INFINITE HORIZON EQUILIBRIUM WITH INCOMPLETE MARKETS

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1. Introduction

This paper examines competitive equilibrium in an infinite horizon model with incomplete markets. The class of economies we study is relatively broad: it allows production, both infinitely lived agents and overlapping generations, and market participation which is either random or deterministic. In addition to the traditional overlapping generations model and the finite horizon incomplete markets model, this class of models includes as a special case the monetary model of Bewley [1980, 1983], applications of which can be found in Scheinkman and Weiss [1986], and Levine [1986a, b]. Cash-in-advance constraints of the type studied by Lucas and Stokey [1984] can also be modeled in this framework by careful limitation of market participation.

The goal of this paper is to characterize equilibria in the infinite horizon as limits of finite horizon truncated equilibria. The central focus is on a condition called extensibility. In an incomplete market model, this condition says that if short sales constraints have been satisfied through a particular date, then, at equilibrium prices in the next period, it is possible to trade in such a way as to satisfy the short sales constraints again. We show by counterexample that without this condition there can be limits of finite horizon equilibria that are not infinite horizon equilibria, and show that if it is satisfied then it is both necessary and sufficient for an infinite horizon equilibrium to be the limit of finite horizon truncated equilibria.

The characterization of infinite horizon equilibria as limits of finite horizon truncated equilibria can be used to prove existence theorems as Balasko and Shell [1980], Wilson [1981], Muller and Woodford [1983] and Spear [1985] do in studying various special cases of this model. By assigning weights to terminal stocks of assets such as money it can also be used to prove the existence of specific types of equilibria, as Bewley [1980, 1983] and Levine [1986a, 1986b] do in showing that money has value. In all of these models the extensibility criterion is satisfied, and the limit of truncated equilibria is shown to be an infinite horizon equilibrium. The converse - that every infinite horizon equilibrium arises this way - is either trivial in the case of pure overlapping generations models, or has not been examined in the case of models with production or infinitely lived agents.

The complete characterization of infinite horizon equilibria as the limit of truncated equilibria has been examined in a game theoretic setting by Fudenberg and Levine [1983], and Harris [1985]. This study is very much in the spirit of those. There is, however, one important difference: because the game models do not involve budget constraints, there is no need for a criterion such as extensibility.

One complication that is common to the game theoretic setting and the one here is the fact that there may be infinite horizon equilibria that are not limits of finite horizon equilibria. In the game theoretic setting, for example, the tit-for-tat equilibrium of the infinitely repeated prisoner's dilemma is not an equilibrium of the finitely repeated game. In the current setting, an equilibrium in which money has value is not an equilibrium of a finite horizon economy. The game theoretic solution is to relax the solution concept by allowing agents in the truncated games to only get within ϵ of the optimium, and to require that epsilon approaches zero as the truncation horizon increases. The solution here is to relax the solution concept by assigning terminal stocks of assets weight ϕ in the utility function, and to require that there be an upper bound on the present value of the terminal stocks that approaches zero as the truncation horizon increases. The proof that such a "transversality condition" works is more complicated than the straightforward game theoretic proof that every infinite horizon equilibrium may be approximated by finite horizon ϵ-equilibria with ϵ small.

Section 2 of the paper introduces the model. Section 3 considers a simple example which illustrates the definitions and shows how extensibility is needed to get infinite horizon equilibria from finite horizon limits. It also relates the model of this paper to some of the literature on incomplete markets with a finite horizon. Section 4 states and proves the main theorems about approximating infinite horizon equilibria with truncated equilibria. Section 5 considers the conditions under which equilibria may be guaranteed to be extensible.

2. The Model

We study an economy in which there are countably many markets $s \in S$. Each market s meets at a specific time $t(s) \in \{1,2,\ldots\}$. Only finitely many markets meet at any given time. Different markets meeting at the same time represent markets at different locations, or contingent markets which meet only in certain states of nature. An example of a market structure of this type has finitely many states of nature η_t in each period. A market is identified with a finite history of the states of nature $s = (\eta_1,\eta_2,\ldots,\eta_t)$. The time at which the market meets, $t(s)$, is the length of the finite history. Taking η_1 to be historically given, the set of markets form a tree. In the more general case, we do not make this assumption.

The objects that are traded in markets are called claims. There are countably many claims $c \in C$. A claim may represent a direct claim to current consumption, or it may represent a claim to recieve other claims in subsequent markets: it may be either a commodity or an asset. In general, claims both figure into the utility function, and provide a future return.

Each claim is traded in exactly one market, and only finitely many claims are traded in any given market. We write $c \in s$ if c is traded in market s. We also define $t(c) \equiv t(s)$ to be the time at which c is traded.

Claims are held and traded by <u>agents</u>. There are countably many agents $a \in A$, but only finitely many agents may trade in any given market. We write $a \in s$ if a is allowed to trade in market s. Some finitely-lived agents may only participate in finitely many markets. Other infinitely-lived dynasties may participate in infinitely many markets.

It should be emphasized that the markets meeting at a given date, the claims traded in a given market, and the agents who participate in a given market are all finite.

The planned holding by agent a of claim c is denoted by $x_c^a \in \mathfrak{R}$. The infinite vector of planned holdings of all claims by agent a is denoted by x^a. We may think of x^a as an element of $\mathfrak{R}^\infty$, the space of all sequences of real numbers. This is a topological space in the product topology. Agent a's preferences over claims is given by a utility function $U^a\colon \mathfrak{R}^\infty \to \mathfrak{R} \cup \{-\infty\}$. This satisfies

(A.1) U^a is weakly quasi-concave and weakly monotone. For $U^a(x^a) > -\infty$, U^a is uniformly continuous in the product topology.

