SIMPLE MACROECONOMIC MODELS WITH

VERY COMPLICATED DYNAMICS

by

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and

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1. Introduction

For many years economists have been interpreting the business cycle as a non-equilibrium phenomenon. In fact, they have regarded it as the prime example of inconsistencies between the prediction of Walrasian models and the reality of capitalist market economies. More recently, various students of the business cycle have shown that simple, aggregate, competitive models may be able to explain some relevant features of observed time-series, provided one introduces stochastic forces that displace the economy from its stable stationary position. Lucas [1987] contains an excellent overview of this line of literature.

Here we adopt the modern framework of intertemporal competitive equilibrium, but at the same time dispense with the _need_ for stochastic forces in order to produce and sustain oscillations in aggregate variables over time. We obtain endogenous oscillations despite the very restrictive framework we adopt. Our economy consists of identical consumers that maximize the discounted value of consumption over an infinite time horizon. Production is carried out by two neoclassical industries, one producing consumption goods, and one producing capital goods. Markets for factors and commodities clear at all points in time, and equilibrium prices are perfectly known at the beginning of time. With these assumptions, our model has a unique competitive equilibrium. The local indeterminacy of equilibria that infects many overlapping generations models used for macroeconomic purposes is thus avoided.

The competitive equilibrium path for our abstract economy is summarized by a dynamical system $k_{t+1} = \tau_\delta(k_t)$ that describes the evolution of the aggregate capital stock over time. We show that for certain parameter values the dynamical system τ_δ exhibits chaotic orbits. Such orbits may

be regarded as formal equivalents (in deterministic models) of business cycle behavior.

That oscillating paths may appear in standard models of competitive equilibrium over time has already been demonstrated by Benhabib and Nishimura [1985], for the case of period-2 cycles, and by Deneckere and Pelikan [1986], Boldrin and Montrucchio [1986], and Boldrin [1988] for the chaotic case. What distinguishes the current paper from the research in Deneckere and Pelikan [1986], and Boldrin and Montrucchio [1986], is that we do not construct "artificial" economies that exhibit a pre-chosen dynamics in equilibrium. Rather, we start with a specification of technology and preferences, and derive the implied dynamics. This was also the object in Boldrin's [1988] study. The two-sector technology is there analyzed in its relationship to cyclic and chaotic paths. While the author, building on previous works of Benhabib-Nishimura [1985], Boldrin-Montrucchio [1986] and Deneckere-Pelikan [1986], provide criteria for assertimg the existence of chaotic paths in an abstract, two-sector model, he does not provide a complete analysis and proof of the existence of such paths for the parameterized example he proposes. We take up from where that paper left off: by using a functional form for the two production technologies that is a special parameterization of the one Boldrin uses, we carry out a complete parametric analysis of the resulting dynamics. This makes the model's predictions potentially testable. We take a preliminary step in this direction in Section 5.

Out model is a special parameterization of that of Boldrin [1988] because, while we retain a Leontief production function for the capital good sector we impose a Cobb-Douglas function for the consumption sector instead of a more general CES. This choice does not seem to alter the final results

in any relevant way but simplifies considerably the computations. In particular, all the results about cycles of a finite periodicity can be obtained for both models with very similar techniques (see Boldrin [1988] for this). We have also verified that the presence of chaos carries over to the CES case. Indeed, chaotic orbits are obtained in that case for values of the discount factor closer to one than those we present here, if one is willing to assume that the elasticity of substitution between labor and capital in the consumption good sector is much smaller than one. We do not present these numerical analyses here as they do not seem to add anything qualitatively relevant to the analysis. We do want to stress, though, that when the named elasticity of substitution is very low, then chaos originates for values of the discount factor that are between 3 and 4 times larger than those we present here. Finally, we would like to point out that Scheinkman [1984] was the first to conjecture that a model exactly like the one we use here could produce optimal chaotic paths.

What economic forces are necessary in order to produce chaotic dynamics? As the analysis below will reveal, both a high degree of impatience (i.e., small discount factors), and a huge difference in the productivity of the two factors (labor and capital) are required. More precisely, we find chaotic solutions when capital is very productive in both sectors, but labor is not. This productivity difference accounts for the huge variations of the capital stock over the course of a "typical" cycle. It does not, however, explain why the economy finds it profitable to oscillate, rather than to proceed along a smooth averaged path, exploiting the concavity in production. In order to understand the economic forces that drive oscillations, we need to analyze the relative profitability of the two sectors at each point in time. This profitability, in turn, depends

on the relative capital-labor intensities of the consumption and investment sector. Since capital-labor intensity reversals play a critical role in our analysis, we will describe the process in some detail.

In our model, the technology of the investment sector is of the constant coefficients type. The other one is Cobb-Douglas. The efficient capital/labor ratio in the investment sector is thus fixed at some level $0 < \gamma < 1$, whereas it is free to move between zero and infinity for the consumption sector. Labor is supplied exogenously and normalized to one at all times. The aggregate capital-labor ratio is thus equal to the aggregate capital stock. Static efficiency and full employment of inputs require that during periods in which the aggregate capital stock is less than γ, the consumption industry must have a capital-labor ratio less than the aggregate one, and, therefore, smaller than γ. Thus, when k_t, the aggregate stock of capital, is between zero and γ, the investment sector is more capital intensive than the consumption good industry. In fact, similar reasoning shows that the reverse must be true when k_t exceeds γ.

Let us now compare two different periods during which the aggregate stocks k and k' are less than γ, with $k' > k$. If product prices are held constant, a straight application of the Rybczynski theorem shows that we should observe a higher investment output in the period associated with k' than in the period associated with k. Leaving product prices free to move will reduce this effect, but not eliminate it. If k and k' are the stocks of the two adjacent periods, this process will yield a path of increasing capital stocks. This explains the rising portion of the cycle.

After a finite number of periods, the aggregate capital stock will exceed γ. At that point the same efficiency and market clearing conditions will result in the consumption sector being more capital intensive. The

substitution effect along the aggregate production possibility frontier then makes it profitable to increase the consumption output and decrease the investment output. This "recession" phase will continue until the aggregate stock falls below γ. At that stage, the cycle is complete.

Oscillations of the type described above may be exactly periodic, of some finite period n, or totally aperiodic (chaotic), depending on the parameter values. We show that any of these cases is, in fact, possible.[1] We should observe that because of the strict concavity of the production possibility frontier the wandering of k_t will imply huge variations in relative prices and rates of return. The high level of discounting is needed in order to eliminate the arbitrage possibilities that would otherwise emerge. This, in turn, suggests that one may avoid "unrealistic" levels of discounting if appropriate portions of increasing returns are introduced in the aggregate technology. This point is elaborated upon in Deneckere and Pelikan [1984], and is also discussed further in the conclusion.

