PATHS OF OPTIMAL ACCUMULATION IN TWO-SECTOR MODELS*

Michele Boldrin

Department of Economics

University of California, Los Angeles

Los Angeles, CA 90024

UCLA Dept. of Economics
Working Paper #464
January 1988

1. Introduction

The 1980s have witnessed an increasing attention on the part of the profession to endogenous, deterministic explanations of the erratic dynamic behavior of many macro variables. Examples of this new line of research are Benhabib and Day [1982], Benhabib and Nishimura [1979, 1985], Grandmont [1985] and most of the works collected in Grandmont (ed.) [1986].

Central attention is placed on the notion of chaos as an appropriate qualitative description of the observed oscillations. Empirically oriented efforts (especially Brock-Sayers [1986] and Le Baron-Scheinkman [1986]) seem to suggest that, indeed, some strange attractors may well be behind certain data.

A sound economic theory for the emergence of chaos in pure exchange OLG economies is provided both in Benhabib-Day [1982] and Grandmont [1985]. On the other hand no reasonable economic arguments have been provided to justify chaotic competitive dynamics in optimal growth models with a representative consumer. Benhabib and Nishimura [1979, 1985] provide an argument for periodic cycles but do not address the existence of more complicated phenomena. That they are possible and perfectly consistent with the generally maintained assumptions is, however, an implication of the general theory developed in Boldrin-Montrucchio [1984, 1986].

This paper attempts to fill this gap by providing a rationale to the existence of chaotic Competitive Equilibrium paths within the context of a simple aggregated optimal growth model with two sectors and an infinitely living consumer. I show that the emergence of chaotic orbits is linked to dynamic changes in the profitability conditions between the two sectors which, in turn, have their origins in the technological structure of the economy. In particular, it is proved that under certain simplifying assumptions, such conditions can be expressed in terms of factor-intensity reversal and high discounting. In the more general case a high degree of impatience is still needed whereas the reversal assumption may be replaced with the hypothesis of a uniformly larger capital/labor ratio in the consumption sector together with a relatively high sensitivity of the price of the investment good to changes in the output of the same sector for low values of the aggregated capital stock. An analytic example that uses a CES and a Leontief production function is used to illustrate the working of the general theorems.

A few words of caution should be added about the notion of chaos we are using. The two main theorems give sufficient conditions for "topological chaos" (see below Definition 3). It is well-known that such behavior may well be unobservable. This is a point well stressed, for example, in Grandmont [1987]. The existence of "observable chaos" (see below Definition 4) cannot be generally proved for this class of models due to unavoidable computability constraints that are briefly discussed in the main text. I provide a corollary which shows that, under regularity assumptions, observable chaos may be derived.

The paper goes as follows: Section 2 describes the model and characterizes the Dynamic Competitive Equilibrium. Section 3 derives a standard Dynamic Programming problem from which the Competitive Equilibrium can be computed. Section 4 contains the main theorems as well as the applications to our example.

2. Intertemporal Competitive Equilibrium in a Two-Sector Economy

2.1 The Model

We deal with a competitive economy where a pure consumption good and a pure capital good are produced and traded over time. Only one representative agent exists. He lives forever and takes as given the sequence of triples $\{w_t, r_t, q_t\}$, $t = 0,1,2,\ldots$ denoting the labor wage rate, the gross capital rental and the price of capital in every period t. They are expressed in units of the consumption good which has the price fixed at one in all periods. Our price system will then be a current-value one at every time t. We assume perfect foresight.

The preferences of the consumer are described by a standard utility function $u(c_t)$ depending on the current level of consumption:

U1) u is an increasing, concave function from $[0,\infty)$ into $[0,\infty)$, C^2 on $(0,\infty)$.

In each period the consumer is endowed with one unit of labor time, which he supplies inelastically at the current wage rate, and with an amount k_t of capital stock which is left over from previous consumption-saving decisions and that he supplies inelastically to the productive sectors. At any point in time his budget constraint is then:

$$c_t + q_t[k_{t+1} - (1-\mu)k_t] = r_t k_t + w_t \tag{1}$$

where μ is the capital depreciation rate. Given the initial capital stock k_o the problem of the consumer amounts to pick up sequences of consumption $\{c_t\}$ and gross saving $\{k_{t+1} - (1-\mu)k_t\}$ to maximize the present value of his lifetime utility under the period-by-period budget constraint (1). Formally we write:

$$\text{(PC)} \qquad \text{Max} \sum_{t=0}^{\infty} u(c_t)\delta^t \quad \text{s.t. (1) all } t = 0,1,2,\ldots \text{ and a given } k_o.$$

Here δ denotes the discount factor.

On the production side we assume the existence of two industries which are distinguished because of the different technologies available and different outputs. They can be imagined as being composed of a large number of perfectly identical competitive firms. Constant returns to scale are assumed to hold in both sectors. We summarize this with two production functions:

$$(2) \qquad y^1 = F^1(k^1, \ell^1), \quad y^2 = F^2(k^2, \ell^2)$$

where the superscript "1" denotes the consumption sector and "2" denotes the capital good sector; k^i, ℓ^i, $i = 1,2$ are the quantities of capital and labor used as inputs in either one of the two industries. With regard to (2) we state:

T1) F^i, $i = 1,2$ is linear homogeneous in its arguments. Also $F^2(0,x) = 0$ for all x in R_+.

T2) F^i, $i = 1,2$ is an increasing and concave function from $[0,\infty) \times [0,\infty)$ into $[0,\infty)$, of class C^2 in the interior of its domain. F^i is strictly concave in each separate factor.

The following hypothesis is also useful:

T3) There exists a $\bar{k} \in (0,\infty)$ such that $F^2(k,1) < \mu k$ for all $k > \bar{k}$ and $F^2(k,1) > \mu k$ for all $k < \bar{k}$.

Firms take the price sequence $\{w_t, r_t, q_t\}$ as given. Their optimal decision problems reduce to the choice of factors-demand sequences $\{k^i_t, \ell^i_t\}$ which maximize the present discounted value of the stream of future profits.

Therefore the consumption good sector solves:

$$(PF^1) \qquad \text{Max } y_t^1 - r_t k_t^1 - w_t \ell_t^1 \qquad \text{s.t.} \quad y_t^1 \le F^1(k_t^1, \ell_t^1), \quad \text{all } \quad t,$$

and the capital good solves:

$$(PF^2) \qquad \text{Max } q_t y_t^2 - r_t k_t^2 - w_t \ell_t^2 \qquad \text{s.t.} \quad y_t^2 \le F^2(k_t^2, \ell_t^2), \quad \text{all } \quad t.$$

Given this description of agents' behavior it is natural to define a Competitive Equilibrium (with perfect foresight) in the following way:

<u>Definition 1</u>: An Intertemporal Competitive Equilibrium (ICE) is given by price sequences $\{w_t, r_t, q_t\}$ and quantity sequences $\{y_t^1, y_t^2, k_t^1, k_t^2, \ell_t^1, \ell_t^2, c_t, k_t\}$ such that:

a) $\{c_t\}$ and $\{k_t\}$ solve (PC) given $\{w_t, r_t, q_t\}$;

b) y_t^1, k_t^1 and ℓ_t^1 solve (PF^1) given $\{w_t, r_t\}$, all $t = 0,1,2,\ldots$;

c) y_t^2, k_t^2 and ℓ_t^2 solve (PF^2) given $\{w_t, r_t, q_t\}$, all $t = 0,1,2,\ldots$;

d) $c_t = y_t^1, y_t^2 = k_{t+1} - (1-\mu)k_t, \ k_t = k_t^1 + k_t^2, \ 1 = \ell_t^1 + \ell_t^2,$

all $t = 0,1,2,\ldots.$

The existence of such ICE can be proved by standard arguments. Moreover we can write down an infinite-horizon maximization problem whose solutions are ICE for our two-sector economy.