Utility need not be strictly monotone: claims which are not claims to consumption, but merely rights to future claims, do not enter into utility at all. The assumption that U^a is uniformly continuous in the product topology implies that consumption in the far distant future makes little difference to current utility. It implies that for any given $\epsilon > 0$ there is a T such that if any two plans both yield some utility, and coincide at and before T, the difference in utility between the plans is no more than ϵ. This insures that preferences are "continuous at infinity" in the sense of Fudenberg and Levine [1983], and makes it possible to draw inferences about infinite horizon equilibrium from the finite horizon case.

We make one other technical assumption about utility.

(A.2) If $U^a(x^a) > -\infty$, $U^a(y^a) > -\infty$, and $z_c^a = x_c^a$ for $t(c) \le T$, $z_c^a = y_c^a$ for $t(c) > T$, then $U^a(z^a) > -\infty$.

This says that if two plans both yield some utility, then the plan that consists of combining the earlier part of one plan with the later part of the other also yields some utility.

As an example of a utility function satisfying (A.1) and (A.2), let x_s^a be the subvector of x^a for which the corresponding claims $c \in s$, and let $u_s^a(x_s^a)$ be a real valued continuous function on the non-negative orthant weakly concave, weakly monotone and bounded above by $\bar{u}_s^a \ge 0$, below by $\underline{u}_s^a \le 0$. Assume that $\Sigma_{s \in S} \bar{u}_s^a < \infty$, and $\Sigma_{s \in S} \underline{u}_s^a > -\infty$. Define

$$U^a(x^a) = \begin{cases} \sum_{s \in S} u_s^a(x_s^a) & x_c^a \ge 0 \quad \text{all} \quad c \in C \\ -\infty & x_c^a < 0 \quad \text{some} \quad c \in C. \end{cases}$$

It may easily be shown that this function satisfies (A.1), and it obviously satisfies (A.2).

Some claims are claims to consumption and enter into the utility function. Other claims return future claims. We let θ_{cd} denote the units of claim c an agent will receive if he holds one unit of claim d. The return $\theta_{cd} \neq 0$ only if $t(c) > t(d)$: a claim can return only claims which are traded subsequently. This rules out the possibility that the amount of claims available for trade could be affected by trades which are consumated later.

Let $w^a \in \mathfrak{R}^\infty$ denote agent a's endowment of claims. The total amount of claim $c \in s$ available to a for trading at s is

$$\sum_{t(d)<t(c)} \theta_{cd} x_d^a + w_c^a,$$

that is, it adds the endowment to the returns on claims held in earlier markets. The <u>no trade portfolio</u> $\bar{x}^a$ is the holding that results if a passively roles over his portfolio without trading. This is defined recursively. If $t(c) = 1$, then $\bar{x}_c^a = w_c^a$. At time t, for claims with $t(c) = t$, we may define

$$\bar{x}_c^a = \sum_{t(d)<t(c)} \theta_{cd} \bar{x}_d^a + w_c^a.$$

Since $t(d) < t(c)$, $\bar{x}_d^a$ has already been defined inductively, and this definition makes sense.

Let p_c denote the price of claim c. If agent a is not allowed to trade c, then his holding of c is given passively by

$$\text{(E.1)} \quad x_c^a \leq \sum_{t(d)<t(c)} \theta_{cd} x_d^a + w_c^a \qquad a \notin s,\ c \in s.$$

If a is allowed to trade at s, then he must satisfy the budget constraint

$$\text{(E.2)} \quad \sum_{c \in s} p_c x_c^a \leq \sum_{c \in s} p_c \left(\sum_{t(d)<t(c)} \theta_{cd} x_d^a + w_c^a \right) \quad a \in s.$$

Notice that both versions of the budget constraint, (E.1) and (E.2) allow the possibility of free disposal.

The rate of return on claims is non-negative:

(A.3) $\theta_{cd} \geq 0$ all $c, d \in C$.

This is a convention similar to the fact that utility is monotone: more current claims can yield only additional future claims. It can be argued, as Geanakaplos and Polemarchakis [1985] do, that it is not unreasonable to allow negative returns on insurance contracts. However, Geanakoplos and Polemarchakis make a the weaker assumption that there is at least one portfolio yielding strictly positive returns. This can be shown to imply that there is an equivalent set of claims (spanning the same set of returns) with nonnegative returns. In understanding (A.3), it is important to keep in mind that agents can construct portfolios with negative returns by selling short.

It is generally useful to distinguish between those claims which figure directly into utility, and those merely present a claim to future claims. We refer to claims that do figure into (some agent's) utility function as <u>consumption claims</u>. We can recursively define a <u>backed</u> claim to be either a consumption claim, or a claim which pays out a positive amount of a backed claim. Conversely <u>unbacked</u> claims pay out only other unbacked claims, and never lead to an eventual payout of a consumption claim, no matter how many times the portfolio rolled over. Money is a typical example of an unbacked claim.

In addition to the budget constraints, there can be constraints on short sales. Each agent a who can participate in market s is subject to a single <u>borrowing constraint</u> X_s^a. This is a subset of $\Re^\infty$, and we require that

(E.3) $x^a \in X_s^a$ for all $a \in s$.

The set X_s^a must satisfy

(A.4) X_s^a is a closed convex set;

if $x^a \in X_s^a$ and $y_c^a = x_c^a$ for $t(c) \leq t(s)$, then $y^a \in X_s^a$;

if $x^a \in X_s^a$ and $y_c^a \geq x_c^a$ for $c \in C$, then $y^a \in X_s^a$.

The second part of this assumption states that the constraint X_s^a constrains only holding at or before the market s. The third part of this assumption assures that if a plan is feasible, then having more of everything is as well. Like in the case of rates of return, this is a convention that claims are unambiguously good: more claims can only make it easier to satisfy the short sales constraints. However, the set X_s^a may be all of $\Re^\infty$, so that this formulation is consistent with the absence of any short sales constraints. Notice that large holdings of claims in earlier markets may make it easier to satisfy the short sale constraints in the current market. Since these earlier claims may have a return that will not be realized until a future market, they may well be used as security for current indebtedness.

