THE DYNAMIC INVESTMENT BEHAVIOR OF FIRMS AND INDUSTRIES IN PERFECT FORESIGHT COMPETITIVE EQUILIBRIUM OVER TIME*

by

Michele Boldrin
Department of Economics
University of California, Los Angeles
Los Angeles, CA 90024

and

Luigi Montrucchio
Dip. di Matematica
Politecnico di Torino
Torino, Italy 10129

UCLA Dept. of Economics
Working Paper #457
November, 1987

*This research was completed while the first author was a Visiting Fellow at the Dept. of Economics, University of Chicago. He is very grateful to all the members of that Department for the warm hospitality and the support provided. Financial support from the A.P. Sloan Foundation through a Dissertation Fellowship is also gratefully acknowledged.

The second author's research was partially supported by funds (40%) from the Italian Ministero della Pubblica Istruzione.

1. Introduction

Most empirical analyses (either at the aggregated or disaggregated level) of dynamic factors-demand use a specification of the equilibrium levels of inputs utilized which implicitly or explicitly assumes the dynamic stability of the (unique) steady state level of such inputs. It is often derived, in fact, as a linear-quadratic approximation around such steady-state, where the displacement from the stationary position is explained by relative-price variations and lags in the adjustment sequence. To quote only a few, at random, from a very large literature: Abel [1979], Meese [1980], Sargent [1978], Shapiro [1986], etc. As most researchers assume markets are in competitive equilibrium over time, an often cited (and natural) microeconomic justification for such a model is provided by the neoclassical theory of the firm's optimization process over time when costs of adjusting the input levels are present. Nevertheless we are not aware of any complete theoretical analysis of the neoclassical model showing that, indeed, global asymptotic stability of the steady state is a logical consequence of the basic assumptions.

Dynamic stability is fairly easy to obtain in the one-dimensional case (i.e., when there is only one variable input that induces costs of adjustment), but is not a trivial consequence in the general, n-dimensional case. In the latter setup global stability may be a consequence of the well-known Turnpike Theorems when the factor of intertemporal discounting is close to one (i.e., "small interest rate"), but is not true in general.

The object of the present paper is to provide an abstract analysis of such a problem and to prove that, indeed, asymptotic stability is obtained at any level of discounting in force of the specific economic assumptions that are made on the production and cost of adjustment functions. The results we obtain guarantee, so to speak, logically coherent dynamic microfoundations to a vast area of applied research that would, otherwise, have to rely on ad-hoc assumptions and/or "intuition".

It is customary to distinguish, in this contest, between "internal" and "external" costs of adjustment (see Section 3 and 4 for a detailed discussion). We consider both cases in our analysis and it turns out that, interestingly enough, they are not exactly equivalent from a dynamic point of view. In fact while the internal costs hypothesis always implies global stability, the same result is not guaranteed in the external-costs case. Even if we are, at the moment, unable to provide a counterexample where optimal cycles or chaos occur for such a model, still we conjecture that such an outcome is indeed possible. We prove, in any case, that stability is guaranteed when an additional simplifying assumption is added; the latter amounts either to exclude relevant interactions among factors in the cost-of-adjustment function or to assume that all inputs depreciate at the same rate per period of time. Finally, we consider also the full Competitive-Equilibrium for the whole industry where output and input prices vary over time according to market clearing equations. We show by a routine argument that such a Competitive Equilibrium over time indeed exists and is unique. With a technique similar to the one used in the individual firm problem we are able to prove that the Competitive Equilibrium sequence of prices and quantities will asymptotically converge to a steady state vector. The picture appears, at that point, remarkably complete and solid, at least within the basic assumptions taken here.

The remainder of the paper is organized as follows: Section 2 contains an introduction to the mathematics of the problem and an exposition of the main stability theory as previously proved in Boldrin-Montrucchio [1988].

Section 3 discusses the internal-costs case for the individual firm in the first part and the external-costs case in the second part. In Section 4 we consider the industry-wide Competitive Equilibrium. Finally, Section 5 compares our result with the ones existing in the literature and draw a few conclusions.

2. Intertemporal Optimizing and Dynamic Stability

In this paper we will consider various models of firm behavior which are all special cases of a more abstract optimization problem. We like to study the latter at the very beginning in order to prove our main stability theorem once-and-for-all. The subsequent discussion will then be devoted to show how and when the general result applies to specific economic models. Assume that an infinitely lived decision maker faces the following objective:

$$\text{(P)} \qquad W_\delta(x) = \text{Max} \sum_{t=0}^{\infty} V(x_t, x_{t+1})\delta^t$$

$$\text{s.t.} \quad x_{t+1} \in \Gamma(x_t)$$

$$x_0 = x, \quad \text{given in } X.$$

Here the n-dimensional vector x_t describes the relevant state of the world at time $t = 0,1,2,\ldots$ and is constrained to some compact and convex set $X \subset \mathbb{R}^n$. $V(x_t, x_{t+1})$ is the instantaneous return function in reduced form: its value is the maximum attainable level of satisfaction when the state is x_t today and will be x_{t+1} in the next period. $\Gamma(x_t)$ is a correspondence describing the subset of X which is achievable tomorrow as a function of the present state x_t. Finally $\delta \in [0,1)$ is a discount factor and $x_0 = x$ specifies the initial condition we start from.