<u>Proposition 1</u>: Consider the economy described by (PC), (PF^1) and (PF^2) under assumptions U1) and T1)-T2). Consider a set of quantity sequences satisfying Definition 1. Then they also solve the following problem:

$$(P1) \qquad \text{Max } \sum_{t=0}^{\infty} u(c_t)\delta^t \qquad \text{s.t.} \quad c_t \le T(k_t, y_t^2)$$

$$k_{t+1} = (1-\mu)k_t + y_t^2$$

where:

$$\text{(T)} \qquad T(k_t,y_t^2) = \text{Max } y_t^1 = F^1(k_t^1,\ell_t^1) \qquad \text{s.t.} \quad y_t^2 \le F^2(k_t^2,\ell_t^2)$$

$$k_t \ge k_t^1 + k_t^2, \quad 1 \ge \ell_t^1 + \ell_t^2, \quad \ell_t^i \ge 0, \quad k_t^i \ge 0, \quad i = 1,2.$$

The reciprocal is also true.

We omit the proof of this statement. It can be easily derived by adopting the arguments of Becker [1981].

2.2 Optimal Sequences

Before moving ahead in the analysis of the intertemporal problem (P1) let us discuss the nature of problem (T). The existence of a solution is guaranteed by standard arguments. The nature of such a solution has been extensively studied in the literature and we will make heavy use of the existing results in the sequel. The reader is referred to Kuga [1972], Hirota and Kuga [1971] and Benhabib and Nishimura [1979] for the proofs.

Let's denote the chosen input levels with: $k^1(k,y)$, $\ell^1(k,y)$, $k^2(k,y)$, $\ell^2(k,y)$. These will be continuously differentiable functions under the maintained assumptions. The Production Possibility Frontier (PPF) $T(k,y)$ is then equal to $F^1[k^1(k,y),\ell^1(k,y)]$. It turns out to be concave and, under some weak technical conditions, twice continuously differentiable. In particular, for interior values of k^i,ℓ^i we have

$$\text{(3)} \qquad T_1(k,y) = F_1^1(k^1,\ell^1) = qF_1^2(k^2,\ell^2) \ge 0$$

$$\text{(4)} \qquad T_2(k,y) = -F_j^1(k^1,\ell^1)/F_j^2(k^2,\ell^2) \le 0 \quad j = 1,2.$$

Additional information on the nature of T can be obtained by exploiting some duality relations. As this is also very standard we summarize here only the essential results.

From problem (T) one has:

$$(5) \qquad r(k,y) = T_1(k,y) \quad \text{and} \quad q(k,y) = -T_2(k,y)$$

where, we recall, the price of the consumption good is taken to be the numeraire. The Hessian matrix of T can then be written as:

$$(6) \qquad \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \partial r/\partial k & \partial r/\partial y \\ -\partial q/\partial k & -\partial q/\partial y \end{bmatrix}$$

Define the cost function which solves the dual of T as: $p = p(\omega)$, where $p = [1,q]$ and $\omega = w/r$. In extensive form we can write:

$$(7) \qquad \begin{aligned} 1 &= a_{11}(\omega)r + a_{21}(\omega)w \\ q &= a_{12}(\omega)r + a_{22}(\omega)w \end{aligned}$$

where the a_{ij}'s indicate the cost-minimizing input coefficients of the two sectors. By totally differentiating (7), using the necessary conditions for cost minimization and the fact that $d\omega/\omega = dw/w - dr/r$ in order to simplify, we get:

$$(8) \qquad \frac{dq}{d\omega} = [a_{22}a_{11} - a_{12}a_{21}]r^2$$

From the assumption $T \in C^2$ we have $T_{12} = T_{21}$, and therefore:

$$(9) \qquad \frac{\partial r}{\partial y} = -\frac{\partial q}{\partial k} = -\frac{\partial q}{\partial \omega}\frac{\partial \omega}{\partial k} .$$

Therefore (6) can be written as:

$$(10) \qquad \begin{bmatrix} T_1 & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial k} & -\frac{\partial q}{\partial \omega}\frac{\partial \omega}{\partial k} \\ -\frac{\partial q}{\partial \omega}\frac{\partial \omega}{\partial k} & -\frac{\partial q}{\partial y} \end{bmatrix}$$

which can now be completely signed:

$$(11) \qquad T_{11} = \partial r/\partial k \le 0,\ T_{22} = -\partial q/\partial y \le 0,\ T_{11}T_{22} - (T_{12})^2 \ge 0$$

and, given that $\partial\omega/\partial k > 0$ (see Intriligator [1971] p. 240):

$$T_{12} = -(\partial q/\partial\omega)(\partial\omega/\partial k) \lesseqgtr 0 \quad \text{according to (8).} \tag{12}$$

The interpretation of the a_{ij}'s implies that T_{12} will be positive when the capital good sector is more capital intensive than the consumption good sector and negative in the opposite case. T_{12} will vanish when both sectors have the same capital intensity for a given factor-price ratio. Assumptions T1)-T3) do not restrict our economy to either of the two patterns. Which one of the two situations will turn out true at a given point in time depends on the relative degrees of convexity of the unit isoquants of F^1 and F^2 together with the prevailing factor-price ratio at that time. Because the latter depends, in turn, on k and y in the general dynamic problem the sign of T_{12} can change along the chosen optimal accumulation path.

The case in which T_{12} can be either positive or negative is in fact considered in our example.

Example: In our simple example we will use:

$$F^1(k^1,\ell^1) = \left[ak_1^\rho + (1-a)\ell_1^\rho\right]^{\frac{1}{\rho}} \quad a \in (0,1), \quad \rho \in (-\infty,1)$$

$$F^2(k^2,\ell^2) = \min\left\{\ell^2, \frac{k_2}{\gamma}\right\} \quad \gamma \in (0,1).$$

Notice that F^2 is not of class C^2, this choice has been dictated by computability reasons. Had we chosen F^2 to be, for example, a second CES we would have not been able to work out an explicit form for T. The sake of simplicity also recommends: $u(c_t) = c_t$. Problem (T) now is

$$\max \left[ak_1^\rho + (1-a)\ \ell_1^\rho\right]^{\frac{1}{\rho}} \qquad \text{s.t.} \quad y \le \min\left\{1-\ell_1, \frac{k-k_1}{\gamma}\right\}$$

The straightforward solution gives a PPF of the type:

$$T(k,y) = \left[a(k-\gamma y)^{\rho} + (1-a)(1-y)^{\rho}\right]^{\frac{1}{\rho}} \tag{13}$$

Such a T is of class C^2 on the interior of its domain, which is the set: $D = \{(k,y) \in [0,1] \times [0,1];\ \text{s.t.}\ 0 \leq y \leq k/\gamma\}$. It is also strictly concave in its second argument everywhere on D but the vertical line $x = \gamma$.