It is assumed that

(A.5) If x^a satisfies (E.3) for all s with $a \in s$, then

$U^a(x^a) > -\infty$.

This amounts to assuming that the borrowing constraints are the only constraints on the portfolio. Utility of $-\infty$ is effectively a constraint; it is assumed that these "extra" constraints do not bind if the borrowing constraints are satisfied.

If we assume that no agent can be forced to trade, then the no-trade portfolio $\bar{x}^a$ defined above must satisy the short sale constraints. More strongly, it is assumed that

(A.6) $\bar{x}^a \in$ interior(X_s^a) for all $a \in s$,

so that the no-trade portfolio strictly satisfies the short sales constraints. This is closely related to the assumption made in ordinary general equilibrium theory that endowments are strictly interior, and could be weakened in much the same way. Notice that there is no requirement that agents have non-trivial endowments in markets in which they do not participate.

The production side of the economy is represented in each market s by a transformation matrix A_s with as many rows as there are claims traded at s. This serves to convert current claims into current claims of different types. This does not mean that there is no intertemporal production. Converting a claim to current consumption into a claim for future consumption is a form of investment. However, intertemporal production is possible only by producing intermediate goods (claims) which are owned by specific individual agents: firms themselves operate only contemporaneously.

In a given market the amount of claims available prior to production is given by adding the returns on previous claims held by agents who can participate in that market to their endowments. Let $\gamma_s \geq 0$ be the levels at which activities are operated. Social feasibility requires that demand not exceed supply:

$$\text{(E.4)} \quad \sum_{a \in s} x_s^a \le A_s \gamma_s + \sum_{a \in s} \left[w_s^a + \sum_{t(c)<t(s)} \theta_{sc} x_c^a \right] \quad \text{for some} \quad \gamma_s \ge 0.$$

Here θ_{sc} is the vector composed of θ_{dc} with $d \in s$.

Because more claims are unambiguously better by (A.1), (A.3) and (A.4), and because there is free disposal, we may assume that prices are nonnegative, so that $p_c \ge 0$. Moreoever, within a single market it is clear that only relative prices matter. This leaves us free to adopt the convention that prices within each market lie on the unit simplex. Throughout the remainder of the paper we adopt this convention:

$$\sum_{c \in s} p_c = 1, \qquad p_c \ge 0.$$

An <u>equilibrium</u> of this economy is a vector of consumption plans $\hat{x}$, production plans $\hat{\gamma}$, and prices $\hat{p}$ which satisfy the social feasibility condition (E.4) and the individual rationality conditions

(E.5) For each agent U^a is maximal subject to the constraints (E.1), (E.2) and (E.3). Profits are maximal in each market.

Because there are constant returns to scale, in equilibrium, profits are necessarily zero.

In this setting of incomplete markets, the assumption that firms act to maximize profits is controversial. Ekern and Wilson [1974] and Radner [1974], consider stockholder unanimity as an alternative criterion for firm decision making. The argument is that alternative production plans yield different patterns of returns across states, effectively creating a market for a different type of claim. Consequently, owners may actually be willing to take a current loss of profit in exchange for a better pattern of returns across states. In the model here, we avoid this issue by allowing only contemporaneous production, and forcing claims to be held by individual agents, rather than in the form of shares of firms: firms purchase claims from agents, convert them into different claims, and sell them back to agents. Alternative production plans do not alter the set of markets perceived by price taking agents: agents always perceive that they can buy unlimited claims of any type traded in markets in which they participate. Roughly, the traditional "problem" with production, involves agents who do not act competitively: they effectively realize that by changing the firm's production plan to produce alternative claims, these claims will be available at a more attractive price.

Our interest is not only in equilibria of the full model. We also are interested in finite horizon <u>truncated equilibria</u>, and how they are related to infinite horizon equilibra. We begin by defining an economy truncated at time T. We let x^a and $\tilde{x}^a$ be contingent plans for holding claims by agent a, and let ϕ^a_{cT} be nonnegative weights for $c \le T$. The truncated utility recieved by a if he follows the plan x^a, and the economy is truncated according to $\tilde{x}^a$ and ϕ^a_T is defined as

$$(2.1) \qquad U^a_T(x^a,\tilde{x}^a,\phi^a_T) = U^a(z^a) + \sum_{t(c)\le T} \phi^a_{cT} x^a_c,$$

where z is defined by

$$(2.2) \qquad z^a_s = \begin{cases} x^a_s & \text{if } t(c) \le T \\ \tilde{x}^a_s & \text{if } t(c) > T \end{cases}$$

To avoid degeneracy, it is always assumed that $U^a(\tilde{x}^a) > -\infty$. Because of (A.2), this implies that if $U^a(x^a) > -\infty$, so is $U^a_T(x^a,\tilde{x}^a,\phi^a_T)$. The weights ϕ_T^a are used to capture the fact that claims held prior to truncation may have a valuable return beyond the truncation horizon. They are essential in constructing truncated equilibria in which unbacked claims have positive value.

An equilibrium is a vector of consumption plans $\hat{x}$, production plans $\hat{\gamma}$, and prices $\hat{p}$ which satisfy (E.1) through (E.5) in all markets for the original utility function. An equilibrium truncated at T with weights ϕ_T and future consumption $\tilde{x}$ also is a vector of consumption plans $\hat{x}$, production plans $\hat{\gamma}$, and prices $\hat{p}$ which satisfy (E.1) through (E.5). Now however the constraints (E.1) to (E.4) are imposed only for markets with $t(s) \leq T$, the utility function in (E.5) is the truncated one, and profits must be maximized in (E.5) only in those markets occuring at or before T. Notice that a truncated equilibrium is by convention an infinite vector of plans and prices. However, none of the components of these vectors occuring after T is at all relevant to the equilibrium. In this sense the model is really a finite horizon model: only the finitely many components of plans and prices occuring at or before T matter.

