Non-Atomic Economies and the Boundaries of Perfect Competition*

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UCLA Economics Department Working Paper No. 502
(Revised) September 1991

Abstract

This paper explores the distinction between nonatomicity and thick markets as the source of perfect competition. The focus of the paper is the construction of a model of an imperfectly competitive economy with a non-atomic continuum of traders and a continuum of differentiated commodities, for which Walrasian equilibria exist. The failure of perfect competition in this instance can be identified in two ways: *individuals can affect prices* and *the core is strictly larger than the set of Walrasian allocations*. By contrast, it is shown that, when markets are *physically* or *economically thick* (or both), then individuals cannot affect prices and the core coincides with the set of Walrasian allocations.

JEL classification nos. : C62, C71, C72, D51.

Keywords: economically thick markets, physically thick markets, core equivalence, withholding test, Walrasian equilibrium.

*We thank Robert Anderson, Elaine Bennett, Bryan Ellickson, Neil Gretsky, Larry Jones, Andreu Mas-Colell, Mario Pascoa and Leo Simon for numerous helpful conversations, Harrison Cheng for pointing out an error in a previous version, and the National Science Foundation for financial support.

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1. Introduction

Since Aumann [1964] introduced a nonatomic general equilibrium model and exhibited its connections to perfect competition, the hypotheses of nonatomicity and perfect competition have been joined. However, when there is an infinite number of commodities, the two hypotheses are distinct. The distinction is based on a division of nonatomic models into *thick* markets economies, which are (typically) perfectly competitive, and *thin* markets economies which are typically not perfectly competitive. The purpose of this paper to make the point that it is thickness of markets rather than nonatomicity which leads to perfect competition.

Attention is confined here to exchange economies, so that individuals differ only in their tastes and endownments. In such model, there are two kinds of market thickness (Gretsky and Ostroy [1985]): *physical* thickness—a large number of potential buyers and sellers of each physically identical commodity; and, *economic* thickness—physical differences are superceded by conditions of economic substitutability among commodities. For markets to be thin, they must be neither physically nor economically thick, i.e., there must not be a large number of buyers and sellers of physically identical commodities *and* different commodities must not be good substitutes for one another.

We do not attempt to model thick and thin markets in complete generality. We insist on having a model in which it is possible to show that Walrasian equilibria exist, and this imposes a certain discipline. The central question on which we focus is: given a nonatomic model in which the existence of Walrasian equilibrium can be demonstrated, how far can we be from perfect competition? Our answer is: far enough to admit thin markets economies, and therefore far enough to exhibit the need for one or the other thick markets hypothesis to reestablish the connection between nonatomicity and perfect competition.

We call attention to a restriction on allowable preferences associated with this discipline which we call *preference for local diversity.* It leads to an asymmetry in the origins of imperfect competition because it permits a kind of one-sided thinness compatible with the possible monopoly power of sellers, but it precludes the sort of preferences that would lead to monopsony power of buyers. Therefore, our model understates the full range of thin market phenomena. Presumably, there is an alternative to this assumption which would permit the possibility of unexploited monopsony as well as monopoly power within a model for which Walrasian equilibrium could be demonstrated, but we have not found it. (See Section 6 for comparisons with models of monopolistic competition.)

How is perfect competition distinguished from the existence of Walrasian equilibrium? Several tests have been suggested:

Does the core coincide with the set of Walrasian equilibria (Edgeworth [1881], Shubik [1959], Debreu and Scarf [1963], Aumann [1964])?

Is a Walrasian equilibrium a non-cooperative equilibrium (Gabszewicz and Vial [1972], Roberts and Postlewaite [1976], Novshek and Sonnenschein [1978], Dubey, Mas-Colell, Shubik [1980])?

Is a Walrasian equilibrium a no-surplus allocation (Ostroy [1980], Makowski [1980])?

These different tests enjoy a considerable overlap, so that if a model passes one of them it is likely to pass the others, at least "generically".[1] In this paper, we adopt the first of these tests and a variant of the second and third that we call the *withholding test.* The spirit of the withholding test is to ask whether an individual actually faces perfectly elastic demand and supply opportunities. We take the answer to this question to be "yes" (and the Walrasian equilibrium to be perfectly competitive) if individuals cannot influence prices by withholding some of their endowments (more precisely, if small groups can have only a small effect on prices by withholding some of their endowments).

The withholding test is related to the (lower hemi-)continuity of the Walrasian correspondence. It is easy to see why. When a small group withholds part of its endowment, this act likes a small perturbation of the data of the economy. The withholding test requires that, following such a perturbation, equilibrium prices not change too much, i.e., the equilibrium price correspondence should be continuous. By contrast, it is well-known (for economies with a finite number of commodities) that core equivalence need not imply continuity of the Walrasian correspondence. We show here that the same thick markets conditions leading to core equivalence are also responsible for the generic continuity of the Walrasian correspondence. (See Section 3, Remark 2, for additional comments on these tests.)

Our formulation of a model in which Walrasian equilibrium is or is not perfectly competitive owes much to the work of Bewley [1973], Mas-Colell [1975], Jones [1983, 1984], Ostroy [1984a], Gretsky and Ostroy [1985], Aliprantis, Brown and Burkinshaw [1985], Zame [1986], Rustichini and Yannelis [1987], Cheng [1987], Khan-Yannelis [1990] and Podczeck [1990]. By linking Walrasian equilibrium to one or the other of the above tests of perfect competition, these authors have helped to delineate the thick/thin markets distinction drawn here.

The model we use is a variant of one constructed by Mas-Colell and Jones as a model of an infinite degree of product differentiation. The space of pure commodities is a compact metric space, and commodity bundles are represented as measures on this space of pure commodities. Our point of departure is to allow for less substitution between commodity bundles than that assumed by Mas-Colell and Jones, while retaining the possibility that initial holdings can be widely varied.

In Section 2, we describe the model and discuss its more important assumptions. In Section 3, the main results are stated. In Section 4, we give several examples to illustrate the results of Section 3. In Section 5, some brief remarks about monopolistic competition are made. Finally, proofs are contained in Section 6.

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[1] Since the first version of this paper was written, we have found a class of economies, called nonatomic assignment models, in which the core test can be easier to pass than the other two. See Gretsky, Ostroy and Zame [1991].

2. The Model

The *space of traders* (T, λ) is the unit interval, equipped with Lebesgue measure; we usually write s, t for individual traders. The *set of pure commodities* is a compact metric space X; *commodity bundles* are positive (Borel) measures on X. We write $M(X)$ for the space of measures on X and $M^+(X)$ for the cone of positive measures; we use Greek letters α, β, γ for commodity bundles and Roman letters x, y, z for points of X. To avoid confusion, we write δ_x (the Dirac measure at x) when we refer to the pure commodity x. For $\alpha \in M(X)$ we write α^+, α^- for the *positive* and *negative* parts of α and $|\alpha| = \alpha^+ + \alpha^-$ for the *absolute value* of α. The *norm* of α is $\|\alpha\| = |\alpha|(X)$.

Recall that $M(X)$ is the dual of the space $C(X)$ of all continuous real-valued functions on X. The *weak star topology* (*w^* topology*) on $M(X)$ is the weakest topology for which the mapping $(\varphi, \alpha) \to \varphi \cdot \alpha = \int \varphi(x) d\alpha(x)$ is continuous for every $\varphi \in C(X)$.

We fix a *reference bundle* μ in $M^+(X)$. This reference bundle provides a scale against which other commodity bundles may be measured. We make the following assumptions about μ:

- μ is nonatomic
- supp $\mu = X$.

Therefore, the set of pure commodities is infinite. We have in mind the canonical case where $X = [0, 1]$ and μ is Lebesgue measure.

An *allocation* is a weak star (Gelfand) integrable function $f: T \to M^+(X)$. (This means that for each $\varphi \in C(X)$, there is a measure $\alpha \in M^+(X)$ such that the real-valued function $t \to \varphi \cdot f(t)$ is Lebesgue integrable and $\int \varphi \cdot f(t) d\lambda(t) = \varphi \cdot \alpha$.) We denote the *space of allocations* by $\mathcal{A}$; the *distance* between two elements of $\mathcal{A}$ is $d(f, g) = \int \|f(t) - g(t)\| d\lambda(t)$. (Recall that, for α a positive measure, $\|\alpha\| = \alpha(X) = \int \mathbf{1} d\alpha$, so that $\int \|f(t)\| d\lambda(t)$ is finite for each allocation f and $d(f, g) \leq \int \|f(t)\| d\lambda(t) + \int \|g(t)\| d\lambda(t)$.) Equipped with this distance function, $\mathcal{A}$ is a complete metric space.

We denote the *initial allocation* (or *endowment*) by $e: T \to M^+(X)$, and write $\omega = \int e(t) d\lambda(t)$ for society's *mean endowment*. We require that:

(E) There exist positive numbers c_1, c_2 such that $c_1\mu(B) \leq \omega(B) \leq c_2\mu(B)$, for every Borel set $B \subset X$, (i.e., ω and μ are mutually boundedly absolutely continuous).

Since μ has full support, (E) implies in particular that ω also has full support; in this sense, all commodities are represented. Since ω is absolutely continuous with respect to μ, the Radon-Nikodym theorem implies that there is an integrable function S such that $\omega(E) = \int_E S d\mu$ for each Borel set E, and $\int \varphi d\omega = \int \varphi S d\mu$ for each integrable function φ. (E) means that $c_1 \leq S \leq c_2$. We will frequently refer to S as *(mean) supply* and write $\omega = S\mu$. More generally, if γ is a positive measure and F is a positive function, then we write $F\gamma$ for the measure $F\gamma(B) = \int_B F d\gamma$. Note that $\|F\gamma\| = \int \mathbf{1} dF\gamma = \int F d\gamma = F \cdot \gamma$.

Prices are bounded Borel functions $p: X \to \mathbf{R}$. We denote the space of all prices by $B(X)$ and the cone of positive prices by $B^+(X)$. Within $B(X)$ we distinguish the subspace of continuous functions $C(X)$ and of positive continuous functions $C^+(X)$. Given a price p and a commodity bundle α, the value assigned to α by p is $p \cdot \alpha = \int p(x) d\alpha(x)$.

We shall assume that preferences $\preceq_t$ of individual traders satisfy the following standard assumptions (for each $t \in T$):

(P.1) $\preceq_t$ is complete, reflexive, transitive, and convex;

(P.2) the irreflexive part $\prec_t$ is strictly monotone (i.e., if α, β are positive measures and $\beta \neq 0$ then $\alpha \prec_t \alpha + \beta$).

Remark 1: (*Convexity in the individualistic vs. distributional formulation.*) Of these assumptions, the only one which requires comment is convexity of preferences. One of the remarkable properties of the finite dimensional non-atomic model is that convexity of individual preferences is superfluous, because of the "convexifying effect of large numbers" (manifested in the Lyapunov convexity theorem or Fatou's lemma). However, in infinite dimensional commodity spaces both the Lyapunov convexity theorem and Fatou's lemma can fail to hold (see Rustichini [1989]); for this reason, we find it necessary to assume convexity of individual preferences. Nevertheless, approximate versions of the Lyapunov convexity theorem and Fatou's lemma are sometimes valid in infinite dimensional spaces, and we shall make use of them in order to establish — for physically or economically thick markets — that the core coincides with the set of Walrasian allocations. For those purposes, we shall find that convexity of individual preferences is indeed superfluous. However, approximate versions of the Lyapunov convexity theorem or of Fatou's lemma are not strong enough to guarantee the *existence* of Walrasian equilibrium.

The comments above apply to what we call the "individualistic" formulation of a nonatomic model as in Aumann [1964] that we follow here. Were we to adopt the "distributional" formulation (as in Hart, Hildenbrand and Kohlberg [1974], Mas-Colell [1975], and Jones [1983]), it would be possible to establish the existence of Walrasian equilibrium without the convexity hypothesis. However, the distributional formulation brings with it many complications. Indeed, even to formulate the core equivalence and withholding tests of perfect competition would require additional constructions (see the distinctions among the "core", the "core of the convex space of agents", and the "distributional core" in Gretsky and Ostroy [1985] and Gretsky, Ostroy and Zame [1991]). We have preferred to adopt the individualistic formulation (even though it requires convexity) because it avoids these complications and makes our points more transparently.

Continuity properties of individual preferences are a key factor in controlling substitution properties between commodities. We shall assume that (for each $t \in T$):

(P.3) $\preceq_t$ is continuous in the norm topology of $M(X)$

(P.4) $\preceq_t$ is upper semi-continuous in the weak star topology of $M(X)$.

Since the norm topology of $M(X)$ is stronger than the weak star topology, our continuity requirements are less stringent than the requirement of weak star continuity of preferences (compare Mas-Colell [1975], Jones [1983, 1984], and Podczeck [1990]). As we shall see, the difference is real and significant. Indeed, the possibility that preferences may fail to be weak star continuous provides the opening to construct non-atomic models which are economically thin.

We assume that the family of preferences is measurable in the following sense:

(P.5) if f, g are allocations, then $\{t \in T : f(t) \preceq_t g(t)\}$ is a (Lebesgue) measurable set.

This assumption may be compared with the usual measurability assumption for non-atomic economies in the finite dimensional setting (see Aumann [1964, 1966]), which would require that for every $\alpha, \beta \in M^+(X)$, the set $\{t \in T : \alpha \preceq_t \beta\}$ is measurable. Our assumption is stronger in that we allow for the comparison of arbitrary allocations and not simply for the comparison of constant allocations. In the finite dimensional setting, a fairly straightforward argument shows the two assumptions to be equivalent. In our setting (and in the presence of our other assumptions, especially the ones below), it may also be shown that the two assumptions are equivalent. However, the argument is not at all straightforward; we shall simply adopt the formulation which is convenient (and leave the unpleasant technicalities to the interested reader).

The final, and crucial, assumptions about preferences concern (marginal) rates of substitution between commodities. As the work of a number of authors (especially Mas-Colell [1986]) has made clear, the need for such assumptions represents a clear distinction between the finite and infinite dimensional settings. Moreover, in our setting, the form of these assumptions is somewhat delicate: they must be strong enough to allow for the demonstration of Walrasian equilbrium, while remaining weak enough to permit markets which are not necessarily economically thick.

We make two assumptions about rates of substitution; the first simply says that all rates of substitution are bounded. This is a rather strong assumption; we use it because it is easy to understand and substantially simplifies several arguments, without interfering with our main aims: the formulation and elaboration of the of the differences between thick and thin markets nonatomic models economies.

(B) There is a constant M such that: if α, β, γ are positive measures, if $\gamma - \alpha + \beta \geq 0$ and $M\|\alpha\| < \|\beta\|$, then $\gamma \prec_t \gamma - \alpha + \beta$ for each $t \in T$.

For our second rate of substitution assumption, we introduce some notation. For the commodity bundle γ and a Borel set Y of X, write $\gamma|Y$ for the restriction of γ to Y, so that $(\gamma|Y)(B) = \gamma(B \cap Y)$ for each Borel set $B \subset X$. Note that $\gamma = \gamma|Y + \gamma|(X \setminus Y)$. We shall consider alternatives to γ on sets Y of small diameter, specifically a measure $F\mu$ restricted to Y. Define $\mathrm{Ex}\,(F|Y) = [\mu(Y)]^{-1} \int_Y F d\mu$, the conditional expectation of F on Y; and $\mathrm{Var}\,(F|Y) = [\mu(Y)]^{-1} \int_Y (F - \mathrm{Ex}\,(F|Y))^2\, d\mu$, the conditional variance of F on Y.

Suppose that for a given γ, $\gamma|Y$ is replaced by $F\mu|Y$. The following assumption, called *preference for local diversity* (PLD), stipulates sufficient conditions for the replacement to be preferred.

(PLD) For each $r > 1$ there is a $\delta > 0$ and a $d > 0$ such that, if

(i) diameter $(Y) \leq \delta$,

(ii) $\text{Var}\,(F|Y)/\text{Ex}\,(F|Y) \leq d$,

(iii) $\|[(F\mu - \gamma)|Y]^+\| \geq r\|[(F\mu - \gamma)|Y]^-\|$,

then

$$\gamma|Y + \gamma|(X\backslash Y) \prec_t F\mu|Y + \gamma|(X\backslash Y).$$

The first condition says that the assumption only applies to nearby commodities (i.e., is 'local'). The second condition says that the replacement $F\mu$ is nearly evenly spread out on Y (i.e., a desire for 'diversity'). The third condition says that the total mass of the commodities received exceeds the total mass of commodities given up by at least r.

Note that we require (PLD) uniformly across traders and commodities: there is a single reference measure μ and a single metric (the diameter of subsets of X) such that all individuals prefer small diameter changes in commodity bundles that are nearly evenly μ-spread out (provided that the spread out bundle has greater total mass).

The benchmark to which we compare (PLD) is the hypothesis of *uniform substitutability* (US) used by Jones [1983, 1984] (and closely related to a notion used by Mas-Colell [1975]). Informally, (US) says that, for nearby commodities, consumers prefer *any* feasible trade in which the terms are favorable.

(US) Given $r > 1$ there exists $\delta > 0$ such that, if diameter $(Y) < \delta$ and β is any commodity bundle with $\beta(Y) \geq \gamma(Y)$, then

$$\gamma|Y + \gamma|(X\backslash Y) \prec_t r\beta|Y + \gamma|(X\backslash Y).$$

To see that (US) implies (PLD), note that if β is if the form $F\mu$, then (US) may be written as (PLD) *without* any reference to the constant d and the variability of F. To see that (PLD) does not imply (US), suppose $X = [0,1]$, $\mu =$ Lebesgue measure, $Y = [0,\delta]$, and $\gamma =$ mass point at 0: then (PLD) guarantees only that $\gamma \prec_t r(1/\delta)(\mu|[0,\delta])$, while (US) guarantees both that $\gamma \prec_t r(1/\delta)(\mu|[0,\delta])$ *and* $(1/\delta)(\mu|[0,\delta]) \prec_t r\gamma$. Evidently, (US) is a stronger assumption since it stipulates the desirability of many more trades. As we shall see, although (PLD) suffices for the existence of Walrasian equilibrium, the stronger requirement (US) leads to quite different conclusions about perfect competition and will be the basis of our definition of economically thick markets. (See especially Theorems 2 and 3 and Examples 1 and 4.)