The reader should not be tempted to believe that factor intensity reversals are typical only of the specific technologies we have chosen. On the contrary, the same phenomenon will be present with any pair of distinct constant elasticity of substitution production functions, as long as they are not both Cobb-Douglas, linear, or input-output. Our choice has been motivated solely by concerns of tractability. One last observation: we have stressed capital-intensity reversals as the driving force behind cyclical movements. We made this choice because it makes the underlying economic forces very transparent. But, as the discussion in Section 2 suggests, this is not a strictly necessary condition. The case in which the

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[1]Scheinkman [1984] was the first to conjecture that the parametric example we study here could produce chaotic optimal paths.

consumption sector is always more capital intensive and the capital stock does not depreciate instantaneously seems alto to be able to produce chaotic dynamics (see also Section 5 and Boldrin [1988]).

The rest of the paper proceeds as follows: In Section 2 we set up the general model, and recall some useful results spread about in the literature. Section 3 introduces the specific parameterization, and studies the dynamical system $k_{t+1} = \tau_\delta(k_t)$. In Section 4 we introduce a related map h_δ, which allows us to prove some global asymptotic stability results. We also derive analytical expressions for τ_δ in certain regions of the parameter space. In Section 5 we analyze a version of the model without instantaneous depreciation. Section 6 reports on some numerical simulations, focusing on the covariance structure of equilibrium paths. Section 7 concludes the paper.

2. A Competitive-Two Sector Economy

Below, we study the dynamic behavior of a simple aggregate model of competitive equilibrium over time. We depart from the traditional Cass-Koopmans one-sector model of growth only in assuming a nonlinear transformation frontier between consumption and investment. This is equivalent to the Uzawa-Jones framework, where consumption and capital are different commodities produced in different sectors. The formal model will be briefly outlined in this section; a more detailed account may be found in Boldrin [1988].

Let k_t denote the stock of capital at date t (t = 0,1,2,...): k_t also represents the aggregate capital-labor ratio. The aggregate amount of labor supplied is thus exogenous, and normalized to one for all t. Capital evolves according to the equation $k_{t+1} = \mu k_t + y_t$, where y_t is the output from the investment sector, and $(1-\mu)$ is the capital depreciation rate

$(0 \leq 1 - \mu \leq 1)$. Consumption is produced according to the process $c_t = F^1(\ell_t^1, k_t^1)$. Similarly, for investment, $y_t = F^2(\ell_t^2, k_t^2)$. Finally, there is a single (representative) agent which owns the entire capital stock, supplies labor and capital to both industries via competitive markets, and maximizes the sum of discounted consumption over his (infinite) lifetime. We make the following assumption on the production functions:

H1: For each i (i = 1,2), $F^i: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is continuous, concave, homogeneous of degree one, and increasing in both arguments.

Standard results on the equivalence of competitive equilibria and Pareto optima establish that the output of the consumption sector may be expressed as a function of the existing capital stock and the output of the investment good: $c = T(x,y)$. The production possibility frontier T is the solution to:

$$\text{Max } F^1(\ell^1, k^1)$$

(T) $\quad$ s.t. $\quad F^2(\ell^2, k^2) \geq y$

$$\ell^1 + \ell^2 \leq 1$$

$$k^1 + k^2 \leq x$$

$$\ell^1, k^1, \ell^2, k^2 \geq 0.$$

Assumption H1 implies that T is concave, increasing in x, and decreasing in y. It will also be of class C^2 on its domain when each F^i belongs to that class and the solution to (T) is interior. We also assume that as an effect of decreasing returns:

H2: There exists a $\bar{k} > 0$ such that $F^2(1,k) < (1-\mu)k$ for all $k > \bar{k}$, and $F^2(1,k) > (1-\mu)k$ for all $k < \bar{k}$. In addition, $F^2(\ell^2,0) = 0$ for all $\ell^2 \in \mathbb{R}_+$.

Let $f(k)$ denote the solution to $T(k,y) = 0$. Then H2 implies that for $k > \bar{k}$, $f(k) < (1-\mu)k$, and for $k < \bar{k}$, $f(k) > (1-\mu)k$. Thus, no capital stock exceeding $\bar{k}$ can be sustained, and, without loss of generality, we may restrict our analysis to the feasible set D:

$$D = \{(k_t, k_{t+1}) \in \mathbb{R}_+^2, \quad \text{s.t.} \quad 0 \le k_t \le \bar{k} \quad \text{and} \quad \mu k_t \le k_{t+1} \le f(k_t) + \mu k_t\}.$$

D is convex, and has nonempty interior as long as f is not identically zero on $[0,\bar{k}]$. The competitive equilibrium sequences $\{c_t, y_t, q_t, r_t, w_t, \ell_t^1, \ell_t^2, k_t^1, k_t^2\}_{t=0}^{\infty}$, where q_t is the price of capital, w_t the wage rate, and r_t the rental rate (with consumption taken as a numeraire), may then be derived from the sequence of optimal capital stocks $\{k_t\}_{t=0}^{\infty}$ that solve:

(P)
$$W_\delta(k_0) = \text{Max} \sum_{t=0}^{\infty} V(k_t, k_{t+1})\delta^t$$

$$\text{s.t.} \quad (k_t, k_{t+1}) \in D$$

$$V(k_t, k_{t+1}) = T(k_t, k_{t+1} - \mu k_t)$$

$$k_0 \quad \text{given in} \quad [0,\bar{k}],$$

using the following relations (which hold either by definition or as a condition for equilibrium:

(1) $$c_t = V(k_t, k_{t+1})$$

(2) $$y_t = k_{t+1} - \mu k_t$$

(3) $$q_t = \delta W_\delta'(k_{t+1}) = -V_2(k_t, k_{t+1})$$

(4) $$r_t = V_1(k_t, k_{t+1})$$

(5) $$w_t = V(k_t, k_{t+1}) + q_t(k_{t+1} - \mu k_t) - r_t k_t$$

(6) $$\ell_t^i = \ell^i(k_t, k_{t+1} - \mu k_t), \quad i = 1,2$$

(7) $$k_t^i = k^i(k_t, k_{t+1} - \mu k_t), \quad i = 1.$$

The functions $k^i(\cdot)$ and $\ell^i(\cdot)$ above are solutions to (T), and W_δ: $[0,\bar{k}] \to \mathbb{R}$ is the value function associated with (P). As $W'_\delta(\cdot)$ denotes the derivative of this function, in (3) we are implicitly assuming that T is at least C^1, and applying the result of Benveniste and Scheinkman [1979]. In fact we will sometimes make the stronger assumption:

H3: $V: D \to \mathbb{R}_+$ is of class C^2 on int(D).

Let $\tau_\delta(\cdot): [0,\bar{k}] \to [0,\bar{k}]$ be the policy function associated with $W_\delta(\cdot)$, i.e.,

(8) $$\tau_\delta(x) = \text{Arg max } \{V(x,y) + \delta W(y); \quad (x,y) \in D\}$$

The optimal sequence $\{k_t\}$ is generated by the discrete dynamical system $k_{t+1} = \tau_\delta(k_t)$, starting at k_0. While $\tau_\delta(\cdot)$ and $W_\delta(\cdot)$ become analytically intractable as soon as V acquires (even mild) nonlinearities, a qualitative analysis can nevertheless be performed. To this end, we first collect some theoretical results.