3. An Example

In this section we consider a model with a single state of nature and location in which a single trader must determine how to hold a single perishable consumption good, and a single asset - a one period commodity bond - over time. The unique equilibrium requires that prices be such that the trader is willing to hold his endowment. After using this example to demonstrate some of the notation introduced in the previous section, we show how a sequence of truncated equilibria may converge to a non-equilibrium. We use this to motivate the condition of extensibility which we use in the next section to characterize infinite horizon equilibria as limits of finite horizon ones.

Markets meet sequentially, so $S = \{1, 2, \ldots\}$, and $t(s) = s$. In each market two types of claims are traded: consumption claims $c(s)$, and bonds $b(s)$. There is a single agent who can participate in all markets; the remaining agents cannot participate in any market. The utility of the single agent who matters depends only on consumption, and has the form:

$$U(x) = \begin{cases} \sum_{s \in S} \delta^{t(s)} u(x_{c(s)}), & x_{c(s)} \geq 0 \quad \text{all} \quad c \in C \\ -\infty, & x_{c(s)} < 0 \quad \text{some} \quad c \in C. \end{cases} \tag{3.1}$$

where u is strictly concave, bounded above and below, and differentiable, and $1 > \delta > 0$. The consumption claim has no rate of return, so $\theta_{cc(s)} = 0$ for all claims c. The bonds are one period consumption bonds paying off one unit of next period consumption, so $\theta_{c(s+1)b(s)} = 1$, and for $c \neq c(s+1)$, $\theta_{cb(s)} = 0$.

The agent who matters is endowed with a single unit of consumption, $w_{c(s)} = 1$, and no bonds, $w_{c(s)} = 0$. There are potentially two short sales constraints in the market s defining the set X_s: holdings of consumption are constrained to be nonnegative, $x_{c(s)} \geq 0$, and short holdings of bonds are limited by $x_{b(s)} \geq \underline{x}$, where $\underline{x} < 0$, and we allow the possibility that $\underline{x} = -\infty$. The budget constraint in market s is

$$p_{c(s)} [x_{c(s)} - 1 - x_{b(s-1)}] + p_{b(s)} x_{b(s)} \leq 0, \tag{3.2}$$

where $x_{b(0)} = 0$ by convention. There is no production. As a result a plan x is socially feasible if consumption is no greater than one, and bond holdings are nonpositive.

In the truncated case, we take the weights $\phi_T = 0$, and the truncated consumption plan $\tilde{x}$ is irrelevant, since (3.1) is additively separable.

Regardless of the value of $\underline{x}$ there is a unique truncated equilibrium. In the final period, T, it must be that $p_{c(T)T} = 1$, and $p_{b(T)T} = 0$, for otherwise the agent would try to sell bonds short in the final period, and use the proceeds to purchase more than the single unit of consumption available to the economy. In earlier markets, $t(s) < T$, prices must be given by $p_{c(s)T} = 1/(1 + \delta)$, and $p_{b(s)T} = \delta/(1 + \delta)$, in order that the agent be willing to hold his endowment of a single unit of consumption and no bonds.

The truncated equilibria converge to a limit: the plan is to hold the endowment of consumption and bonds, and the prices are $p_{c(s)} = 1/(1 + \delta)$, and $p_{b(s)} = \delta/(1 + \delta)$ in all markets s. Whether or not this is an equilibrium, however, depends on $\underline{x}$. If $\underline{x} = -\infty$, then the limit is not an equilibrium. In this case a variety of Ponzi schemes can improve on the equilibrium level of utility: short sales of bonds can be used to finance extra consumption, and financed in turn by future short sales of bonds. If, however, $\underline{x} > -\infty$, then such Ponzi schemes are impossible, because the debt, which must grow at the rate $1/\delta$, will eventually exceed the limit on short sales of bonds. In this case the limit is an equilibrium. Notice the difference between the finite and infinite horizon cases: in the finite horizon it is possible make short sales of bonds useless in the final period by having their price equal to zero, while in the infinite horizon case there are no final period prices to manipulate. This also makes clear how the infinite horizon model differs from the finite horizon incomplete market models studied by Hart [1975], Werner [1985], Duffie [1985], Geanakoplous and Mas-Colell [1985], Geanakoplos and Polemarchakis [1985], and Duffie and Shafer [1986]. Although those models study what happens when there are no short sales constraints, from our perspective there is a short sales constraint: in the final period no short sales are possible. Of course this may equally well be arranged by making the prices of final period assets equal to zero.

The reason that the limit of truncated equilibria is not an equilibrium without short sales constraints is that the limiting budget constraint is very unlike the budget constraints approaching the limit. To understand why, let $B^a(p)$ denote the budget constraint set in the infinite horizon when prices are p: these are the points which satisfy both the short sales constraints (E.3), and the budget constraints (E.1) and (E.2). Let $B^a_T(p)$ denote the T period truncated budget constraint when prices are p: here (E.1), (E.2) and (E.3) only apply in markets at and before period T. Fix T, and consider t bigger than T. Then in $B_T(p_t)$ there are no restrictions on short sales of bonds, while in $B_t(p_t)$ short sales of bonds are limited by the fact that no bonds may be sold short in the final period. Of course the limit reflects the behavior of $B_T(p_t)$ as $p_t \to p$, and in the limit, unlimited short sales are possible.