Let us now turn our attention to the intertemporal problem (P1). Given any initial condition a unique optimal solution to (P1) exists, which is also supported by a sequence of competitive prices. McKenzie [1986] gives a proof of this under conditions much weaker than ours. For heuristic purposes we consider here the associated Lagrangian:[1]

$$L = \sum_{t=0}^{\infty} \{u(c_t) + p_t^1[T(k_t,y_t) - c_t] + p_t^2[(1-\mu)k_t + y_t - k_{t+1}]\}\delta^t \tag{14}$$

The first order conditions for interior solutions are:

$$u'(c_t) = p_t^1 \tag{15a}$$

$$p_t^1 T_2(k_t,y_t) = -p_t^2 \tag{15b}$$

$$p_{t+1}^2 = p_t^2/\delta(1-\mu) - [p_{t+1}^1 T_1(k_{t+1},y_{t+1})]/(1-\mu) \tag{15c}$$

$$c_t = T(k_t,y_t) \tag{15d}$$

$$k_{t+1} = (1-\mu)k_t + y_t \tag{15e}$$

It is easy to see that (15a-d) together with the "dated" version of (5) will completely "price" our competitive economy. By construction these prices will be current value prices. Set $p_t^1 = 1$ and $p_t^2 = q_t$ all t, then the sequence of Lagrange multipliers $\{w_t,r_t,q_t\}$ solving (P1) and (T) will be the competitive prices associated to the ICE quantities

$\{c_t, k_t, y_t, k_t^1, k_t^2, \ell_t^1, \ell_t^2\}$. Finally, by massaging (15a-e) the Euler equation for the interior optimal paths $\{k_t\}$ can be derived:

$$(16) \qquad u'[T(k_t, k_{t+1} - (1-\mu)k_t)]\, T_2(k_t, k_{t+1} - (1-\mu)k_t) +$$
$$+ \delta u'[T(k_{t+1}, k_{t+2} - (1-\mu)k_{t+1})] \cdot$$
$$\cdot \{T_1(k_{t+1}, k_{t+2} - (1-\mu)k_{t+1}) - (1-\mu)T_2(k_{t+1}, k_{t+2} - (1-\mu)k_{t+1})\} = 0$$

We conclude this part by stating (without proof) the following well-known proposition:

<u>Proposition 2</u>: Let u, F^1 and F^2 satisfy conditions U1) and T1)-T2). If there exists sequences $\{c_t, y_t, k_t\}$ satisfying (15a-e) plus the transversality condition: $\lim \delta^t q_t k_t = 0$ for $t \to \infty$, then $\{c_t, y_t, k_t\}$ solves (P1).

3. <u>Dynamic Programming</u>

3.1 <u>The Value Function and the Optimal Policy Function</u>

The previous discussion should have clarified the economic content of the problem:

$$(P) \quad W_\delta(k_o) = \text{Max} \sum_{t=0}^{\infty} V(k_t, k_{t+1})\delta^t \qquad \text{s.t.} \quad (k_t, k_{t+1}) \in D, \quad k_o \text{ given in } K$$

where:

A1) D is a compact and convex subset of R_+^2. We assume: $(0,0) \in D$ and $(0,y) \notin D$ all $y > 0$.

A2) V: $D \to R_+$ is a concave function, strictly concave in its second argument, continuous on D and C^2 on the interior of D. We assume $V_1(x,y) \geq 0$ and $V_2(x,y) \leq 0$ all $(x,y) \in D$.

A3) The discount factor δ lies in $[0,1)$.

Let us relate A1)-A2) to U1) and T1)-T3). T3) implies that for all $k > \bar{k}$: $T(k,\mu k) < 0$, and for all $k < \bar{k}$: $T(k,\mu k) > 0$, hence production of capital will be limited to the interval $K = [0,\bar{k}]$. The feasible values (k_t, k_{t+1}) of initial stock-final stock pairs will belong to $D \subset K \times K$. Denote as $y = f(k)$ the solution to $T(k,y) = 0$, the set D is defined as:

$$D = \{(k,k') \in K \times K, \quad \text{s.t.} \quad (1-\mu)k \leq k' \leq (1-\mu)k + f(k)\}$$

which is obviously compact and convex. The second part of A1) follows directly from T1). Finally A2) comes from setting $V(k,k') = u[T(k,k'-(1-\mu)k)]$ and U1) and T2). As we want to concentrate our analysis on the features of the model which are implications of different technological assumptions we make the simplifying hypothesis:

$\tilde{\text{U}}$1) The utility function has the linear form: $u(c_t) = c_t$.

It is worth stressing that $\tilde{\text{U}}$1) is of no harm to the generality of the analysis. All the results can be replicated, with minor changes by adopting a generic utility function. The Dynamic Programming approach to the study of (P) considers the equivalent problem:

$$(17) \qquad W_\delta(k_o) = \text{Max } \{V(k_o,k_1) + \delta W_\delta(k_1), \qquad \text{s.t.} \quad (k_o,k_1) \in D\}$$

A solution to (17) is a map τ_δ: $K \to K$, with graph contained in D. We call this map the (optimal) policy function of (P). The whole sequence of optimal capital stocks solving (P) will then be described as the Dynamical System: $k_{t+1} = \tau_\delta(k_t)$. We will study the asymptotic properties of the accumulation paths by means of the map τ_δ.

To enhance the economic significance of what follows it is worth pointing out that the knowledge of the optimal sequence $\{k_t\}$ is enough to deduce the paths over time of all the price-sequences and the quantity-sequences listed in the Definition of an ICE in Chapter 2. In particular, one would like to know how the capital stock price evolves over time. In fact stock prices are much more easily observable than quantities, which makes it possible to test the implications of the model. The problem has a simple solution.[2]

Definition 2: Let $f: X \to X$ and $g: Y \to Y$ be two maps, with X and Y any pair of topological spaces. We say that f and g are topologically conjugate if there exists a homeomorphism $h: X \to Y$ such that: $h \cdot f = g \cdot h$. The homeomorphism h is called a topological conjugancy.

Theorem 1: Assume $\tau: K \to K$ is an interior solution to (P) under A1)-A3). Let $\theta: [0,\infty) \to [0,\infty)$ be such that $q_{t+1} = \theta(q_t)$, with q_t defined as in (7) above. Then τ and θ are topologically conjugate.

Proof: By the definition of q_t and the result of Benveniste and Scheinkman [1979] it is obvious that:

$$q_t = \delta W'(k_{t+1}),$$

where W' is the first derivative of the value function. Also $q_{t-1} = \delta W'(k_t)$, and being k_{t+1} optimal given k_t we can write:

$$q_{t-1} = \delta W'(k_t), \quad q_t = \delta W'[\tau(k_t)]$$

Because W is strictly concave, W' is a homeomorphism from K into $[0,\infty)$. Hence

$$q_t = \theta(q_{t-1}), \quad \text{with} \quad \theta = \delta W' \cdot \tau \cdot (W'\delta)^{-1} \qquad \text{Q.E.D.}$$

Corollary 1: Under A1)-A3) if k^* is an Optimal Steady State (OSS) for τ_δ, then $q^* = \delta W'(k^*)$ is an OSS for θ_δ, and if $\{k_t(k_o)\}$ is an orbit of τ_δ with initial condition k_o, then $\{q_t\} = \delta\{W'[k_t(k_o)]\}$ is an orbit for θ_δ. In short, the dynamics of k_t over K is identical, up to a monotonically decreasing homeomorphism, to that of q_t over R_+.

Proof: It follows from Theorem 1 and Proposition (A.1) in Appendix A.