We offer the following example of (PLD) and its relationship with (US). (Several of the examples in Section 4 are homogenous versions of this one).

Example: Take $X = [0,1], \mu =$ Lebesgue measure. Let $u\colon [0,\infty) \to [0,\infty)$ be a concave, increasing, differentiable function such that $u'(0) < \infty$ and $u'(x) \to 0$ as $x \to \infty$. For each real number $\rho > 0$, define the utility function U_ρ by:

$$U_\rho(\gamma) = \int u(\rho^{-1}\gamma[x-\rho, x])\, d\mu(x) + \gamma[0,1],$$

and define U_0 by

$$U_0(\gamma) = \int u(g(x))d\mu(x) + \gamma[0,1],$$

(where we have written $\gamma = g\mu + \gamma_s$ with $g\mu$ absolutely continuous and γ_s singular with respect to μ). It is not hard to see that, for each $\rho > 0$, the utility function U_ρ is weak star continuous and satisfies (US), and hence (PLD). On the other hand, the utility function U_0 is weak star upper-semicontinuous but not weak star continuous, and it satisfies (PLD) but not (US). (Since $U_\rho(\gamma) \to U_0(\gamma)$ for each γ, the utility functions U_ρ might be interpreted as "averaged" versions of U_0. See Jones [1983] for related discussion.)

To obtain some additional insight into (PLD) and its relationship to (US), it is useful to consider the interpretation of these assumptions in the contexts of consumption under uncertainty and consumption through time. In the former context, let us view points of X as indexing consumption of a single commodity over different states of the world and interpret μ as the objective probability distribution over states; we restrict our attention to consumption patterns absolutely continuous with respect to μ. Then (PLD) is implied if agents prefer (local) portfolio diversification and so corresponds to (local) risk aversion. By contrast, (US) corresponds to (local) risk neutrality.

To illustrate, suppose an agent's state-independent expected utility function is of the form $U(g) = \int u(g(x))d\mu(x)$, where $\gamma = g\mu$. To be consistent with assumption (B) we assume that the derivative of u is bounded away from 0 and ∞. Also assume $u(0) = 0$. A standard argument shows that the concavity of u (i.e., risk aversion) implies (PLD). However, if $U(\cdot)$ is of the above separable form and (US) holds, then u must be linear (risk neutrality). We defer the demonstration to Section 6.

If we view $X = [0,1]$ as indexing consumption of a single commodity at different points of time, then a mass point is a "gulp" while an absolutely continuous measure is a "sip". In this context, (US) requires that (over small time intervals) gulps and sips be nearly perfect substitutes, while (PLD) requires only that sips substitute for gulps, but not necessarily vice versa. (See Kreps and Huang [1989] and Hindy and Huang [1989] for a closely related discussion.)

3. Results

An *economy* is a pair $\mathcal{E} = \{(\prec_t), e\}$, consisting of a family of preferences and an initial allocation, satisfying the assumptions (E), (P.1)-(P.5), (B), and (PLD) above.

An allocation f is in the *core* of $\mathcal{E}$ if it is feasible and there does not exist an allocation g and a coalition $T' \subset T$ of positive measure such that $\int_{T'} g(t)d\lambda(t) = \int_{T'} e(t)d\lambda(t)$ and $f(t) \prec_t g(t)$ for almost every $t \in T'$.

A *Walrasian equilibrium* for $\mathcal{E}$ is a pair (p, f) where $p \in B(X)$ is a non-zero price and $f: T \to M^+(X)$ is an allocation such that:

(1) for almost every $t \in T$, $p \cdot f(t) = p \cdot e(t)$;

(2) for almost every $t \in T$, if $\alpha \in M^+(X)$ and $p \cdot \alpha \leq p \cdot e(t)$, then $\alpha \preceq_t f(t)$;

(3) $\int f(t)\, d\lambda(t) = \int e(t)\, d\lambda(t)$ (f is *feasible*).

By definition, prices are bounded Borel functions, and are defined everywhere. However, there remains a certain unavoidable indeterminacy of equilibrium prices on sets of measure zero. (The phrase "measure zero" should always be interpreted with respect to to the reference measure μ or the initial endowment ω; since ω and μ are mutually absolutely continuous, they have the same sets of measure zero). This indeterminacy stems from the fact that, although the support of ω, the initial endowment, is—by assumption (E)—equal to the entire commodity space X, nonatomicity of μ (and hence ω) entails that $\omega(\{x\}) = 0$ for all points $x \in X$. Because the story is a bit subtle, we defer it to the discussion preceding Lemma 2, in Section 6. The summary conclusion is that we may, without loss, identify prices (bounded Borel functions) which agree almost everywhere; the set of equivalence classes is $L_\infty(\mu) = L_\infty(\omega)$. With the usual abuse of notation, we frequently ignore the distinction between a bounded Borel function and the equivalence class it represents. The norm of a price p is $\|p\| = \text{ess sup } p$.

Although we allow for prices which are arbitrary bounded Borel functions (or equivalence classes in $L_\infty(\mu)$), we shall in fact prove that equilibrium prices necessarily enjoy certain continuity properties. Roughly speaking, we shall show that equilibrium prices are as continuous as the Radon-Nikodym derivative of society's mean endowment ω with respect to the reference bundle μ, i.e., mean supply. The following discussion makes this notion precise.

Let $\varphi: X \to \mathbf{R}$ be a bounded Borel function, let $Y \subset X$ be a Borel subset of X and let $x \in X$ be a point of X. Write $\varphi|Y$ for the restriction of φ to Y. We say that the *essential limit* of $\varphi|Y$ at x is $a \in \mathbf{R}$, and write $\text{ess lim}_x(\varphi|Y) = a$, if there is a subset $Y_0 \subset Y$ with $\mu(Y_0) = 0$ such that $\varphi|(Y - Y_0)$ has the limit a at x (in the usual sense). Note that φ need not have an essential limit at any point. On the other hand, Lusin's theorem asserts that for every $\epsilon > 0$, there is a compact subset $K \subset X$ such that $\mu(Y - K) < \epsilon$ and $\varphi|K$ is continuous, and in particular, has an essential limit at every point of supp $(\mu|K)$.

We say that the bounded Borel function ψ is *essentially as continuous as* φ if $\psi|Y$ has an essential limit at x whenever $\varphi|Y$ has an essential limit at x. (Note that this relation between ψ and φ depends only on their equivalence classes in $L_\infty(\mu)$.) If φ is continuous, this means that ψ has an essential limit at every point, and in particular, differs from a continuous function only on a set of measure zero (see Lemma 5, Section 6). (In keeping with our intent to identify prices which agree almost everywhere, we usually say simply that ψ is continuous.) We write $C_\varphi(X)$ for the space of functions which are essentially as continuous as φ (in contrast to $C(X)$, the space of continuous functions on X); we regard $C_\varphi(X)$ as a subspace of $B(X)$ or of $L_\infty(\mu)$ as convenient.

Our starting point is:

Theorem 1 (Existence of Walrasian Equilibrium)

(a) Walrasian equilibria exist.

(b) All equilibrium prices belong to $C_S(X)$ (where $\omega = S\mu$ is the mean societal endowment

and S is mean supply).

(c) The set of equilibrium prices of norm 1 is a norm compact subset of $L_\infty(\mu)$.[2]

Theorem 1 says that equilibrium prices are "at least as continuous" as the mean supply S. In particular, if the mean supply of commodities depends continuously on commodity names, then the prices of commodities will also depend continuously on commodity names. Because of its connection with imperfect competition, it is important to keep in mind that if mean supply *fails* to depend continuously on commodity names, then price may also fail to depend continuously on commodity names.

To see the connection with imperfect competition, consider the canonical case of differentiated commodities in which $T = X = [0,1]$, and each trader is endowed with exactly one unit of his named good; i.e., $e(t) = \delta_t$, the Dirac measure at t. In this case, the mean endowment $\omega = \int e(t)d\lambda(t) = \lambda$ is Lebesgue measure on X. If we take the reference measure $\mu = \lambda$ also, then Theorem 1 yields a price $p \in C(X)$. Now suppose that a small group of traders, say those in the interval $[0,\epsilon]$, withhold half their endowment from the market; call the resulting allocation e'. The Radon-Nikodym derivative S' (with respect to $\mu = \lambda$) of the mean endowment ω' now has a jump discontinuity at ϵ. We can conclude from Theorem 1 that the equilibrium price p' is continuous on the interval $[0,\epsilon)$ and on the interval $(\epsilon, 1]$ but it is possible that p' has a jump discontinuity at ϵ. Moreover, the *size of this jump* might not approach 0 as ϵ tends to 0. In other words, traders in the interval $[0,\epsilon]$ may face downward sloping demand curves no matter how small the group; and, hence, individuals may not be perfect competitors.

To conclude that a given Walrasian equilibrium (p, f) is perfectly competitive, we should be able to say that if $\{T_n\}$ is a sequence of small groups (whose size converges to 0), and $\{e_n\}$ is a corresponding sequence of allocations (tending to e) at which the group T_n withholds some of its endowment, then there should exist corresponding Walrasian equilibria (p_n, f_n) such that the equilibrium prices p_n converge uniformly to p. This is, of course, a way of saying that no small group can affect prices, and it is a definition of perfect competition which has a lot in common with the no-surplus definition (although the no-surplus definition would have the group withhold all of its endowment); see Ostroy [1984].

To formalize this test, fix preferences and regard the economy as parametrized by initial allocations.

Definition: The Walrasian equilibrium (p, f) (corresponding to the initial allocation e) *passes the withholding test* provided that: Given a sequence T_n of sets of traders such that $\lambda(T_n) > 0$ and $\lambda(T_n) \to 0$, and given a sequence e_n of initial allocations such that $e_n(t) = e(t)$ for $t \notin T_n$ and $e_n(t) \leq e(t)$ for $t \in T_n$, there exist Walrasian equilibria (p_n, f_n) corresponding to the initial allocations e_n such that $\{p_n\}$ converges to p uniformly on X.

Note that the withholding test refers to convergence of Walrasian *prices* and not Walrasian *allocations*. This is as it should be: although Walrasian *allocations* are not uniquely

[2]Taking price to have norm = 1 is a normalization. Alternatively, we could normalize so that $p \cdot \mu = 1$ or $p \cdot \omega = 1$, but these would prove less convenient for our purposes.

determined by prices, the corresponding *utilities* are; thus, although the allocations f_n need not converge to f the corresponding utilities *will* converge.

It would be too much to ask that *every* Walrasian equilibrium pass the withholding test; this need not be the case even for non-atomic economies with two commodities (See Ostroy [1980]). In applying this test we should, rather, take the generic point of view; i.e., we should ask that the withholding test be satisfied for a generic set of Walrasian equilibria. Actually, it makes sense to ask for a bit more. Let us say that the initial allocation e *passes the withholding test* if *every* Walrasian equilibrium corresponding to e passes the withholding test. For perfect competition, we shall insist that a generic set of initial allocations pass the withholding test. (By "generic" we shall mean residual, or second category; i.e., the interesection of a countable number of dense open sets. Recall that the Baire category theorem says that generic subsets of complete metric spaces are dense.)

For our other test of perfect competition, we shall use the familiar Edgeworth test that the core coincides with the set of Walrasian allocations. More precisely, we say that the initial allocation e *passes the core equivalence test* if the core (relative to the initial allocation) coincides with the set of Walrasian allocations.

The following result gives a more precise statement of the possible disjunction between nonatomicity and perfect competition mentioned in the Introduction. It is demonstrated by Example 4 in Section 4.

Theorem 2 (Existence of Imperfectly Competitive Economies) *There exist economies which fail the core equivalence test and the withholding test of perfect competition. More precisely, there exist a family of preferences and an open set of initial allocations in which all Walrasian equilibria fail the core equivalence and the withholding tests of perfect competition.*

To explore the boundary between the perfectly competitive and the imperfectly competitive territories, we ask: What additional restrictions on economies guarantee that Walrasian equilibria are perfectly competitive? In the remainder of this Section, we show that perfect competition will result if markets are physically or economically thick.

We deal first with economic thickness, which we wish to interpret as strong substitutability between commodities. We say that

Definition: *Markets are economically thick* if preferences are weak star continuous and satisfy the Uniform Substitutability assumption (US).

Theorem 3 (Perfectly Competitive Equilibrium in Economically Thick Markets) *If markets are economically thick, then:*

(a) all equilibrium prices belong to a norm compact subset of $C(X)$

Moreover,

(b) every initial allocation passes the core equivalence test,and

(c) a generic set of initial allocations pass the withholding test.

Existence of Walrasian equilibrium for economically thick markets has been demonstrated by Mas-Colell [1975] and Jones [1983] for exchange economies and by Podczeck [1990] for economies with production; core equivalence is also established by Mas-Colell.

A key difference between Theorems 1 and 3 is the conclusion about the continuity of equilibrium prices. When markets are economically thick, equilibrium prices are continuous whether or not S (recall that $\omega = S\mu$) is continuous. (In fact, prices lie in an equicontinuous set.) The substitution possibilities among commodities have the effect of "smoothing out" discontinuities in equilibrium prices which might arise from discontinuities in mean supply. Recalling our previous example in which $e(t) = \delta_t$, again suppose that the group of agents in $[0, \epsilon]$ withhold half their endowment from the market. As noted above, the result is that mean supply has a jump discontinuity at ϵ; however, if markets are economically thick, equilibrium prices will remain continuous.

Examples illustrating that core equivalence does not imply that the withholding test is passed in economically thick markets—i.e., that (b) and (c) of Theorem 3 are distinct—may be found in Gretsky, Ostroy and Zame [1991].

There are several possible expressions of the idea that markets are physically thick. For our purposes, the following version is appropriate:

Definition: Markets are ***physically thick*** if the initial allocation e satisfies: there is a constant K such that

$$e(t) \leq K\mu \quad \text{for almost every } t \in T.$$

(Since the reference measure μ and society's mean endowment ω are mutually boundedly absolutely continuous, it would be equivalent to require that there be a constant K' such that $e(t) \leq K'\omega$ for every $t \in T$.)

Notice that whereas economic thickness is defined by restrictions on preferences, physical thickness is defined by restrictions on endowments. Physical thickness implies that there are many (potential) sellers of each commodity. The monotonicity and (PLD) assumptions on preferences guarantee that there also are many potential buyers.

An alternative, more in the spirit of Ostroy [1984] and Gretsky and Ostroy [1985], would be to identify physical thickness of markets with Bochner integrability of the initial allocation e.[3] This would yield a more general notion of physical thickness of markets.[4] However, as may be seen from the example below, Bochner integrability is consistent with the possibility that small groups of consumers have a corner on the market for small sets of commodities. This is a situation we wish to exclude. (See also Example 3 in Section 4.)

Example: For each n, let λ_n be the restriction of Lebesgue measure λ on $[0, 1]$ to the

[3] The allocation e is *Bochner integrable* if it is norm measurable and $\int \|e(t)\| d\lambda(t) < \infty$.

[4] It is not hard to show that if e is weak* measurable and $e(t) \leq K\mu$ for all t, then e is in fact norm measurable. Hence, thick markets allocations are Bochner integrable.

interval $I_n = [(2^n - 2)/2^n, (2^n - 1)/2^n)]$. Define the allocation $e : T \to M^+(X)$ by:

$$e(t) = \begin{cases} 2^n \lambda_n, & \text{for } t \in I_n; \\ \delta_1, & \text{for } t = 1. \end{cases}$$

The allocation e is Bochner integrable and $\int e(t)d\lambda(t) = \lambda$. Moreoever, for each $t \neq 1$, there is a number K_t such that $e(t) \leq K_t\lambda$. However, there is no *uniform* bound on $e(t)$ in comparison with λ, and e is not a thick markets allocation. Note that small groups of traders (traders in the interval I_n) have a corner on the market for small sets of commodities (commodities in the interval I_n).

Our final result says that physical thickness also leads to perfect competition. To establish the sense in which the two tests of perfect competition are similar in their generic conclusions with respect to physically thick markets, the following metric is defined.

If e_1, e_2 are thick markets allocations, then their mean societal endowments ω_1, ω_2 are boundedly absolutely continuous with respect to μ; write S_1, S_2 for the corresponding Radon-Nikodym derivatives, which belong to $L_\infty(\mu)$. We define the *distance* between e_1 and e_2 as:

$$d(e_1, e_2) = \int \|e_1(t) - e_2(t)\| d\lambda(t) + \|S_1 - S_2\|_\infty.$$

Equipped with this metric, the space $\mathcal{T}$ of thick markets allocations is a complete metric space.

Theorem 4 (Perfectly Competitive Equilibrium with Physically Thick Markets) *If markets are physically thick, then:*

(a) All equilibrium prices belong to $C_S(X)$

Moreover,

(b) every initial allocation passes the core equivalence test, and

(c) a generic set of initial allocations pass the withholding test.

Theorem 4 arrives at the same conclusions about perfect competition as Theorem 3, but for different reasons. When markets are not economically thick, equilibrium prices are only as smooth as mean supply (Theorem 1). Hence, a change in the initial allocation by a small group which effects a large (in norm) change in mean supply—a large change in the supply of some commodities—can result in a large (in norm) change in equilibrium prices—a large change in the prices of some commodities. Physical thickness of markets means that small groups of agents can effect only small changes in the mean supply of all commodities by withholding part of their endowment, leading, in general, to only a small change in equilibrium prices. Individuals have no market power because there are numerous suppliers of identical commodities. Economic thickness of markets means that the kind of large change in mean supply of some commodities which can be caused by a small group withholding will be smoothed to ensure little effect on equilibrium prices. In the absence of either kind of market thickness, a nonatomic model may fail to be perfectly competitive.