Turning to the general case, we can rule out the possibility that $B^a_T(p_t)$ is unlike $B^a_t(p_t)$ by defining the prices p_t to be <u>extensible</u> at or before t if for every $T \le t$ and every agent a, whenever $x^a \in B^a_T(p_t)$ there is $y^a \in B^a_{T+1}(p_t)$ such that $y^a_c = x^a_c$ for $t(c) \le T$. In other words, it is never possible to trade into a position which will force the violation of future constraints. This condition is clearly violated in the example when there are no short sales constraints: there are points in $B^a_{t-1}(p_t)$ in which more than a single bond is sold short; no such point can admit an extension to $B^a_t(p_t)$ since the agent cannot repay more than a single unit of debt in the final period. It is a consequence of the results in the next section that if each truncated equilibrium price vector is extensible, then any limit of the truncated equilibria must also be an equilibrium.

Extensibility is not entirely adequate as a criterion for when truncated equilibria converge to an infinite horizon equilibrium, however. While any short sales constraint $\underline{x} > -\infty$ will restore the continuity between the finite and infinite horizon model, if $\underline{x} < 1$ the truncated equilibria are not extensible, since no more than a single unit of debt can be repayed in the final period. There are two cases that, taken together, show how to weaken the extensibility criterion. First, when $\underline{x} < -1/(1 - \delta)$ the short sales constraint is misleading. The present value of the future endowment at the limiting prices p is equivalent to $1/(1 - \delta)$ units of bonds. If, in the infinite horizon, the agent attempts to sell short more bonds than this, he cannot possibly repay the debt, and will, in some future period, be forced to violate the short sales constraint. The fact that plans in $B_T(p_t)$ that involve selling short more than $1/(1 - \delta)$ bonds cannot be extended to $B_t(p_t)$ is irrelevant, since such plans are ruled out in the limit anyway. As a result, in considering extensibility, we can restrict our attention to those plans which are (approximately) feasible in the infinite horizon limit.

If $-1/(1 - \delta) \leq \underline{x} < -1$ then the extensibility criterion is still violated. However the violation is not too serious in the sense that if t is much larger than T, then for any plan in $B_T(p_t)$ we can find a slightly different plan, also in $B_T(p_t)$ which can be extended. The reason for this is that we can extend any plan for which short sales are slightly less than $1/(1 - \delta)$: the effect of prohibiting short sales at the end point (t) is gradually reduced as we work our way backwards in time (to T).

The criterion that captures these two case is called <u>approximate</u>

<u>extensibility</u>. Fix an $\epsilon_o > 0$. We say that a sequence of prices $p_t \to p$ is approximately extensible if for any T and ϵ there exists a $T' > T$ such that for $t \geq T'$ and $x_T^a \in B_T^a(p_t)$, $x^a \in B^a(p)$ with $\sup_{t(c)\leq T}|x_{cT} - x_c| \leq \epsilon_o$ there exists $x_t^a \in B_T^a(p_t)$, $B_t^a(p_t)$ with $\sup_{t(c)\leq T}|x_{cT} - x_{ct}| \leq \epsilon + \sup_{t(c)\leq T}|x_{cT} - x_c|$. In other words, if x_T^a is within ϵ_o of a point in $B^a(p)$, there is a point x_t^a within ϵ of the distance to that point which admits an extension to $B_t^a(p_t)$. In the example, this property is always satisfied when $\underline{x} > -\infty$, and is not satisfied when $\underline{x} = -\infty$. In the next section we show that extensibility of truncated equilibria is both necessary and sufficient for the limit to be an equilibrium. Notice that extensibility implies approximate extensibility.

4. <u>Finite Horizon Approximations to Equilibrium</u>

We now study the relationship between extensible equilibria in the infinite horizon model, and those in finite horizon truncations of the model. Our goal is to prove that infinite horizon equilibria can be completely characterized as limits of finite horizon equilibria.

Our first step is to show that $B_T^a(p)$ is lower hemi-continuous for each T.

<u>Lemma (4.1)</u>: If $p_t \to p$ and $x^a \in B_T^a(p)$, then there is $x_t^a \in B_T^a(p_t)$ with $x_t^a \to x^a$.

<u>Proof</u>: This fact is also implicitly used by Radner [1972] in his outline of an existence proof when there are multiple budget constraints; the proof is a minor variation on Debreu's [1959] proof with a single budget constraint. In case x^a strictly satisfies the many budget constraints, the proof is trivial. If a certain finite subset of budget constraints bind, then like

Debreu, we draw a straight line between x^a, and the no trade plan $\bar{x}^a$, and take x_t^a to be the point where this line first exactly satisfies one of the binding budget constraints at the prices p_t. Q.E.D.

Next, we must give a transversality condition linking equilibrium holding of claims to the weights ϕ_T^a. For given weights and prices, define $\chi_T^a(\phi_T^a, p_T)$ to be the minimum of $\Sigma_{t(c)\le T}\, \phi_{cT}^a x_c^a$ subject to $x^a \in B_T^a(p_T)$. The sequence of weights ϕ_T satisfy the transversality condition with repect to the sequence of prices p_T^a and plans x_T^a provided that

(4.1) $$\Sigma_{t(c)\le T}\, \phi_{cT}^a x_{cT}^a - \chi_T^a(\phi_T^a, p_T) \le \epsilon_T \quad \text{where} \quad \lim_{T\to\infty} \epsilon_T = 0.$$

Our goal is to prove

Theorem 4.2: A triple $\hat{x}$, $\hat{\gamma}$ and $\hat{p}$ are an equilibrium if and only if they are the limit (in the product topology) of approximately extensible equilibria $\hat{x}_T$, $\hat{\gamma}_T$ and $\hat{p}_T$ truncated with respect to $\tilde{x}$ and weights ϕ_T which satisfy the transversality condition.

We begin by proving that limits of truncated equilibria are equilibria.