Fact 1: Let $(x,y) \in \text{Int}(D)$ be a point on the policy function, i.e., $y = \tau_\delta(x)$. Then if $V_{12}(x,y) > 0$ (<0), the policy function is locally increasing (decreasing) at (x,y). Furthermore if $(y,\tau_\delta(y)) \in \text{Int}(D)$, then $V_{12}(x,y) > 0$ ($<0,=0$) implies the policy function is strictly increasing (strictly increasing, constant) at (x,y).

Proof: See Benhabib and Nishimura [1985].

Fact 2 (Turnpike): If $V_{12}(x,y) \geq 0$ for all $(x,y) \in \text{Int}D$, then any optimal sequence k_t converges to some $k*$ as $t \to \infty$, where $k* \in \text{Fix}(\tau_\delta)$. Furthermore, for every given strictly concave V (and associated feasible set D), there exists a $\bar{\delta} < 1$ such that for all $\delta \in [\bar{\delta},1)$ the dynamical

system $k_{t+1} = \tau_\delta(k_t)$ is globally asymptotically stable, i.e., there exists a unique k* such that $k_t(k_0) \to k*$ as $t \to \infty$, for every $k_0 \neq 0$.

Proof: See McKenzie [1986], Scheinkman [1976], Dechert and Nishimura [1983], and Deneckere and Pelikan [1986].

Fact 3: Assume $V_{12}(x,y) < 0$ for $(x,y) \in \tilde{D} \subset int(D)$. Let $(x*(\delta),x*(\delta)) \in \tilde{D}$ satisfy $\tau_\delta(x*(\delta)) = x*(\delta)$ for $\delta \in [\delta^-,\delta^+] \subset (0,1)$. Assume there exists $\delta^0 \in (\delta^-,\delta^+)$ such that:

(i) $V_{22}(x*(\delta^0),x*(\delta^0)) + \delta^0 V_{11}(x*(\delta^0),x*(\delta^0))$

$\qquad - (1+\delta^0)V_{12}(x*(\delta^0),x*(\delta^0)) = 0$

(ii) $V_{22}(x*(\delta),x*(\delta)) + \delta V_{11}(x*(\delta),x*(\delta))$

$\qquad - (1+\delta)V_{12}(x*(\delta),x*(\delta)) > 0$, for $\delta \in [\delta^-,\delta^0)$

(iii) $V_{22}(x*(\delta),x*(\delta)) + \delta V_{11}(x*(\delta),x*(\delta))$

$\qquad - (1+\delta)V_{12}(x*(\delta),x*(\delta)) < 0$, for $\delta \in (\delta^0,\delta^+]$.

Then there exists a period-2 orbit for τ_δ for all δ in some (right or left) neighborhood of δ^0.

Proof: See Benhabib and Nishimura [1985].

Fact 4: Under hypothesis H1 and H3, we have:

(i) $T_{12}(x,y) > 0$ for all (x,y) such that

$\qquad (k^1/\ell^1)(x,y) < (k^2/\ell^2)(x,y)$;

(ii) $T_{12}(x,y) < 0$ for all (x,y) such that

$\qquad (k^1/\ell^1)(x,y) > (k^2/\ell^2)(x,y)$.

Furthermore, if $(\bar{x},\bar{y}) \in D$ is a feasible pair such that $T_{12}(\bar{x},\bar{y}) = 0$ then

$T_{12}(\bar{x},y) = 0$ for all y feasible from $\bar{x}$.

Proof: See Benhabib and Nishimura [1985] and Boldrin [1988].

Fact 5: If τ_δ is monotonic (either increasing or decreasing) over all of $[0,\bar{k}]$ then the most complicated orbit is a cycle of period 2.

Proof: See Deneckere and Pelikan [1986].

As pointed out in the Introduction, the objective of this paper is to study conditions under which our simple two-sector competitive economy displays persistent and endogenous oscillations. We will appeal to the mathematical notion of "chaos" to describe such phenomena.

Definition 1: We say that τ_δ: $[0,\bar{k}] \rightarrow [0,\bar{k}]$ has topological chaos when there exists an orbit of period-3 for τ_δ, i.e., $\exists x \in (0,\bar{k}]$: $\tau_\delta^3(x) = x$, and $x \notin \text{Fix}(\tau_\delta)$. By Sarkovskii's theorem (see, for example Guckenheimer and Holmes [1983, p. 311]), this implies that τ_δ has orbits of period n for any natural number n. It also implies (see Li and Yorke [1975]), that there exists a nondenumerable set $S \subset [0,\bar{k}]$ such that all orbits of τ_δ with initial conditions $k_0 \in S$ exhibit aperiodic behavior. More formally this means there exists $\epsilon > 0$ such that for every pair of points x and y in S with $x \neq y$:

$$\limsup_{n \to \infty} |\tau_\delta^n(x) - \tau_\delta^n(y)| \geq \epsilon$$

$$\liminf_{n \to \infty} |\tau_\delta^n(x) - \tau_\delta^n(y)| = 0.$$

and for every $y \in \text{Per}(\tau_\delta)$ and $x \in S$:

$$\limsup_{n \to \infty} |\tau_\delta^n(x) - \tau_\delta^n(y)| \geq \epsilon$$

As is well-known from the dynamical systems literature, "topological chaos"

is a weak form of chaos because it might happen to be essentially unobservable, i.e., the (Lebesgue) measure of S may be zero. A stronger notion of chaos, more difficult to verify, is that of "ergodic chaos".

<u>Definition 2</u>: Let $f: X \to X$ determine the dynamical system $x_{t+1} = f(x_t)$. A measure ν on X is invariant with respect of f if $\nu(A) = \nu(f^{-1}(A))$ for every (Borel) measurable subset A of X. We say that the map f displays "<u>ergodic chaos</u>" if it has a unique invariant probability measures which is absolutely continuous with respect to the Lebesgue measure and also ergodic.

For further details on these delicate matters the reader should consult Collet and Eckmann [1980], Devaney [1986], or Guckenheimer and Holmes [1983].

The last formal result we need gives a set of computable sufficient conditions for the existence of topological chaos in a two-sector economy.

<u>Fact 6</u>: Assume there exists a $k* \in (0,\bar{k})$ such that $V_{12}(k*,\cdot) = 0$. Then τ_δ has topological chaos for all $\delta \in (0,1)$ that satisfy the following conditions:

(i) $V_2(x,k*) + \delta V_1(k*,\cdot) = 0$, has two roots, $k_1 \in (0,k*)$ and $k_2 \in (k*,\bar{k}]$;

(ii) $V_2(x,k_1) + \delta V_1(k_1,k*) = 0$, has a root $k_3 \in [k*,\bar{k}]$;

(iii) $V_2(x,k_3) + \delta V_1(k_3,k_1) = 0$, has at least one real root.