Remark 1: The assumptions $V \in C^1$ and τ_δ interior are critical to obtain the result. In fact, if differentiability of the value function W_δ is not guaranteed we have to use the supergradient set of $W_\delta(k_t)$ to obtain the price q_t. But the supergradient correspondence, even if monotonic, does not need to be lower-hemicontinuous. Therefore, we cannot claim the existence of a continuous, monotonic selection that realizes the topological conjugancy between prices and quantities in the non-differentiable case.

Example: Problem (P) for our model economy is:

$$\text{Max} \sum_{t=0}^{\infty} [a(k_t(1+\gamma-\gamma\mu) - \gamma k_{t+1})^\rho + (1-a)(1-k_{t+1} - (1-\mu)k_t)^\rho]^{\frac{1}{\rho}} \delta^t$$

$$\text{s.t.} \quad (k_t, k_{t+1}) \in D$$

where:

$$D = \{(x,y) \in [0,1] \times [0,1], \quad \text{s.t.} \quad (1-\mu)x \leq y \leq (1-\mu)x + \frac{x}{\gamma}\}$$

It is apparent that we will not be able to put it in the form (17), essentially because the nonlinear structure of our V makes it impossible (to the author) to find an explicit form for the value function $W_\delta(k_o)$. Consequently, we will not compute the policy function for our example. This is generally the case for this class of problems when non-linearities are introduced. It explains why, in the theoretical part, we search for information on the τ_δ that can be directly computed from the V, in particular from the associated Euler equation.

4. Exotic Dynamics

4.1 Optimal Cycles and Optimal Chaos

It is well-known that the dynamical system τ_δ which solves (P) exhibits regular behaviors for certain parameter values. The Turnpike theorems (see McKenzie [1986]) assure that, for any V satisfying (A.2) there exists a value of the discount factor close enough to one that guarantees that all optimal paths, from any initial condition, will eventually converge to a unique steady state. The point of our research is to show that, in general, this is not the case and provide conditions under which very irregular dynamics are optimal. The work of Benhabib and Nishimura [1985] gives sufficient conditions for the existence of optimal period-two cycles. In the two-sector setup they amount to a consumption-good sector which is more capital/labor intensive than the investment sector in an appropriate neighborhood of the steady state, together with an appropriate value of δ. This is Theorem 1 in their paper. It exploits the fact that the τ_δ is downward sloping, with a slope at the steady state that crosses the value -1 when the discount factor passes through a threshold. This creates the period-two cycle by means of a flip bifurcation. In any case a monotonic policy function cannot produce orbits more complicated than that, as can be easily checked.

Therefore, we need to know what determines the slope of the policy function. The answer is again provided by Benhabib and Nishimura [1985] in their Theorem 2:

<u>Theorem 2</u>: Let $\{k_t\}$ be an optimal path. Let $(k_t, k_{t+1}) \in \text{int}(D)$ and let A1)-A3) hold. Then:

i) If $V_{12}(x,y) > 0$ for all $(x,y) \in \mathrm{int}(D)$, $k_t < k_{t+1}$ implies $k_{t+1} \leq k_{t+2}$. If $(k_{t+1},k_{t+2}) \in \mathrm{int}(D)$, $k_t \leq k_{t+1}$ implies $k_{t+1} < k_{t+2}$ (i.e., τ_δ is strictly increasing on interior segments of K).)

ii) If $V_{12}(x,y) < 0$ for all $(x,y) \in \mathrm{int}(D)$, $k_t < k_{t+1}$ implies $k_{t+1} \geq k_{t+2}$. If $(k_{t+1},k_{t+2}) \in \mathrm{int}(D)$, $k_t < k_{t+1}$ implies $k_{t+1} > k_{t+2}$ (i.e., τ_δ is strictly decreasing on interior segments of K).

Proof: See Benhabib and Nishimura [1985] Theorem 2.

In our model we have:

(18) $$V_{12}(k_t,k_{t+1}) = T_{12}(k_t,k_{t+1}-(1-\mu)k_t) - (1-\mu)T_{22}(k_t,k_{t+1}-(1-\mu)k_t)$$

Equation (12) implies that we have to assume either a factor intensity reversal and a large negative magnitude of T_{12} relative to $(1-\mu)T_{22}$, or a more capital intensive consumption sector with relative magnitudes of T_{12} and $(1-\mu)T_{22}$ such that (18) has opposite signs over different subsets of the interior of D. For the sake of simplicity we will take the first road in its most extreme version:

A4) The PPF T is derived from F^1 and F^2 satisfying T1)-T2) and such that there exists one, and only one, factor-intensity reversal. Moreover, the depreciation factor μ equals one.

Under $(\tilde{U}1)$ and A4) we have $T(k_t,k_{t+1}) = V(k_t,k_{t+1})$. Then we can prove:

Lemma 1: Under $(\tilde{U}1)$ and A4) if $(k^*,k^{*\prime}) \in D$ are such that $V_{12}(k^*,k^{*\prime}) = 0$, then $V_{12}(k^*,k') = 0$ for all k' feasible from k^*.

Proof: See Appendix A.

All in all we have:

<u>Proposition 3</u>: Under A1, A2), A4) and ($\tilde{U}$1) the following is true: τ_δ is increasing on $[0,k^*]$ and decreasing on $[k^*,\bar{k}]$ for all $\delta \in [0,1)$.

<u>Proof</u>: By Lemma 1 $V_{12} = 0$ only along the vertical line $k_t = k^*$ in the (k_t,k_{t+1}) plane. Using Theorem 2 and the fact that $\tau_\delta(0) = 0$ from A1), if τ_δ is not identically zero on K it must increase in $[0,k^*]$ and decrease in $[k^*,\bar{k}]$. Q.E.D.

<u>Example</u>: Our pair of production functions clearly satisfy the first part of A4); in fact any pair of distinct CES production functions exhibit factor intensity reversal as long as they are not both Leontief, Cobb-Douglas or linear. If we set also $\mu = 1$ then our $V(k_t,k_{t+1})$ becomes

$$[a(k_t-\gamma k_{t+1})^\rho + (1-a)(1-k_{t+1})^\rho]^{\frac{1}{\rho}}.$$

The second derivative V_{12} is zero for $k_{t+1} = 1$ or for $k_t = \gamma$, positive on the interior of D for all $k_t \in [0,\gamma)$ and negative for all $k_t \in (\gamma,1]$. Lemma 1 is therefore also satisfied with $k^* = \gamma$, (the boundary value $k_{t+1} = 1$ does not matter here). Simple manipulations of the Euler equation:

$$(19)\quad -[a(k_{t-1}-\gamma k_t)^\rho + (1-a)(1-k_t)^\rho]^{\frac{1}{\rho}-1}\cdot\{(1-a)(1-k_t)^{\rho-1} + a\gamma(k_{t-1}-\gamma k_t)^{\rho-1}\} +$$

$$+ \delta a[a(k_t-\gamma k_{t+1})^\rho + (1-a)(1-k_{t+1})^\rho]^{\frac{1}{\rho}-1} \cdot (k_t-\gamma k_{t-1})^{\rho-1} = 0$$

will show that the unique, interior steady state $k(\delta)$ can be expressed as

$$(20)\qquad k(\delta) = \left\{1 + (1-\gamma)\left[\frac{1-a}{a(\delta-\gamma)}\right]^{\frac{1}{1-\rho}}\right\}^{-1}$$

Therefore for $\delta \in [0,\gamma]$ we have no interior state and for $\delta \in (\gamma,1)$ we have a unique interior steady state which is on the upward sloping branch of

τ_δ for $\delta < \gamma(1 + (1\text{-}a/a)\cdot\gamma^\rho)$ and on the downward sloping one for $\delta > \gamma(1 + (1\text{-}a)/a\cdot\gamma^\rho)$, (see Appendix B for the computations). Some additional algebra will show that there exists a pair of values of δ greater than $\gamma(1 + (1\text{-}a)/a\gamma^\rho)$ but smaller than one at which Theorem 1 of Benhabib-Nishimura [1985] is verified, so that our model economy exhibits dynamic competitive equilibria that are cycles of period-two. Once again, the algebra is in Appendix B. We start our search for chaos by defining it:

Definition 3: We say that τ_δ: K $\rightarrow$ K has topological chaos when there exists a period-three cycle for τ_δ on K.