Proof that limits are sufficient: Convergence of $\hat{x}_T$, $\hat{\gamma}_T$ and $\hat{p}_T$ in the product topology to $\hat{x}$, $\hat{\gamma}$ and $\hat{p}$ means that $\hat{x}_{cT} \to \hat{x}_c$, $\hat{\gamma}_{sT} \to \hat{\gamma}_s$, and $\hat{p}_{cT} \to \hat{p}_c$ for each c and s. It follows directly that $\hat{x}$ is both socially and individually feasible at the prices $\hat{p}$. The fact that $\hat{\gamma}$ maximizes profits at $\hat{p}$ and that $\hat{x}$ is optimal for any finitely lived consumer is an immediate consequence of the fact that continuous finite horizon optimization problems are upper-hemi-continuous. The crucial step is to show that $\hat{x}$ is optimal at $\hat{p}$ for an agent who participates in infinitely many markets.

Suppose in fact that $z^a \in B^a(\hat{p})$ and $U^a(z^a) = U^a(\hat{x}^a) + \delta$ where

$\delta > 0$. Choose ϵ so that $9\epsilon < \delta$. Since U^a is uniformly continuous in the product topology, we may choose T sufficiently large that if y^a is any plan satisfying the short sale constraints through T, and $t \geq T$

$$(4.2)\qquad \begin{aligned} &|U_t^a(y^a,\tilde{x}^a,0) - U^a(y^a)| \leq \epsilon \\ &|U_T^a(y^a,\tilde{x}^a,0) - U_t^a(y^a,\tilde{x}^a,0)| \leq \epsilon . \end{aligned}$$

Next we may choose $t \geq T$ sufficiently large that

$$(4.3)\qquad |U_T^a(\hat{x}_t^a,\tilde{x}^a,0) - U_T^a(\hat{x}^a,\tilde{x}^a,0)| \leq \epsilon ,$$

since $\hat{x}_t^a \to \hat{x}^a$ in the product topology and T is fixed. Since $\hat{p}_t \to \hat{p}$ in the product topology, by Lemma 4.1, we may also assume that there is a $z_T^a \in B_T^a(\hat{p}_t)$ with

$$(4.4)\qquad |U_T^a(z^a,\tilde{x}^a,0) - U_T^a(z_T^a,\tilde{x}^a,0)| \leq \epsilon .$$

We may also assume from (4.1) and $9\epsilon < \delta$ that

$$(4.5)\qquad \epsilon_t < \delta - 9\epsilon ,$$

where ϵ_t is from the transversality condition (4.1).

Finally, by approximate extensibility and (4.4), we may assume that t has been chosen sufficiently large that there is an approximate extension $z_t^a \in B_t^a(\hat{p}_t)$ with

$$(4.6)\qquad |U_T^a(z^a,\tilde{x}^a,0) - U_T^a(z_t^a,\tilde{x}_t^a,0)| \leq 2\epsilon .$$

This last step is the crucial one. Lemma 4.1 does not assert that we can find $z_t^a \in B_t^a(\hat{p}_t)$ close to z^a, it says only that for fixed T that we can find $z_T^a \in B_T^a(\hat{p}_t)$, close to z^a. However, we need the former conclusion rather than the latter, because we do not know that $\hat{x}_T^a$ or $\hat{x}_t^a$ is optimal in $B_T^a(\hat{p}_t)$, while we do know that $\hat{x}_t^a$ is optimal in $B_t^a(\hat{p}_t)$.

As we just remarked, since $\hat{p}_t$ and $\hat{x}_t$ are a truncated equilibrium and $z_t^a \in B_t^a(\hat{p}_t)$, it must be that

$$(4.7) \qquad U_t^a(\hat{x}_t^a, \tilde{x}^a, \phi_t^a) \geq U_t^a(z_t^a, \tilde{x}^a, \phi_t^a).$$

We shall now show that (4.7) contradicts $U^a(z^a) = U^a(\hat{x}^a) + \delta$.

From (4.6) and (4.2) we see that

$$(4.8) \qquad |U_t^a(z^a, \tilde{x}^a, 0) - U_t^a(z_t^a, \tilde{x}^a, 0)| \leq 4\epsilon,$$

while from (4.3) and (4.2) we see that

$$(4.9) \qquad |U_t^a(\hat{x}_t^a, \tilde{x}^a, 0) - U_t^a(\hat{x}^a, \tilde{x}^a, 0)| \leq 3\epsilon.$$

Moreover, $U^a(z^a) = U^a(\hat{x}^a) + \delta$ and (4.2) imply

$$(4.10) \qquad U_t^a(z^a, \tilde{x}^a, 0) - U_t^a(\hat{x}^a, \tilde{x}^a, 0) \geq \delta - 2\epsilon.$$

Combining this with (4.8) and (4.9) yields

$$(4.11) \qquad U_t^a(z_t^a, \tilde{x}^a, 0) - U_t^a(\hat{x}_t^a, \tilde{x}^a, 0) \geq \delta - 9\epsilon.$$

It follows that

$$\begin{aligned}(4.12) \qquad & U_t^a(z_t^a, \tilde{x}^a, \phi_t^a) - U_t^a(\hat{x}_t^a, \tilde{x}^a, \phi_t^a) \geq \\ & \delta - 9\epsilon + \sum_{t(c)\leq t} \phi_{ct}^a z_{ct}^a - \sum_{t(c)\leq t} \phi_{ct}^a \hat{x}_{ct}^a \geq \\ & \delta - 9\epsilon + \chi_t^a(\phi_t^a, \hat{p}_t) - \sum_{t(c)\leq t} \phi_{ct}^a \hat{x}_{ct}^a,\end{aligned}$$

where the last line follows from the definition of $\chi_t^a(\phi_t^a, \hat{p}_t)$. From the transversality condition (4.1) we have

$$(4.13) \qquad U_t^a(z_t^a, \tilde{x}^a, \phi_t^a) - U_t^a(\hat{x}_t^a, \tilde{x}^a, \phi_t^a) \geq \delta - 9\epsilon - \epsilon_t > 0,$$

with the final inequality from (4.5). This, however, contradicts (4.7).

Q.E.D.

Next we prove that every equilibrium is the limit of truncated equilibria.