Mathematically this is described by the following hypothesis:

(A1) $\Gamma: X \to X$ is a continuous and compact-valued correspondence with a convex graph and such that $x \in \Gamma(x)$ for all $x \in X$.

(A2) $V: D \to \mathbb{R}$ is a continuous and concave function defined on $D = \{(x,y) \in \mathbb{R}^{2n}$ s.t. $x \in X$ and $y \in \Gamma(x)\}$. $V(x,\cdot)$ is strictly concave for every given x.

We summarize here some well-known properties of (P) that are useful for our purposes (see Lucas, Prescott and Stokey [1986] for the details). The function $W_\delta: X \to \mathbb{R}$ is called the value function of (P): it gives the maximum achievable total reward as a function of the initial condition $x_0 = x$. It is (strictly) concave and continuous and it satisfies the relation:

(1) $$W_\delta(x) = \text{Max}\{V(x,y) + \delta W_\delta(y);\ \text{s.t. } y \in \Gamma(x)\}$$

which is the celebrated Bellman Equation. Define as $\tau_\delta: X \to X$ the continuous map solving (1), i.e.:

(2) $$W_\delta(x) = V(x,\tau_\delta(x)) + \delta W_\delta(\tau_\delta(x))$$

We call τ_δ the (optimal) policy function of (P). Using the Bellman Optimality Principle one can show that $\{x_t\}_{t=0}^{\infty}$ is a feasible sequence realizing the maximum in (P) if and only if it satisfies: $x_{t+1} = \tau_\delta(x_t)$, $x_0 = x$. The shape of τ_δ will depend, <u>ceteris paribus</u>, on the magnitude of δ. More formally: for given x, Γ and V the map $\delta \to \tau_\delta$ from the interval $[0,1)$ into the space $C^0(X;X)$ is itself continuous (in the uniform topology). This implies that the dynamical system described on X by the iterates of τ_δ may have different qualitative features at different values of δ. In particular, for suitable forms of V, periodic and even aperiodic trajectories can be produced by τ_δ at certain magnitudes of δ.

As a matter of fact, there are some economic problems which belong to the class (P) and for which it is counterintuitive, or even against empirical evidence, to theorize the optimality of irregular and oscillatory behaviors. The family of models of the firm we are addressing here seems to belong to this group, at least when we see it as the General Eqilibrium foundation of the empirical researches we referred to in the Introduction. We would like, therefore, to specify (P) in such a way that the predicted policy functions generates "simple dynamics" for all $\delta \in (0,1)$ and for all dimensions of the state space. The notion of "simple" we are using is formally stated in the next two Definitions.

Definition 1: Let $f: X \to X$ be a continuous function from X into itself defining the dynamical system $x_{t+1} = f(x_t)$. The non-wandering set $\Omega(f)$ associated to it is defined as: $\Omega(f) = \{x \in X,$ s.t. for every neighborhood U of x and $T > 0$, $\exists t \geq T$ such that $f^t(U) \cap U \neq \phi\}$. Here f^t denotes the t^{th} iterate of f, i.e., $f^t(x) = f(f^{t-1}(x))$, $f^0(x) = x$.

We find it useful to point out that, in general, the structure of $\Omega(f)$ can be incredibly complicated: it includes all the steady states, the periodic orbits, the strange attractors, etc. As the asymptotic behavior of the dynamical system $x_{t+1} = f(x_t)$ is described by $\Omega(f)$ it is clear that the former can, in general be very complex. Therefore our notion of simple dynamics will be stated in terms of a simple non-wandering set:

Definition 2: A dynamical system $f: X \to X$ is "simple" if $\Omega(f) = \text{Fix}(f)$, where $\text{Fix}(f) = \{x \in X,$ s.t. $x = f(x)\}$.

The intuition behind this should be clear: every observable trajectory of a simple dynamical system will asymptotically move toward some steady state.

Consequently the vectors $\{x_t\}$ solving (P) will stay within smaller and smaller neighborhoods of some point as time goes by and will make the agent's behavior more and more predictable.

The reader should also observe that if $\mathrm{Fix}(\tau_\delta)$ results in a singleton, for some or even all values of δ, then the strongest form of stability, global asymptotic stability, will be obtained. This is the case in which comparative static exercises are meaningful as the sensitivity of the unique asymptotic state $x^*(\delta)$ to variations in δ could be evaluated.

In Boldrin-Montrucchio [1988] we proved that a set of sufficient conditions on V can be found in order to guarantee a simple τ_δ for every level of discounting.

<u>Theorem 1</u>: Let $\tau_\delta: X \to X$ solve (P) under (A-1)-(A-2). Then τ_δ is simple for every $\delta \in [0,1)$ if V satisfies:

$$(*) \qquad \sum_{t=1}^{N} V(x_t, x_t) \geq \sum_{t=1}^{N} V(x_t, x_{t+1})$$

for any finite sequence $\{x_1, \ldots, x_N\}$ of points in X; (the position $x_{N+1} = x_1$ is understood in (*)).

