WSD^*, then it can be represented by (6.2) with concave f (Propositions 7.1-7.3). Then I show that if it can be represented by (6.2) with concave f, then it satisfies WSD^* (Proposition 7.4). The proof for the WSD_*-convex f case is similar.

<u>Proposition 7.1</u>: Let $x < y \leq x' < y'$ such that $\nu([x,y] \times [0,1]) = \nu([x',y'] \times [0,1])$. For every p,q, and γ such that $0 \leq p < p + \gamma \leq q <$

$q + \gamma \leq 1$, $\nu([x,y] \times [p,p+\gamma]) \leq \nu([x',y'] \times [q,q+\gamma])$.

Proof: Let X,Y L_1 and $\delta \in \Delta$ such that $X^o \subset Y^o$, $(X,\delta),(Y,\delta) \in \Psi$, $X \sim (x,1)$, $(X^o \cup \delta)^+ \sim (y,1)$, and $Y \sim (x',1)$. Since ν is a measure, $\nu(Y^o \cup \delta) - \nu(Y^o) = \nu(X^o \cup \delta) - \nu(X^o) = \nu([x,y] \times [0,1]) = \nu([x',y'] \times [0,1])$, hence $(Y^o \cup \delta)^+ \sim (y',1)$.

Let S, $T \in L_1$ such that $T \succsim (y',1),(x',1) \succsim S \succsim (y,1)$ and let s and t such that $S \sim (s,1)$ and $T \sim (t,1)$. By WSD* it follows that

$(0,p;X,\gamma;S,q-p-\gamma;(Y^o \cup \delta)^+,\gamma;T,1-q-\gamma) \succsim (0,p;(X^o \cup \delta)^+,\gamma;S,q-p-\gamma;Y,\gamma;T,1-q-\gamma) \Rightarrow$

$(0,\alpha;x,\gamma;s,q-p-\gamma;y',\gamma;t,1-q-\gamma) \succsim (0,\alpha;y,\gamma;s,q-p-\gamma;x',\gamma;t,1-q-\gamma) \Rightarrow$

$\nu([x',y'] \times [q,q+\gamma]) \geq \nu([x,y] \times [p,q])$.

Proposition 7.2: Let x,y,x', and y' be as in Proposition 7.1. For every $0 \leq p < q \leq 1$, $\nu([x,y] \times [p,q]) = \nu([x',y'] \times [p,q])$.

Proof: By Proposition 7.1 it follows that for every n and $i \leq n-2$, $\nu([x,y] \times [p + \frac{i}{n}(q-p),p + \frac{i+1}{n}(q-p)]) \leq \nu([x',y'] \times [p + \frac{i+1}{n}(q-p), p + \frac{i+2}{n}(q-p)])$, hence for every n, $\nu([x,y] \times [p,q - \frac{1}{n}(q-p)]) \leq \nu([x',y'] \times [p + \frac{1}{n}(q-p),q])$, and by the continuity of $\succsim$ it follows that $\nu([x,y] \times [p,q]) \leq \nu([x',y'] \times [p,q])$.

Similarly, $\nu([x,y] \times [0,p]) \leq \nu([x',y'] \times [0,p])$ and $\nu([x,y] \times [q,1]) \leq \nu([x',y'] \times [q,1])$. Since $\nu([x,y] \times [0,1]) = \nu([x',y'] \times [0,1])$, it follows that $\nu([x,y] \times [p,q]) = \nu([x',y'] \times [p,q])$.

Define $u(x) = \nu([0,x] \times [0,1])$.

Proposition 7.3: There is a concave function f such that $\nu([x,y] \times [p,q]) = [u(y) - u(x)][f(1-p) - f(1-q)]$.

Proof: By the definition of u, $\nu([x,y] \times [0,1]) = u(y) - u(x)$. By Proposition 2, if $u(y) - u(x) = \frac{m}{n}[u(y') - u(x')]$, then $\nu([x,y] \times [p,q]) = \frac{m}{n}\nu([x',y'] \times [p,q])$. Hence, by the continuity assumption, $\nu([x,y] \times [p,q]) = \theta(p,q)[u(y)-u(x)]$. Define $f(p) = \theta(1-p,1)$ and as ν is a measure, $\nu([x,y] \times [p,q]) = [u(y)-u(x)][f(1-p)-f(1-q)]$.