<u>Proof</u>: See Boldrin [1988].

Fact 2 is the classical turnpike theorem: we will use it just to note that there exists an upper bound on the set of δ that may produce oscillating behavior. Fact 3 shows that when τ_δ is downward sloping around an

optimal stead state (OSS), then a cycle of period-2 may bifurcate from the OSS when it loses stability. Since the information necessary to verify the hypothesis of this result is local in nature, we can use it to detect orbits of period-2. This will turn out to be the first (or last) step of a bifurcation cascade leading to chaos in our specific model. A simple generalization of Fact 3 will also allow us to detect the existence of orbits of period-4 and, potentially, of any orbit with period-2^n. Fact 5 shows that a nonmonotonic r_δ is necessary, although not sufficient, in order to obtain complicated dynamics. Facts 1 and 4 link the slope of r_δ to factor-intensity conditions. For the case in which $\mu = 0$ (i.e., the capital stock lasts only one period) the causal relation is clear: r_δ is increasing when the investment sector has a higher capital/labor ratio than the consumption sector, decreasing in the opposite case, and flat at the reversal points. When $\mu \neq 0$ we can easily see that:

$$(9) \qquad V_{12}(k_t, k_{t+1}) = T_{12}(k_t, k_{t+1} - \mu k_t) - \mu T_{22}(k_t, k_{t+1} - \mu k_t)$$

The slope of r_δ depends, therefore, also on μ and the sensitivity of the price of capital to variations in the output of the investment sector. The critical point of r_δ, when it exists, will not necessarily coincide with a factor-intensity reversal, and will not be independent of δ, as in the case when $\mu = 0$. On the other hand, note that r_δ may now be nonmonotonic even in the absence of a capital-intensity reversal: this is true if T_{12} is negative everywhere (i.e., the consumption sector is always more capital intensive) and if both μ and T_{22} are "large enough" for small values of k_t.

Finally, Fact 6 tells us how to check for chaos when the critical point k* is independent of δ (and thus of r_δ(k*)). Obviously, this is the case when $\mu = 0$. In the parameterized model we study below, the result is

also applicable to the case $\mu > 0$, but that is a consequence of the functional forms chosen.

3. **The Model**

In this section (and in Section 4), we analyze the case $\mu = 0$, essentially because it is computationally more tractable. Analogous results for the case $\mu > 0$ are contained in Section 5. Most of the calculations are also relegated to the Appendix.

3.1 Technology and Preferences

Assume that F^1 is of the Cobb-Douglas variety and that F^2 is Leontief:

$$(10) \qquad c_t = F^1(\ell_t^1, k_t^1) = (\ell_t^1)^\alpha \, (k_t^1)^{1-\alpha} \qquad \alpha \in (0,1),$$

$$(11) \qquad y_t = F^2(\ell_t^2, k_t^2) = \min\{\ell_t^2, k_t^2/\gamma\}, \qquad \gamma \in (0,1).$$

Since $k_{t+1} = y_t$ and $\ell_t^1 + \ell_t^2 = 1$ for all t, the feasible set reduces to:

$$(12) \qquad D = \{(x,y) \in [0,1] \times [0,1], \quad \text{s.t.} \quad 0 \leq y \leq \min\{1, x/\gamma\}\}$$

The PPF is easily derived as:

$$(13) \qquad T(k_t, k_{t+1}) = (1-k_{t+1})^\alpha \, (k_t - \gamma k_{t+1})^{1-\alpha}$$

Straightforward economic intuition suggests that $k^* = \gamma \in (0,1)$ is the unique value of k where a capital/labor intensity reversal takes place: as the (efficient) k^2/ℓ^2 ratio is fixed at γ, an economy-wide capital/ labor ratio less than γ will make the consumption sector less capital intensive and the opposite will be true for k_t larger than γ. Thus, τ_δ is increasing on $[0,\gamma]$ and decreasing on $[\gamma,1]$. One can check this formally using equation (19) below. Also, from (11), note that $\tau_\delta(0) = 0$

for all δ, and that any choice $k_{t+1} \in [0,1]$ is feasible when $k_t \geq \gamma$.

We take utility to be linear in consumption. Since labor is supplied inelastically, we have $V(k_t, k_{t+1}) = T(k_t, k_{t+1})$.

3.2 The Dynamic Equilibria

As shown in Section 2, the optimal sequence $\{k_t\}$ must solve:

(14)
$$\text{Max} \sum_{t=0}^{\infty} \delta^t (1-k_{t+1})^{\alpha} (k_t - \gamma k_{t+1})^{1-\alpha}$$

$$\text{s.t.} \quad 0 \leq k_{t+1} \leq \min(1, k_t/\gamma),$$

$$k_0 \text{ given in } [0,1].$$

By computing the first partial derivatives of V:

(15)
$$V_1(x,y) = (1-\alpha) \, [(1-y)/(x-\gamma y)]^{\alpha} > 0$$

(16)
$$V_2(x,y) = -\left(\frac{1-y}{x-\gamma y}\right)^{\alpha} \left[\alpha\left(\frac{x-\gamma y}{1-y}\right) + \gamma(1-\alpha)\right] < 0$$

we see that:

$$\lim_{y \uparrow \gamma^{-1}x} V_1(x,y) = +\infty \quad \text{for} \quad x < \gamma,$$

$$\lim_{y \uparrow 1} V_1(x,y) = 0 \quad \text{for} \quad x > \gamma,$$

$$V_1(\gamma,y) = (1-\alpha)\gamma^{-\alpha}.$$

Also,

$$\lim_{y \uparrow \gamma^{-1}x} V_2(x,y) = -\infty \quad \text{for} \quad x < \gamma,$$

$$\lim_{y \uparrow 1} V_2(x,y) = -\infty \quad \text{for} \quad x > \gamma,$$

$$V_2(\gamma,y) = -\gamma^{1-\alpha}.$$

This implies that $(x, r_\delta(x)) \in D$ for all $x \in (0,1]$, except possibly at $x = \gamma$. Concerning the concavity of V, we have:

$$(17) \qquad V_{11}(x,y) = -\alpha(1-\alpha) \left(\frac{1-y}{x-\gamma y}\right)^\alpha (x-\gamma y)^{-1} \le 0, \quad \text{for} \quad (x,y) \in D$$

$$(18) \qquad V_{22}(x,y) = -\alpha(1-\alpha) \left(\frac{1-y}{x-\gamma y}\right)^\alpha \left[\frac{(x-\gamma)^2}{(1-y)^2(x-\gamma y)}\right] \le 0, \quad \text{for} \quad (x,y) \in D,$$

$$(19) \qquad V_{12}(x,y) = \alpha(1-\alpha) \left(\frac{1-y}{x-\gamma y}\right)^\alpha \left[\frac{\gamma - x}{(x-\gamma y)(1-y)}\right] \gtrless 0, \quad \text{as} \quad x \lessgtr \gamma.$$

From (17)-(19) we see that V loses strict concavity in x along the upper boundary of D, and strict concavity in y along the vertical line $x = \gamma$. Also, $V_{11}V_{22} - V_{12}^2 = 0$ for all $(x,y) \in D$.