3. <u>The Cost-of-Adjustments Model of the Firm</u>

Stripped down to its bare essentials, the problem we have in mind is based on the following story.[1]

A representative firm is considered which produces a single output by means of a vector of inputs and a time invariant technology. The critical feature of the model comes from the existence of some inputs which are quasi-fixed: a variation in their utilized quantities will entail positive costs for the firm over and above the payment of the pure market price.

These "adjustment costs" depend on the magnitude of the change.

In our formal treatment we will assume that all inputs belong to this category: the case in which they can be split into two subgroups, one being composed of freely adjustable factors is easily tractable as a special version. Denote the vector of inputs at time $t = 0,1,2,\ldots$ with $x_t \in \mathbb{R}^n_+$.

All the $(n+1)$ spot markets in which the output and inputs are traded, are assumed to be perfectly competitive: prices are therefore taken parametrically by the firm. The firm is infinitely lived and it perfectly foresees the sequences of prices $\{p_t\}$ and $\{q_t\}$ $p_t \in \mathbb{R}_+$, $q_t \in \mathbb{R}^n_+$, $t = 0,1,\ldots$ that will clear the markets.

The technology is described by a production function $f(x_t)$ and a cost-of-adjustment function $g(x_t,x_{t+1})$ which are both time-invariant and have values expressed in units of output: $g(x_t,x_{t+1})$ is then the output foregone in period t in order to adjust the input level from x_t to x_{t+1} for the next period. Because quasi-fixed factors depreciate, we denote with Σ the $n \times x$ diagonal matrix with diagonal elements $0 \le \sigma_i \le 1$, $i = 1,\ldots,n$ representing the rates of depreciation for each coordinate of x. Finally, we denote with $I_t \in \mathbb{R}^n_+$ the gross purchases of quasi-fixed factors during period t. The firm's total cash-flow in any period can then be written as:

(3) $$G(x_t,x_{t+1},p_t,q_t) = p_t[f(x_t) - g(x_t,x_{t+1})] - \langle q_t,I_t \rangle$$

The assumption of perfect capital-markets closes the model: a profit-maximizing firm will then act according to the objective:

(P1) $$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} G(x_t,x_{t+1},p_t,q_t)\delta^t$$

$$\text{s.t. } x_{t+1} = (I-\Sigma)x_t + I_t \ge 0$$

x_0 given in $\mathbb{R}^n_t$

$(\{p_t\},\{q_t\})$ given for all $t = 0,1,2,\ldots$

To get things going we assume that the price sequences $\{p_t\}$ and $\{q_t\}$ are constant over time and use the output as the numeraire: $p_t = 1$ and $q_t = q \in \mathbb{R}^n_+$ all $t = 0,1,\ldots$. In Section 4 we show how this hypothesis can be removed, and the central stability results retained, by using a technique suggested first in Lucas-Prescott [1971]. A second simplifying assumption constrains the input vectors within a compact and convex set $X \subset \mathbb{R}^n_+$. This, again, is of no harm to the generality of the argument: convexity is just natural in this context and compactness may be derived endogenously by using standard assumptions on f and g, see Brock-Scheinkman [1974, pp. 5-8] for the details.

The firm's maximum problem now becomes:

$$\text{(P2)} \qquad W(x) = \text{Max} \sum_t^{\infty} [f(x_t) - g(x_t,x_{t+1}) - \langle q,x_{t+1}-(I-\Sigma)x_t\rangle]\ \delta^t$$

$$\text{s.t.} \quad x_t \in X \quad \text{all} \quad t$$

$$x_0 = x \in X, \quad \text{given;}$$

under the hypothesis:

(I) f-g: $X \times X \to \mathbb{R}_+$ is a continuous function. $f(x) - g(x,y)$ is concave over $X \times X$ and strictly concave in y for every $x \in X$.

It is clear that (P2) is just another way of writing (P): this means that the assumptions we have made so far are too weak for our purposes. In fact, they are consistent with policy functions τ_δ of almost any type.[2]

The culprit here is the cost function: we have assumed, not even explicitly, that g has some degree of convexity but nothing more. We will show in the following which kind of additional structure can be put on g in order to apply Theorem 1.

3.1 The Internal Costs Hypothesis

This is the case studied, among others, by Lucas [1967], Mortensen [1973], Scheinkman [1978].

Assumption (I) is supplemented with:

(II) $g(x_t, x_{t+1})$ is convex and satisfies: $g(x_t, x_{t+1}) = g(x_{t+1} - x_t)$.

These are the so-called internal costs and their fundamental source is technological: installation of new equipment and/or training of new labor forces cause a temporary reduction in production as resources must be devoted to these activities. These are opportunity costs for the use of resources: the convexity hypothesis is therefore justified by the classical non-increasing returns arguments, strict convexity in the second argument is added to assure a unique solution to the maximization problem. The new problem is:

$$\text{(P3)} \qquad W(x) = \text{Max} \sum_{t=0}^{\infty} [f(x_t) - g(x_{t+1} - x_t) - \langle q, x_{t+1} - (I-\Sigma)x_t \rangle] \, \delta^t$$

$$\text{s.t. } x_t \in X \text{ all } t, \quad x_0 = x \text{ given.}$$

Denote with $x_{t+1} = \tau_\delta(x_t)$ the unique solution to (P3). The following is true:

<u>Lemma 1</u>: The concave function $-g(x_{t+1} - x_t)$ satisfies (*) of Theorem 1.