Finally, since for every n $\nu([x,y] \times [p,p+\frac{1}{n}]) < \nu([x,y] \times [p+\frac{1}{n},p+\frac{2}{n}])$ it follows that $f(1-p) - f(1-p-\frac{1}{n}) < f(1-p-\frac{1}{n}) - f(1-p-\frac{2}{n})$, hence f is concave.

Proposition 7.4: If $\succsim$ satisfies the independence axiom and can be represented by the anticipated utility functional (6.1) with concave f, then it also satisfies WSD*.

Proof: Let $A = (X_1,\frac{1}{n};\ldots;X_n,\frac{1}{n})$ and $B = (X_1,\frac{1}{n};\ldots;(X_i^o \cap X_j^o)^+,\frac{1}{n};\ldots;(X_i^o \cup X_j^o)^+,\frac{1}{n};\ldots;X_n,\frac{1}{n})$, that is,

$$B = (\underset{(1)}{X_1,\tfrac{1}{n}};\ldots;\underset{(\ell)}{X_\ell,\tfrac{1}{n}};\underset{(\ell+1)}{(X_i^o \cap X_j^o)^+,\tfrac{1}{n}};\underset{(\ell+2)}{X_{\ell+1},\tfrac{1}{n}};\ldots;\underset{(i)}{X_{i-1},\tfrac{1}{n}};\underset{(i+1)}{X_{i+1},\tfrac{1}{n}};\ldots;\underset{(j-1)}{X_{j-1},\tfrac{1}{n}};$$

$$\underset{(j)}{X_{j+1}^{1},\tfrac{1}{n}};\ldots;\underset{(k-1)}{X_k,\tfrac{1}{n}};\underset{(k)}{(X_i^o \cup X_j^o)^+,\tfrac{1}{n}};\underset{(k+1)}{X_{k+1},\tfrac{1}{n}};\ldots;\underset{(n)}{X_n,\tfrac{1}{n}})$$

It is sufficient to prove that the value of B is greater than that of A. By using the independence axioms and (6.1) it follows that the value of A is given by

$$W(A) = V(X_1) + \sum_{m=2}^{n} [V(X_m)-V(X_{m-1})]f(\frac{n-m+1}{n})$$

Hence,

$$W(B) - W(A) = -\sum_{m=\ell+2}^{i-1} [V(X_m) - V(X_{m-1})][f(\frac{n-m+1}{n}) - f(\frac{n-m}{n})] +$$

$$\sum_{m=j+2}^{k} [V(X_m) - V(X_{m-1})][f(\tfrac{n-m+2}{n}) - f(\tfrac{n-m+1}{n})] -$$

$$[V(X_{\ell+1}) - V((X_i^o \cap X_j^o)^+)][f(\tfrac{n-\ell}{n}) - f(\tfrac{n-\ell-1}{n})] -$$

$$[V(X_i) - V(X_{i-1})][f(\tfrac{n-i+1}{n}) - f(\tfrac{n-i}{n})] +$$

$$[V(X_{j+1}) - V(X_j)][f(\tfrac{n-j+1}{n}) - f(\tfrac{n-j}{n})] +$$

$$[V((X_i^o \cup X_j)^+) - V(X_k)][f(\tfrac{n-k+1}{n}) - f(\tfrac{n-k}{n})] =$$

$$\sum_{m=j+1}^{k} [V(X_m) - V(X_{m-1})][f(\tfrac{n-m+2}{n}) - f(\tfrac{n-m+1}{n})] +$$

$$[V((X_i^o \cup X_j^o)^+) - V(X_k)][f(\tfrac{n-k+1}{n}) - f(\tfrac{n-k}{n})] -$$

$$\sum_{m=\ell+2}^{i} [V(X_m) - V(X_{m-1})][f(\tfrac{n-m+1}{n}) - f(\tfrac{n-m}{n})] -$$

$$[V(X_{\ell+1}) - V((X_i^o \cap X_j^o)^+)][f(\tfrac{n-\ell}{n}) - f(\tfrac{n-\ell-1}{n})] >$$

$$\sum_{m=j+1}^{k} [V(X_m) - V(X_{m-1})][f(\tfrac{n-j+1}{n}) - f(\tfrac{n-j}{n})] +$$

$$[V((X_i^o \cup X_j^o)^+) - V(X_k)][f(\tfrac{n-j+1}{n}) - f(\tfrac{n-j}{n})] -$$