If τ_δ has period three then, by the Sarkovskij theorem it has also cycles of any other period, and by Li and Yorke [1975], there exists a nondenumerable set $S \subset K$ and an $\epsilon > 0$ such that for every pair x and y in S with $x \neq y$:

$$\limsup |\tau_\delta^n(x) - \tau_\delta^n(y)| \geq \epsilon$$

and

$$\liminf |\tau_\delta^n(x) - \tau_\delta^n(y)| = 0.$$

The latter is a weak form of sensitive dependence on initial conditions. It is weak because: a) given S we do not need to be able to pick x and y arbitrarily close to each other, and b) even if uncountable the set S can have Lebesgue measure zero in K. A stronger form of chaos,[3] that we may call "observable chaos," will satisfy:

Definition 4: τ_δ: K $\rightarrow$ K has observable chaos if there exists a probability measure μ on K, which is invariant with respect to τ_δ, absolutely continuous and ergodic.

Unfortunately, it is almost impossible to check if τ_δ satisfies Definition 4 without knowledge of its functional form. Because this is the case in most of the applications of our theory, we will concentrate our attention on topological chaos (however see below Corollary 2). The following Lemma is crucial to our analysis. It is a well known result form the theory of one-dimensional discrete dynamical systems.

<u>Lemma 2</u>: Let $\tau_\delta\colon K \to K$ be continuous. Then τ_δ has cycles of period three if and only if there exist distinct intervals $K^1 \subset K$ and $K^2 \subset K$, such that:

$$\tau_\delta(K^1) \supset K^2 \quad \text{and} \quad \tau_\delta(K^2) \supset (K^1 \cap K^2).$$

Our strategy is that of looking for computable conditions on V such that Lemma 2 is satisfied by the associated policy function. In the first case we need to replace A1) with the following:

$\tilde{\text{A}}$1) D is a compact and convex subset of $K \times K$, such that

i) $(0,0) \in D$, and $(0,y) \in D$ implies $y = 0$.

ii) $(k,0) \in D$ for all $k \in K$.

iii) There exists a $k^1 \leq k^*$ such that $(k,\bar{k}) \in D$ for all $k \geq k^1$.

The implications of $\tilde{\text{A}}$1) on F^1 and F^2 should be clear. The only stringent condition is iii).

We need three Lemmas.

<u>Lemma 3</u>: A path $\{k_t^*\}$ such that Rel int $\Gamma(k_{t-1}^*) \cap$ Rel int $\Gamma(k_{t+1}^*) \neq \phi$ for all t, with Γ defined as: $\Gamma(x) = \{y \in K,\ \text{s.t. } (x,y) \in D\}$ is an optimal solution to problem (P) if and only if:

$$V(k_{t-1}^*, y) + \delta V(y, k_{t+1}^*)$$

is maximized at $y = k_t^*$ for all $t = 0,1,\ldots.$

Proof: See Appendix A.

Lemma 4: Under $\tilde{A}1)$, A2), A3) and A4): $\tau_\delta(\bar{k}) = 0$ if and only if:

$$V(\bar{k},k) + \delta V(k,0) \text{ is decreasing in } k.$$

Proof: Apply Lemma 3.

Lemma 5: Assume $\tau_\delta(\bar{k}) = 0$ for given δ. Then under $\tilde{A}1)$, A2), A3) and A4) $\tau_\delta(k^*) = \bar{k}$ if and only if:

$$V(k^*,k) + \delta V(k,0) \text{ is increasing in } k.$$

Proof: Apply Lemma 3.

Remark 2: The assumption of $V \in C^2$ implies that Lemma 4 is satisfied for all $\delta \leq |V_2(\bar{k},0)|/V_1(0,0)$, whereas Lemma 5 is satisfied together with Lemma 4 if $\delta \in C(V)$, which is defined as:

$$C(V) = \{\delta \in (0,1), \text{ s.t. } |V_2(k^*,\bar{k})|/V_1(\bar{k},0) \leq \delta \leq |V_2(\bar{k},0)|/V_1(0,0)\}$$

It is easy to check that strict concavity of V in either one of its two arguments will bound $C(V)$ away from $\underline{\delta} = |V_2(k^*,0)|/V_1(0,0)$, which is the largest value of δ which makes τ_δ identically zero. At the same time there is nothing in our model that assures $C(V)$ is not empty. Hence we must assume it explicitly:

A5) V satisfies A2), A4) and is such that $C(V)$ is not empty.

We summarize all this in the following form:

Theorem 3: Under $\tilde{A}1)$, A2), A3), A4) and A5) a map τ_δ which solves (P) is topologically chaotic if $\delta \in C(V)$.

<u>Proof</u>: Lemma 4 implies $\tau_\delta(\bar{k}) = 0$, Lemma 5 implies $\tau_\delta(k*) = \bar{k}$. We know that $\tau_\delta(0) = 0$ in any case. $\tilde{A}1)$ guarantees that graph(τ_δ) $\subset$ D. Set $K^1 = [0,k*]$ and $K^2 = [k*,\bar{k}]$. The theorem follows from Lemma 2. Q.E.D.

<u>Corollary 2</u>: Under the Assumptions of Theorem 3 and the regularity hypothesis that τ_δ has a negative Schwartzian derivative everywhere on K for $\delta \in C(V)$, the dynamical system τ_δ exhibits observable chaos.

<u>Proof</u>: Notice that $\tau_\delta[\tau_\delta(k*)] = 0$ for $\delta \in C(V)$, that is the unique critical point is mapped onto the unstable fixed point at the origin. For a theorem of Collet and Eckmann[4] this implies that τ_δ has no stable periodic orbits and in fact displays a unique invariant, ergodic and absolutely continuous measure on K. Q.E.D.