<u>Proof</u>: Let $\{x_t, \ldots, x_N\}$ be any feasible sequence. We need to show that:

$$-Ng(0) \geq - \sum_{t=1}^{N} g(x_{t+1} - x_t)$$

holds, with $x_{N+1} = x_1$. This follows from concavity of $-g$, i.e.:

$$-g(0) = -g\left[\frac{1}{N} \sum_{t=1}^{N} (x_{t+1}-x_t)\right] \geq -\frac{1}{N} \sum_{t=1}^{N} g(x_{t+1}-x_t) \qquad \text{Q.E.D.}$$

We have therefore proved:

Proposition 1: Under (I) and (II) the optimal investment policy of the firm is a simple dynamical system for every level of the market interest rate.

Proof: By Theorem 1 we need only to show that the short-run return function $V(x,y) = f(x) - g(y-x) - \langle q, y-(I-\Sigma)x \rangle$ satisfies (*). The linear part trivially satisfies it with equality. The rest follows from Lemma 1. Q.E.D.

3.2 The External-Costs Hypothesis

This is the second relevant hypothesis in the literature, classical references are Brock/Scheinkman [1974] and Gould [1968]. Assumption (II) is replaced with:

(III) The cost function satisfies $g(x,y) = g(y-(I-\Sigma)x)$ and it is convex.

The story behind external-costs relies on the existence of some market imperfections: the presence of monopolistic elements in the factor-markets could explain (III). In this case the equilibrium price for the quasi-fixed factors will be an increasing function of the amount demanded by the single firm: q is not taken parametrically any more, and the linear term $\langle q, y-(I-\Sigma)x) \rangle$ will be incorporated in g. The reader should note that this idea is rather at odds with the hypothesis of a competitive firm: we need to assume different technologies and some firm-specific factors which happen to be in limited supply. Notice, anyhow, that the latter hypothesis would only imply the nonlinear (and increasing) shape of g, not its convexity.

This second requirement is therefore substantially arbitrary from a theoretical point of view. It is also clear that we need it to make the problem concave.

Taking this caveat as understood we can consider the new objective:

$$\text{(P4)} \qquad W(x) = \text{Max} \sum_{t=0}^{\infty} [f(x_t) - g(x_{t+1} - (I-\Sigma)x_t)]\delta^t$$

$$\text{s.t. } x_t \in X \quad \text{all} \quad t, \quad x_0 = x \quad \text{given.}$$

Once again strict convexity guarantees the existence of a policy function describing the optimal program: $x_{t+1} = \tau_\delta(x_t)$. Unfortunately the argument of the previous paragraph cannot be replicated here. In fact we have:

<u>Proposition 2</u>: Under assumptions (I) and (III) the return function: $V(x,y) = f(x) - g(y-(I-\Sigma)x)$ may not satisfy (*) for some feasible sequence $\{x_1,\ldots,x_N\}$.

<u>Proof</u>: We give an example of cost function g satisfying (I) and (III) but not (*) for some simple sequence. Set $g(z) = z'Az$ with:

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \quad z = [z_1,z_2] \in [0,1] \times [0,1], \quad z = y-(I-\Sigma)x \quad \text{and}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad 0 \le \sigma_i \le 1, \quad i = 1,2.$$

The cost function clearly satisfies (III) and will satisfy (I) for any standard concave production function. Some tedious but straightforward computations, given in the Appendix, show that (*) is not satisfied even for simple sequences of the form $\{x_1,x_2\}$. Q.E.D.

In fact it is likely that an optimal cyclic investment policy could be derived from the previous example, even if it appears hard to work out the details. Nevertheless, with regard to (P4), there are two very significant and empirically important cases in which we can establish a stability result.

Proposition 3: Consider (P4) under (I) and (III) and assume that: $\Sigma = \sigma I$ is satisfied for some scalar $\sigma \in [0,1]$, where I is the $n \times n$ identity matrix. Then the policy function τ_δ is simple.

Proof: See Appendix.

We can also prove:

Corollary: Under the same assumptions of Proposition 3, τ_δ is simple even if convexity of g is replaced by quasi-convexity, whenever τ_δ is unique.

Proof: See Appendix.

Our second case is:

Proposition 4: Consider (P4) under (I) and (III), and assume that for any vector $z = (z_1, \ldots, z_n)$ the cost function also satisfies $g(z) = \Sigma_{i=1}^{n} g_i(z_i)$, with g_i convex for all $i = 1, \ldots, n$. Then the policy function is simple.

Proof: See Appendix.

Under the rationale for external costs we have given above, it should be clear that assuming this kind of separability in the cost function amounts to assuming that the different markets for inputs are separated and cross-elasticities negligible.

4. The Industry Competitive Equilibrium

There are many good reasons for which the dynamic models of the firm we have studied in the previous section may look unsatisfactory. The hypothesis of constant prices seems particularly inconsistent with the economic intuition, considering that quantities will in general be changing along an optimal path.

We devote this section to show how to relax this assumption without harming the stability results. To keep things relatively simple and self-contained we will consider only interior paths by assuming enough steepness of f and g on the boundaries. In addition only the internal-costs case will be worked out in full detail: the case of external-costs does not present any additional complications, besides those already addressed in Section 3.2, and can therefore be treated similarly.