$$\sum_{m=\ell+2}^{i} [V(X_m) - V(X_{m-1})][f(\tfrac{n-i+1}{n}) - f(\tfrac{n-i}{n})] -$$

$$[V(X_{\ell+1}) - V((X_i^o \cap X_j^o)^+)][f(\tfrac{n-i+1}{n}) - f(\tfrac{n-i}{n})] =$$

$$[V((X_i^o \cup X_j^o)^+) - V(X_j)][f(\tfrac{n-j+1}{n}) - f(\tfrac{n-j}{n})] -$$

$$[V(X_i) - V((X_i^o \cap X_j^o)^+)][f(\tfrac{n-i+1}{n}) - f(\tfrac{n-i}{n})] >$$

$$\nu(X_i^o \setminus X_j^o)\{[f(\tfrac{n-j+1}{n}) - f(\tfrac{n-j}{n})] - [f(\tfrac{n-i+1}{n}) - f(\tfrac{n-i}{n})]\} > 0$$

The proof for convex f and WSD_* is similar. Q.E.D.

Strict first order stochastic dominance and irrelevance are essential in the proofs of Lemma 6 and Theorem 7, as is demonstrated by the following examples.

Example 1: $\succsim$ can be represented by the function

$$V(X) = \max\{x: 1-x > F_X(x)\}$$

$\succsim$ is continuous, satisfies first order stochastic dominance and irrelevance. It even satisfies SD^* and SD_*, but it does not satisfy strict first order stochastic dominance. Obviously, V is a measure, but $\succsim$ cannot be represented by a product measure.

Example 2: $\succsim$ can be represented by the function

$$V(X) = \mu(X^o) + \mu(X^o \cap [0,1] \times [0,1])\mu(X^o \cap [1,\infty) \times [0,1])$$

where μ denotes the Lebesgue's measure. $\succsim$ is continuous, satisfies strict first order stochastic dominance and WSD^*, but does not satisfy the irrelevance axiom. Indeed, it cannot be represented by a measure.

7. The Shape of the Function f

Theorem 7 suggests two possible shapes for the anticipated utility decision-weight function f — either convex or concave. These results have clear behavioral relevance. Consider the case where the cumulative distribution function F is continuous and differentiable. By using Machina's results for his local utility function (Machina (1982)), Segal (1985a) proved that $X \succsim Y$ whenever Y differs from X by a mean preserving increase in risk if and only if

(7.1) $$u''(x)f'(1-F(x)) - u'(x)f''(1-F(x))F'(x) < 0.$$

As this inequality must hold for all F and x, it follows that the decision maker is risk averse (lover) if and only if u is concave and f convex (u is convex and f concave). For similar results see Chew, Karni, and Safra (1985) and Yaari (1985b).

The significance of the convexity of f is wider than simple risk aversion. It turns out that some phenomena that have nothing to do with the standard definition of risk aversion lead to the conclusion that f is convex. For example, the common response to the Allais paradox, and especially to the generalized Allais paradox, implies that f is convex (Segal (1985a), but see also Quiggin (1982) for a different opinion). The rejection of the probabilistic insurance leads to the conclusion that f is convex (Segal (1984)). Yaari (1985b) found that if f is convex, decision makers prefer stable prices. The convexity of f, which is necessary for risk aversion, enlarge the set of problems which can be analyzed by using the concept of risk rejection.

One interesting implication of the convexity of f is its connection to quasi convex preferences.

Definition: The preference relation $\succsim$ is quasi convex (quasi concave) iff for every $X,Y \in L_1$ and $\alpha \in (0,1)$, the lottery $A = (X,\alpha;Y,1-\alpha)$ is not strictly better than both X and Y (is not strictly worse than both X and Y).

For the importance of this concept see for example Green (1984). Obviously, if $\succsim$ satisfies the independence axiom, then it is both quasi convex and quasi concave. The next proposition deals with preference relations satisfying the reduction of compound lotteries axiom.

Proposition 8: If $\succsim$ can be represented by the anticipated utility functional and satisfies the reduction of compound lotteries axiom, then it is quasi convex (quasi concave) if and only if f is convex (concave).