Clearly, there is no hope of finding analytical expressions for either W_δ or r_δ. Nevertheless we can use the Euler equation to extract some information on the local behavior of r_δ for those sets of parameters at which its graph is interior to D. The Euler equation is:

$$(20) \qquad \delta(1-\alpha) \left[\frac{1-k_{t+1}}{k_t - \gamma k_{t+1}}\right]^\alpha = \left[\frac{1-k_t}{k_{t-1} - \gamma k_t}\right]^\alpha \left[\gamma(1-\alpha) + \alpha\left(\frac{k_{t-1} - \gamma k_t}{1-k_t}\right)\right]$$

From (20) we may, first of all, conclude that there exists at most one OSS different from zero, namely:

$$(21) \qquad k^* = \frac{(\delta-\gamma)(1-\alpha)}{(\delta-\gamma)(1-\alpha) + \alpha(1-\gamma)}$$

The position of k^* relative to γ is of some importance, as r_δ is increasing on $[0,\gamma)$ and decreasing on $(\gamma,1]$. Observe that

$$(22) \qquad k^* \lessgtr \gamma \quad \text{if and only if} \quad \delta \lessgtr \gamma/(1-\alpha).$$

The next proposition uses this information to state a global asymptotic stability result for $\delta \le \gamma/(1-\alpha)$.

<u>Proposition 1</u>: For $0 \le \delta \le \gamma$ the optimal path k_t converges to zero for any initial condition k_0 in $[0,1]$, and no interior OSS exists. For $\gamma < \delta \le \gamma/(1-\alpha)$, there exists a unique interior OSS $k*$ defined by (21) and the optimal path converges to $k*$, for any k_0 in $(0,1]$.

<u>Proof</u>: The first part is obvious. Since 0 is the only fixed point for τ_δ, we have $k_{t+1} = \tau_\delta(k_t) < k_t$, and thus $k_t \to 0$ as $t \to \infty$. If, on the other hand, $\delta \in (\gamma, \gamma/(1-\alpha))$, then $k*$ lies in $(0,\gamma]$ and the origin is unstable. Thus, for $x \in (0,k*)$, we have $\tau_\delta(x) > x$, and the trajectory from x converges to $k*$. Conversely, if $x \in (k*,1]$, $\tau_\delta(x) < x$, and thus the entire interval $[k*,1]$ is attracted to $k*$. ∎

Observe that Proposition 1 implies global asymptotic stability if $\alpha \ge 1-\gamma$ independently of δ! From (21), we may also derive some comparative statics results:

<u>Proposition 2</u>: The OSS level of the capital stock defined by (21):

a) increases with the discount factor δ;

b) decreases with the labor productivity factor α;

c) decreases with γ, the capital/labor ratio in the investment sector.

Let us now turn to the case where $\delta > \gamma/(1-\alpha)$. It is well known from the literature on optimal growth theory (see, for example, McKenzie [1986] and Scheinkman [1976]), that when $k*$ is locally stable for τ_δ, i.e., $|\partial \tau_\delta(k)/\partial k|_{k=k*} < 1$, the second order system produced by the Euler equation (20) has a local saddle point structure at $k*$. This means that of the two eigenvalues of the characteristic polynomial:

$$(23) \qquad \delta V_{12}(k*,k*)\lambda^2 + [V_{22}(k*,k*) + \delta V_{11}(k*,k*)]\lambda + V_{12}(k*,k*) = 0$$

associated with the linearization of (20), one lies inside, and one lies

outside the unit circle. In fact, the smaller eigenvalue corresponds to $\tau_\delta'(k*)$ (see Deneckere and Pelikan [1984]). For our example, (23) reduces to:

$$(24) \qquad \delta[(\gamma-k*)/(1-k*)]\lambda^2 - [\delta+(\gamma-k*)^2/(1-k*)^2]\lambda + (\gamma-k*)/(1-k*) = 0$$

from which we may compute:

$$(25) \qquad \begin{aligned} &\lambda_1 = (1-k*)/(\gamma-k*) < 0 \\ &\lambda_2 = [(\gamma-k*)/(1-k*)]\delta^{-1} < 0 \end{aligned}$$

The signs of the expressions in (25) follows from the fact that $k* > \gamma$ whenever $\delta > \gamma/(1-\alpha)$. We can easily reduce (25) to:

$$(26) \qquad \lambda_1 = \alpha/[\gamma-(1-\alpha)\delta] < 0, \quad \lambda_2 = [\gamma-(1-\alpha)\delta]/\alpha\delta < 0$$

Thus:

$$(27) \qquad \begin{aligned} &\lambda_1 \in (-\infty,-1) \quad \text{for} \quad \delta \in (\gamma/(1-\alpha), (\alpha+\gamma)/(1-\alpha)), \quad \text{and} \\ &\lambda_1 \in (-1,0) \quad \text{for} \quad \delta > (\alpha+\gamma)/(1-\alpha) \end{aligned}$$

$$(28) \qquad \begin{aligned} &\lambda_2 \in (-1,0) \quad \text{for} \quad \delta \in (\gamma/(1-\alpha), \gamma(1-2\alpha)), \quad \text{and} \\ &\lambda_2 \in (-\infty,-1) \quad \text{for} \quad \delta > \gamma/(1-2\alpha), \quad \text{if} \quad \alpha < 1/2, \quad \text{and} \\ &\lambda_2 \in (-1,0) \quad \text{for} \quad \delta \in (\gamma/(1-\alpha),1) \quad \text{if} \quad \alpha \geq 1/2. \end{aligned}$$

Hence, we have proven:

<u>Proposition 3</u>: The OSS $k*$ is locally asymptotically stable when $\alpha \geq 1/2$, and when $\alpha < 1/2$ it is stable for parameter values δ in $(\gamma/(1-\alpha), \gamma/(1-2\alpha)) \cup ((\alpha+\gamma)/(1-\alpha),1)$.

An immediate corollary to Proposition 3 is:

<u>Corollary</u>: The OSS $k*$ is locally asumptotically stable for all $\delta \in (\gamma/(1-\alpha),1)$ when $\alpha \geq (1-\gamma)/2$.