Remark 2: (*The two tests of perfect competition.*) Core equivalence and withholding appear to be rather different tests of competitivity and, indeed, their methods of proof are distinct. A summary of their differences is:

- There are separate traditions behind each: core equivalence has been the most intensively studied formal standard of competitivty; nevertheless, the withholding test is more closely aligned with the ordinary meaning of perfect competition as "the inability of individuals to influence prices".
- In game-theoretic terminology, the core represents a cooperative while the withholding test more nearly represents a non-cooperative approach.[5]
- The core starts from a more primitive position of bargaining in which prices may emerge whereas the withholding test takes prices and Walrasian equilibria as given and asks, in effect, whether individuals really face conditions of perfectly elastic demand and supply.

Our position is that the two tests are complementary, with each having something to offer over and above the other. Consider our results that the two tests produce generically similar conclusions, although the withholding test is the more stringent. In the language of mechanism design, the withholding test is related to the non-manipulability of the Walrasian mechanism (see, for example, Roberts and Postlewaite [1976]). But, unlike the definitions from that literature, the withholding test does not distinguish between (utility-)favorable and unfavorable price manipulations, i.e., any manipulation of equilibrium prices represents a violation. It is useful to comment on the implications of this property (of perfectly elastic demand and supply).

When Walrasian equilibrium permits favorable manipulation, that is a clear indication of imperfect competition and it is expected that this would be accompanied by core *in*equivalence. See Example 4, below. While inequivalence may be accompanied by favorable manipulation, core equivalence is not necessarily accompanied by non-manipulability or unfavorable manipulation (see Ostroy [1980, Example 2] and Gretsky, Ostroy and Zame [1991]).

Any manipulation, favorable or unfavorable, contradicts the ordinary definition of perfect competition, although within the framework of a fixed Walrasian mechanism, unfavorable manipulation could be ignored. However, in a bargaining framework, while individuals might not directly benefit themselves, their ability to change prices would help/harm others and this opens up a potential profit opportunity; individuals could threaten to withhold to obtain compensation for not doing so. It is well-known that the core does not take account of threat possibilities. Thus, the core equivalence test of perfect competition might be passed because it does not allow *individuals* to threaten. Theorems 3 and 4 show that since the two tests (generically) agree, when there is core equivalence, core bargaining is (generically) immune from such threat possibilities.

[5] See Mas-Colell [1983] for an a non-cooperative perspective on core equivalence.

4. Examples

Examples are given here illustrating several of the conclusions stated in the previous section. In the discussion to follow some claims will be sketched, but most will simply be asserted. In the latter category are: preferences and endowments satisfy assumptions (E), (P.1-5), (B), and (PLD) of the model.

In all cases below, the space of commodities X coincides with the space of agents $T = [0,1]$. Lebesgue measure is denoted by λ on T and by μ in X; the latter is the reference measure on $M^{+}(X)$. Also, in all cases the aggregate initial endowment (before applying the withholding test) will be $\int e d\lambda = \mu$. Of course, the way this aggregate endowment is distributed will determine whether or not markets are physically thick. Equilibrium prices belong to $B(X)$; in fact, for the economy whose total allocation is μ, equilibrium prices will always be the characteristic function of X, denoted by $\mathbf{1}$.

In all but the last example individual preferences are identical and representable by a concave, positively homogeneous function $u: M^{+}(X) \to \mathbf{R}$ (which varies among the examples). This will, of course, lead to substantial simplifications and short-cuts, e.g., even though endowments differ among individuals, in computing Walrasian prices we may pretend that the entire economy is a single individual. Also, in all but the last example, the unique Walrasian equilibrium allocation will be a f such that $f(t) = \mu$ for each t.

The purpose of these examples is to show how substitution possibilities among individuals of the kind that permit/preclude perfect competition are jointly determined by (1) substitution possibilities among commodities by consumers and (2) substitution possibilities among the suppliers of those commodities. To illustrate these phenomena we find it useful below to highlight the properties of the directional derivative of the utility function (see Hart [1979] and especially Jones [1984] for a similar point of view). The directional derivative of u at the point α in the direction β is

$$u'(\alpha;\beta) = \lim_{h \to 0_{+}} \frac{u(\alpha + h\beta) - u(\alpha)}{h}$$

The sense in which $u'(\alpha;\beta)$ is or is not continuous in α and β will be emphasized below.

In the simplified setting of the first two examples there is a test which, if passed, will suffice to satisfy the core equivalence and withholding tests. We shall outline the construction and the reasons why it yields a two-for-one result here.

Letting $\{T_n\}$ be a sequence of sets with $\lambda(T_n) \to 0$ used in the withholding test, suppose that T_n withholds all of its endowment, i.e., T_n *withdraws*. Let p_n be equilibrium prices after T_n withdraws. If we can conclude that $\|p_n - \mathbf{1}\| \to 0$ after T_n withholds *all* of its endowment, we would reach the same conclusion if T_n withholds only some.

Withdrawal is closer to the core: for a coalition to block in an exchange economy it must do better after the complementary coalition has withdrawn. Because u is homogeneous, the game-theoretic characteristic function associated with the economy (u, e) is $v(S) = u(\int_S e)$ where $v(S)$ is the "value" of coalition S, a Borel set in T; i.e., the homogeneity of u allows us to pretend that the game is one of transferable utility. (Briefly, the reason is that all Pareto-optimal allocations for the subeconomy (u, e_S) consist of allocations $f(t) = w(t) \int_S e$

where $w : S \to \mathbf{R}_+$ and $\int_S w d\lambda = 1$.

To demonstrate that a Walrasian allocation is the only one in the core it suffices to show that $\lim \|p_n - \mathbf{1}\| = 0$, where p_n is a Walrasian price vector after T_n withdraws. Then the Walrasian allocation f satisfies the condition

$$\lim \frac{u(\int_{T\setminus T_n} f) - v(T \setminus T_n)}{\lambda(T_n)} = 0$$

This is an application of the no-surplus test of the competitiveness of Walrasian Equilibrium. In transferable utility models, the above condition is known to imply that the core consists of a single allocation.

The summary conclusion is that to pass the core equivalence and withholding tests, it suffices in our simplified setting to establish that $\|p_n - \mathbf{1}\| \to 0$, where p_n is a Walrasian price after T_n withdraws.

EXAMPLE 1: Perfect Competition with Economically Thick Markets

Initial endowments are given by $e(t) = \delta_t$, i.e., everyone is the unique supplier of their own-name good.

Preferences are given by the function

$$u_t(\gamma) = u(\gamma) = \left(\int [\|\gamma\| + \gamma_\rho(x)]^{1/2}\right)^2 + m\|\gamma\|$$

where $\gamma_\rho(x) = \rho^{-1}\gamma[x-\rho, x]$ and $[x-\rho, x]$ is the interval modulo $\mathbf{1}$ of length ρ. This kind of function as well as its various properties described below are found in Jones [1984].

What makes this an economically thick markets example — certainly it is thin markets in terms of endowments — is the substitution possibilities among commodities. First, u is weak star lower-semi-continuous. To illustrate, consider a sequence of measures with finite support $\gamma_n = n^{-1}\sum \delta_{x_k}$, $x_k = k/n$, $k = 1, \ldots, n$. (Note: $\|\gamma_n\| = 1$.) This sequence has the property that for any $x, \gamma_n([x-\rho, x]) \to \mu([x-\rho, x]) = \rho$, and $\gamma_n \to \mu$ in the weak star sense. Thus, $u(\gamma_n) \to \left\{\int (\|\mu\| + \mu_\rho(x))^{1/2} d\mu(x)\right\}^2 + m\|\mu\| = u(\mu) = 2 + m$. Even though μ represents consumption of all commodities in $[0,1]$, because commodities are good substitutes the utility of such an allocation can be approximated by consumption of a finite number.

A calculation shows that

$$\begin{aligned} u'(\gamma; \delta_z) &= m + \left(\int [\|\gamma\| + \gamma_\rho(x)]^{1/2}\, d\mu(x)\right) \cdot \left(\int_X [\|\gamma\| + \gamma_\rho(x)]^{-1/2}\right. \\ &\qquad \left. + \rho^{-1} \int_{[z-\rho, z]} [\|\gamma\| + \gamma_\rho(x)]^{-1/2}\, d\mu(x)\right), \end{aligned}$$

where $[z, z+\rho]$ is an interval in $[0,1]$ modulo 1.

The function u' is uniformly weak star continuous in $\delta_z, z \in X$. This means that for any $r > 1$ there exists a $\delta > 0$ such that if $|y - y'| < \delta$, then $u'(\gamma; r\delta_y) > u'(\gamma; \delta_{y'})$; so Jones uniform substitutability condition holds. (See also Hart [1979].)

It is readily verified that if $p = \mathbf{1}$, each individual having wealth $\mathbf{1} \cdot \delta_t = 1$ will maximize utility by purchasing the bundle μ. Further, if p were not equal to $\mathbf{1}$, all individuals would wish to purchase more of the lower-priced than the higher-priced goods which would contradict market clearing since $\int e = \mu$. So, $(\mathbf{1}, f)$, where $f(t) = \mu$, is the unique Walrasian allocation.

To see that this example satisfies the core equivalence and withholding tests we appeal to the remarks above asserting that we need only show the following: if $\{T_n\}$ is a sequence of small groups subject to the above restrictions, p_n is a Walrasian price when T_n withdraws, and $\|p_n - \mathbf{1}\| \to 0$, then both tests are passed.

The price vector p_n is an equilibrium for the single individual with utility function u and endowment $\int_{T \setminus T_n} e = \mu^n$. Equilibrium prices will satisfy the condition that prices are proportional to marginal utilities, i.e.,

$$p_n(x) = cu'(\mu^n; \delta_x).$$

The function $u'(\gamma; \delta_x)$ is jointly weak star continuous in γ and x. This implies, in particular, that for any $r > 1$, there exists a $\delta > 0$ such that if $|x - x'| < \delta$, then $u'(\gamma; r\delta_x) > u'(\gamma; \delta_{x'})$. (See Jones [1983].) Thus, p_n is continuous, and by the joint continuity of u', $\{p_n\}$ is uniformly continuous. Since $u'(\mu; \delta_x) = 1 + m$ and $\sup\{|u'(\mu^n; \delta_x) - u'(\mu; \delta_x)| x \in X\} \to 0$ as $\mu^n \to \mu$, we may put $c = (1+m)^{-1}$ to establish that $\|p_n - \mathbf{1}\|_\infty \to 0$.

Not only is the withholding test satisfied at e, but by a similar argument it can be shown that there exists an $\epsilon > 0$ such that if $\int \|e' - e\| d\lambda < \epsilon$, then the withholding test is satisfied at e'.

EXAMPLE 2: Perfect Competition With Physically Thick Markets

The utility function of Example 1 evaluates a bundle γ by taking a continuously rolling average of the amounts in each interval $[x - \rho, x]$. The fact that $\rho > 0$ is the source of its weak star continuity (more precisely, its weak star lower-semi-continuity). The following function puts $\rho = 0$:

$$u_t(\gamma) = u(\gamma) = \left\{ \int (\|\gamma\| + g(x))^{1/2} d\mu \right\}^2 + m\|\gamma\|$$

where $g(x)$ is the μ-derivative of the μ-continuous part of γ.

To verify that u is not weak star lower-semi-continuous, consider again the sequence $\gamma_n = n^{-1} \sum \delta_{x_k}$, where $\gamma_n \to \mu$ in the weak star sense. Then $u(\gamma_n) = (1+m)$ but $u(\mu) = (2+m)$. Similarly, examination of the directional derivative of this function,

$$u'(\gamma; \delta_z) = (1+m),$$

reveals that it is not weak star continuous either in γ or in z. However, $u'(\gamma; \beta)$ is jointly continuous in the norm topology for γ and β. The norm continuity of directional derivatives will suffice for this example provided markets are physically thick.

The physically thick markets condition requires that there exist an K such that $e(t) \leq K\mu$ where again $\mu = \int e$. To make the results even more transparent assume $\mathbf{1} \cdot e(t) = 1$.

Given the symmetry of the utility function, the aggregate endowment equal to μ, and the identity of individual wealths at $p = \mathbf{1}$, it readily follows that $(\mathbf{1}, f)$ where $f(t) \equiv \mu$ is the unique Walrasian equilibrium.

Again we subject the economy to a withdrawal test by T_n. Now, because endowments are not personalized, when T_n is withdrawn $\operatorname{supp} \mu^n = X$ for n sufficiently large. (Recall $\mu^n = \int_{T \setminus T_n} e$.) It follows from the hypotheses on physically thick markets that $\liminf \mu(\operatorname{supp} \int_{T_n} e) > \delta$. This should be compared to the physically thin markets Example 1 where

$$\lim \mu(\operatorname{supp} \int_{T_n} e) = \lim \mu(\operatorname{supp} \mu | T_n) = 0.$$

For thick markets we have the conclusion that $\| \int_{T_n} e\|_\infty = \|\mu - \mu^n\|_\infty \to 0$. Inspection of the function $u(\gamma)$ reveals that if p_n is a Walrasian price when T_n withdraws — i.e., p_n is a Walrasian price for the single individual with utility u having initial allocation μ^n then $\|p_n - \mathbf{1}\| \to 1$. This convergence of prices implies the passing of the core and withholding tests.

Finally, on the question of genericity, there is an $\epsilon > 0$ such that if e' is another initial allocation *and* e' is also a physically thick markets allocation with $d(e, e') < \epsilon$, then the economy with initial allocation e' will also exhibit properties similar to e.

EXAMPLE 3: A Single Monopolist

Suppose tastes are the same as in Example 2 and that $e : T \to M^+(X)$ is Bochner integrable. It is well-known that for any $\epsilon > 0$ there exists K_ϵ such that $\lambda\{t : e(t) \leq K_\epsilon \mu\} < \epsilon$, i.e., the market is *almost thick.* Recall the example in Section 3 where

$$\begin{aligned} e(t) &= 2^n(\mu | I_n) \quad t \in I_n \\ e(1) &= \delta_1, \end{aligned}$$

and $\mu | I_n$ is the restriction of μ to $I_n = [(2^n - 2)/2^n, (2^n - 1)/2^n]$.

The unique Walrasian equilibrium for this example is the same as the previous ones: equilibrium prices are $\mathbf{1}$ and each individual spends his wealth $\mathbf{1} \cdot e(t) = 1$ to purchase $f(t) = \mu$.

The withholding test reveals that individual 1, and only individual 1, has monopoly power. Suppose $T_n = [n - 1/n, 1]$. If T_n withholds 1/2 of its endowment, the total endowment is $\int_{T \setminus T_n} e + (1/2) \int_{T_n} e = \mu_{|T \setminus T_n} + 1/2 \mu_{|T_n} = \mu^n$. Exploiting the homogeneity of u, equilibrium prices for the single individual with endowment μ^n are proportional to the unique gradient (Gateaux derivative)

$$q_n(z) = \begin{cases} a_n(\|\mu_n\| + 1)^{1/2} & \text{if } z \in T \setminus T_n \\ a_n(\|\mu_n\| + 1/2)^{1/2} & \text{if } z \in T_n \end{cases}$$

where $a_n = \int (\|\mu_n\| + g_n(x))^{1/2} d\mu(x)$ and $\mu_n = g_n \mu$.

Letting $p_n = (1 + m)^{-1} q_n$, so that p_n is the equilibrium price for μ_n such that $\int p_n d\mu = 1$, it is clear that $\lim \|p_n - \mathbf{1}\| \not\to 0$. However, for any other $[T_n]$, say $T_n = [(n-2)/2n, 1/2]$, there is an K such that for $t \in T_n$, $e(t) \leq K\mu$, and the analysis and conclusions follow those of Example 2.

The core equivalence test of perfect competition does not catch the monopoly power of individual 1: if we were to suppose another allocation f' were in the core, there would be a set $S \subset T$ of positive λ-measure such that on $S, f'(t) \neq f(t)$ and the presence or absence of individual t=1 in this set would be irrelevant. Hence, this example passes the core equivalence test.

EXAMPLE 4: A World of Monopolists

Here we combine the endowments of the economically thick markets example with the tastes of the physically thick markets example, obtaining a situation that is neither physically nor economically thick, a recipe for monopolistic competition. A variant of this example appears in Ostroy [1973, 1984]. Pascoa [1986b] has several extensions which include economies with production. Romer [1987] uses a similar example. A version with large but finite numbers of individuals and commodities was used by Dixit and Stiglitz [1977] to exhibit monopolistic competition.

Let $e(t) = \delta_t$ and $u_t = u$ be as in Examples 2 and 3. Again $\mathbf{1} \cdot e(t) = 1$ and therefore, by an argument familiar from the previous examples $(\mathbf{1}, f)$ where $f(t) = \mu$, is the unique Walrasian equilibrium.

In this example we find that exactly the same argument we used to show that individual 1 had monopoly power in Example 3 can be duplicated to show that *every individual* has monopoly power. Thus, if $T_n = [(n-2)/2n, 1/2]$, then $p_n(x) = 1, x \notin T_n$ and

$$p_n(x) = [a_n(\|\mu_n\| + 1/2)^{1/2} + m]/(1+m)$$

is the Walrasian price when T_n withholds one-half of its allocation. Evidently, $\|p_n - 1\| \not\to 0$.