Proof: Assume first that f is convex. Let $X = (x_1,p_1;\ldots;x_n,p_n)$, $Y = (x_1,q_1;\ldots;x_n,q_n)$ and let $\alpha \in (0,1)$. Let $Z = (x_1,\alpha p_1 + (1-\alpha)q_1;\ldots; x_n,\alpha p_n + (1-\alpha)q_n)$. By RCLA, $Z \sim (X,\alpha;Y,1-\alpha)$. By (6.1)

$$V(Z) = u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{j=i}^{n} [\alpha p_j+(1-\alpha)q_j]) \leq$$

$$\alpha[u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{j=i}^{n} p_j)] +$$

$$(1-\alpha)[u(x_1) + \sum_{i=2}^{n} [u(x_i)-u(x_{i-1})]f(\sum_{j=i}^{n} q_j)] = \alpha V(X) + (1-\alpha)V(Y)$$

Hence either $X \succsim Z$ or $Y \succsim Z$.

The proof for the case where f is concave is similar. Q.E.D.

Theorem 7 gives some more insight into the concept of risk aversion. Naturally, if a decision maker does not like risk, then he should be concerned with the possibility of receiving less than what he already has. Therefore, he prefers a lottery that reduces the probability of receiving less than each possible outcome (the SD_* axiom.) Risk lover, on the other hand, is more interested in the possibility of receiving more than what he already has, and will therefore prefer a lottery that increases the probability of winning more than each possible prize (the SD^* axiom.) Theorem 7 shows indeed that SD_* implies risk aversion while SD^* implies risk loving.

8. Some Remarks on the Independence Axiom

One of the common vindications of the expected utility theory (besides its usefulness) is that it is based upon normatively appealing assumptions.

Special attention was given to the independence axiom, which became almost synonymous to the theory itself. It was first formulated by P. Samuelson in a lecture he gave in Paris in 1952, and in an article in Econometrica in the same year. Its formal presentation, as suggested by Samuelson, is as follows:

> Strong Independence: If lottery ticket $(A)_1$ is (as good or) better than (B_1), and lottery ticket $(A)_2$ is (as good or) better than $(B)_2$, then an even chance of getting $(A)_1$ or $(A)_2$ is (as good or) better than an even chance of getting $(B)_1$ or $(B)_2$.
>
> This is simply a version of what Dr. Savage calls the "sure thing principle." Whether heads or tails comes up, the A lottery ticket is better than the B lottery ticket; hence it is reasonable to say that the compound (A) ticket is definitely better than the compound (B)
>
> [1952, p. 672]

Note that this is exactly the independence axiom I used throughout this paper. Moreover, Samuelson himself was aware of the fact that it requires a different reduction axiom and outlined it separately (p. 671, Sec. 4).

For a long time it was believed that Savage's sure thing principle and the independence axiom are equivalent (this is explicitly claimed by Samuelson himself). The sure thing principle, when adjusted to money-probabilities lotteries (rather than outcomes-events lotteries) states that

$$(x_1,p_1;\ldots;x_n,p_n;z_1,r_1;\ldots;z_\ell,r_\ell) \succsim (y_1,q_1;\ldots;y_m,q_m;z_1;r_1;\ldots;z_\ell,r_\ell) \Leftrightarrow$$

$$(x_1,p_1;\ldots;x_n,p_n;w_1,s_1;\ldots;w_k,s_k) \succsim (y_1,q_1;\ldots;y_m,q_m;w_1,s_1;\ldots;w_k,s_k)$$

As the sure thing principle compares lotteries in L_1 and the independence axiom compares two-stage lotteries, these two axioms are equivalent only at the presence of the reduction of compound lotteries axiom. This confusion led to it that today the independence axioms is unjustly rejected on normative and descriptive grounds, while normative arguments and empirical evidence prove it to be the most natural decision rule to be used for two-stage lotteries.

The best known evidence against the expected utility hypothesis is the Allais paradox. Allais (1953) claimed that most people prefer $X_1 = (0,0.9;5000000,0.1)$ to $Y_1 = (0,0.89;1000000,0.11)$, but $Y_2 = (1000000,1)$ to $X_2 = (0,0.01;1000000,0.89;5000000,0.1)$ while by expected utility theory $X_1 \succsim Y_1$ iff $X_2 \succsim Y_2$. Such a behavior certainly contradicts the sure thing principle (see Savage (1954) and Section 6 above). It is sometimes argued that it also contradicts the independence axiom (Machine, p. 287). Indeed, let $X = (0,\frac{1}{11};5000000,\frac{10}{11})$, $Y = (1000000,1)$, and $Z = (0,1)$. By the independence axiom, $A_1 = (X,0.11;Z,0.89) \succsim B_1 = (Y,0.11;Z,0.89)$ iff $A_2 = (X,0.11;Y,0.89) \succsim B_2 = (Y,0.11;Y,0.89)$, while by Allais paradox $A_1 \succ B_1$ but $B_2 \succ A_2$. It is, however, beyond doubt that this argument crucially depends on the reduction of compound lotteries axiom. Indeed, anticipated utility theory, which may be consistent with the independence axiom is not contradicted by the Allais paradox.