We need to consider the competitive equilibrium over time for the output market of an industry composed by a fixed number of M different firms, producing the same kind of good. Retain the previous notation and denote with f_j, g_j, Σ_j, $y_{jt} = [f_j(x_{jt}) - g_j(x_{jt+1}-x_{jt})]$ the elements of a generic firm $j = 1,\dots,M$. Note that the input vector x_{jt} has coordinates x^i_{jt}, $i = 1,\dots,n$ some of which may be zero. We also need the additional notation:

$$I^i_{jt} = x^i_{jt+1} - (1-\sigma^i_j)x^i_{jt}, \quad I^i_t = \sum_{j=1}^{M} I^i_{jt}, \quad i = 1,\dots,n.$$

$$y_t = \sum_{j=1}^{M} y_{jt} \quad \text{and} \quad z_t = [x^1_{1t},\dots,x^n_{1t},\dots,x^i_{jt},\dots,x^n_{Mt}] \in \mathbb{R}^{nM}.$$

Finally, let $\tilde{X}$ be the convex and compact subset of $\mathbb{R}^{nM}$ to which z_t

belongs for all t, obtained by natural product of the M feasible sets X_j of the participating firms. Now assume that the market equilibrium prices for output and inputs obey to:

$$p_t = \phi(y_t),\ \phi' \le 0,\ \phi > 0$$

$$q_t^i = H_i(I_t^i),\ H_i' \ge 0,\ H_i > 0,\quad i = 1,\dots,n$$

where ϕ is the inverse demand function for output and the H_i are the inverse supply functions of the n inputs. Each firm solves:

$$\text{(Pj)}\quad \text{Max} \sum_{t=0}^{\infty} \delta^t \{p_t[f_j(x_{jt}) - g_j(x_{jt+1}-x_{jt})] - \langle q_t, x_{jt+1}-(I-\Sigma_j)x_{jt}\rangle\}$$

s.t. $x_{jt} \in X_j \subset \mathbb{R}^n$, X_j convex and compact

x_{jo} given.

<u>Definition</u>: A set of sequences $\{x_{1t},\dots,x_{Mt}\}_{t=0}^{\infty}$ and $\{p_t,q_t\}_{t=0}^{\infty}$ with $x_{jt} \in X_j$ all j and t, $p_t \in \mathbb{R}_{++}$, $q_t \in \mathbb{R}_{++}^n$ all t is a P.F.C.E. for this industry if the following are satisfied:

(i) $\{x_{jt}\}_{t=0}^{\infty}$ solves (Pj) for the given $\{p_t,q_t\}_{t=0}^{\infty}$, all $j = 1,\dots,M$.

(ii) $$p_t = \phi\left(\sum_{j=1}^{M} y_{jt}\right) \quad \text{all } t; \qquad \text{(ii)}$$

(iii) $$q_t^i = H_i\left(\sum_{j=1}^{M} I_{jt}^i\right) \quad \text{all } t \text{ and all } i = 1,\dots,n.$$

The M problems (Pj) are clearly non-autonomous with respect to the time variable. However, it is a routine exercise to use the consumer-surplus device introduced by Lucas-Prescott [1971] to derive an autonomous dynamic programming problem, which unique solution for every initial condition is the one and only P.F. Competitive Equilibrium sequence $\{x_{1t},\ldots,x_{Mt}\}_{t=0}^{\infty}$.

In order to accomplish this, we need a few more definitions. Set:

$$F(z_t, z_{t+1}-z_t) = \int_0^{y_t} \phi(L)dL, \quad \text{with} \quad y_t = \sum_{j=1}^{M} [f_j(x_{jt}) - g_j(x_{jt+1}-x_{jt})]$$

$$H(z_{t+1} - (I-\bar{\Sigma})z_t) = \sum_{i=1}^{n} \int_0^{I_t^i} H_i(L)dL.$$

where $\bar{\Sigma}$ is the $(n{\times}M) \times (n{\times}M)$ diagonal matrix having the M matrices Σ_j along the diagonal.

Notice that $F\colon \tilde{X} \times \tilde{X} \to \mathbb{R}$ and $H\colon \tilde{X} \times \tilde{X} \to \mathbb{R}$, are well defined and continuous. We have:

<u>Lemma 2</u>: The function F defined above is concave over $\tilde{X} \times \tilde{X}$ while H is convex over $\tilde{X} \times \tilde{X}$. Also F is strictly concave when all the g_j are strictly convex.

<u>Proof</u>: The statement for F follows easily either by taking derivatives and computing the Hessian D^2F when smoothness is assumed or by noting that F is a monotonic transformation of the f_j and g_j that preserves concavity. For **H** we need only to note that it is the sum of n functions:

$$\tilde{H}_i = \int_0^{I_t^i} H_i(L)dL$$

and that each one of them is convex, because all the H_i are increasing.

Q.E.D.