<u>Example</u>: Even if Theorem 3 cannot be applied directly to our V, a simple limit argument given in Appendix B can be used to prove that, indeed, its conclusions apply when ρ is negative and the triple of parameters satisfy the relation $[a-a^2(1-\gamma^\rho)]^{\left|\frac{1}{\rho}\right|}[1-a(1-\gamma)] > 1$. As shown in the Appendix the set C(V) in this case is:

$$C(V) = \{\delta \in (0,1), \quad \text{s.t.} \quad (1-a+a\gamma^\rho)^{\frac{1}{\rho}} \le \delta \le a^{\left|\frac{1}{\rho}\right|}[1-a(1-\gamma)]\}.$$

Notice that, for a given a, the upper bound may be made very large if γ is very close to zero and ρ converges to minus infinity. For "non-extreme" values of (a,γ,ρ) the following intervals may be computed:

$\rho = -1,\quad a = .6,\quad \gamma = .1: \ .156 \le \delta \le .276$

$\lambda_1(.25) = -2.67,\quad \lambda_2(.25) = -1.49$

$\rho = -2, \quad a = .6, \quad \gamma = .1\text{: } .128 \le \delta \le .356$

$\lambda_1(.35) = -1.6, \quad \lambda_2(.35) = -1.77$

$\rho = -5, \quad a = .6, \quad \gamma = .2\text{: } .221 \le \delta \le .469$

$\lambda_1(.46) = -1.52, \quad \lambda_2(.46) = -1.42$

$\rho = -10, \quad a = .5, \quad \gamma = .2\text{: } .214 \le \delta \le .559$

$\lambda_1(.55) = -1.41, \quad \lambda_2(.55) = -1.28$

$\rho = -20, \quad a = .4, \quad \gamma = .2\text{: } .209 \le \delta \le .649$

$\lambda_1(.64) = -1.3, \quad \lambda_2(.64) = -1.1$

The pattern should be clear enough. In particular it is striking to find out that for economies with low elasticity of substitution (i.e., $\rho < 0$ and large in magnitude), chaos may be obtained for non-extreme values of the other relevant parameters.

Nevertheless the conditions of Theorem 3 are not always easy to compute. It may be surprising to discover that linear homogeneity of the production functions simplifies the search for chaotic accumulation paths. We have:

<u>Lemma 6</u>: Under Assumptions A1)-A4) if:

$$V_2(k,k^*) + \delta V_1(k^*,k') = 0$$

has a solution $k = G(k^*,k')$, then G is independent from k'.

<u>Proof</u>: By the Implicit Function theorem: $\frac{\partial k}{\partial k'} = -\frac{\partial V_{12}(k^*,k')}{V_{22}(k,k^*)}$. By Lemma 2 $V_{12}(k^*,\cdot) = 0$ independently of the second argument. Q.E.D.

<u>Remark 3</u>: A way of restating the Lemma is: the preimage of the critical point k^* under the policy function τ_δ is independent from the image of

k^* under τ_δ.

We can now state our last theorem.

<u>Theorem 4</u>: Consider problem (P) under A1)-A4). Assume there exists an interval $C(V) \subset (0,1)$ such that for all $\delta \in C(V)$ the following conditions are satisfied at such δ:

i) $V_2(x,k^*) + \delta V_1(k^*,\cdot) = 0$ has a solution $k_1 \in (0,k^*)$

ii) $V_2(x,k_1) + \delta V_1(k_1,k^*) = 0$ has a solution $k_2 \in (k^*,\bar{k}]$

iii) $V_2(x,k_2) + \delta V_1(k_2,k_1) = 0$ has a solution $k_3 \in \mathbb{R}_+$

Then τ_δ has topological chaos.

<u>Proof</u>: Notice that if k^* has any pre-image other than itself, $k_1 < k^*$ must hold. Therefore i) implies that: $M = \tau_\delta(k^*) > k^*$. We do not know if $k_2 \le M$. This can be detected from iii). In fact, $k_2 > M$ would imply that k_2 has no pre-image. Hence $k_1 = \tau_\delta(k_2) > \tau_\delta(M)$. Now set $I_1 = [k_1,k^*]$ and $I_2 = [k^*,M]$. Then: $\tau_\delta(I_1) = [k^*,M] = I_2$ and $\tau_\delta(I_2) = [\tau_\delta(M),M] \supset [k_1,M] = (I_1 \cup I_2)$. The theorem then follows from Lemma 2. Q.E.D.

<u>Remark 4</u>: This theorem is not much more than a computational device; nevertheless a very useful one when the direct criterion of Theorem 3 fails. Our example clearly fits both theorems so that we will not go through other computations here. If we set $\rho = 0$ then we have the Cobb-Douglas case, which does not fit the conditions of Theorem 3. José Scheinkman [1984] first conjectured that chaos may exist in such a case. The computations contained in Boldrin-Deneckere [1987] show that he was right, as it satisfies the conditions of Theorem 4.

5. Conclusions

In this paper we study the Dynamic Competitive Equilibrium of an aggregate, neoclassical, two-sector model under the hypothesis that all markets clear at each point in time and that all agents are identical maximizers with infinite perfect foresight. We concentrate on the technological side of the model and show that a source for endogenous sustained oscillations may be found in the changes of relative profitability between the two sectors. This, in turn, is brought about by shifts in the relative capital/labor intensities. We adopt the idea of chaotic dynamics as a qualitative representation of sustained, apparently stochastic, oscillations. Two sets of conditions are obtained under which the optimal accumulation paths (and the related competitive prices and quantities sequences) are chaotic. The theoretical results are applied to a simple example: it is remarkable that, even in such a restricted and highly stylized framework, chaos appears for parameter values which cannot be rejected as "unrealistic" on a priori grounds. In particular, we show that the idea according to which oscillations are optimal only when the discount factor is extremely small (i.e., the "rate of interest" is of the thousands-percent magnitudes) is not true in general. In fact as the degree of factors substitutability in the economy decreases, chaos and cycles appear for discount factors "fairly large".

It is clear, nevertheless, that such an abstract and overly-simplified setup cannot be proposed as a complete macro-model. It simply shows that there is room for an endogenous explanation of Competitive Business Cycles.

Footnotes

*A preliminary version of this paper was circulated in August 1986 under the same title as Technical Report #502 of the Institute of Mathematical Studies in the Social Sciences, Stanford University, Stanford, CA. I am very grateful to the Institute and to Professor Mordecai Kurz in particular, for the kind hospitality and financial support provided during the Summer 1986. Financial support from the A.P. Sloan Foundation through a Dissertation Fellowship is also gratefully acknowledged. I am indebted to Jess Benhabib, Raymond Deneckere, Lional McKenzie, Luigi Montrucchio, Paul Romer and José Scheinkman for several helpful discussions, criticisms and suggestions that greatly improved the article.

[1] See Dechert [1978] for an exact justification of the use of this formalism in our class of models.

[2] The problem was posed to me by Jose Scheinkman.

[3] See Grandmont [1987] for a discussion of the relevance of these two different forms of chaos in economic modelling.

[4] I do not report here either the text of the Theorem nor the intuition behind it in order to avoid a long technical digression. As a matter of fact the results we are using come from the work of various authors, among whom are the named Collet and Eckmann. A statement, without proof, can be found in Grandmont [1987] as Theorem D.1.9.

APPENDIX A

Proposition A.1: If two maps $f: I \to I$ and $g: J \to J$ are topologically conjugate through the homeomorphism h then for every invariant set $C \subset I$, $h(C) \subset J$ is invariant and for every invariant set $C' \subset J$, $h^{-1}(C') \subset I$ is invariant.

Proof: Trivial. Q.E.D.

Corollary 1 follows from the fact that OSS's and orbits of τ_δ are invariant sets for τ_δ.

Proof of Lemma 1: Consider an Edgeworth box for problem T when the capital stock is k^*, and there exists a level of future capital stock k^{*1} such that $V_{12}(k^*,k^{*1}) = 0$. Let $c^* = T(k^*,k^{*1})$ and $y^* = k^{*1} - (1-\mu)k^*$. The two isoquants of F^1 and F^2 associated to c^* and y^* are tangent at a point (k^1,ℓ^1), $(k^*-k^1,1-\ell^1)$ at which the two sectors have the same capital intensity. Such a point must be on the diagonal of the box. Linear homogeneity implies that all the points on the diagonal are points of tangency for some isoquants (c,y) with $0 \leq c \leq T(k^*,0)$ and $0 \leq y \leq f(k^*)$. The PPF then coincides with the diagonal in this case and, therefore, $V_{12}(k^*,y+(1-\mu)k^*) = 0$ for all y. Q.E.D.