Some empirical results show that people are willing to accept the independence axiom, but they usually reject the reduction axiom. Kahneman and Tversky (1979) found that most people prefer $(3000,1)$ to $(0,0.2;4000,0.8)$, but $(0,0.8;4000,0.2)$ to $(0,0.75;3000,0.25)$. Consider now the compound lotteries $A = ((3000,1),0.25;0.0.75)$ and $B = ((0,0.2;4000,0.8),0.25;0,0.75)$. By the independence axiom $A \succ B$, but by the reduction axiom, $B \succ A$. Kahneman and Tversky found that most people prefer A to B. Other evidence for the rejection of the reduction axiom, especially when time between the stages is involved, can be found in Ronen (1971) and Snowball and Brown (1979).

Another common error is the claim that Savage's sure thing principle and the independence axiom have the same normative justification. This is false, as becomes apparent from the explanations Savage and Samuelson gave for their axioms. Savage claimed that if X and Y differ only on S, then the common outcome on "not S" should not effect the preference order between X and Y. Samuelson said that if X is preferred to Y, it should be preferred to Y even when receiving X or Y becomes uncertain, and other prizes are possible. This argument cannot justify the sure thing principle, as there is no initial preference order between half lotteries like (0,0.01;5000000,0.1;-) and (1,0.11;-). Similarly, we usually assume that $(x_1,x_2,\ldots,x_n) \succsim (x_1',x_2,\ldots,x_n)$ iff $x_1 > x_1'$, because there is a well defined natural order on quantities of commodities. However, we do not necessarily assume that $(x_1,x_2,x_3,\ldots,x_n) \succsim (x_1',x_2',x_3,\ldots,x_n)$ iff $(x_1,x_2,y_3,\ldots,y_n) \succsim (x_1',x_2',y_3,\ldots,y_n)$, because there is no initial natural order on the bundles $(x_1,x_2,-)$.

In this paper I interpret the independence axiom as a mechanism that transforms two-stage lotteries into one-stage lotteries. This is done by using the certainty equivalents of the possible outcomes in the compound lotteries. According to this approach, the independence axioms and the reduction axiom cannot be used together. Indeed, if the decision maker uses the reduction axiom, then the independence axiom becomes meaningless, as he is never really concerned with two-stage lotteries. Similarly, if the decision maker transforms two stage lotteries into simple lotteries by using the certainty equivalent mechanism, then he can no longer use the reduction axiom. However, using the stochastic dominance axioms is not ruled out by the independence axiom, because they do not change the structure of a compound lottery. (These stochastic dominance axioms become redundant at the presence of the reduction axiom, as follows from Theorem 4). I therefore believe that Theorem 5 gives a better normative basis for expected utility theory than the standard IA-RCLA one.

9. Concluding Remarks

This paper showed that alternatives to expected utility theory can be developed even if the independence axiom is accepted, provided one is willing to forgo the reduction of compound lotteries axiom. As an alternative to this last axiom I suggest several different concepts of stochastic dominance for two-stage lotteries. I believe that in the context of lotteries over time, these stochastic dominance axioms are more acceptable than the reduction of compound lotteries axiom. Moreover, by using these stochastic dominance axioms it becomes evident why anticipated utility theory is a natural extension of expected utility theory.

Some questions were not answered by this paper. For example, it is not clear whether WSD^* and SD^* (and WSD_* and SD_*) are equivalent. Although Theorem 6 needs only the weaker versions WSD^* and WSD_*, the question whether anticipated utility theory satisfies SD^* and SD_*, which have a clear normative appeal, is still open. Another interesting question is what preference relations satisfy WSD, together with IA and irrelevance.