We may now consider the following problem:

$$\text{(P.0)} \qquad \text{Max} \sum_{t=0}^{\infty} [F(z_t, z_{t+1}-z_t) - H(z_{t+1}-(I-\tilde{\Sigma})z_t)]\delta^t$$

$$\text{s.t.: } z_t \in \tilde{X} \text{ all } t,$$

$$z_o \text{ given}$$

which solution is a map: $\tau_\delta(z_t) = z_{t+1}$. In force of the interiority and compactness assumptions we have made and of the strict concavity of (F-H) proved in Lemma 2, it is a routine exercise to conclude that the unique sequence (z_t) that solves (P.O.) is a Competitive Equilibrium for our industry. The reverse is also true and we may therefore conclude that the global behavior of quantities and prices in Competitive Equilibrium for the whole industry will be simple if such is the solution to (P.O.).

The reader should notice that, even if we have assumed only internal adjustment costs, the process of endogenizing the input-prices has created an external-costs problem for the whole industry. This is only natural, as the Pareto optimal solution cannot disregard the effects of the individual firm demands on the inputs market prices.

Anyhow: problem (P.O.) can now be handled by means of the technique illustrated in Section 2. Let's begin by picking a numeraire: $p_t = 1$ all t, so that the demand function will satisfy $\phi(y_t) = 1$ all t and therefore the function $F(z_t, z_{t+1}-z_t)$ simplifies to:

$$(4) \qquad F(z_t, z_{t+1}-z_t) = \int_0^{\sum_{j=1}^{M} [f_j(x_{jt})-g_j(x_{jt+1}-x_{jt})]} dL =$$

$$= \sum_{j=1}^{M} [f_j(x_{jt})-g_j(x_{jt+1}-x_{jt})$$

We have:

Lemma 3: The concave function $F: \tilde{X} \times \tilde{X} \to \mathbb{R}$ defined in (4), satisfies condition (*) of Theorem 1.

Proof: Let the sequence $\{z_1, \ldots, z_N\}$ be given. We want to show that:

$$\sum_{t=1}^{N} \left\{ \sum_{j=1}^{M} \left[f_j(x_{jt}) - g_j(0) \right] \right\} \geq \sum_{t=1}^{N} \left\{ \sum_{j=1}^{M} \left[f_j(x_{jt}) - g_j(x_{jt+1} - x_{jt}) \right] \right\}$$

holds true for any such sequence. The latter inequality reduces to:

$$- \sum_{j=1}^{M} \left[N g_j(0) - \sum_{t=1}^{N} g_j(x_{jt+1} - x_{jt}) \right] \geq 0$$

which is obviously satisfied because g_j is convex for all j (see Lemma 1).

Q.E.D.

Lemma 4: The concave function $H: \tilde{X} \times \tilde{X} \to \mathbb{R}$ satisfies condition (*) of Theorem 1 when one of the following occurs:

1) $\Sigma_j = \Sigma$ all $j = 1, \ldots, M$ (depreciation matrices are invariant across firms)

2) for each $\tilde{H}_i = \int_0^{I_t^i} H_i(L) dL$ there exists M functions $h_j^i : X_j \to \mathbb{R}$ such that $\tilde{H}_i(I_t^i)$ can be written as: $\tilde{H}_i(I_t^i) = \sum_{j=1}^{M} h_j^i(I_{jt}^i)$ for all $i = 1, \ldots, n$.

Proof: When hypothesis 1) is realized then H is the sum of n concave functions:

$$\int_0^{\sum_{j=1}^{M}(x^i_{jt+1}-(1-\sigma^i)x^i_{jt})} H_i(L)dL = G_i(z^i_{t+1}-(1-\sigma^i)z^i_t)$$

where z^i_t denotes the M-dimensional vector $[x^i_{1t},\ldots,x^i_{Mt}]$, for each $i = 1,\ldots,n$. Then G_i belongs to the same class of functions considered in Proposition 3, so that condition (*) applies to each one of them and, a fortiori, applies to their sum.

When hypothesis 2) is realized then H is totally separable and so each of the functions $\tilde{H}_i$ belongs to the class of functions considered in Proposition 4 and (*) applies once again. Q.E.D.

<u>Proposition 5</u>: Under the maintained hypothesis, the Competitive Equilibrium sequence (z_t, p_t, q_t) generated by the optimal solution to (P.O.), will exhibit a simple dynamic, if either condition 1) or condition 2) in Lemma 4 is satisfied.

<u>Proof</u>: It follows from Lemmas 3 and 4 and Theorem 1. Q.E.D.

Notice that, while condition 1) is easy to accept as a first approximation, condition 2) imposes overly strong restrictions on the factor markets. It amounts to saying that each factor is traded in M different markets, one for each firm, with no interdependence across markets.

5. Relations With the Previous Literature

The problem of global dynamic stability was almost never taken into consideration in the papers explicitly dedicated to the firm problem. It has obviously been studied in the turnpike literature, but only in a few cases the special functional form of V has been exploited.