<u>proof that limits are necessary</u>: Given an equilibrium $\hat{x}$, $\hat{\gamma}$, $\hat{p}$ and a truncation date T, we must construct $\hat{x}_T$, $\hat{\gamma}_T$, $\hat{p}_T$, together with a truncation plan $\tilde{x}$ and weights ϕ_T which are a truncated equilibrium and satisfy the transversality condition. We take the truncated plans and prices to be the same as the equilibrium plans, $\hat{x}_T = \hat{x}$, $\hat{\gamma}_T = \hat{\gamma}$, $\hat{p}_T = \hat{p}$, $\tilde{x} = \hat{x}$, and show how to construct weights which satisfy the transversality condition. Notice that the proposed truncated equilibrium is obviously both socially and individually feasible, and that profits are at a maximum. To show that the truncated equilibria are approximately extensible, take the approximation to be the given nearby plan in $B^a(\hat{p})$. What we must show is that, relative to the weights we construct, agents are indeed maximizing their truncated utility.

The method of finding truncation weights is closely related to the dynamic programming method used by Weitzman [1973] in establishing a similar tranversality condition in a somewhat simpler model. Let $\Omega_T \equiv \{(\omega_c) | t(c) \le T,\ \exists \tilde{x},\ x^a \in B^a(\hat{p}),\ \omega_c = x^a_c - \tilde{x}^a_c\}$. Notice that this is a convex subset of a finite dimensional space. Define $V^a_T\colon \Omega_T \to \Re$ by

$$(4.14) \qquad V^a_T(\omega) = \max_{\substack{x^a, \tilde{x}^a \in B^a(\hat{p}) \\ x_c - x^a_c \le \tilde{\omega}^a_c,\ t(c) \le T}} U^a_T(\tilde{x}^a, x^a, 0)$$

Since $B^a(\hat{p})$ is a convex set, this is a concave function. Notice also, by the monotonicity assumptions (A.1), (A.2) and (A.4), that an argmax satisfying $x^a_c - \tilde{x}^a_c = \omega_c$ for $t(c) \le T$ always exists. Moreover, since $\bar{x}^a$ satisfies all the short sales constraints strictly, there is $\bar{y}^a \ll \bar{x}^a$

which does so as well. Since $\bar{x}^a$ exactly satisfies the budget constraints, $\bar{y}^a$ strictly satisfies them. It follows that $0 \in \text{interior}\ (\Omega_T)$. Consequently, there are weights ϕ^a_T (non-negative by monotonicity and free disposal) such that

$$(4.15) \qquad V^a_T(\omega) - \sum_{t(c)\leq T} \phi^a_{cT}\omega_c \leq V^a_T(0).$$

Since $\hat{x}^a$ maximizes U^a in $B^a(p)$

$$(4.16) \qquad V^a_T(0) = U^a(\hat{x}^a).$$

First set $x^a = \hat{x}^a$, and let $\omega_c = \hat{x}^a_c - \tilde{x}^a_c$. Then, by definition of V^a_T

$$V^a_T(\omega) \geq U^a_T(\tilde{x}^a, \hat{x}^a, 0).$$

We conclude from (4.15) and (4.16) that

$$U^a(\hat{x}^a) = V^a_T(0) \geq U^a_T(\tilde{x}^a, \hat{x}^a, 0) - \sum_{t(c)\leq T} \phi^a_{cT}(\hat{x}^a_c - \tilde{x}^a_c),$$

or, in other words,

$$\left(U^a(\hat{x}^a) + \sum_{t(c)\leq T} \phi^a_{cT}\hat{x}^a\right) - \left(U^a_T(\tilde{x}^a, \hat{x}^a, 0) + \sum_{t(c)\leq T} \phi^a_{cT}\tilde{x}^a_c\right) \geq 0,$$

so $\hat{x}^a$ is in fact optimal relative to the weights ϕ^a_T.

Next, set $\tilde{x}^a = \hat{x}^a$, and $\omega_c = x^a_c - \hat{x}^a_c$. Then, by the definition of V^a_T,

$$V^a_T(\omega) \geq U^a_T(\hat{x}^a, x^a, 0).$$

We conclude from (4.15) and (4.16) that

$$U^a(\hat{x}^a) = V_T^a(0) \geq U_T^a(\hat{x}^a, x^a, 0) - \sum_{t(c)\leq T} \phi_{cT}^a(x_c^a - \hat{x}_c^a),$$

or, in other words,

$$\epsilon_T \geq U^a(\hat{x}^a) - U_T^a(\hat{x}^a, x^a, 0) \geq \sum_{t(c)\leq T} \phi_{cT}^a(\hat{x}_c^a - x_c^a).$$

Since U^a is continuous in the product topology we may assume $\epsilon_T \to 0$. Since this holds for all $x^a \in B^a(\hat{p})$, by letting $\Sigma_{c\leq T}\, \phi_{cT}^a x_c^a \to \chi_T^a(\phi_T^a, \hat{p})$, we get

$$\epsilon_T \geq \sum_{t(c)\leq T} \phi_{cT}^a \hat{x}_c^a - \chi_T^a(\phi_T^a, \hat{p}).$$ Q.E.D.

5. Extensibility of Equilibria

We now consider the existence of infinite horizon equilibria. We must show that truncated equilibria exist, and that there is a convergent sequence of truncated equilibria that is approximately extensible. The existence of truncated equilibria is relatively straightforward - we indicate what is known on the subject, and outline conditions sufficient for existence in the current context. These conditions also guarantee the existence of a convergent sequence of truncated equilibria. Approximate extensibility is more complex. However, there are a variety of sufficient conditions which guarantee that every truncated equilibrium is extensible.

First we consider the existence of truncated equilibria, extensible or not. When there is no production, markets form a tree, and all assets return the same mix of good or money in a particular market, Werner [1985], Duffie [1985], Geanakoplous and Mas-Colell [1985], and Geanakoplos and Polemarchakis [1985] have shown in this model that there is a truncated equilibrium in which short sales constraints can be chosen not to bind.