Finally, a natural possible extension of the anticipated utility functional is a general measure on $\mathbb{R}_+ \times [0,1]$. (Recall that anticipated utility theory assumes a product measure and expected utility theory assumes a product measure with a linear measure on the probabilities axis.) Lemma 6 shows that such an extension can be obtained from the irrelevance axiom.

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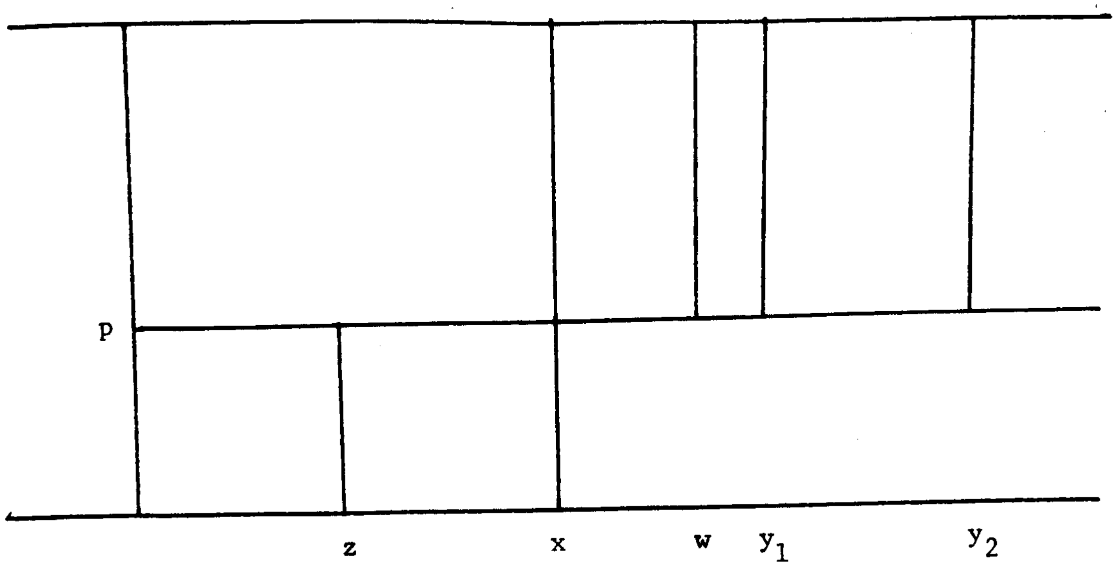