<u>Proof</u>: If $\alpha \geq 1/2$, $\lambda_2 \in (-1,0)$, and if $(1-\gamma)/2 \leq 1/2$, $\lambda_2 \in (-1,0)$ as well, since then $\gamma/(1-2\alpha) \geq 1$. ∎

One should note that Proposition 3 states a local result only. Intuition suggests that $k*$ may in fact be globally asymptotically stable, but a proof of this requires additional analysis (see Section 4).[2]

A natural question now arises: what happens when $\delta \in [\gamma/(1-2\alpha), (\alpha+\gamma)(1-\alpha)]$? A partial answer is the following proposition:

<u>Proposition 4</u>: Let $\alpha < (1-\gamma)/2$. Then the policy function τ_δ has a cycle of period-2 in a neighborhood of $\delta^- = \gamma(1-2\alpha)$ and $\delta^+ = (\alpha+\gamma)/(1-\alpha)$. These cycles are locally stable when they exist for $\delta \in (\delta^-,\delta^+)$, and unstable in the other cases.

<u>Proof</u>: To get existence one needs only to apply Fact 3 to our model. For our model the sign of $B(\delta) = V_{22}(\delta) + \delta V_{11}(\delta) - (1+\delta)V_{12}(\delta)$, where $V_{ij}(\delta) = V_{ij}(k(\delta),k(\delta))$ evaluated at $k*$ is opposite to that of:

$$[(k*-\gamma)^2 + \delta(1-k*)^2 + (1+\delta)(\gamma-k*)(1-k*)],$$

Therefore we have:

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[2] Our discussion above also implies the following stability intervals with respect to the other parameters of the model: $k* \in (\gamma,1)$ is stable for $\alpha \in (0,(\delta-\gamma)/(1+\delta)) \cup ((\delta-\gamma)/2\delta, (\delta-\gamma/\delta)$, and for $\gamma \in (0,(1-\alpha)\delta-\alpha) \cup (\delta(1-2\alpha), \delta(1-\alpha))$.

$$
B(\delta) \ : \ \begin{cases} >0 & \text{for} \quad \delta < \delta^- \\ =0 & \text{for} \quad \delta = \delta^- \\ <0 & \text{for} \quad \delta^- < \delta < \delta^+ \\ =0 & \text{for} \quad \delta = \delta^+ = (\alpha+\gamma)/(1-\alpha) \\ >0 & \text{for} \quad \delta > \delta^+ \end{cases}
$$

Therefore cycles of period-2 exist both around δ^- and δ^+. Stability follows from standard results in dynamical systems theory, a simple proof of which can be found in Benhabib and Nishimura (1985, Corollary 1). ∎

The reader should note that, given $\delta < 1$, it is always possible to find α and γ in $(0,(1-\gamma)/2)$ and $(0,1)$ respectively, such that $\gamma/(1-2\alpha) = \delta$. This means that, at every level of discounting, we can always find some technology that has optimal two cycles. In fact, the dynamic behavior of this economy for $\delta \in (\delta^-, \delta^+)$ may become very complicated. Our contention is that, for suitable a, ρ and γ, there exists an interval $(\delta*, \delta**) \subset (\delta^-, \delta^+)$ at which τ_δ has (at least) topological chaos. We demonstrate this at the end of this section. We also believe that the emergence of chaos follows the classical "period-doubling bifurcation pattern" as $\delta \to \delta*$ from the left or $\delta \to \delta**$ from the right. Without complete knowledge of τ_δ this claim cannot be proven. Some supporting evidence from the simulations we have run is available from the authors. We content ourselves here to show that a second "flip bifurcation" (see Devaney [1986] for a technical treatment) may lead to an orbit of period 4. We use the same logic behind Fact 3 and Proposition 3.

Proposition 5: Let $x(\delta), y(\delta)$ denote a period-2 interior orbit of τ_δ, for δ values in (δ^-, δ^+). Assume there exists an interval $[\delta^{--}, \delta^{++}] \subset$

(δ^-,δ^+) and a $\delta^0 \in (\delta^{--},\delta^{++})$ such that the function

$$G(\delta) = V^*_{22}V_{22} + \delta^2 V^*_{11}V_{11} + \delta(V^*_{11}V^*_{22}-V^{*2}_{12}) + \delta(V_{11}V_{22}-V^2_{12}) + (1+\delta^2)V^*_{12}V_{12}$$

where $V^*_{ij} = V_{ij}(x(\delta),y(\delta))$, $V_{ij} = V_{ij}(y(\delta),x(\delta))$, $i,j = 1,2$ satisfies:

$$G(\delta) \begin{cases} > 0 & \text{for } \delta \in [\delta^{--},\delta^0) \\ = 0 & \text{for } \delta = \delta^0 \\ < 0 & \text{for } \delta \in (\delta^0,\delta^{++}] \end{cases}$$

Then there exists a period-4 orbit for τ_δ bifurcating from $(x(\delta),y(\delta))$ at $\delta = \delta^0$.

Proof: See the Appendix.

We may now prove the following corollary:

Corollary: Let $(x(\delta),y(\delta))$ be a period-2 point for our model, and suppose it exists for all δ in (δ^-,δ^+) with $x(\delta) < \gamma$ and $y(\delta) > \gamma$. Then a cycle of period-4 exists for all values of $\delta^0 \in (\delta^-,\delta^+)$ at which either one of the following two equations hold:

(i) $(x(\delta^0) - \gamma)/(1-x(\delta^0)) = -(\delta^0)^2(1-y(\delta^0))/(y(\delta^0)-\gamma)$

(ii) $(x(\delta^0)-\gamma)/(1-x(\delta^0)) = -(1-y(\delta^0))/(y(\delta^0)-\gamma)$.

Proof: See the Appendix.

To verify the presence of such bifurcations in our model, consider the example $\alpha = .03$, $\gamma = .09$. Proposition 3 implies that the steady state k^* is locally stable when δ lies in the interval $[.0928,.0957]$. For discount factors in $[.0957,.0974]$ stable period-2 orbits are present, verifying Proposition 4. At $\delta = .0974$, the period-2 orbit $x^* = .0738$, $y^* = .3980$ bifurcates into a stable period-4 orbit, which exists for $\delta \in [.0974,.0978]$.

In fact, our simulation results reveal that successive bifurcations eventually lead to chaos when δ reaches the value .099. This chaos exists for $\delta \in [.099,.112]$, as can be checked directly by applying Fact 6 of Section 2 to our model. Figure 1 describes the evolution of the policy function τ_δ for $\alpha = .03$ and $\gamma = .09$, as δ moves in $(0,1)$, and Figure 2 depicts the set of (α,γ) parameters for which topological chaos is present for some $\delta \in (0,1)$. As expected, the concavity of V implies that only extreme values of the parameters can yield chaotic dynamics.