To demonstrate core *in*equivalence, note that the symmetry of the example plus the homogeneity of u implies that the characteristic function v can be written as $v(S) = u(\int_S e)$ and, in fact, by abuse of notation, since $v(S) = v(S')$ whenever $\lambda(S) = \lambda(S')$ we can write $v(\lambda(S))$. It is readily verified that for each non-null S,

$$\nu(\lambda(S)) = u(\int_S e) = u(\mu|S) < u(\lambda(S)\mu),$$

where $u(\lambda(S)\mu)$ is the utility of the allocation to all the members of S in a Walrasian allocation. Under these conditions, it can be shown that the core coincides with the Walrasian allocation if and only if

$$\lim_{[1-\lambda(S)]\to 0} \frac{u(\lambda(S)\mu) - u(\mu|S)}{[1-\lambda(S)]} = 0.$$

And, it is precisely because prices do *not* converge that this condition does not hold and the set of allocations in the core includes more than the Walrasian allocation $f(t) = \mu$, for each t.

Neither the core inequivalence nor the failure to satisfy the withholding test are "knife-edge" properties. I.e., if e' is another initial allocation such $d(e', e) = \int \|e'(t) - e(t)\| d\lambda(t) <$

1, it may be verified that the economy defined by e' would also exhibit core inequivalence and failure of the withholding test. This establishes the validity of Theorem 2.

EXAMPLE 5: A Perfectly Competitive Continuum of Edgeworth Boxes

A continuum of Edgeworth boxes suggests that trading relations reduce to a continuum of bilateral monopolies. However, the hypothesis that an individual likes only two (pure) commodities contradicts (PLD). We present an example in which *equilibrium* trades are bilateral — as in the continuum of Edgeworth boxes — but because of commodity substitution there is perfect competition. This is a variant of an example in Zame [1986].

Let $h(t) = t + 1/2$ (modulo 1) and let endowments be given by $e(t) = \delta_{h(t)}$; each individual's endowment consists of one unit of the commodity that is "one-half unit to the right of his name."

Preferences of individual t are given by

$$u_t(\gamma) = \int [g(x)]^{k_t(x)} d\mu(x) + \gamma_s(t) + m\|\gamma\|.$$

$\gamma_s(t)$ is the μ-singular part of γ evaluated at t and $g(x)$ is the derivative of the μ-continuous part of γ. The function $k_t(x) = 1 - (t - x)^2$. (A similar utility function is given in Jones [1984].)

The marginal utility of the good having the same name as the individual is $u_t'(\gamma; \delta_t) = (1 + m)$, while the marginal utility of any other good is $u_t'(\gamma; \delta_s) = m, s \neq t$. For a μ-continuous measure described by its derivative g, notice that for fixed $\|g\|$, $u_t'(\gamma; g)$ is increasing to $(1 + m)\|g\|$ as g becomes more concentrated around t.

We assert that the unique Walrasian equilibrium (p, f) is given by $p = \mathbf{1}$ and $f(t) = \delta_t$. In terms of net trades, notice that $f(t) - e(t) = (\delta_t, -\delta_{h(t)}) = -(-\delta_t, \delta_{h(t)}) = -[f(h(t)) - e(h(t))]$, i.e., t and $h(t)$ form an Edgeworth pair in which they can execute all their trades.

A summary description of this example suggests it should have properties similar to the previous one in which monopoly power is ubiquitous. Markets are physically thin and preferences are not similar to the weak star continuous preferences which permitted markets to be economically thick. Nevertheless, this example is perfectly competitive.

Heuristically, it is not difficult to see why. In equilibrium, individual t is the only buyer of commodity $x = t$, but he has no monopsony power because individual $h(t)$, the only supplier of commodity t, could sell his one unit in small amounts to others who would be willing to pay virtually as much per unit. Also, individual sellers have no monopoly power, because if a small group T_n were to withhold one-half their endowment, then even though the new Walrasian equilibrium would require a significant shift in the equilibrium allocation in which buyer t did not concentrate his purchases in his most preferred commodity but in a small cluster (with μ-non-null measure) near t, this would nevertheless lead to equilibrium prices that were nearly unity. The complete withdrawal of a small group would have similar consequences and this would lead by the kind of argument given above to core equivalence.

More formally, let us look at the properties of the directional derivative $u_t'(\gamma; \beta)$. At most points u' is weak star continuous in *neither* γ *nor* β. However, if we look at the equilibrium allocation where $f(t) = \delta_t$, we find that $u'_t(\delta_t; \delta_t)$ *is* weak star continuous in both variables at that point. Thus, *at equilibrium* this example shares the same important continuity property that is universally true in Example 1.

5. CONCLUDING REMARKS

Research on nonatomic economies has emphasized its role as the natural setting for the display and analysis of perfect competition. Such an emphasis ignores the possibility of monopolistic competition, i.e., environments in which small scale traders possess monopoly power. Recent examples of monopolistically competitive nonatomic models are Hart [1985a,b], Pascoa [1986a,b], and Romer [1987].[6]

The goal of this paper has been to provide a nonatomic framework in which either perfectly competitive or monopolistically competitive environments might occur. This has been accomplished using the traditional idea that "perfect competition is a special case". By adopting a sufficiently general setting to permit monopolistic competition, we then ask what additional conditions are necessary for perfect competition. Our answer is that, over and above nonatomicity, markets must be physically or economically thick.

The canonical case of a physically thick market is Aumann's model of a nonatomic economy with a finite number of (physically identical) commodities. For many commodities (e.g., skilled labor, housing, etc.), the hypothesis of physical thickness does not apply. In these cases, the question of competivity is: are markets economically thick, i.e., do individual preferences satisfy the uniform substitutability condition with respect to the available set of commodities? To address interesting and important instances of monopolistic competition, the answer must be "no".

We have adopted the assumption of (PLD) as *one* possible alternative to, and extension of, uniform substitutability that is compatible with monopolistic competition. The origins of an individual's monopoly power in the supply of a particular commodity are first, the market must be physically thin and, second, the typical buyer must find the commodity supplied to be a complement rather than a substitute for other commodities. Under (PLD), this complementarity can only arise if the typical buyer has a *distinct* preference for diversity, i.e., for commodity bundles whose support have positive μ measure compared with commodity bundles with null support. Therefore, in a monopolistically competitive setting, the typical buyer will consume (small quantities of) a positive fraction of the available supply of commodities. There is, however, no possibility of monopsony power because (PLD) precludes that a few buyers could have a 'corner' on the demand for any commodity. We note that the same two features of a distinct preference for diversity and the absence of monopsony power are also present in the models of Romer [1987] and Pascoa [1986b].

(PLD) may be contrasted with the crucial *non-neighboring goods* (NNG) assumption of Hart [1985] and extended by Pascoa [1986a]. In terms of the above, (NNG) implies that individuals exhibit a distinct aversion to diversity; a typical individual is only interested in

[6] The first two papers are inspired by the ideas of Chamberlin [1933] while the third follows the ideas of Chamberlin's teacher Young [1928].

commodity bundles with μ-null (in fact, finite) support. For example, indexing houses by their location, (NNG) says that nearby houses would not be good substitutes.

It seems that to have a more flexible and valuable model of monopolistic competition, something like a convex combination of (NNG) and (PLD) is needed, i.e., a description of preferences in which nearby commodities does have some meaning in terms of substitution possibilities and yet not necessarily the same meaning for all individuals. Such an hypothesis would also be useful in the theory of production for describing substitution among differentiated inputs.

6. Proofs

In Section 2, the following was claimed: *If preferences satisfy* (US) *and are represented by* $U(g) = \int u(g(x))d\mu(x)$, *where* μ *is nonatomic, then* u *is linear.* To demonstrate, suppose the contrary. Therefore the derivative of u is not constant and we may choose two values $s_1 < s_2$ for which $u'(s_1) > u'(s_2)$. Fix r such that $u'(s_1)/u'(s_2) > r > 1$; and choose disjoint sets Y_1, Y_2 such that diameter $(Y_1 \cup Y_2)$ is less than the δ corresponding to that r (in the statement of (US)). Let g_i be the characteristic function of Y_i, $i = 1, 2$. According to (US),

$$\begin{aligned} U(s_1 g_1 + s_2 g_2) &= u(s_1)\mu(Y_1) + u(s_2)\mu(Y_2) \\ &< u([rs_1 \frac{\mu(Y_1)}{\mu(Y_2)} + s_2])\mu(Y_2) \\ &= U([rs_1 \frac{\mu(Y_1)}{\mu(Y_2)} + s_2]g_2) \end{aligned}$$

Because the derivative of a concave function is decreasing, and $u(0) = 0$, we obtain first the inequality,

$$U(s_1 g_1 + s_2 g_2) \geq s_1 u'(s_1)\mu(Y_1) + U(s_2 g_2);$$

and, then since $u'(s_1)/u'(s_2) > r$,

$$\begin{aligned} U([rs_1 \frac{\mu(Y_1)}{\mu(Y_2)} + s_2]g_2) &\leq rs_1 u'(s_2)[\frac{\mu(Y_1)}{\mu(Y_2)}]\mu(Y_2) + U(s_2 g_2) \\ &< s_1 u'(s_1)\mu(Y_1) + U(s_2 g_2). \end{aligned}$$

Since this is a contradiction, we conclude that u must be linear and that (US) entails risk-neutrality. (Similar conclusions contrasting (PLD) and (US) may be obtained if u is state-dependent, provided that the derivative of u depends continuously on the state.)∎

The following lemma records the fact that the value a price p assigns to society's mean allocation is (as we wish it to be) the average of the values it assigns to each trader's individual allocation.

Lemma 1 *If* p *is a price and* f *is an allocation, then*

$$p \cdot \int f(t)d\lambda(t) = \int p \cdot f(t)d\lambda(t).$$

Moreover, if $\nu = \int f(t)d\lambda(t)$ *and* $A \subset X$ *is a Borel set such that* $\nu(A) = 0$ *then* $f(t)(A) = 0$ *for almost all* $t \in T$.

Proof: Fix a Gelfand integrable function $g : T \to M^+(X)$ and write B_g for the set of bounded Borel functions q with the property that $q \cdot g(t)$ is measurable and $\int q \cdot g(t) d\lambda(t) = q \int g(t) d\lambda(t)$; we want to show that B_g contains all bounded Borel functions.

Recall that $B(X)$ is the smallest space of functions which contains all continuous functions and is closed under the formation of pointwise limits of bounded sequences. The definition of Gelfand integrability means that every continuous function belongs to B_g, so we need to show that if $q \in B(X)$ is the pointwise limit of a bounded sequence $\{q_n\}$ of functions in B_g, then q also belongs to B_g. To see this, note first that, for each t, an application of the Lebesgue bounded convergence theorem yields:

$$\begin{aligned} q \cdot g(t) = \int q(x) dg(t)(x) &= \int [\lim q_n(x)] dg(t)(x) \\ &= \lim \int q_n(x) dg(t)(x) \\ &= \lim [q_n \cdot g(t)]. \end{aligned}$$

Since the functions q_n belong to B_g, the functions $[q_n \cdot g(t)]$ are measurable; this means in particular that $q \cdot g(t)$ is the limit of a sequence of measurable functions, and is therefore measurable. Combining the above equalities with two further applications of the Lebesgue bounded convergence theorem yields:

$$\begin{aligned} \int q \cdot g(t) d\lambda(t) &= \int \lim [q_n \cdot g(t)] d\lambda(t) \\ &= \lim [\int q_n \cdot g(t) d\lambda(t)] \\ &= \lim [q_n \cdot \int g(t) d\lambda(t)] \\ &= [\lim q_n] \cdot \int g(t) d\lambda(t)] \\ &= q \cdot \int g(t) d\lambda(t). \end{aligned}$$

Hence $q \in B_g$. We conclude that B_g contains all continuous functions, and is closed under the formation of pointwise limits of bounded sequences; this means that B_g consists of all bounded Borel functions, as desired.

To obtain the second statement, note that $\nu(A) = \chi_A \cdot \nu$ for every measure $\nu \in M(X)$ (where χ_A is the characteristic function of A). From the above we conclude that, for every Borel set $A \subset X$,

$$\gamma(A) = \chi_A \cdot \gamma = \int \chi_A \cdot g(t) d\lambda(t) = \int g(t)(A) d\lambda(t).$$

Since each of the measures $g(t)$ is positive, the last integral is 0 if and only if $g(t)(A) = 0$ for almost all $t \in T$, as desired. ∎

We turn next to a discussion of the sense in which it is possible to identify prices which agree almost everywhere. There are three issues: (1) How are equilibrium *prices* affected by a price change on a set (of commodities) of measure zero? (2) How are equilibrium *allocations* affected by a price change on a set of measure zero? (3) How can we recognize equilibrium prices independent of their values on a set of measure zero?

Let us first record a useful observation: If p and q are prices which agree almost everywhere (i.e., except on a set of measure zero), then $p \cdot \omega = q \cdot \omega$. It follows from Lemma 1 that, for any feasible allocation f (and in particular, for f equal to the initial allocation e), $p \cdot f(t) = q \cdot f(t)$ for almost all traders t.

(1) To see how equilibrium *prices* are affected by a price change on a set of measure zero, let (p, f) be a Walrasian equilibrium and let q be a price for which $q = p$ almost everywhere; we ask whether (q, f) must also be a Walrasian equilibrium. This is of course a question of wealth, of expenditures, and of the costs of desirable commodity bundles. As noted above, the set of traders whose wealth differs at p and at q constitutes a set of measure zero. Similarly, the set of traders whose expenditure differs at p and at q also constitutes a set of measure zero. Since the notion of equilibrium is insensitive to any null set of traders, these effects are of no importance. However, if $q(x) = q \cdot \delta_x < p \cdot \delta_x = p(x)$ for some $x \in X$ then the pure commodity δ_x is certainly cheaper at q than at p. Since this commodity might be desirable, it might be the case that *every* trader (or at least every trader in some set of positive measure) would wish to consume additional quantities of δ_x and could afford to do so; in this circumstance, q *will not* be an equilibrium price. However, if $q = p$ almost everywhere, and $q \geq p$ everywhere, then no commodities are cheaper at q than at p; since wealth and expenditures are affected only for a null set of traders, in this circumstance, q *will* be an equilibrium price.

(2) To see how equilibrium *allocations* are affected by a price change on a set of measure zero, let us suppose that (p, f) and (q, g) are Walrasian equilibria corresponding to the same initial allocation e, and that $p = q$ almost everywhere; we ask for the relationship between the Walrasian allocations f and g. Since optimal consumption choices are not necessarily unique, there is no reason to suppose that $f = g$ almost everywhere. However, our observation above yields that $q \cdot f(t) = p \cdot f(t) \leq p \cdot e(t) = q \cdot e(t)$ for almost all traders t, and similarly that $p \cdot g(t) = q \cdot g(t) = q \cdot g(t) \leq q \cdot e(t) = p \cdot e(t)$ for almost all traders t; hence (p, g) and (q, f) are also Walrasian equilibria. That is, p and q admit the same equilibrium allocations.

(3) Finally, we come to the question of recognizing equilibrium prices, independently of their values on a set of measure zero. Let p, q be prices which agree almost everywhere; as we have already noted and used several times, $p \cdot e(t) = q \cdot e(t)$ for almost all traders. Moreover, if α is a measure which is absolutely continuous with respect to μ, then $p \cdot \alpha = q \cdot \alpha$. In combination, this means that, for almost all traders t, the budget set at the price p and the budget set at the price q contain the same absolutely continuous measures. The key to the following lemma, which enables us to recognize an equilibrium price, independently of its values on a set of measure zero, is that the absolutely continuous measures in the budget set determine its optimal elements, even if the optimal elements are not themselves absolutely continuous.

Lemma 2 *Let f be a feasible allocation and let p be a bounded Borel function. The following statements are equivalent:*

(i) there is a bounded Borel function p^* such that $p^* = q$ almost everywhere and (f, p^*) is a Walrasian equilibrium;

(ii) for almost all traders t, if $\alpha \in M^+(X)$ is absolutely continuous with respect to ω and $f(t) \prec_t \alpha$ then $p \cdot e(t) < p \cdot \alpha$.

In view of this discussion, we shall identify prices which agree almost everywhere. Similarly, we shall frequently not distinguish between a price (which is a bounded Borel function on X) and its equivalence class (which is an element of $L_\infty(\mu)$).

The proof of Lemma 2 requires a number of constructions and some preliminary lemmas which will also be useful in the proofs of the Theorems. The first order of business is to construct a sequence $\{\Pi_n\}$ of partitions of X and a corresponding sequence $\{\Phi_n\}$ of "averaging operators" mapping $M(X)$ onto finite dimensional subspaces.

Fix the reference measure μ, an initial allocation e, and the societal endowment ω. Let S be the supply function (i.e., the Radon-Nikodym derivative of ω with respect to μ); by assumption (E), there are constants c_1, c_2 such that $0 < c_1 \le S \le c_2 < \infty$. For E a subset of X, we write $\text{var}(S, E)$ for the *variation* of S on E and $\text{essvar}(S, E)$ for the *essential variation*; i.e.,

$$\begin{aligned} \text{var}(S, E) &= \sup_{x \in E} S(x) - \inf_{x \in E} S(x) \\ \text{essvar}(S, E) &= \operatorname*{ess\,sup}_{x \in E} S(x) - \operatorname*{ess\,inf}_{x \in E} S(x) \end{aligned}$$

Using an inductive procedure, we can construct a sequence $\{\Pi_n\}$ of partitions of X with the following properties:

(a) Π_n is a partition of X into a finite number of measurable sets of positive measure;

(b) Π_{n+1} is a refinement of Π_n (i.e., every set in Π_{n+1} is contained in some set in Π_n);

(c) every set in Π_n has diameter less than 2^{-n};

(d) for each set $E \in \Pi_n$, $\text{essvar}(S, E) < 2^{-n}$.