Proof of Lemma 3: Necessity is obvious. We will prove sufficiency by showing that our condition implies that there exists a price sequence (q_t) such that:

$$V(k_t^*,k_{t+1}^*) + \delta q_{t+1}k_{t+1}^* - q_t k_t^* \geq V(x,y) + \delta q_{t+1}y - q_t x \tag{A1}$$

for all $(x,y) \in D$. The latter, together with the transversality condition which is trivially satisfied in our model, has been shown in McKenzie [1974] to be sufficient for optimality of $\{k_t^*\}$, extending a result of Weitzman [1973] to the discounted case.

From our hypothesis we have that:

$$V(k_{t-1}^*, k_t^*) + \delta V(k_t^*, k_{t+1}^*) \geq V(k_{t-1}^*, y) + \delta V(y, k_{t+1}^*)$$

for all $y \in \text{Rel int } \Gamma(k_{t-1}^*) \cap \text{Rel int } \Gamma^{-1}(k_{t+1}^*) \neq \phi$.

Set $G(x) = V(k_{t-1}^*, x) + \delta V(x, k_{t+1}^*)$. The last inequality means that zero is a supergradient of G at k_t^*, i.e., $0 \in \partial G(k_t^*)$. By Theorem 23.8 of Rockafellar [1970], we have also that $0 \in [\partial_2 V(k_{t-1}^*, k_t^*) + \delta\partial_1 V(k_t^*, k_{t+1}^*)]$ and therefore there exists a sequence of vectors $\{q_t\}$ such that $-q_{t+1} \in \partial_2 V(k_{t-1}^*, k_t^*)$, $q_t \in \delta\partial_1 V(k_t^*, k_{t+1}^*)$. This in turn implies: $(q_t/\delta, -q_{t+1}) \in \partial V(k_t^*, k_{t+1}^*)$ and by definition:

$$V(k_t^*, k_{t+1}^*) \geq V(x,y) + q_t(k_t^* - x) - \delta q_{t+1}(k_{t+1}^* - y)$$

which is equivalent to (A1). Q.E.D.

APPENDIX B

The first and second order partial derivative of our PPF are easy to compute. We leave it to the reader. The Euler equation at an interior steady state k reduces to:

$$[a(k-\gamma k)^\rho + (1-a)(1-k)^\rho]^{\frac{1}{\rho}-1} [(\delta-\gamma)a(k-\gamma k)^{\rho-1} - (1-a)(1-k)^{\rho-1}] = 0$$

from which (20) may be obtained after a few manipulations. We are interested in parameter values at which the steady state $k(\delta)$ lies on the downward sloping arm of τ_δ, i.e., it is larger than γ. This is necessary in order to obtain cycles and chaos. From (20) one has:

$$k(\delta) \in \begin{cases} (0,\gamma) & \text{for} \quad \gamma < \delta < \gamma\left[1+ \dfrac{1-a}{a\gamma^{\rho}}\right] \\ \\ (\gamma,1) & \text{for} \quad \gamma\left[1+ \dfrac{1-a}{a\gamma^{\rho}}\right] < \delta < 1 \end{cases}.$$

In order to guarantee that $\gamma(1 + (1-a)/a\gamma^{\rho})$ is less than one, we need to impose $\gamma^{\rho}(1-\gamma)/\gamma > (1-a)/a$, which is understood from now on.

To consider the local stability of $k(\delta)$ and the possibility of period-two cycles bifurcating around it, we have to compute the pair of eigenvalues associated with the linearized Euler equation in a neighborhood of $k(\delta)$. These are the roots λ or: $\delta V_{12}\lambda^2 + (V_{22}+\delta V_{11})\lambda + V_{12} = 0$, where $V_{ij} = V_{ij}(k(\delta),k(\delta))$, $i,j = 1,2$. In our case this reads:

$$\delta \frac{\lambda-k(\delta)}{1-k(\delta)} \lambda^2 - \left[\left(\frac{\gamma-k(\delta)}{1-k(\delta)}\right)^2 +\delta\right]\lambda + \frac{\gamma-k(\delta)}{1-k(\delta)} = 0.$$

The two eigenvalues are:

$$\lambda_1 = (1-k(\delta))/(\gamma-k(\delta)); \quad \lambda_2 = (\gamma-k(\delta))/(1-k(\delta))\delta.$$

Substituting (20) in the latter will give:

$$\lambda_1 = \frac{(1-a)^{\frac{1}{1-\rho}}}{\gamma(1-a)^{\frac{1}{1-\rho}} - [a(\delta-\gamma)]^{\frac{1}{1-\rho}}}; \quad \lambda_2 = \delta^{-1}\left\{\gamma - \left[\frac{a(\delta-\gamma)}{1-a}\right]^{\frac{1}{1-\rho}}\right\}$$

For values of δ larger than γ, λ_1 behaves as:

$$\lambda_1 \in \begin{cases} (\frac{1}{\gamma'} + \infty) & \text{for} \quad \gamma < \delta < \gamma\left[1 + \dfrac{1-a}{a\gamma^{\rho}}\right] \\ (-1,0) & \text{for} \quad \gamma + (1+\gamma)^{1-\rho} \dfrac{1-a}{a} < \delta \\ (-\infty,-1) & \text{for} \quad \gamma\left[1 + \dfrac{1-a}{a\gamma^{\rho}}\right] < \delta < \gamma + (1+\gamma)^{1-\rho} \dfrac{1-a}{a} \end{cases}$$

The behavior of λ_2 is analogous even if not all the critical values may be explicitly computed. Anyhow λ_2 is a decreasing function of δ in the interval $[\gamma,1)$, with $\lambda_2(\gamma) = 1$ and $\lambda_2(1) < -1$ for $a > [1 + (1-\gamma)/(1-\gamma)^{1-\rho}]^{-1}$. Assuming that (a,γ,ρ) satisfy the latter inequality and denoting with δ^{oo} the unique solution to $\lambda_2(\delta) = -1$ we have:

$$\lambda_2 \epsilon = \begin{cases} (0,1) & \text{for } \gamma < \delta < \gamma\left(1 + \dfrac{1-a}{a\gamma^{\rho}}\right) \\ (-1,0) & \text{for } \gamma\left(1 + \dfrac{1-a}{a\gamma^{\rho}}\right) < \delta < \delta^{oo} \\ (-\infty,-1) & \text{for } \delta^{oo} < \delta \end{cases}$$

To prove that there exists a period-two cycle for some value of $\delta > \gamma(1 + (1-a)/a\gamma^{\rho})$ we adopt the sufficient conditions given in Theorem 1 of Benhabib-Nishimura [1985]. They amount to showing that there exists a δ^{o} and an interval $[\delta^{-},\delta^{+}]$ in $[\gamma(1 + (1-a)/a\gamma^{\rho}),1)$ such that $B(\delta) = V_{22}(\delta) + \delta V_{11}(\delta) - (1+\delta)V_{12}(\delta)$, satisfies:

$$B(\delta): \begin{cases} >0 & \text{for } \delta \in [\delta^{-},\delta^{o}) \\ =0 & \text{for } \delta = \delta^{o} \\ <0 & \text{for } \delta \in (\delta^{o},\delta^{+}] \end{cases}$$

where $V_{ij}(\delta) = V_{ij}(k(\delta),k(\delta))$, $i,j = 1,2$.