Without the special assumption on asset returns, Hart [1975] showed that there may be no equilibrium. However, Duffie and Shafer [1986] show that even in this case an equilibrium exists in generic economies.

In the infinite horizon, to insure the absence of Ponzi schemes, we observed that some short sales constraints are needed (although they need not bind). If we take short sales constraints such that asset holding is bounded below, Radner [1972] showed that we can dispense with special return structure assumptions and generic economies. In the model of this paper, which allows production, and drops the assumption that markets form a tree, add the assumption that the production technology and short sales constraints are such that social feasibility implies bounds on individual holdings of claims. Then it can be shown that a truncated equilibrium exists. Following Debreu [1959], we truncate the budget constraints using bounds on individual holding that cannot bind in equilibrium. Combined with Lemma 4.1, which states that truncated budget constraints are lower-hemi-continuous, this implies that demand is a upper-hemi-continuous convex valued correspondence. Since social feasibility implies _a priori_ bounds on individual holdings of claims, the activity levels γ are also bounded. The fact that there are separate price simplices and versions of Walras's Law in each market leads to only notational complications: the remainder of the proof can follow any standard argument for the existence of competitive equilibrium. An extra bonus of this approach is that equilibrium claim holding and production plans are bounded in each period, so that set of truncated equilibria lie in compact subset of $\Re^{\infty}$. This implies that the sequence of truncated equilibria as $T \to \infty$ has a convergent subsequence.

The most productive approach to approximate extensibility is to impose conditions which guarantee that every equilibrium is extensible. Suppose that if short sales constraints are satisfied prior to a certain time T, when rolled to period $T+1$ they are satisfied again without trading. For example, since rates of return are non-negative, this is true if the only short sales constraints are that non-negative amounts of every claim must be held in every market. In this case, since passive rolling over of the portfolio automatically satisfies the budget constraint, budget constraints are extensible, regardless of prices. The case of non-negativity restrictions is important in practice: this is the type of restriction in the Bewley money model, for example.

The condition requiring the feasibility of rolling over the portfolio can be weakened if a priori bounds on relative prices can be established. Suppose, for example, that there are two consumption goods, food and shelter. Suppose that bonds pay off in food only, and consider an agent whose endowment consists only of shelter, not food. If this individual sells bonds short, he must trade in order to have non-negative food holdings in the next period. If we can establish a priori using the marginal conditions for food and shelter a equilibrium bound on the relative price of food to shelter, the agent is guaranteed that he can trade his shelter for a minimal amount of food, and use this to support short sales of bonds. In this case also, the budget constraint, at equilibrium prices, will be extensible.

References

Balasko, Y. and K. Shell [1980], "The Overlapping Generations Model, I: The Case of Pure Exchange Without Money," Journal of Economic Theory, 23, 281-306.

Bewley, T. [1980], "The Optimum Quantity of Money," in Models of Monetary Economics, J. Karaken and N. Wallace (eds.), Federal Reserve Bank of Minneapolis, January.

__________ [1983], "A Difficulty with the Optimum Quantity of Money," Econometrica, 51 #5, September, 1485-1504.

Debreu, G. [1979], A Theory of Value, Cowles Foundation.

Duffie, D. [1985], "Stochastic Equilibrium with Incomplete Financial Markets," mimeo, Graduate School of Business, Stanford.

__________ and W. Shafer [1986], "Equilibrium in Incomplete Markets," in two parts, mimeo, MSRI, UC-Berkeley.

Ekern, S. and R. Wilson [1974], "On the Theory of the Firm in an Economy with Incomplete Markets," Bell Journal 171-183.

Fudenberg, D. and D. Levine [1983], "Subgame-Perfect Equilibria of Finite- and Infinite-Horizon Games," Journal of Economic Theory, 31 #2, December.

Geanakoplos, J. and A. Mas-Colell [1985], "Real Indeterminacy with Financial Assets," mimeo, Yale.

__________ and H. Polemarchakis [1985], "Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete," mimeo, Yale.

Hart, O. [1975], "On the Optimality of Equilibrium when the Market Structure is Incomplete," Journal of Economic Theory, 11, 418-443.

Harris, C. [1985], "Perfect Equilibrium in Infinite Horizon Games," Journal of Economic Theory, 37, 99-125

Kehoe, T., and D. Levine [1985], "Markov Equilibria of Stationary Rational Expectations Models," mimeo, UCLA.

Levine, D [1986a], "Efficiency and the Value of Money," mimeo, UCLA.

__________ [1986b], "Aggregate Borrowing Constraints and Expansionary Monetary Policy," mimeo, UCLA.

Lucas, R. and N. Stokey [1984], "Money and Interest in a Cash in Advance Economy," mimeo, Northwestern.

Muller, W. and M. Woodford [1983], "Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers: I. Existence of Equilibrium," mimeo, MIT.

Radner, R. [1972], "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets," Econometrica, 40, March.

__________ [1974], "A Note on Unanimity of Stockholders Preferences Among Alternative Production Plans: A Reformulation of the Ekern-Wilson Model," Bell Journal, 181-184.

Scheinkman, J. and L. Weiss [1986], "Borrowing Constraints and Aggregate Economic Activity," Econometrica, January.

Spear, S. [1985], "Rational Expectations in the Overlapping Generation Model," Journal of Economic Theory, 35, no. 2.

Weitzman, M. [1973], "Duality Theory for Infinite Horizon Convex Models," Management Science, 19 #7, March.

Werner, J. [1985], "Equilibrium in Economics with Incomplete Financial Markets," Journal of Economic Theory, 36(1), 110-119.

Wilson, C. [1981], "Equilibrium in Dynamic Models with an Infinity of Agents," Journal of Economic Theory, 24, 95-111