For each n, we write M_n for the finite dimensional linear subspace of $M(X)$ spanned by the measures $\mu(E)^{-1}(\mu|E)$, for $E \in \Pi_n$. It is easily checked that these (normalized) restriction measures form an order basis for M_n. In particular, the dimension of M_n is the cardinality $c(n)$ of Π_n, M_n is a sublattice of $M(X)$, and M_n is isomorphic (as a vector lattice) to $\mathbf{R}^{c(n)}$ (by an isomorphism which takes the measures $\mu(E)^{-1}(\mu|E)$ in M_n to the coordinate vectors of $\mathbf{R}^{c(n)}$). In addition, $\mu \in M_n$ and $M_n \subset M_{n+1}$ for each n.

Define the mappings $\Phi_n : M(X) \to M(X)$ by:

$$\Phi_n(\alpha) = \sum_{E \in \Pi_n} \alpha(E)[\mu(E)^{-1}(\mu|E)],$$

We sometimes call each of these mappings an *averaging operator*, and the sequence $\{\Phi_n\}$ an *averaging sequence*. The following result records the basic properties of the maps Φ_n.

Lemma 3: *Each of the averaging operators Φ_n has the properties:*

(a) Φ_n is a positive linear mapping of $M(X)$ onto M_n;

(b) $\Phi_n(\alpha) = \alpha$ for each $\alpha \in M_n$;

(c) $\|\Phi_n(\alpha)\| \leq \|\alpha\|$ for each α, and $\|\Phi_n(\alpha)\| = \|\alpha\|$ if $\alpha \geq 0$;

(d) for each $E \in \Pi_n$ and each $\alpha \in M^+(X), \alpha(E) = \Phi_n(\alpha)(E)$.

The sequence $\{\Phi_n\}$ of averaging operators has the properties:

(e) for each $\alpha \in M(X)$, $\Phi_n(\alpha) \to \alpha$ in the weak star topology;

(f) for each $\beta \in M(X)$ which is absolutely continuous with respect to $\mu, \Phi_n(\beta) \to \beta$ in the norm topology.

Proof: The verfications of (a)-(d) are straightforward and are left to the reader. To obtain (e), fix a measure $\alpha \in M(X)$, a continuous functions $q \in C(X)$ and a real number $\epsilon > 0$. By considering the positive and negative parts of α separately, we may, without loss of generality, assume that α is positive. Continuity of q means that we can find a $\delta > 0$ such that $|q(x) - q(x')| < \epsilon$ whenever $d(x, x') < \delta$. We assert that, if $2^{-n} < \delta$, then $|\int q d\alpha - \int q d\Phi_n(\alpha)| < 2\epsilon\alpha(X)$. To see this, choose and fix, for each set E in the partition Π_n, a point $x_E \in E$. Then $\int q d\alpha = \sum \int_E q d\alpha$, where the sum extends over all sets $E \in \Pi_n$. Since the diameter of each such E is at most $2^{-n} < \delta$, we see that $|\int_E q d\alpha - q(x_E)\alpha(E)| < \epsilon\alpha(E)$. Hence $|\int q d\alpha - \sum q(x_E)\alpha(E)| < \epsilon\alpha(X)$. Similarly, $|\int q d\Phi_n(\alpha) - \sum q(x_E)\Phi_n(\alpha)(E)| < \epsilon\Phi_n(\alpha)(E)$. Since $\alpha(E) = \Phi_n(\alpha)(E)$ for each $E \in M_n$ and $\alpha(X) = \Phi_n(\alpha)(X)$, combining these two estimates yields that $|\int q d\alpha - \int q d\Phi_n(\alpha)| < 2\epsilon\alpha(X)$, as asserted. Since ϵ and q are arbitrary, this yields (e).

To obtain (f), it is convenient to first treat a special case. Let α be a positive measure which is absolutely continuous with respect to μ, and which has the property that the Radon-Nikodym derivative ψ of α with respect to μ is a positive *continuous* function. Let $\epsilon > 0$ be a positive real number, and choose a $\delta > 0$ with the property that $|\psi(x) - \psi(x')| < \epsilon$ whenever $d(x, x') < \delta$. We assert that, if $2^{-n} < \delta$, then $\|\alpha - \Phi_n(\alpha)\| < \epsilon[\alpha(X) + \mu(X)]$. To see this, fix a subset $A \subset X$, and for each set E in the partition Π_n, choose and fix a point $x_E \in A \cap E$. We obtain (sums running over all sets $E \in \Pi_n$):

$$\begin{aligned} \alpha(A) = \sum \alpha(A \cap E) &= \sum \int_{A \cap E} 1 d\alpha = \sum \int_{A \cap E} 1 d(\psi\mu) \\ &= \sum \int_{A \cap E} \psi d\mu. \end{aligned}$$

Expanding $\Phi_n(\alpha)(A)$ in a similar way yields:

$$\begin{aligned} \Phi_n(\alpha)(A) &= \sum \alpha(E)[\mu(E)^{-1}\mu(A \cap E)] \\ &= \sum (\int_E 1 d\alpha)[\mu(E)^{-1}\mu(A \cap E)] \\ &= \sum [\mu(E)^{-1}\mu(A \cap E)] \int_E \psi d\mu. \end{aligned}$$

Our choice of points E and the fact that $2^{-n} < \delta$ implies that

$$|\int_{A\cap U} \psi d\mu - \psi(x_E)\mu(A \cap E)| < \epsilon\alpha(A \cap E)$$

and that $|\int_E \psi d\mu - \psi(x_E)\mu(E)| < \epsilon\alpha(E)$. Putting all this together yields

$$|\alpha(A) - \Phi_n(\alpha)(A)| < \epsilon[\alpha(X) + \mu(X)].$$

Since A is arbitrary, this means that $\|\alpha - \Phi_n(\alpha)\| \leq \epsilon[\alpha(X) + \mu(X)]$, as asserted. Since $\epsilon > 0$ is arbitrary, this yields (f) in the special case that α is positive and its Radon-Nikodym derivative ψ is continuous.

To obtain the general case, fix a measure α which is absolutely continuous with respect to μ, with Radon-Nikodym derivative ψ; there is no loss of generality in assuming that α (and hence ψ) is positive. Fix $\epsilon > 0$. Absolute continuity of α implies that we can find a real number $\rho > 0$ such that $\alpha(A) < \epsilon$ whenever $\mu(A) < \rho$. By Lusin's theorem, we can find a compact subset $X' \subset X$ such that $\mu(X - X') < \epsilon$ and $\psi|X'$ is continuous; write m for the maximum of ψ on X'. Choose an open set U containing X' such that $\mu(U - X') < \epsilon/m$, and choose a continuous function $\tilde{\psi}$ on X which agrees with ψ on X', is bounded by m, and vanishes outside U. Set $\tilde{\alpha} = \tilde{\psi}\mu$; it is easily seen that $\|\alpha - \tilde{\alpha}\| < \epsilon$. The argument above shows that $\|\Phi_n(\tilde{\alpha}) - \tilde{\alpha}\| < \epsilon$ for n sufficiently large; combining this with (c), the estimate $\|\alpha - \tilde{\alpha}\| < \epsilon$, and the triangle inequality yields that $\|\Phi_n(\alpha) - \alpha\| < 3\epsilon$ for n sufficiently large. This completes the proof. ∎

By way of introduction to the next Lemma, consider a trader t and commodity bundles α, β such that $\alpha \prec_t \beta$. The averaging sequence $\{\Phi_n\}$ has the property that $\Phi_n(\gamma) \rightarrow \gamma$ (weak star) for every $\gamma \in M(x)$, so weak star upper semi-continuity of preferences tells us that, for all sufficiently large n, $\Phi_n(\alpha) \prec_t \beta$. However, weak star upper semi-continuity tells us nothing at all about trader $t's$ preferences over the bundles α and $\Phi_n(\beta)$. The following lemma, which is the first critical application of (PLD), fills this gap.

Lemma 4 (Averaging Property): Let $\alpha, \beta \in M^+(X)$ and let $t \in T$. If $\alpha \prec_t \beta$, then there is an n_0 such that $\alpha \prec_t \Phi_n(\beta)$ for each $n \geq n_0$.

Proof: Continuity of preferences allows us to find a real number $r > 1$ such that $\alpha \prec_t r^{-1}\beta \prec_t \beta$; write $\gamma = r^{-1}\beta = g\mu + \gamma_s$. Let δ be the number given in (PLD) and choose n_0 so that $2^{-n_0)} < \delta$. Fix $n \geq n_0$ and fix a set $E \in \Pi_n$. Since α_s is a singular measure, we can find a subset $E' \subset E$ such that $\gamma_s(E') = \gamma_s(E)$ and $\mu(E') = 0$. Set $c_E = r\gamma(E)/\mu(E)$, $A_E = \{x \in (E - E') : g(x) > c_E\}$ and $B_E = \{x \in (E - E') : g(x) \leq c_E\}$.

We claim that $\alpha \preceq_t r\Phi_n(\gamma)$. To see this, write

$$\gamma = \sum\{(\gamma|E') + (g\mu|A_E) + (g\mu|B_E)\}$$

(summation over $E \in \Pi_n$) so that:

$$\begin{aligned}\Phi_n(r\gamma) &= \sum c_E(\mu|E) \\ &= \sum\{(\alpha|E) - [(\gamma|E') + (rg\mu - c_E)(\mu|A_E) + (c_E - rg\mu)(\mu|B_E)\}.\end{aligned}$$

Our construction guarantees that (PLD) can be applied to each of the terms in curly brackets, so that $\gamma \preceq_t \Phi_n(r\gamma)$. Recalling that $\gamma = r^{-1}\beta$ and $r\Phi_n(\gamma) = \Phi_n(\beta)$, and that $\alpha \prec_t r^{-1}\beta$, and applying transitivity of preferences yields the desired conclusion. ∎

The following lemma justifies a remark made in the discussion of essentially continuous functions, at the end of Section 2.

Lemma 5: *Let X be a compact metric space and let μ be a positive measure on X with* $\operatorname{supp}\mu = X$*. If φ is a bounded Borel function on X which has an essential limit at each point, then there is a continuous function ψ on X which agrees with φ almost everywhere.*

Proof: For each $x \in X$ define

$$\psi(x) = \operatorname*{ess\,lim}_{y \to x} \varphi(y).$$

To see that ψ is continuous, fix a point $x \in X$ and a sequence $\{x_n\}$ converging to x, and suppose that $\psi(x_n) \not\to \psi(x)$. Passing to a subsequence, we may find a $\delta > 0$ such that $|\psi(x_n) - \psi(x)| \geq \delta$. For each $\epsilon > 0$ and each n we may find a set A_n of positive measure such that $d(y, x_n) < \epsilon$ and $|\varphi(y) - \psi(x_n)| < 1/2\delta$ for each $y \in A_n$. However, this provides sets of positive measure arbitrarily close to x on which φ differs from $\psi(x)$ by more that $1/2\delta$; this contradicts the fact that $\psi(x) = \operatorname{ess\,lim} \psi(y)$. We conclude that ψ is continuous.

If ψ and φ differ on a set of positive measure, we can find a $\delta > 0$ and a set $B \subset X$ of positive measure such that $|\varphi(y) - \psi(y)| \geq \delta$ for each $y \in B$. We can then find a sequence $\{B_n\}$ of subsets of B, each of positive measure, having the property that $\operatorname{diam}(B_n) \to 0$. Compactness of X implies that there is an $x \in X$ with the property that every open neighborhood of x contains infinitely many B_n's. This is incompatible with the facts that $\psi(x) = \operatorname{ess\,lim} \varphi(y)$ and that ψ continuous at x so we conclude that $\varphi = \psi$ almost everywhere. ∎

We now have all the ingredients necessary to prove Lemma 2.

Proof of Lemma 2: To see that (i) implies (ii), we need only note that, if $p = p^*$ almost everywhere, then $p \cdot \alpha = p^* \cdot \alpha$ for every measure α which is absolutely continuous with respect to μ, and $p \cdot e(t) = p^* \cdot e(t)$ for almost every t (Lemma 1).

To see that (ii) implies (i), we show first that, at the price p, almost every trader's consumption is in his budget set. Since f is feasible, Lemma 1 yields:

$$\begin{aligned} \int [p \cdot f(t)] d\lambda(t) &= p \cdot \int f(t) d\lambda(t) \\ &\leq p \cdot \int e(t) d\lambda(t) \\ &= \int [p \cdot e(t)] d\lambda(t). \end{aligned}$$

Hence, if there is a set of traders of positive measure for which $p \cdot f(t) > p \cdot e(t)$, there must also be a set of traders$T' \subset T$ of positive measure for which $p \cdot f(t) < p \cdot e(t)$. Monotonicity of preferences and the averaging property imply that for each $t \in T'$ we may find a rational number $\rho_t > 1$ and an index n_t such that

$$\begin{aligned} p \cdot [\rho_t f(t)] &< p \cdot e(t) \\ \Phi_n[\rho_t f(t)] &\succ_t f(t) \end{aligned}$$

for each $t \in T'$, $n \geq n_t$. Hence, we can find a single index n^*, a single rational number ρ^* and a set of traders T^* of positive measure such that

$$\begin{aligned} p \cdot [\rho^* f(t)] &< p \cdot e(t) \\ \Phi_n[\rho^* f(t)] &\succ_t f(t) \end{aligned}$$

for each $t \in T^*$, $n \geq n^*$. Since f is a feasible allocation, $\gamma^* = \int_{T^*} f(t) d\lambda(t)$ is absolutely continuous with respect to μ. Hence, $\Phi_n(\gamma^*) \to \gamma^*$ in norm, so that $p \cdot \Phi_n(\gamma^*) \to p \cdot \gamma^*$. Three applications of Lemma 1 yield that

$$\begin{aligned} p \cdot \gamma^* &= \int_{T^*} p \cdot f(t) d\lambda(t) \\ \Phi_n(\gamma^*) &= \int_{T^*} \Phi_n(f(t)) d\lambda(t) \\ p \cdot \Phi_n(\gamma^*) &= \int_{T^*} p \cdot \Phi_n(f(t)) d\lambda(t) \end{aligned}$$

Write $\omega^* = \int_{T^*} e(t) d\lambda(t)$, and note that $\int_{T^*} p \cdot [\rho^* f(t)] d\lambda(t) < \int p \cdot e(t)\, d\lambda(t) = p \cdot \omega^*$.Together with the equalities above and convergence of $p \cdot \Phi_n(\gamma^*)$ to $p \cdot \gamma^*$, this means we may choose an $\bar{n} \geq n^*$ so large that

$$p \cdot [\rho^* \Phi_n(\gamma^*)] < p \cdot \omega^*.$$

Hence we can find a set of traders $T^{**} \subset T$ of positive measure such that $p \cdot [\rho^* \Phi_n(f(t))] < p \cdot e(t)$ for every $t \in T^{**}$. Since $\rho^* \Phi_n(f(t)) \succ_t f(t)$ and $\rho^* \Phi_n(f(t))$ is absolutely contintuous with respect to μ, this violates (ii). We conclude that almost every trader is in his budget set, as desired.

The next task is to show that p is essentially as continuous as S. If this were not so, we could find a Borel set $Y \subset X$ and a point $z \in X$ which belongs to $\text{supp}(\mu|Y)$, such that $S|Y$ has an essential limit at z but $p|Y$ does not. In fact there is no loss of generality in assuming that $S|Y$ has a limit at z, say $\lim_z(S|Y) = s$; (E) guarantees that $s \neq 0$. To say that $p|Y$ does not have an essential limit at z means that there are sets $A, B \subset Y$ such that:

(a) $\mu(A) > 0, \mu(B) > 0$,

(b) $z \in \text{supp}\,(\mu|A) \cap \text{supp}\,(\mu|B)$

(c) $p_A = \inf\,(p|A) > \sup\,(p|B) = p_B$

Set $\rho = p_A/p_B$ and choose r with $\rho > r > 1$. For this r, (PLD) yields corresponding numbers $\delta > 0$ and $d > 0$.

Write $f(t) = g_t\mu + \eta_t$, where η_t is singular with respect to μ. We claim that for almost every t, $\eta_t(A) = 0$ and $(1+d)g_t(x) \leq g_t(y)$ for almost all $x \in A, y \in B$. To see this, write

$T_1 = \{t \in T : \eta_t(A) \neq 0\}$ and $T_2 = \{t \in T$: there are sets of positive measure $A' \subset A, B' \subset B$ such that $(1+d)g_t(x) > g_t(y)$ for $x \in A', y \in B'\}$. A straightforward argument shows that T_1 and T_2 are measurable sets; we wish to see that they have measure 0.

If T_1 has positive measure, let $t \in T_1$. Since η_t is singular with respect to μ, we can find a measurable set $A_t \subset A$ such that $\eta_t(A_t) = \eta_t(A) > 0$ and $\mu_t(A) = 0$. Note that $f(t)(A_t) = \eta_t(A) = \eta_t(A)$. Let $B_t \subset B$ be a set of positive measure (to be chosen later), and set

$$\gamma_t = f(t)|_{X \setminus (A_t \cup B_t)} + r\left[\frac{f(t)(A_t \cup B_t)}{\mu(A_t \cup B_t)}\mu|_{(A_t \cup B_t)}\right].$$

We may choose B_t so that $f(t)(B_t)$ and diameter $(B_t \cup \{z\})$ are as small as we like. For an appropriate choice, (PLD) guarantees that $\gamma_t \succ_t f(t)$ and the fact that $\rho > r > 1$ guarantees that $p \cdot \gamma_t < p \cdot f(t)$. We have already shown that

$$p \cdot f(t) < p \cdot e(t)$$

for almost all t. It T_1 has positive measure then, arguing just as before, we can find a subset $T_1' \subset T_1$ of positive measure and an index n such that $\Phi_n(\gamma_t) \succ_t f(t)$ and $p \cdot \Phi_n(\gamma_t) < p \cdot e(t)$ for each $t \in T_1'$. Since this contradicts (ii), we conclude that T_1 has 0 measure.