1) The most relevant one is Scheinkman [1978], where the continuous-time version of the internal-costs model is considered. He shows that every optimal path either converges to the boundary of the feasible set or to the unique, interior OSS. His basic argument is totally different from ours and cannot be applied to the external-costs model. He exploits the fact that in his model the Hamiltonian is separable in the state and co-state variables. Differentiability is also used. Further, he provides a solution to the industry-competitive equilibrium problem along the same lines we have used here, but he keeps input prices fixed over time.
2) Brock-Scheinkman [1974] is the only global analysis of the discrete-time version of the problem. They take in explicit consideration only the external-costs, but the internal-costs is also covered by their technique. They add differentiability to our assumptions and obtain a global stability result for the interior solutions as a consequence of the negative quasi-definitiveness of a certain Hessian matrix. The latter is satisfied, in general, for values of the discount factor close to one. Their theorem does not, therefore, exclude different behaviors for smaller values of δ.
3) Again for the discrete-time problem, an interesting local analysis has been recently carried on by Dasgupta [1985] (see also Dasgupta-McKenzie [1983]). His main concern is the relation between local stability and regularity of the optimal steady states. Here regularity means that the value of the steady state capital stocks $<q(\delta),k^*(\delta)>$ increases when δ increases. Theorem 4.2 of the first paper shows that, for the internal-costs case, the OSS's are locally asymptotically stable.

To prove this, one needs differentiability, but a strong turnpike result is gained when this theorem is combined with our Proposition 1:

there exists a unique globally asymptotically stable optimal steady state. This holds independently of δ. The same result can be obtained, without differentiability, by imposing strong concavity and convexity assumptions on f and g respectively. This can be used to prove that $\mathrm{Fix}(\tau_\delta)$ is a singleton at each δ which, together with Proposition 1, yields the named conclusion.

Returning to Dasgupta's result, it is worth noting that it cannot be extended to the external-costs case. We have already seen this in our example of Section 3.2.

Furthermore, some interesting comparative statics and dynamics propositions are implied. For example, Remark 4.1 in Dasgupta [1985] implies that regularity and local stability are equivalent for both the internal and external-costs models. Also by Theorem 9 of Dasgupta-McKenzie [1983] we may deduce that all optimal paths are "dynamically regular" in both models. In fact, we have shown that they always converge to an optimal steady state. In fact in the external-costs case there is a (measure zero) set of investment paths which are not dynamically regular: they are the unstable steady states which, from the previous remark, are not even regular.

4) Finally, the n-dimensional version of the monopolist optimal pricing problem studied by Brock-Dechert [1985] turns out to be a case of external-costs and our stability results may be applied.

APPENDIX

Proof of Proposition 2: Take a sequence in $[0,1] \times [0,1]$ defined as $\{x,y\} = \{[x_1,x_2], [y_1,y_2]\}$. Then (*) is satisfied if $g(y-(I-\Sigma)y) + g(x-(I-\Sigma)x) \leq g(y-(I-\Sigma)x) + g(x-(I-\Sigma)y)$ which in this case is:

$$(\sigma_1 y_1)^2 + (\sigma_2 y_2)^2 + \sigma_1\sigma_2 y_1 y_2 + (\sigma_1 x_1)^2 + (\sigma_2 x_2)^2 + \sigma_1\sigma_2 y_1 y_2 \leq$$

$$\leq (y_1-(1-\sigma_1)x_1)^2 + (y_2-(1-\sigma_2)x_2)^2 + (y_1-(1-\sigma_1)x_1)(y_2-(1-\sigma_2)x_2) +$$

$$+ (x_1-(1-\sigma_1)y_1)^2 + (x_2-(1-\sigma_2)y_2)^2 + (x_1-(1-\sigma_1)y_1)(x_2-(1-\sigma_2)y_2)$$

To simplify computations set $\sigma_1 = 0$, $\sigma_2 = 1$. This entails no loss of generality given the continuity of g with respect to Σ. The inequality above boils down to:

$$(x_1-y_1)[x_2-y_2 + 2(x_1-y_1)] \geq 0$$

which should hold for any pair of points $[x_1,x_2]$ and $[y_1,y_2]$ in the non-negative unit square. This is not the case. Q.E.D.

Proof of Proposition 3: We only need to show that (*) is satisfied, i.e., that:

$$\sum_{i=1}^{N} g(\sigma x_i) \leq \sum_{i=1}^{N} g(x_{i+1}-(1-\sigma)x_i)$$

holds for all sequences $\{x_1,\ldots,x_N\}$ and all $\sigma \in [0,1]$. For given sequence and σ define the convex combination:

$$\sigma x_{i+1} = (1-\sigma)\sigma x_i + \sigma[x_{i+1}-(1-\sigma)x_i]$$

for $i = 1,\ldots,N$ and $N+1 = 1$.

Then the convexity of g implies: $g(\sigma x_{i+1}) \leq (1-\sigma)g(\sigma x_i) + \sigma g[x_{i+1}-(1-\sigma)x_i]$ all i. Summing up from $i = 1$ to $i = N$ we get, after simplification:

$$\sigma \sum_{i=1}^{N} g(\sigma x_i) \leq \sum_{i=1}^{N} g(x_{i+1} - (1-\sigma)x_i) \qquad \text{Q.E.D.}$$

Proof of Corollary to Proposition 3: Using the same convex combination as in the proof of the proposition, quasi-convexity implies: $g(\sigma x_{i+1}) \leq \text{Max}[g(\sigma x_i),\ g(x_{i+1} - (1-\sigma)x_i)]$ $i = 1,\ldots,N$. Hence there exists $\mu \in [0,1]$ such that

$$g(\sigma x_{i+1}) \leq (1-\mu)g(\sigma x_i) + \mu g(x_{i+1} - (1-\sigma)x_i)$$

$i = 1,\ldots,N$. Summing up and simplifying yields the desired inequality.