One might suspect that part of the reason that such extreme values of the parameters are needed stems from the fact that the elasticity of substitution between capital and labor in the consumption good sector is fairly large. To investigate this issue, we also ran simulations for a tractable generalization of our model, suggested by Boldrin [1988]. This generalization retains the Leontief technology for the investment sector, but allows for a CES in the consumption sector. Thus, $F^1(\ell^1,k_1) = [\alpha(\ell^1)^\rho + (1-\alpha)(k^1)^\rho]^{1/\rho}$, which approaches (10) as $\rho \to 0$. The elasticity of substitution, σ, for the CES is equal to $1/(1-\rho)$; negative values of ρ thus permit much smaller values of σ. Our simulations reveal that chaotic optimal paths do arise for this model as well, and that when σ is fairly low, chaos appears for values of the discount factor roughly three times larger than those found for the Cobb-Douglas model. A typical example has $\alpha = .2$, $\gamma = .2$, $\delta = .25$, and $\rho = -.5$. Since the values for the discount factor at which chaos appears in the Cobb-Douglas model are themselves approximately 100 times larger than the ones found in the artificial economies constructed by Boldrin and Montrucchio [1986] and Deneckere and Pelikan [1986], no definite conclusion can be drawn, at this stage, as to whether a model of this type could produce chaotic dynamics at more reasonable values

of the discount factor.

4. The First-Order Difference Equation h_δ

4.1 The Relationship Between r_δ and h_δ

Recall from (20) above that the Euler equation for our model is given by:

$$(20) \qquad \delta(1-\alpha) \left[\frac{1-k_{t+1}}{k_t - \gamma k_{t+1}}\right]^\alpha = \left[\frac{1-k_t}{k_{t-1} - \gamma k_t}\right]^\alpha \left[\gamma(1-\alpha) + \alpha \frac{k_{t-1} - \delta k_t}{1-k_t}\right]$$

Consider the ratio $(1-k_{t+1})/(k_t - \gamma k_{t+1}) = z_t$ for $t = 0,1,2,\ldots$ From equations (6), (7), (10) and (11), we see that $z_t = \ell_t^1/k_t^1$, i.e., z_t is the time t labor/capital ratio in the consumption sector. The advantage of working with the variable z_t is that this reduces (20) from a second-order difference in equation k_t to a first-order difference equation in z_t:

$$(29) \qquad z_t = z_{t-1}\left[\frac{\gamma}{\delta} + \frac{\alpha}{\delta(1-\alpha)} z_{t-1}^{-1}\right]^{1/\alpha}$$

More compactly, we may write $z_t = h_\delta(z_{t-1})$.

The purpose of this section is to investigate if any useful information on the dynamics of the unknown policy function r_δ can be derived from the study of h_δ. Before elaborating on the relationship between r_δ and h_δ, we must first establish some properties of the map h_δ.

To simplify notation, let $a = \gamma/\delta$ and $b = \alpha/((1-\alpha)\delta)$, so that $h(z) = [a+bz^{-1}]^{1/\alpha}$. Since we have already characterized r_δ for δ values less than γ, we will confine our attention to the case $a < 1$. It is apparent that h is decreasing on $(0,1/\gamma)$ and increasing on $(1/\gamma,\infty)$, with $\lim_{z\to\infty} h(z) = \infty$. In fact, if we let $\mu = (\gamma/\delta)^{1/\alpha} < 1$, we see that $\lim_{z\to\infty} [h(z)-\mu z] = 0$, so that h asymptotes to μz as z approaches infinity. Furthermore, h has a unique fixed point $z* = h(z*)$, which satisfies

$z* = \alpha[(\delta-\gamma)(1-\gamma)]^{-1}$. A typical graph of h (with $\delta \geq \gamma/(1-\alpha)$) is depicted in Figure 3.

Proposition 6: The map $h_\delta: \mathbb{R}_+ \to \mathbb{R}_+$ is of at least class C^3 on any compact set of $\mathbb{R}_+$. It has negative Schartzian derivative for $0 < \alpha < \frac{1}{2}$.

Proof: See Appendix.   ∎

For $\delta \leq \gamma/(1-\alpha)$, we have $z* \geq 1/\gamma$, and thus $h^n(z) \to z*$ for all z $\in (0,\infty)$. This confirms Proposition 1 of Section 3. Henceforth, we will confine our attention to values of δ exceeding $\gamma/(1-\alpha)$, so that $z* < 1/\gamma$ and $h(1/\gamma) < 1/\gamma$. Consider now the interval $I = [h(1/\gamma), h^2(1/\gamma)]$. It is easy to see that I is an invariant set for h. Moreover, the basin of attraction of I under h is $(0,\infty)$, i.e., for every $z^0 \in (0,\infty)$ there exists N sufficiently large such that $h^n(z^0) \in I$, for all $n \geq N$. Thus, we may restrict our study of h to the interval I, i.e., consider the dynamical system $h_\delta: I \to I$, for $\delta \in (\gamma/(1-\alpha),1)$.

Let us now turn to the relationship between r_δ and h_δ. For any initial point k_0 in $[0,1]$, pick an arbitrary k_1 that is feasible from k_0. The pair (k_0,k_1) corresponds to a choice of $z_0 \in (0,\infty)$. With these initial conditions, we may run (20) forward to produce a candidate optimal path $\{k_t\}$, or equivalently, run (29) forward to produce a candidate optimal path $\{z_t\}$. Because z_t will be in I for sufficiently large t, the associated sequence $\{k_t\}$ will be uniformly bounded and therefore satisfy the transversality condition $\lim_{t\to\infty} \delta^t q_t k_t = 0$. Hence we cannot rule out nonoptimal $\{k_t\}$ (or $\{z_t\}$) by appealing to the transversality condition. Since interior optimal paths must satisfy the Euler equation (20) (or (29)), and since optimal paths are unique, we conclude that all but

one choice of k_1 from k_0 will induce a sequence $\{z_t\}$ that corresponds to a sequence $\{k_t\}$ which is infeasible, i.e., that does not satisfy $0 \leq k_{t+1} \leq \min\{1, k_t/\gamma\}$. We will not give a complete characterization of the relationship between the policy function and the map h_δ here. Rather, we will content ourselves to observing that when the graph of r_δ is interior, Per $r_\delta \subset$ Per h_δ. Because period-2 points play a central role in describing the dynamics of one-dimensional systems, we will study those in detail below.

<u>Proposition 7</u>: Let (w,z) be a period-2 orbit for h, i.e., $h(w) = z$ and $h(z) = w$. Then the pair (x,y) defined by:

$$x = \frac{1-\gamma w-z}{(1-\gamma z)(1-\gamma w)-wz}, \quad y = \frac{1-\gamma z-w}{(1-\gamma z)(1-\gamma w)-wz}$$

is an optimal cycle for r_δ if and only if the pair (w,z) satisfies one of the following restrictions:

(i) $0 < w < 1/(1+\gamma)$,   and $0 < z < 1/(1+\gamma)$

(ii) $1/(1+\gamma) < w \leq \min\{1/\gamma, h^2(1/\gamma)\}$,   and $1/(1+\gamma) < z \leq h^2(1/\gamma)$.