IF T_2 has positive measure, let $t \in T_2$. Using the definition of T_2 and nonatomicity of μ, we can find subsets $A_t' \subset A$ and $B_t' \subset B$ with $\mu(A) = \mu(B) > 0$, and $\epsilon.0$ such that, if we write

$$a = \frac{1}{\mu(A_t')}\int_{A_t'} g_t(x)d\mu(x) = \frac{f(t)(A_t')}{\mu(A_t')},$$

and

$$b = \frac{1}{\mu(B_t')}\int_{B_t'} g_t(y)d\mu(y) = \frac{f(t)(B_t')}{\mu(B_t')},$$

then

$$\begin{aligned} a - \epsilon &< g_t(x) < a + \epsilon \quad \textit{for each } x \in A_t' \\ b - \epsilon &< g_t(y) < b + \epsilon \quad \textit{for each } y \in B_t' \\ a(1+d) &> b + \epsilon(1 + r + d). \end{aligned}$$

Set $a' = (b + ra)/((1 + d + r)$ and $b' = b + r(a - a')$ and define the measure

$$\gamma_t' = f(t)|_{X \setminus (A_t' \cup B_t')} + a'\mu|_{A_t'} + b'|_{B_t'}.$$

(PLD) together with some straightforward (but messy) algebra shows that $\gamma_t' \succ_t f(t)$; simple algebra shows that $p \cdot \gamma_t' < p \cdot f(t)$. Arguing as in the previous paragraph, we obtain a subset $T_2' \subset T_2$ of positive measure and an index n such that $\Phi_n(\gamma_t') \succ_t f(t)$ and $p \cdot \Phi_n(\gamma_t') < p \cdot e(t)$ for each $t \in T_2'$. Since this contradicts (ii), we conclude that T_2 has measure 0.

Now fix subsets $A' \subset A, B' \subset B$. Since f is a feasible allocation, $\int f(t)d\lambda(t) = \omega = S\mu$, so that Lemma 1 implies:

$$\omega(A') = \int_{A'} S(x)d\mu(x) = \int_T \left\{\left[\int_{A'} g_t(x)d\mu(x)\right] + \eta_t(A')\right\} d\lambda(t).$$

Since $\eta_t(A') = 0$ for almost all t, we obtain:

$$\begin{aligned}\int_{A'} (1+d)S(x)d\mu(x) &= \int_T \int_{A'} (1+d)g_t(x)d\mu(x)d\lambda(t) \\ &\leq [\mu(A')/\mu(B')] \int_T \int_{B'} g_t(y)d\mu(t)d\lambda(t) \\ &\leq [\mu(A')/\mu(B')] \int_{B'} S(y)d\mu(y).\end{aligned}$$

Since A' and B' are arbitrary, this means that $(1+d)S(x) \leq S(y)$ for almost every $x \in A, y \in B$. However, since we may choose the sets A, B to be arbitrarily close to z, this contradicts the fact that $S|Y$ has a non-zero essential limit at z. We conclude that p is essentially as continuous as S, as desired.

In order to obtain the equilibrium price p^*, we use Lusin's theorem to find a disjoint sequence $\{Y_n\}$ of compact subsets of X such that:

(a) for each n, the restriction $S|Y_n$ is continuous;

(b) for each n, $\mu(Y_n) \neq 0$ and $\text{supp}(\mu|Y_n) = Y_n$;

(c) $\mu(X - \cup Y_n) = 0$.

(The remainder of this construction depends on μ and on the sequence $\{Y_n\}$, but is otherwise independent of the supply function S and the endowment ω.) For each k, set $Z_k = Y_1 \cup \ldots \cup Y_k$, so that $\{Z_k\}$ is an increasing sequence of compact sets and $\cup Z_k = \cup Y_n = Y$. For each k, the restriction $p|Y_k$ is as continuous as $S|Y_k$, and hence (by Lemma 5) may be altered on a set of measure 0 so as to be continuous. Carry out this construction for each k, and call the resulting function p^* (so that p^* is defined at each point of Y); at each point $z \in (X - Y)$, define $p^*(z) = \|p\|_\infty$. It is clear that $p^* = p$ almost everywhere; to verify that p^* is an equilibrium price we need to verify that almost all traders optimize and that the consumption bundle of almost every trader lies in his budget set.

Let T^* be the set of traders for which $f(t)$ is not optimal at the price p^*; i.e., there is a measure $\alpha \in M^+(X)$ (which may depend on t) such that $p^* \cdot \alpha \leq p^* \cdot e(t)$ and $f(t) \prec_t \alpha$ We will show that for each $t \in T^*$, there is an absolutely continuous measure γ such that $p^* \cdot \gamma \leq p^* \cdot e(t)$ and $f(t) \prec_t \gamma$. Continuity and monotonicity of preferences imply that there is a measure $\beta \in M^+(X)$ such that $p^* \cdot \beta \leq p^* \cdot e(t)$ and $f(t) \prec_t \beta$. By the averaging property, we know that there is an index n such that $f(t) \prec_t \Phi_n(\beta)$ for all sufficiently large n. On the other hand, we assert that $\limsup[p^* \cdot \Phi_n(\beta)] \leq p^* \cdot \beta$. To see this, we fix an index k and decompose β into the sum of three measures: $\beta = \beta_k + \gamma_k + \gamma^*$ where $\beta_k = \beta|Z_k$, $\gamma_k = \beta|(Y - Z_k)$ and $\gamma^* = \beta|(X - Y)$. Since p^* is continuous on Z_k and $\Phi_n(\beta_k) \to \beta_k$ (weak star), we conclude that $[p^* \cdot \Phi_n(\beta_k)] \to p^* \cdot \beta_k$ for each k. Since $p^*(z) = \|p^*\|_\infty = \|p\|_\infty$ at each point $z \in (X - Y)$ and $\|\Phi_n(\beta)\| = \|\beta\|$ (because β is a positive measure), it is surely the case that $[p^* \cdot \Phi_n(\gamma^*)] \leq p^* \cdot \gamma^*$ for each n. Finally, $p^* \cdot \Phi_n(\gamma_k)$ and $p^* \cdot \gamma_k$ are very small if k is very large (since the norm of the measure γ_k is very small for large k). Putting all these facts together yields that $\limsup[p^* \cdot \Phi_n(\beta)] \leq p^* \cdot \beta$, as asserted. However, since

$\Phi_n(\beta)$ is absolutely continuous with respect to μ, this implies that, for sufficiently large n, $p\Phi_n(\beta) < p^*\Phi_n(\beta) < p^* \cdot e(t) = p \cdot e(t)$ and $f(t) \prec_t \Phi_n(\beta)$. Since $\Phi_n(\beta)$ is an absolutely continuous measure, our hypothesis implies that T^* has measure zero.

Finally, note that, since p and p^* agree almost everywhere and f is a feasible allocation, Lemma 1 guarantees that $p^* \cdot e(t) = p \cdot e(t)$ and $p^* \cdot f(t) = p \cdot f(t)$ for almost every t. We have already shown that, at the price p, almost every trader's consumption is in his budget set, so the same is true at the price p^*. The argument above shows that, at the price p^* almost all traders are optimizing, so we conclude that (p^*, f) is an equilibrium, as desired.∎

In the proof of Theorem 1, we shall construct an equilibrium for the given economy as the limit of equilibria of approximating finite dimensional economies; we shall need three lemmas about these limits. The first of these is a relative of Ascoli's theorem. In its statement, $\{\Pi_n\}$ is the sequence of partitions constructed above.

Lemma 6: *Let $\{\varphi_k\}$ be a bounded sequence of functions from X to $\mathbf{R}^+$. Assume that for every $\epsilon > 0$, there are indices k^*, n^* such that $\mathrm{var}(\varphi_k, E) < \epsilon$ whenever $k \geq k^*$ and $E \in \Pi_{n^*}$. Then there is a subsequence of $\{\varphi_k\}$ which converges uniformly on X.*

Proof: For each n and each $E \in \Pi_n$ choose a point $x_E \in E$. Since $\{\varphi_k\}$ is a bounded sequence, we may, passing to a subsequence if necessary, assume that for each n and each $E \in \Pi_n$, the sequence $\{\varphi_k(x_E)\}$ of real numbers is convergent; call the limit $\varphi(x_E)$. We want to see that this implies uniform convergence of $\{\varphi_k\}$. To this end, fix $\epsilon > 0$ and choose k^*, n^* as in the hypotheses of the lemma. Since Π_{n^*} is finite, we can choose an index $k^{**} \geq k^*$ such that $|\varphi_k(x_E) - \varphi(x_E)| < \epsilon$ for each $E \in \Pi_{n^*}$ and each $k > k^{**}$. Since $\mathrm{var}(\varphi_k, E) < \epsilon$ for each $k \geq k^*$ and each $E \in \Pi_{n^*}$, combining the triangle inequality with the fact that Π_{n^*} is a partition of X allows us to conclude that $|\varphi_k(x) - \varphi_{k'}(x)| < 3\epsilon$ each $k, k' \geq k^{**}$ and each $x \in X$. In other words, the sequence $\{\varphi\}$ is uniformly Cauchy, and hence uniformly convergent, as desired. ∎

The remaining technical lemmas are of a functional analytic nature and deal with limits of sequences of (weakly) integrable functions. Recall that, for E a locally convex topological vector space with dual space E', the *weak topology* $\sigma(E, E')$ on E is the weakest topology for which the mappings $x \to \varphi \cdot x : E \to \mathbf{R}$ are continuous (for each $\varphi \in E'$). If (T, λ) is a measure space, a weakly measurable function $f : T \to E$ is *weakly integrable* if for each $T' \subset T$ there is a vector $x_{T'} \in E$ such that $\varphi \cdot x_{T'} = \int_{T'} \varphi \cdot f(t) d\lambda(t)$ for each $\varphi \in E'$. We say that a sequence $\{f_n\}$ of weakly integrable functions *converges weakly* to the weakly integrable function f if, for each $\varphi \in E'$, the sequence $\{\varphi \cdot f_n\}$ converges to $\varphi \cdot f$ in the weak topology of $L_1(\lambda)$.

The following lemma is an infinite dimensional version of a result of Artstein [1979]. Similar infinite dimensional results may be found Khan and Majumdar [1986] and Yannelis [1987]; for a related finite dimensional result, see also Simon and Zame [1987].

Lemma 7: *Let E be a locally convex topological vector space, with dual space E' which is separable in the weak star topology, let (T, λ) be a measure space, and let $\{f_n\}$ be a sequence of weakly integrable functions from T to E, converging weakly to the weakly integrable func-*

tion f. For each $t \in T$, let K_t be a weakly compact subset of E. Assume that, for each n, $f_n(t) \in K_t$ for almost every $t \in T$. Then for almost every $t \in T$, $f(t)$ belongs to the weakly closed convex hull of $\{f_n(t)\}$.

Proof: Since E' is separable in the weak star topology $\sigma(E', E)$, it is also separable in the Mackey topology $\tau(E', E)$ (this is the topology of uniform convergence on weakly compact subsets of E); let ξ be a countable (Mackey) dense subset of E'. Write $C(t)$ for the weak closure of $\{f_n(t)\}$ and $C^*(t)$ for the weakly closed convex hull of $C(t)$; $C(t)$ is weakly compact (since it is a weakly closed subset of K_t), but $C^*(t)$ may not be. The separation theorem tells us that a vector in E fails to lie in $C^*(t)$ if and only if it can be separated from $C(t)$ by a weakly continuous linear functional — i.e., by an element of E'. Thus $f(t) \notin C^*(t)$ if and only if there is a functional $\xi_t \in E$ such that $\xi_t \cdot f(t) > 2 > 1 > \xi_t \cdot x$ for each $x \in C(t)$. Since the Mackey topology on E' is the topology of uniform convergence on weakly compact subsets of E weak compactness of $C(t)$ implies that we can choose ξ_t to belong to the set $\{\xi_i\}$. Since $C(t)$ is the weak closure of $\{f_n(t)\}$, we conclude that $f(t) \notin C^*(t)$ if and only if there is an index i such that $\xi \cdot f(t) > 2 > 1 > \xi_i \cdot f_n(t)$ for each n. This displays $T' = \{t : f(t) \notin C(t)\}$ as the countable intersection of measurable sets, so it is measurable. Moreover, if $\lambda(t') > 0$ then we can find a vector ξ_k and a subset $T'' \subset T'$ such that $\xi_k \cdot f(t) > 2 > 1 > \xi_k \cdot f_n(t)$ for every n and every $t \in T''$. Put another way, this means that the functions $\xi_k \cdot f_n$ are bounded above by 1 on T'' while the function $\xi_k \cdot f$ is bounded below by 2 on the same set. This contradicts the assumption that $\{\xi_k \cdot f_n\}$ converges weakly to $\xi_k \cdot f$, and this contradiction establishes the lemma. ▮

Remark: Since the weak limiting set WLS$\{f_n(t)\}$ is the intersection, over all k, of the weak closure of $\{f_n(t)\}_{n=k}$, it follows immediately that, for almost every $t \in T$, $f(t)$ belongs to the weakly closed convex hull of WLS$\{f_n(t)\}$.

We shall use Lemma 7 for the case $E = M(X)$ with the weak star topology. In this case, $E' = C(X)$, the weak star topology on $M(X)$ is just $\sigma(E, E')$ and Gelfand integration coincides with weak integration as defined above. In other circumstances, it would be natural to take E to be a Banach space and E' to be its dual. (In the latter case, weak star separability of E' is equivalent to norm separability of E).

The final technical lemma guarantees that certain weak limits exist.

Lemma 8: *If $\{f_n\}$ is a bounded sequence of Gelfand integrable functions from T into $M^+(X)$, then there is a subsequence which converges weakly (to a Gelfand integrable function).*

Proof: Say that $\|f_n(t)\| \leq R$ for each n, t. For each n, define the (countably additive) vector measure F_n on T, taking values in $M(X)$, by setting $F_n(E) = \int_E f_n(t) d\lambda(t)$. Since (T, λ) is a separable measure space, we can find a countable family $\{E^i\}$ of Borel subsets of T with the property that, for every Borel set E and every $\epsilon > 0$, there is an E^j such that $\lambda(E - E^j) < 0$ and $\lambda(E^j - E) < 0$. For each j, the sequence $\{F_n(E^j)\}$ of elements of $M(X)$ is bounded (because $\|F_n(E^j)\| \leq R\lambda(E^j)$), and hence (by Alaoglu's theorem) has a weak star convergent subsequence. Diagonalizing as necessary, we may assume that, for each j, the sequence $\{F_n(E^j)\}$ converges in the weak star topology; call the limit $F(E^j)$.

In fact, the convergence of each of the sequences $\{F_n(E^j)\}$ entails the convergence of $\{F_n(E)\}$ for every Borel subset $E \subset T$. To see this, fix a Borel set E and an $\epsilon > 0$, and choose a sequence $\{D^k\} \subset \{E^i\}$ such that $\lambda(E - D^k) \to 0$ and $\lambda(D^k - E) \to 0$. For each k, the sequence $\{F_n(D^k)\}$ converges to $F(D^k)$. Moreover, for each n,

$$\|F_n(D^k) - F_n(D^{k'})\| \leq R[\lambda(D^k - D^{k'}) + \lambda(D^{k'} - D^k)],$$

for every k and k', so that

$$\|F(D^k) - F(D^{k'})\| \leq R[\lambda(D^k - D^{k'}) + \lambda(D^{k'} - D^k)]$$

for every k and k'. In other words, the sequence $\{F(D^k)\}$ is Cauchy (in norm) and hence converges (in norm) to an element $F(E) \in M(X)$. Elementary computations now yield that $F_n(E) \to F(E)$ (weak star) for every Borel set $E \subset T$, that F is a countably additive vector measure, and that $\|F(E)\| \leq R\lambda(E)$ for every Borel set $E \subset T$. In particular, F is absolutely continuous with respect to λ (i.e., $F(E) = 0$ whenever $\lambda(E) = 0$).

Since every countably additive vector measure which is absolutely continuous with respect to λ has a weak star Radon-Nikodym derivative (Diestel and Uhl [1977]), it follows that there is a Gelfand integrable function $f: T \to M(X)$ such that $F(E) = \int_E f(t)d\lambda(t)$ (Gelfand integral) for every Borel set E. We claim that $\{f_n\}$ converges weakly to f.

To see this, fix a continuous function $q \in C(X)$ and observe that $|q \cdot f_n(t)| \leq R\|q\|_\infty$ for every t, so that the sequence $\{q \cdot f_n\}$ lies in an order bounded subset of $L_1(\mu)$. Since order bounded subsets of $L_1(\mu)$ are weakly compace, some subsequence of $\{q \cdot f_n\}$ converges weakly to a function $\psi \in L_1(\mu)$; we need to show that ψ is necessarily equal to $q \cdot f$. If this is not so, then we can find a set $A \subset T$, having positive measure, such that $\int_A \psi d\lambda(t) \neq \int_A q \cdot f d\lambda(t)$. On the other hand, the definition of the Gelfand integral, together with the definition of ψ and the facts we have already established, yield:

$$\begin{aligned}
\int_A \psi d\lambda(t) &= \int_T \chi_A \psi d\lambda(t) \\
&= \lim \int_T \chi_A (q \cdot f_n) d\lambda(t) \\
&= \lim\, [q \cdot \int_T (\chi_A f_n) d\lambda(t)] \\
&= \lim\, [q \cdot \int_A f_n d\lambda(t)] \\
&= \lim\, [q \cdot F_n(A)] \\
&= q \cdot F(A) \\
&= q \cdot \int_A f d\lambda(t) \\
&= \int_A (q \cdot f) d\lambda(t).
\end{aligned}$$

This is a contradiction, so we conclude that $\{q \cdot f_n\}$ converges weakly to $q \cdot f$; since $q \in C(X)$ is arbitrary, this completes the proof. ∎

With the preliminaries out of the way, we now turn to the proofs of the main results.