Q.E.D.

Proof of Proposition 4: Here the depreciation factors can be different so $\Sigma = \text{diag}(\sigma_1,\ldots,\sigma_n)$. Let $x_1,\ldots,x_N$ be a sequence of feasible vectors in X, with coordinates $(x_i^1,\ldots,x_i^n)$ for $i = 1,\ldots,N$. Once again set:

$$\sigma_j x_{i+1}^j = (1-\sigma_j)\sigma_j x_i^1 + \sigma_j [x_{i+1}^j - (1-\sigma_j)x_i^j]$$

for every vector $i = 1,\ldots,N$ and each coordinate $j = 1,\ldots,n$.

As we have assumed convexity for every g_j, we have:

$$g_j(\sigma_j x_{i+1}^j) \leq (1-\sigma_j)g_j(\sigma_j x_i^j) + \sigma_j g_j(x_{i+1}^j - (1-\sigma_j)x_i^j) \qquad \forall_i, \forall_j$$

By summing the above inequality along the index i we get:

$$\sum_{i=1}^{N} g_j(\sigma_j x_{i+1}^j) \leq (1-\sigma_j) \sum_{i=1}^{N} g_j(\sigma_j x_i^j) + \sigma_j \sum_{i=1}^{N} g_j(x_{i+1}^j - (1-\sigma_j)x_i^j)$$

for all $j = 1,\ldots,n$. After simplification we get:

$$\sum_{i=1}^{N} g_j(\sigma_j x_i^j) \le \sum_{i=1}^{N} g_j(x_{i+1}^j - (1-\sigma_j)x_i^j)$$

all $j = 1,\ldots,n$. A second summation along the coordinate gives:

$$\sum_{j=1}^{n} \sum_{i=1}^{N} g_j(\sigma_j x_i^j) \le \sum_{j=1}^{n} \sum_{i=1}^{N} g_j(x_{i+1}^j - (1-\sigma_j)x_i^j).$$

Interchanging the order of summation and using the property of g given in the Proposition we get:

$$\sum_{i=1}^{N} g(\Sigma x_i) \le \sum_{i=1}^{N} g(x_{i+1} - (I-\Sigma)x_i)$$

Q.E.D.

FOOTNOTES

[1] Here we are extensively drawing from a very large literature on the argument; to name a few: Brock-Scheinkman [1974], Gould [1968], Lucas [1967], Mortensen [1973], Treadway [1971].

[2] For a proof of this, rather strong, assertion see Boldrin-Montrucchio [1986].

REFERENCES

Abel, A. [1979], Investment and the Value of Capital, New York: Garland.

Boldrin, M., and Montrucchio, L. [1986], "On the Indeterminacy of Capital Accumulation Paths," Journal of Economic Theory, 40, 26-39.

__________ and __________ [1988], "Acyclicity and Stability for Intertemporal Optimization Models," International Economic Review 29, no. 1 (forthcoming).

Brock, W., and Dechert, D.W. [1985], "Dynamic Ramsey Pricing," International Economic Review, 26, 569-590.

__________, and Scheinkman, J. [1974], "On the Long-Run Behavior of a Competitive firm," Report 7441, Center for Mathematical Studies in Business and Economics, University of Chicago.

Dasgupta, S. [1985], "A Local Analysis of Stability and Regularity of Stationary States in Discrete Symmetric Optimal Capital Accumulation Models," Journal of Economic Theory, 36, 302-318.

__________ and McKenzie, L.W. [1983], "The Competitive Statics and Dynamics of Stationary States," Working Paper #83-5, University of Rochester, Dept. of Economics.

Gould, J.P. [1968], "Adjustment Costs in the Theory of Investment of the Firm," Review of Economic Studies, 35, 47-55.

Lucas, R.E. Jr. [1967], "Adjustment Costs and the Theory of Supply," Journal of Political Economy, 75, 321-334.

__________ and Prescott, E.C. [1971], "Investment Under Uncertainty," Econometrica, 39.

__________, __________ and Stokey, N.L. [1986], "Recursive Methods in Economic Dynamics," mimeo, University of Chicago (September).

Meese, R. [1980], "Dynamic Factor Demand Schedules for Labor and Capital Under Rational Expectations," Journal of Econometrics, 14, 141-158.

Mortensen, D. [1973], "Generalized Costs of Adjustment and Dynamic Factor Demand Theory," Econometrica, 41, 657-665.

Sargent, T. [1978], "Estimation of Dynamic Labor Demand Schedules Under Rational Expectations," Journal of Political Economy, 86, 1009-1044.

Scheinkman, J.A. [1978], "Stability of Separable Hamiltonions and Investment Theory," Review of Economic Studies, 45, 559-570.

Shapiro, M. [1980], "Capital Utilization and Capital Accumulation: Theory and Evidence," Journal of Applied Econometrics, 1, 211-234.

Treadway, A. [1971], "The Rational Multivariate Flexible Accelerator," Econometrica, 38, 845-55.