Figure 1

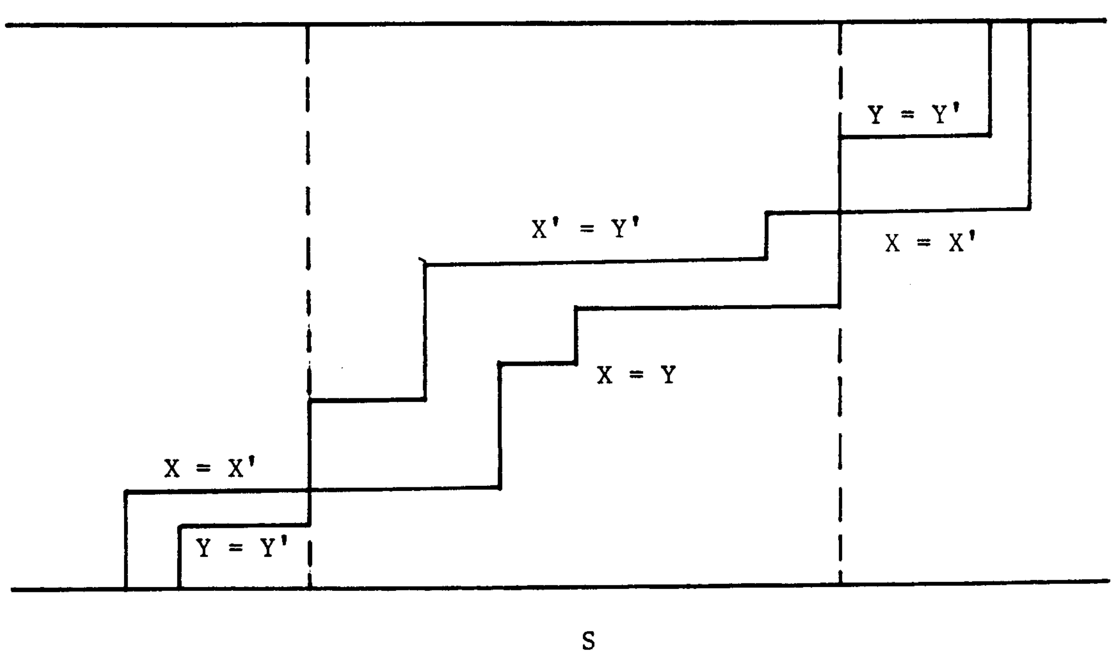

Figure 2

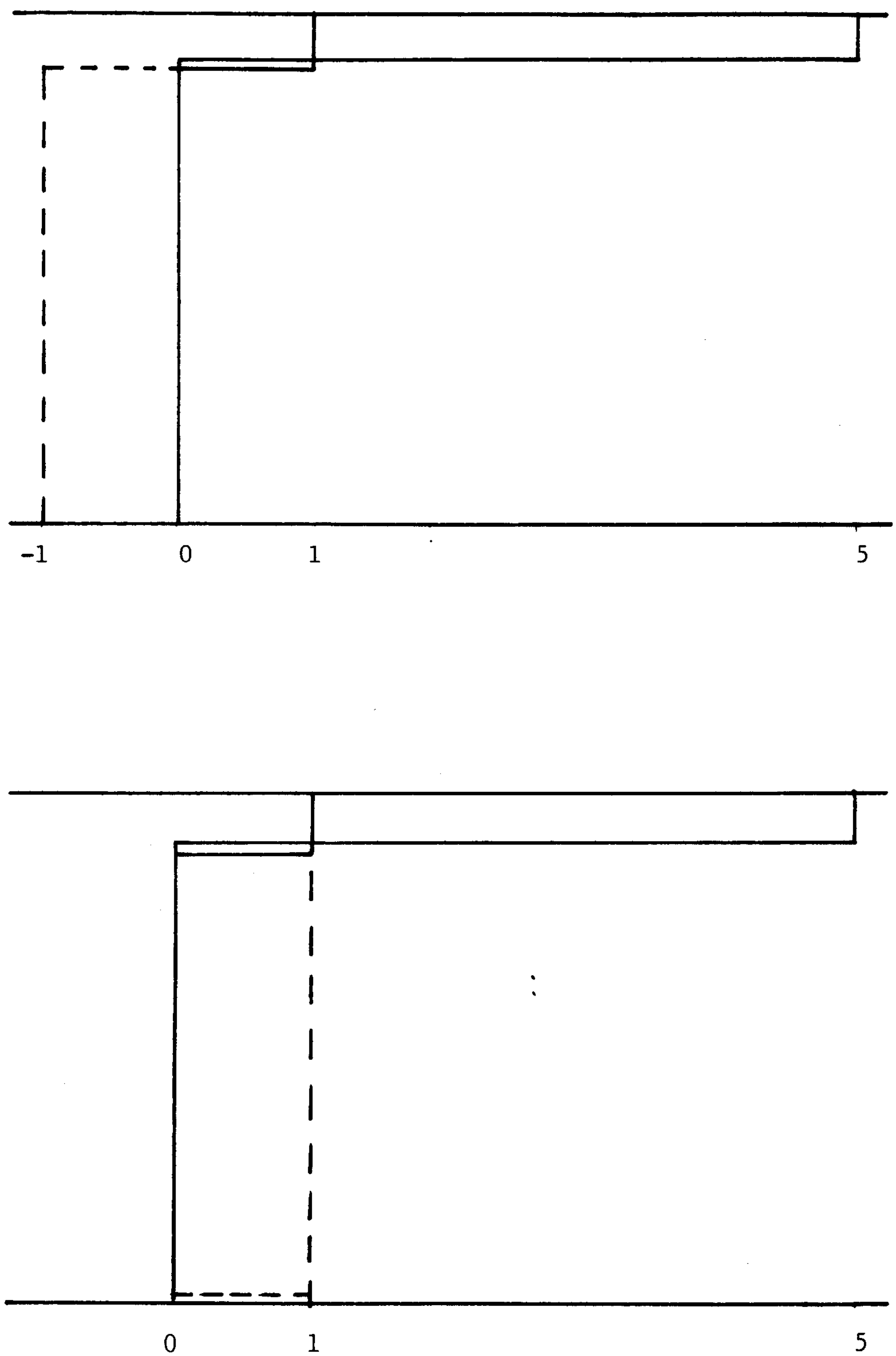

Figure 3