Proof of Theorem 1: We construct an equilibrium for the given economy as the limit of equilibria of approximating finite dimensional economies.

The first step is to use the averaging operator Φ_n to construct these finite dimensional approximations. For each n, we consider the economy with commodity space M_n, with space of traders equal to (T, λ), with preferences the restrictions to M_n of the given preferences on $M^+(X)$, and with intital endowment $e_n = \Phi_n \circ e$ (i.e., $e_n(t) = \Phi_n[e(t)]$ for each t). Set $\omega_n = \Phi_n(\omega)$ (it follows from Lemma 1 that $\omega_n = \int e_n(t)d\lambda(t)$) and write S_n for the Radon-Nikodym derivative of ω_n with respect to μ. This economy has an equilibrium (f_n, p_n) (Aumann [1966]). The price p_n belongs to the dual space of M_n, which we may identify with the space of functions on X which are constant on each of the sets in the partition Π_n; in particular $p_n \in B^+(X)$. (The price p_n is necessarily strictly positive, since preferences are strictly monotone.) Note that $\mu \in M_n$ for each n, by construction; we normalize so that $\|p_n\|_\infty = 1$.

Since (B) tells us that rates of substitution are bounded, the equilibrium nature of p_n implies that , if $\alpha, \beta \in M_n^+$ with $\|\alpha\| = \|\beta\|$, then $p_n \cdot \alpha \leq M(p_n \cdot \beta)$. Since $\|p_n\|_\infty = 1$, this implies in particular that $M^{-1} \leq p_n \leq 1$.

For each n and each $E^n \in \Pi_n, \mathrm{var}(S, E) < 2^{-n}$. Since S_k is obtained by averaging S over sets in Π_k, and Π_k is a refinement of Π_n for $k \geq n$, it follows that $\mathrm{var}(S_k, E) < 2^{-n}$ for $k \geq n$. We may now use (PLD) as in the proof of Lemma 2 to obtain estimates for the variations of the prices p_k; we conclude that:

(*) For each $\epsilon > 0$, there is are indices n^*, k^* such that $\mathrm{var}(p_k, E) < \epsilon$ whenever $k \geq k^*, n \geq n^*$ and $E \in \Pi_n$.

In view of Lemma 6, passing to a subsequence if necessary, we may assume that $\{p_k\}$ converges uniformly to a limit price p on X. Uniform convergence implies that $\|p_n\|_\infty = 1$ and that $M^{-1} \leq p_n \leq 1$. For $\alpha \in M^+(X), \Phi_n(\alpha) \to \alpha$ in the weak star topology, so $p_n \cdot \alpha \to p \cdot \alpha$ and $p_n \cdot \Phi_n(\alpha) \to p \cdot \alpha$.

Having constructed a limit price, we now construct a limit allocation. To this end, fix an index k. For each $t \in T$,

$$p_n \cdot f_n(t) \leq p_n \cdot e_n(t) \leq \|e_n(t)\| \leq \|e(t)\|.$$

Since $M^{-1}\|\mu\|^{-1} \leq p$, we conclude that $\|f_n(t)\| \leq M\|\mu\|^{-1}\|e(t)\|$ For each integer R, set $T_R = \{t \in T : \|e(t)\| \leq R\}$. Applying Lemma 8 to the sequence $\{f_n|T_R\}$ yields a weakly convergent subsequence. Diagonalizing as necessary, we may assume that, for each R the sequence $\{f_n|T_R\}$ converges weakly on T_R. Piecing together the limits of these sequences provides a limit allocation $f\colon T \to M^+(X)$. Since $\int f(t)d\lambda(t) = \lim \int f_n(t)d\lambda(t) \leq \omega_n$ and $\omega_n \to \omega$, it follows that f is in fact Gelfand integrable, that f is a feasible allocation, and that $\{f_n\}$ converges weakly to the limit allocation f.

To show that (f, p) is a Walrasian equilibrium, we verify the hypotheses of Lemma 2. Consider the set T^* consisting of all traders t for which there is a measure $\alpha \in M^+(X)$

which is absolutely continuous with respect to μ and has the properties that $f(t) \prec_t \alpha$ and $p \cdot \alpha < p \cdot e(t)$; It is not hard to show that T^* is a measurable set. If $\lambda(T^*) > 0$, then continuity of preferences together with the fact that $\Phi_n(\alpha) \to \alpha$ in norm whenever α is absolutely continuous, implies that we may find an index i, a subset $T^{**} \subset T^*$ of positive measure, and a positive measure $\beta \in M_i$ such that $f(t) \prec_t \beta$ and $p \cdot \beta < p \cdot e(t)$ for each $t \in T^{**}$ Lemma 7 tells us that (for almost all t) $f(t)$ belongs to the weak star closed convex hull of $\{f_n(t)\}$; since preferences are weak star upper semi-continuous and convex, this implies that for almost all t, $f_n(t)$ for n sufficiently large. In particular this means that there is an index m and a subset T^{***} of T^{**}, having positive measure, such that $f_n(t) \prec_t \beta$ for all $t \in T^{***}$ and all $n \geq m$. On the other hand, we have already noted that $p_n \cdot e_n(t) = p_n \cdot \Phi_n[e(t)] \to p \cdot e(t)$ for almost all t, and $\Phi_n(\beta) = \beta$ for $n \geq i$ (since $\beta \in M_i$ and Φ_n is the identity on M_i for $n \geq i$), so we conclude that, for sufficiently large n, there is a subset T^{****} of T^{***}, having positive measure, with the property that $f_n(t) \prec_t \beta$ and $p_n \cdot \beta < p_n \cdot e_n(t)$, for each $t \in T^{****}$. This contradicts the equilibrium nature of (f_n, p_n) Thus, the supposition that $\lambda(T^*) > 0$ leads to a contradiction. We may therefore apply Lemma 2 to find a bounded Borel function p^*, agreeing with p almost everywhere, such that (f, p^*) is an equilibrium; the argument of Lemma 2 shows that in fact $p = p^*$ so that (f, p) is an equilibrium, as desired.

That all equilibrium prices belong to $C_S(X)$ follows exactly as in the proof of Lemma 2.

It remains only to show that the set $P(e)$ of normalized equilibrium prices is compact in the norm topology. To this end, let $\{(f_n, p_n)\}$ be a sequence of equilibria with prices satisfying the normalization $p_n \cdot \mu = 1$. Arguing exactly as before, we conclude that $\|p_n\| \leq M\|\mu\|^{-1} < \infty$ for each n, and that for each $\epsilon > 0$, there are indices n^*, k^* such that $\mathrm{var}(p_k, E) < \epsilon$ whenever $k \geq k^*, n \geq n^*$ and $E \in \Pi_n$. Lemma 6 implies that some subsequence of $\{p_n\}$ converges uniformly to some price p. As before (and passing to a subsequence if necessary), we see that the allocations $\{f_n\}$ converge weakly to an allocation f, and that (f, p) satisfies the hypotheses of Lemma 2. Hence there is a bounded Borel function p^* which agrees with p almost everywhere such that (f, p^*) is an equilibrium. Since $p_n \to p$ uniformly and p, p^* represent the same class in $L_\infty(\mu)$, it follows that $p_n \to p$ in the $L_\infty(\mu)$ norm. Hence $P(e)$ is a norm compact subset of $L_\infty(\mu)$. This completes the proof of Theorem 1. ∎

Theorems 3 and 4 rest on a result about points of continuity of an upper hemi-continuous correspondence. The usual version of this result, which requires that the range space be compact (see Hildenbrand [1974] for example), would be adequate for Theorem 3, but Theorem 4 requires the stronger version below, which requires only that the correspondence has compact values.

Lemma 9: *If X and Y are complete metric spaces, and $P : X \to\to Y$ is an upper hemi-continuous correspondence with compact values, then the set of points of continuity of P is a residual subset of X.*

Proof: For each compact subset A of Y and each integer k, let A^k denote the open set of points of Y whose distance to A is smaller than 2^{-k}. Since P has compact values, upper hemi-continuity of P at the point $x \in X$ means that for each k there is an open set U containing x such that $P(x') \subset P(x)^k$ whenever $x' \in U$. In the presence of upper hemi-continuity, continuity of P at the point $x \in X$ would mean that for each k there is an open set V containing x such that $P(x') \subset P(x'')^k$ whenever $x', x'' \in V$.[7]

For each k , let W_k be the set of points $x \in X$ for which there is an open set V which contains x and has the property that $P(x') \subset P(x'')^k$ whenever $x', x'' \in V$. It is evident that each of the sets W_k is open. Moreover, the set of points of continuity of P is simply the intersection of all the sets W_k, so to prove the lemma it remains only to show that each W_k is a dense subset of X.

To this end fix a k, and suppose to the contrary that W_k is not dense. There is thus an open set $W \subset X$ with the property that, for each open set $V \subset W$, there are points $x', x'' \in V$ such that $P(x') \not\subset P(x'')^k$. We pick a point $x_1 \in W$ and use upper hemi-continuity of P to find an open subset V_1 of W, containing x_1 and having diameter at most 1, such that $P(x') \subset P(x_1)^{2k}$ for every $x' \in V_1$. Since $V_1 \subset W$ we can find points $x_2, y_2 \in V_1$ such that $P(y_2) \not\subset P(x_2)^k$. We may again use upper hemi-continuity of P to find an open subset V_2 of V_1 containing x_2 and having diameter less than $1/2$, such that $P(x') \subset P(x_2)^{4k}$ for every $x' \in V_2$. We can again find points $x_3, y3 \in V_2$ such that $P(y_3) \not\in P(x_3)^k$. Proceeding by induction, we choose a decreasing sequence $\{V_n\}$ of open sets, and sequences $\{x_n\}, \{y_n\}$ such that: V_n has diameter at most 2^{-n}, the points x_n and y_n belong to $V_n, P(x') \subset P(x_n)^{nk}$ for every $x' \in V_n$, and $P(y_n) \not\subset P(x_n)^n$. Since the sets V_n have diameter at most 2^{-n}, the sequences $\{x_n\}, \{y_n\}$ are Cauchy; completeness of X means that they converge, necessarily to the same limit, call it z. Upper hemi-continuity of P means that $P(y_n) \subset P(z)^{2k}$ for n sufficiently large. On the other hand, our construction guarantees that $P(z) \subset P(x_n)^{nk}$ for every n. Combining these, we obtain that $P(y_n) \subset P(x_n)^k$ for k sufficiently large. This contradicts our supposition that W_k is not dense, and this contradiction completes the proof. ∎

Proof of Theorem 3: Theorem 1 establishes the existence of equilibrium under assumption (E), that the support of the mean societal endowment is all of X. However, when markets are thin, withholding by a small group may yield a mean societal endowment whose support is no longer all of X. Our first task is to show that, with the hypothesis of economic thickness, equilibrium continues to exist even for an initial allocation e' with mean societal endowment ω' with $supp\, \omega' \neq X$. In essence, the difficulty is to find the "correct" reservation prices for commodities in $X \setminus supp\, \omega$.[8]

Choose a positive measure $\hat{\omega}$ for which $\operatorname{supp} \hat{\omega} = X$ and, for each positive integer n, set $e^n(t) = e(t) + (2^{-n})\hat{\omega}$. Note that e^n is an allocation for which the mean societal endowment $\omega^n = \omega + (2^{-n})\hat{\omega}$ has support equal to X. Define the reference bundle $\mu^n = \omega^n$ and observe (as was already noted earlier) that, for this (or any other) choice of reference bundle, (US)

[7] Compactness of the values of P is used only here, to insure the validity of these characterizations of continuity and upper hemi-continuity.

[8] The argument we give for the existence of equilibria is based on our Theorem 1; an alternate argument could be given along the lines of Jones [1983].

implies (PLD), and that the assumption (E) is also satisfied. It thus follows from Theorem 1 that, for the initial allocation e^n, an equilibrium exists. Moreover, since the supply function S^n is identically 1 , every equilibrium price is (essentially) continuous. Indeed, the argument of Lemma 2 shows that the modulus of continuity of all equilibrium prices (of norm 1) may be chosen independently of e and $\hat{\omega}$. Moreover, (B) implies that all equilibrium prices of norm 1 are bounded below by M^{-1}. In particular, the sets $P(e^n)$ of all normalized equilibrium prices for the initial allocation e^n all lie in a bounded, equicontinuous family in $C(X)$. Ascoli's theorem tells us that bounded, equicontinuous subsets of $C(X)$ are relatively compact, so if we define $P(e)$ to be the limiting set of $\{P(e^n)\}$, it follows that $P(e)$ is a non-empty compact subset of $C(X)$, and every price in $P(e)$ has norm 1.

We assert that every price in $P(e)$ is an equilibrium price for the initial allocation e. To see this, let $\{(f^n, p^n)\}$ be a sequence of equilibria (with (f_n, p_n) corresponding to the initial allocation e^n), and such that $\|p_n\| = 1$ for each n. Arguing as in the proof of Theorem 1, and passing to a subsequence if necessary, we may show that the allocations f^n converge weakly to an allocation f, the prices p^n converge uniformly to a price p with $\|p\| = 1$, and that (f, p) is an equilibrium corresponding to the initial allocation e. Since every price in $P(e)$ arises as the limit of such prices p^n, we conclude that every price in $P(e)$ is an equilibrium price for the initial allocation e. In particular, equilibria exist.

To see that all equilibrium prices are continuous on the support of the mean societal endowment, fix an initial allocation e with mean societal endowment ω, and let (f, p) be an equilibrium. Note that, if we restrict attention to commodities in $\operatorname{supp}\omega$, the pair (f, p) remains an equilibrium. If we define the reference bundle $\mu = \omega$, then the supply function S is identically 1 (on $\operatorname{supp}\omega$), so we may simply apply Theorem 1 to conclude that $p|\operatorname{supp}\omega$ is continuous, as asserted. (We do not draw any conclusions about the behavior of an arbitrary equilibrium price on $X - \operatorname{supp}\omega$. However, the equilibrium prices in $P(e)$ enjoy a special status, since, by construction, they are continuous on all of X.)

We turn now to the core equivalence test.[9] Fix an initial allocation e with mean societal endowment ω. Without loss of generality, define the reference bundle $\mu = \omega$, and let f be an allocation which belongs to the core. We construct an equilibrium price by finding a linear functional supporting an appropriate cone. Let us identify (via the Radon-Nikodym theorem), $L_1(\mu)$ as the subspace of M(X) consisting of those measures which are absolutely continuous with respect to μ. Let $\mathcal{G}$ be the space of pairs (T', g) such that T' is a subset of T having positive measure and $g : T \to L_1^+(\mu)$ is a measurable function having the property that $f(t) \prec_t g(t)$ for almost every $t \in A$. Let $\mathcal{P}$ be the preferred net trade set:

$$\mathcal{P} = \left\{ \int_{T'} g(t) d\lambda(t) - \int_{T'} e(t) d\lambda(t) : (T', g) \in \mathcal{G} \right\},$$

and let $\mathcal{P}^*$ be the weak star closure of $\mathcal{P}$, and let C be the cone generated by $\mathcal{P}^*$. We will find a linear functional $p \in L_\infty(\mu)$ that supports the cone $C \cap L_1(\mu)$; an appropriate representative of the equivalence class of p will provide the equilibrium price.

In the usual finite dimensional context, the corresponding results are established by

[9] The argument we give for core equivalence is in the spirit of Gretsky and Ostroy [1985]; an alternate argument could be given along the lines of Mas-Colell [1975].

appeal to the fact that the integral of a correspondence is compact and convex. This fact depends in turn on the Lyapunov convexity theorem, which says that the range of a non-atomic vector measure is compact and convex. As we have already noted, the Lyapunov convexity theorem is not true in infinite dimensional spaces; in particular, the range of a non-atomic vector measure with values in $M(X)$ need not be compact or convex. However, it is true that the weak star closure of the range of such a measure is (weak star) compact and convex. It follows, exactly as in the finite dimensional context, that the weak star closure of the integral of a correspondence is compact and convex. With this change, convexity of C follows in the same way as in the finite dimensional context. (For details, we refer to Gretsky and Ostroy [1985].)

Our next task is to show that the norm closure of $C \cap L_1(\mu)$ is a proper subcone of $L_1(\mu)$. To accomplish this, it suffices to show that $\mathcal{P}^*$ misses a norm open subcone of $L_1(\mu)$ containing $-\omega = -\mu$. In fact, we show that $\mathcal{P}^*$ contains no measures of the form $-c\omega + \nu$, where $c > 0$ and $\|\nu\| < c/(4M)\|\omega\|$. ($M$ is the constant specified in assumption (B).)

Suppose to the contrary that $-c\omega + \nu \in \mathcal{P}^*$. Then there is a sequence $(T^n, g^n) \in \mathcal{G}$ such that

$$\int_{T^n} g^n(t)d\lambda(t) - \int_{T^n} e(t)d\lambda(t) = -c\omega + \nu + \zeta^n,$$

where $\zeta^n \to 0$ (weak star). Write $\gamma^n = \int_{T^n} g^n(t)d\lambda(t)$ and $\omega^n = \int_{T^n} e(t)d\lambda(t)$. Passing to a subsequence if necessary, we way assume that $\gamma^n \to \bar{\gamma}$ and $\omega^n \to \bar{\omega}$ (weak star). Note that $\bar{\gamma} - \bar{\omega} = -c\omega + \nu$. Since $\bar{\gamma}$ and $\bar{\omega}$ are positive and $\|\gamma\| \leq c/(4M)\|\omega\| < c\|\omega\|$, we conclude in particular that $\bar{\omega} \neq 0$.