For our model $B(\delta)$ reads:

$$B(\delta) = f(k(\delta),k(\delta))[(k(\delta)-\gamma)^2 + \delta(1-k(\delta))^2 + (1+\delta)(\gamma-k(\delta))(1-k(\delta))],$$

where the function f is always negative. Therefore we have:

$$
B(\delta) : \begin{cases} >0 & \text{for} \quad \gamma\left(1 + \dfrac{1-a}{a\gamma^{\rho}}\right) \le \delta < \delta^{oo} \\ =0 & \text{for} \quad \delta = \delta^{oo} \\ <0 & \text{for} \quad \delta^{oo} < \delta < \delta^{o} \\ =0 & \text{for} \quad \delta = \delta^{o} = \gamma + (1+\gamma)^{1-\rho}\,\dfrac{1-a}{a} \\ >0 & \text{for} \quad \delta > \delta^{o} \end{cases}
$$

where δ^{oo} solves: $\delta + \gamma = \left[\frac{1-a}{a(\delta-\gamma)}\right]^{\frac{1}{1-\rho}}$, i.e., it is such that $\lambda_2(\delta^{oo}) = -1$.

Therefore cycles of period-two exist both around δ^{o} and δ^{oo}. To verify that our example also satisfies Theorem 3 may seem not immediate given that $[a(x-\gamma y)^{\rho} + (1-a)(1-y)^{\rho}]^{1/\rho}$ is not sub-differentiable at the points $(\gamma,1)$ and $(1,0)$. Nevertheless, one may take a sequence of positive numbers $\{x_n\}$ in a neighborhood of 1 and such that $\lim x_n = 1$ for $n \to \infty$ and consider the sequence of artificial problems:

$$
\text{(Pn)} \qquad \max \sum_{t=0}^{\infty} \left[a(k_t-\gamma k_{t+1})^{\rho} + (1-a)(1-k_{t+1})^{\rho}\right]^{\frac{1}{\rho}} \delta^{t}
$$

$$
\text{s.t.} \quad (k_t, k_{t+1}) \in D = \{(x,y) \in [0,x_n] \times [0,x_n] \ \text{ s.t. } \ 0 \le y \le \tfrac{x}{\gamma}\}
$$

Then it is easy to see that the policy function associated to the n^{th} problem will satisfy $\tau_{n\delta}(\gamma) = x_n$ and $\tau_{n\delta}(x_n) = 0$ for $\delta \in C_n(V)$ where $C_n(V)$ is the appropriate interval associated to (Pn). As $\tau_{n\delta} \to \tau_{\delta}$ uniformly when $x_n \to 1$ we may conclude that also for our model economy there exists a set $C(V)$ of values of δ that imply $\tau_{\delta}(\gamma) = 1$ and $\tau_{\delta}(1) = 0$ for $\delta \in C(V)$. The upper and lower bounds of $C(V)$ can in fact be directly computed as limits of those for $C_n(V)$.

$$C(V) = \{\delta \in (0,1), \quad \text{s.t.} \quad (1-a+a\gamma^{\rho})^{\frac{1}{\rho}} \leq \delta \leq a^{\left|\frac{1}{\rho}\right|} [1-a(1-\delta)]\}$$

for $\rho \in (-\infty,0)$.

Notice that $C(V)$ might well be empty. It is not such, when (a,ρ,γ) satisfy:

$$[a-a^2(1-\gamma^{\rho})]^{\left|\frac{1}{\rho}\right|} [1-a(1-\gamma)] > 1.$$

REFERENCES

Becker, R. [1981], "The Duality of a Dynamic Model of Equilibrium and an Optimal Growth Model: The Heterogeneous Capital Goods Case," Quarterly Journal of Economics 96, 271-300.

Benhabib, J., and Day. R. [1982], "A Characterization of Erratic Dynamics in the Overlapping Generations Model," Journal of Economic Dynamics and Control 4, 37-55.

__________ and Nishimura, K. [1979], "The Höpf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimal Economic Growth," Journal of Economic Theory 21, 421-444.

__________ and __________ [1985], "Competitive Equilibrium Cycles," Journal of Economic Theory 35, 284-306.

Boldrin, M., and Deneckere, R. [1987], "Simple Growth Models With Very Complicated Dynamics," mimeo, University of Chicago and Northwestern University, May.

__________ and Montrucchio, L. [1984], "The Emergence of Complex Dynamics in Models of Optimization Over Time: The Role of Impatience," Rochester Center for Economic Research, University of Rochester, WP No. 7, October.

__________ and __________ [1986], "On the Indeterminacy of Capital Accumulation Paths," Journal of Economic Theory 40, 26-39.

Brock, W. [1980], "Distinguishing Random and Deterministic Systems: Abridged Version," Journal of Economic Theory 40, 168-195.

__________ and Sayers, C. [1986], "Is the Business Cycle Characterized by Deterministic Chaos?" Journal of Monetary Economics forthcoming.

Dechert, D. [1978], "ℓ_∞-Multipliers with an Application to Discrete Time Optimal Control Problems," MRG WP#7826, University of Southern California, August.

Deneckere, R. and Pelikan, S. [1986], "Competitive Chaos," Journal of Economic Theory 40, 13-25.

Devaney, R.L. [1986], "An Introduction to Chaotic Dynamical Systems," Menlo Park, CA: Benjamin/Cummings Publishing Co.

Grandmont, J.M. [1985], "On Endogenous Competitive Business Cycles," Econometrica 53, 995-1096.

__________ [ed. 1986], Nonlinear Economic Dynamics, New York: Academic Press.

__________ [1987], "Non-linear Difference Equations and Sunspots: An Introduction," Lecture Notes, IMSSS, Stanford University, July.

Hirota, M. and Kuga, K. [1971], "On An Intrinsic Joint Production," International Economic Review 12, 87-98.

Intriligator, M.D. [1971], Mathematical Optimization and Economic Theory, Englewood Cliffs, NJ: Prentice Hall.

Kuga, K. [1972], "The Factor-Price Equalization Theorem," Econometrica 40, 723-736.

Li, T. and Yorke, J. [1975], "Period Three Implies Chaos," Amer. Math. Monthly 82, 985-992.

McKenzie, L. [1974], "Turnpike Theorems with Technology and Welfare Function Variable," in J. Los and M.W. Los (eds.), Mathematical Models in Economics, New York: American Elsevier.

__________ [1986], "Optimal Economic Growth and Turnpike Theorems," in K. Arrow and M. Intriligator (eds.), Handbook of Mathematical Economics, vol. III, New York: North-Holland.

Rockafeller, T. [1970], Convex Analysis, NJ: Princeton University Press.

Sarkovskij, A.N. [1964], "Coexistence of Cycles of a Continuous Map of a Line Into Itself," U.R.Mat.Z. 16, 61-71.

Scheinkman, J. [1984], "General Equilibrium Models of Economic Fluctuations: A Survey of Theory," mimeo, University of Chicago, July.

__________ and Le Baron, B. [1986], "Nonlinear Dynamics and Stock Returns," Journal of Business, forthcoming.

Weitzman, M. [1973], "Duality Theory for Infinite Horizon Convex Models," Management Science 19, 783-789.