For each n, we shall construct an allocation h^n that is feasible for the group T^n. We then show that for sufficiently large n, $h^n(t) \succ_t g^n(t)$ for each $t \in T^n$. Transitivity implies $h^n(t) \succ_t f(t)$ for each $t \in T^n$; together with feasibility this will contradict that assumption that f is a core allocation.

We begin by choosing a real number r such that $1 < r < M$, $r < 1 + c/4$, $M(1 - 1/r) < c/4$. Let δ be the corresponding number from the uniform substitutability assumption (US). Choose a finite covering of X by open sets U_i, and let φ_i be a partition of unity subordinate to the cover $\{U_i\}$ (i.e., $\{\varphi_i\}$ is a family of continuous functions from X into $[0, 1]$; support $(\varphi_i) \subset U_i$ for each i, and $\sum \varphi_i = 1$). For each i, n set:

$$\begin{aligned} a_i^n &= \min\{\varphi_i \cdot \omega^n, r\varphi_i \cdot \gamma^n\}, \\ b_i^n &= \varphi_i \cdot \gamma^n - r^{-1}a_i^n \\ \beta_i^n &= \varphi_i\omega^n - \frac{a_i^n}{\varphi_i \cdot \omega^n}\varphi_i\omega^n \\ \beta^n &= \sum_i \beta_i^n. \end{aligned}$$

(Here and below, we follow the notational convention that $0/0 = 0$.) For each $t \in T^n$, set

$$\begin{aligned} h^n(t) = & \sum \left(\frac{\varphi_i \cdot g^n(t)}{\varphi_i \cdot \gamma^n}\right)\left(\frac{a_i^n}{\varphi_i \cdot \omega^n}\right)\varphi_i\omega^n \\ & + \sum \left(\frac{\varphi_i \cdot g^n(t)}{\varphi_i \cdot \gamma^n}\right)\left(\frac{b_i^n}{\varphi_i \cdot \omega^n}\right)\beta^n. \end{aligned}$$

(For motivation, see below.)

A straightforward calculation, keeping in mind that $\int_{T^n} g^n(t)d\lambda(t) = \gamma^n$ and that $\sum \varphi_i = 1$, shows that $\int h^n(t)d\lambda(t) = \omega^n$. That is, h^n is feasible for the group T^n.

We want to see that $h^n(t) \succ_t g^n(t)$ for every $t \in T^n$, provided that n is sufficiently large. To this end, it is useful to make an observation (which is the motivation underlying our construction of h^n). Let $\alpha, \eta, \theta \in M^+(X)$, let φ_j be any element of the partition of unity $\{\varphi_i\}$ chosen above, and let $\succ$ be any preference relation satisfying our assumptions. We ask the following question: For what positive, real values of a, b is

$$(*) \qquad a\varphi_j\alpha + b\eta + (1-\varphi_j)\theta \succ \theta?$$

Since $0 \le \varphi_j \le 1$ and $\theta = \varphi_j\theta + (1-\varphi_j)\theta$, and support (φ_j) has diameter $< \delta$, (US) implies that $(*)$ holds whenever

$$a\varphi_j \cdot \alpha = \|a(\varphi_j\alpha)\| > r\|\varphi_j\theta\| = r(\varphi_j \cdot \theta),$$

(independently of $b \ge 0$). On the other hand, (B) implies that $(*)$ holds whenever

$$b\|\eta\| = \|b\eta\| > M\|\varphi_j\theta\| = M(\varphi_j \cdot \theta),$$

(independently of $a \ge 0$). More generally, if we combine (US) and (B), we can conclude that $(*)$ holds whenever

$$b\|\nu\| > M(\varphi_j \cdot \theta - r^{-1}\min\{a(\varphi_j \cdot \alpha), r(\varphi_j \cdot \theta)\}).$$

To see that $h^n(t) \succ_t g^n(t)$ for each $t \in T^n$, we need only apply this observation sequentially for $j = 1, \ldots, n$. This will yield the desired result, provided that

$$(1) \qquad \|\beta^n\| > M\sum_i(\varphi_i \cdot \gamma^n - r^{-1}a_i^n).$$

To verify that this inequality is valid for sufficiently large n, we use the fact that $\|\alpha_1 + \alpha_2\| = \|\alpha_1\| + \|\alpha_2\|$, if α_1, α_2 are positive measures to write:

$$\begin{aligned}
\|\beta^n\| = \|\sum_i \beta_i^n\| &= \sum_i \|\beta_i^n\| \\
&= \sum_i \|\varphi_i\omega^n - \frac{a_i^n}{\varphi_i \cdot \omega^n}\varphi_i\omega^n\| \\
&= \sum_i \|\frac{\varphi_i\omega^n}{\varphi_i \cdot \omega^n}(\varphi_i \cdot \omega^n - a_i^n)\| \\
&= \sum_i(\varphi_i \cdot \omega^n - a_i^n) \\
&= \sum_i(\varphi_i \cdot \omega^n - \min\{\varphi_i \cdot \omega^n, r\varphi_i \cdot \gamma^n\}).
\end{aligned}$$

Since $\gamma^n \to \bar{\gamma}$ and $\omega^n \to \bar{\omega}$ (weak star), to verify (1) for large n it suffices to verify

$$\sum_i(\varphi_i \cdot \bar{\omega} - \min\{\varphi_i \cdot \bar{\omega}, r\varphi_i \cdot \bar{\gamma}\}) > M\sum_i(\varphi_i \cdot \bar{\gamma} - r^{-1}\min\{\varphi_i \cdot \bar{\gamma}, r\varphi_i \cdot \bar{\gamma}\}).$$

Write $J = \{j : \varphi_j \cdot \bar{\omega} \leq r\varphi_j \cdot \bar{\omega}\}$ and $K = \{1, \ldots, n\} \setminus J$. Simplifying (2) yields

$$\sum_{k \in K} (\varphi_k \cdot \bar{\omega} - r\varphi_k \cdot \bar{\gamma}) > M \sum_{j \in J} (\varphi_j \cdot \bar{\gamma} - r^{-1}\varphi_j \cdot \bar{\omega}).$$

Substitute $\bar{\gamma} = \bar{\omega} - c\bar{\omega} + \nu$, write $\Phi_J = \sum_{j \in J} \varphi_j$ and $\Phi_K = \sum_{k \in K} \varphi_k$, simplify, transpose all terms to the left side, leaving us to verify that the following expression is positive:

$$A = (1-r)\Phi_K \cdot \bar{\omega} - M(1-r^{-1})\Phi_J \cdot \bar{\omega} + rc\Phi_K \cdot \omega + Mc\Phi_J \cdot \omega - r\Phi_K \nu - M\Phi_J \cdot \nu.$$

We now simplify and estimate, using the facts that $r > 1$, $M > r$, $\Phi_J \geq 0$, $\Phi_K \geq 0$, $\Phi_J + \Phi_K = 1$, $0 \leq \bar{\omega} \leq 1$, $\mathbf{1} \cdot \omega = \|\omega\|$, $\mathbf{1} \cdot \nu = \|\nu\|$, $1 < r < 1 + c/4$, $M(r^{-1} - 1) < c/4$, and $\|\nu\| < c/4M$ to obtain:

$$\begin{aligned} A &\geq (1-r)\|\omega\| - M(1-r^{-1})\|\omega\| + c\Phi_K \cdot \omega \\ &\quad + c\Phi_J \cdot \omega - M\Phi_K \cdot \nu - M\Phi_J \cdot \nu \\ &\geq (1-r)\|\omega\| - M(1-r^{-1})\|\omega\| + c\|\omega\| - M\|\nu\| \\ &\geq -(c/4)\|\omega\| - (c/4)\|\omega\| + c\|\omega\| - (c/4)\|\omega\| \\ &\geq (c/4)\|\omega\|. \end{aligned}$$

As we have shown, this inequality yields (1), so that $h^n(t) \succ_t g^n(t)$ and hence that $h^n(t) \succ_t f(t)$ (by transitivity), for all $t \in T^n$, provided n is sufficiently large. Since h^n is feasible for the group T^n, this contradicts our assumption that f is in the core.

We conclude that $\mathcal{P}^*$ indeed misses a norm open subcone of $L_1(\mu)$ containing $-\omega$. Hence, $C \cap L_1(\mu)$ is a proper subcone of $L_1(\mu)$, and $-\omega \notin C \cap L_1(\mu)$. The separation theorem provides a non-zero continuous linear functional p on $L_1(\mu)$ (i.e., an element of $L_\infty(\mu)$) such that $p \cdot \zeta \geq 0$ for each $\zeta \in C \cap L_1(\mu)$ and $p \cdot (-\mu) < 0$ (so $p \cdot \mu > 0$). We assert that some representative of the equivalence class p is an equilibrium price supporting the allocation f. To this end, choose any bounded Borel function $\bar{p}$ representing p. It will suffice (by Lemma 2) to show that, for almost every $t \in T$ there does not exist a measure $\alpha \in L_1^+(\mu)$ such that $f(t) \prec_t \alpha$ and $p \cdot \alpha < p \cdot e(t)$. If this were not so, then separability of $L_1(\mu)$ and continuity of preferences would enable us to find a set T' of positive measure and a measure $\beta \in M^+(X)$ such that $f(t) \prec_t \beta$ and $p \cdot \beta < p \cdot e(t)$ for each $t \in T'$. If we let $g^* : T' \to M^+(X)$ be the function which is identically equal to β, then the pair (T', g^*) belongs to $\mathcal{G}$, so that $\lambda(T')\beta \in C$; since C is a cone, $\beta \in C$ also. Hence $p \cdot \beta \geq 0$, which contradicts the fact that $p \cdot \beta < p \cdot e(t)$ for each $t \in T'$. We conclude that for almost every $t \in T$, there does not exist a measure $\alpha \in L_1^+(\mu)$ such that $f(t) \prec_t \alpha$ and $p \cdot \alpha < p \cdot e(t)$. Lemma 2now implies that there is a bounded Borel function p^* that agrees with $\bar{p}$ almost everywhere and supports f as an equilibrium price. Thus, every initial allocation passes the core equivalence test.

To see that a generic set of initial allocations pass the uniform withholding test, consider the correspondence $P : \mathcal{A} \twoheadrightarrow C(X)$. We have already seen that P has compact values, and a similar argument shows that it is upper hemi-continuous. Lemma 9 implies that the set of points of continuity of P is a residual set. This is not quite enough: If e is an initial allocation with $\operatorname{supp} \omega \neq X$, then $P(e)$ is not the full set of (normalized) equilibrium prices (because there may be many choices for reservation prices of commodities in $X - \operatorname{supp}\omega$, and continuity of P at e will not imply that e passes the withholding test. However, if $\operatorname{supp}\omega = X$, then $P(e)$ is the full set of (normalized) equilibrium prices, and continuity of P at e will imply that e passes the withholding test. In other words, if e is a point of continuity of P and $\operatorname{supp}\omega = X$ then e passes the withholding test. To show that the set of all such initial allocations is a residual set, it suffices (because the intersection of two residual sets is a residual set) to show that the subset $\mathcal{A}_0$ of $\mathcal{A}$ consisting of initial allocations e with $\operatorname{supp}\omega = X$ is a residual subset of $\mathcal{A}$.

To this end, choose a countable dense subset X_0 of X; for each point $x \in X_0$ and each positive integer r, let $B(x,r)$ be the open ball in X of center x and radius r. Since X_0 is a dense subset of X, every open subset of X contains a set in this family. Hence, if $\omega \in M^+(X)$, then $\operatorname{supp}(\omega) = X$ if and only if $\omega(B(x,r)) > 0$ for every $B(x,r)$. Thus, $\mathcal{A}_0$ is the intersection of the sets $\mathcal{A}_{x,r} = \{e : \omega(B(x,r)) > 0\}$, and these sets are open and dense in $\mathcal{A}_0$. Since the family $\{B(x,r)\}$ is countable, we conclude that $\mathcal{A}_0$ is the intersection of a countable family of dense open sets, and is thus residual. This completes the proof of Theorem 3. ∎

Proof of Theorem 4: Fix a thick markets allocation e and a core allocation f. As in the proof of Theorem 3, we construct an equilibrium price as a supporting functional for the cone generated by an appropriate net trade set. In this case, we want to take $\mathcal{G}$ to be the set of all pairs (T', g), where $T' \subset T$ is a subset of positive measure and $g: T' \to L_1^+(\mu)$ is a Bochner integrable function (i.e., a measurable function such that $\int \|g(t)\| d\lambda(t) < \infty$) such that $f(t) \prec_t g(t)$ for every $t' \in T'$. Set

$$\mathcal{P} = \left\{ \int_{T'} g(t) d\lambda(t) - \int_{T'} e(t) d\lambda(t) : (T', g) \in \mathcal{G} \right\}$$

and let C be the norm closed cone generated by $\mathcal{P}$. Note that C is contained in $L_1(\mu)$ since $L_1(\mu)$ is a norm closed subspace of $M(X)$. Since e is a thick markets allocation, it is in particular Bochner integrable, so as in Theorem 3, we may appeal to the usual arguments, together with the fact that the norm closure of the range of a vector measure defined by a Bochner integrable function is convex, to conclude that C is convex. (See Gretsky and Ostroy [1985] or Khan [1986].)

We claim that C is a proper subcone of $L_1(\mu)$. To see this, set

$$Q = \{\Psi \in L_1(\mu) : \|\Psi^+\| < M^{-1} \|\Psi^-\|\}.$$

Since Q is an open cone, to show that C is a proper cone, it suffices to show that $\mathcal{P} \cap Q = \emptyset$.

If this were not so we could find $(T', g) \in \mathcal{G}$ with

$$\int_{T'} g(t) d\lambda(t) - \int_{T'} e(t) d\lambda(t) = \Psi^+ \mu - \Psi^- \mu,$$

and $\|\Psi^+\| < M^{-1} \|\Psi^-\|$. Since Q is open and g is Bochner integrable, there is no loss of generality in assuming that g is a simple function. Since Ψ^+, Ψ^- are disjoint, we conclude that

$$\Psi^+ \mu \leq \int_{T'} g(t) d\lambda(t).$$

The Riesz Decomposition Property (see Schaeffer [1974]) allows us to find a simple function s such that $0 \leq s(t) \leq g(t)$ for each t and

$$\int_{T'} s(t) d\lambda(t) = \Psi^+ \mu.$$

Set $h(t) = g(t) - s(t) + M\|s(t)\|(\Psi^- \mu)$. Direct calculation, making use of the additivity of the norm on the positive cone of $L_1(\mu)$, shows that

$$\int_{T'} h(t) d\lambda(t) - \int_{T'} e(t) d\lambda(t) \leq 0.$$

On the other hand, our assumption (B) implies that $h(t) \succ_t g(t)$, and hence $h(t) \succ_t f(t)$ for each $t \in T'$. This contradicts the fact that f is in the core. We conclude that $\mathcal{P} \cap Q = \emptyset$ and hence that C is a proper convex cone, as desired.

We can now find a norm continuous linear functional p on $L_1(\mu)$ (i.e., an element of $L_\infty(\mu)$) which supports the cone C. As in the proof of Theorem 3, we can then apply Lemma 2 to obtain an equilibrium price for the core allocation f, as desired.

To obtain the withholding test, we will wish again to apply Lemma 9, but for a different space of allocations. We consider the space $\mathcal{T}$ of thick markets allocations; for each $e \in \mathcal{T}$, we let $P(e)$ be the set of normalized equilibrium prices, so that (by Theorem 1), $P(e)$ is a non-empty norm compact subset of $L_\infty(\mu)$. We claim that the correspondence $P : \mathcal{T} \twoheadrightarrow L_\infty(\mu)$ is upper hemi-continuous. To see this, let $\{e_k\}$ be a sequence of thick markets allocations which converge to e (in the metric of $\mathcal{T}$), and let ω_k, ω be the corresponding societal endowments, with supply functions $S_n k, S$. For each k, let (f_k, p_k) be a Walrasian equilibrium for the initial allocation e_k, with $\|p_k\| = 1$. We want to show that some subsequence of $\{(f_n, p_k)\}$ converges to an equilibrium for the initial allocation e.

The definition of convergence in $\mathcal{T}$ implies that $S_n \rightarrow S$ in the $L_\infty(\mu)$ norm. Since S is bounded above and bounded away from zero, we may find constants $c_1, c_2 > 0$ such that $0 < c_1 \leq S \leq c_2 < \infty$. Let $\{\Pi_n\}$ be a sequence of partitions constructed as before so that:

(a) Π_n is a partition of X into a finite number of measurable sets of positive measure;

(b) Π_{n+1} is a refinement of Π_n (i.e., every set in Π_{n+1} is contained in some set in Π_n);

(c) every set in Π_n has diameter less than 2^{-n};

(d) for each set $E \in \Pi_n$, $\text{essvar}(S, E) < 2^{-n}$.

Since $S_k \rightarrow S$ uniformly, it follows that, for each n there is a k^* such that $\text{essvar}(S_k, E) < 2^{-n}$ for each $k \geq k^*$ and each $E \in \Pi_n$. Arguing as in Theorem 1, we conclude that for each $\epsilon > 0$, there are indices n^*, k^* such that $\text{essvar}(S_k, E) < 2^{-n}$ whenever $k \geq k^*, n \geq n^*$ and $E \in \Pi_n$. By Lemma 6, some subsequence of $\{p_k\}$ converges in the $L_\infty(\mu)$ norm to a price p with $\|p\|_\infty = 1$, and some subsequence of $\{f_k\}$ converges weakly to an allocation f. As in the proof of Theorem 1, we conclude that (f, p) is an equilibrium. In particular, P is an upper hemi-continuous corresponcence. By Lemma 9, the set of points of continuity of P is a residual set; it is easily seen that every initial allocation that is a point of continuity of P passes the withholding test. This completes the proof. ∎

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