<u>Proof</u>: If (x,y) is a period-2 point for r_δ, the labor/capital ratios w and z must satisfy: $w = (1-y)/(x-\gamma y)$, and $z = (1-x)/(y-\gamma x)$. Inverting these relationships yields the stated expressions for x and y. Some simple calculations now show that feasibility, i.e., $0 \leq x \leq \min(y/\gamma, 1)$ and $0 \leq y \leq \min(x/\gamma, 1)$ implies either $0 < w,z < 1/(1+\gamma)$ or $z > 1/(1+\gamma)$ and $1/(1+\gamma) < w < 1/\gamma$. Finally, the restrictions $w < \min(1/\gamma, h^2(1/\gamma))$ and $z \leq h^2(1/\gamma)$ in (ii) follow from the fact that (w,z) and (z,w) must lie in $I \times I$.   ∎

Graphically, the two situations are illustrated in Figure 4.

<u>Corollary</u>: Case (i) of Proposition 7 is possible only if $\delta > (\alpha+\gamma)/(1-\alpha)$, in case (ii) is possible only if $\delta \in (\gamma/(1-\alpha), (\alpha+\gamma)/(1-\alpha))$.

<u>Proof</u>: For case (i) to apply, we need $z^* = \alpha/(\delta-\gamma)(1-\alpha) < 1/(1+\gamma)$, and vice versa for case (ii). But this inequality is easily rewritten as $\delta > (\alpha+\gamma)/(1-\alpha)$. Above, we showed that $\delta \leq \gamma/(1-\alpha)$ implies global stability of z^*. This yields the lower endpoint of the interval in case (ii). ∎

4.2 <u>Simple Dynamics for r_δ</u>

In Section 3 we proved that the steady state k^* is globally asymptotically stable when $\delta \leq \gamma/(1-\alpha)$. We also saw that k^* was locally asymptotically stable when $\delta \in [\gamma/(1-\alpha),1)$ and $\alpha \geq (1-\gamma)/2$, and when $\delta \in [\gamma/(1-\alpha), \gamma/(1-2\alpha)) \cup ((\alpha+\gamma)/(1-\alpha),1)$. We will now strengthen some of these results.

<u>Proposition 8</u>: If $\delta \geq \gamma/(1-\alpha)$ and:

$$(1-\alpha) + \alpha[\delta(1-\alpha)/\gamma]^{1/\alpha} < [\delta(1-\alpha)/\gamma]^2$$

then h_δ has simple dynamics.

<u>Proof</u>: Under the conditions of the proposition, $0 < h(1/\gamma) < h^2(1/\gamma) < 1/\gamma$. Thus, the map h_δ confined to the trapping region $I \times I$ is strictly monotone (decreasing), and hence can only display simple dynamics. ∎

<u>Corollary 1</u>: Under the conditions of Proposition 8, the policy function $r_\delta(\cdot)$ has simple dynamics.

<u>Proof</u>: Assume first that graph $r_\delta \subset \text{Int}(D)$, i.e., that $r_\delta(\gamma) < 1$. In that case, the Euler equation (20) must hold along all optimal trajectories. Observe now that the trapping region for r_δ corresponds to the trapping region for h_δ. Observe also that the trapping region for h_δ, $I \times I$, is

entirely located on the downward sloping branch of that function. Now any value of k such that $z(k) = (1-r_\delta(k))/(k-\gamma r_\delta(k)) \in I \times I$ must satisfy k $> \gamma$. We conclude that the trapping region for r_δ is entirely located on the downward sloping branch of that function, and hence that r_δ displays only simple dynamics.

Next, suppose that $r_\delta(\gamma) = 1$; the Euler equation then holds from every value of the initial capital stock, except possibly at $k = \gamma$. Some reflection now shows that the fact that the Euler condition may hold as an inequality at $k = \gamma$ implies that the dynamics of z_t is now given by a "chopped off" version of h_δ, i.e., that $z_{t+1} = \max\{\omega, k_\delta(z_t)\}$ for some choice of ω. The trapping region for this modified function will be a strict (nonempty) subset of the trapping region for h_δ, and hence the same reasoning as above can be applied.   ∎

Corollary 2: The conditions of Proposition 8 are satisfied whenever $\alpha \geq \frac{1}{2}$.

Proof: Let $z = \delta(1-\alpha)/\gamma$. By assumption, $z > 1$.

We may rewrite the condition of Proposition 8 as:

$$\phi(z) = (1-\alpha) + \alpha z^{1/\alpha} - z^2 < 0$$

Observe that $\phi(1) = 0$, and $\phi'(z) = z^{(1/\alpha)-1} - 2z \leq z - 2z < 0$.   ∎

One puzzling feature of the map $h_\delta(\cdot)$ should be noticed: we may easily calculate the slope of h_δ at z^*:

$$(30) \qquad h_\delta'(z^*) = [\gamma-\delta(1-\alpha)]/\alpha\delta$$

The slope of h_δ thus coincides with λ_2, one of the eigenvalues of the Euler equation linearized at the steady state k^*. The second eigenvalue λ_1, defined in (26), apparently gets lost in the transformation (k_t, k_{t+1})

→ z_t. This implies that the (locally) stabilizing effect of λ_1 on the dynamics of k_t when $\delta > (\alpha+\gamma)/(1-\alpha)$ and $\alpha < (1-\gamma)/2$ is not transmitted to the orbits of z_t. In particular, for such parameter values, $r_\delta(\cdot)$ is locally stable around k*, whereas $h_\delta(\cdot)$ is locally unstable around z*. We solve this apparent puzzle in Proposition 9.

<u>Proposition 9</u>: For $\delta \in ((\alpha+\gamma)/(1-\alpha),1)$ the policy function satisfies:

(31) $r_\delta(k_t) = (1-\lambda_1)k* + \lambda_1 k_t$, for all $k_t \in [\gamma,1]$.

Furthermore, the steady state k* is globally asymptotically stable.

<u>Proof</u>: From Proposition 3 in Section 3, we already know that for $\delta \in ((\alpha+\gamma)/(1-\alpha),1)$, r_δ maps the interval $[\gamma,1]$ into itself. From (27) we know that λ_1 is stable in the assumed range for δ. Global stability then follows immediately from the functional form (31).

Using (29) and (27), we may rewrite (31) as

$$k_{t+1} = \frac{(\delta-\gamma)(1-\alpha)}{\delta(1-\alpha)-\gamma} - \frac{\alpha}{\delta(1-\alpha)-\gamma} k_t$$

To show that this is the policy function, we only need to check that the pairs (k_t, k_{t+1}) so defined satisfy the Euler equation for all t. But observe that

$$\frac{1-k_{t+1}}{k_t - \gamma k_{t+1}} = z_t = \frac{\alpha(k_t-\gamma)}{(\delta-\gamma)(1-\alpha)(k_t-\gamma)} = z*$$.

Since any sequence r_δ^n stays in $[\gamma,1]$ for all $k_0 \in [\gamma,1]$ this proves the desired result.   ∎

Proposition 9 allows one to derive an explicit expression for r_δ, not just on $[\gamma,1]$, but actually on all of $[0,1]$. Indeed, let $y \in [\gamma,1]$. Since r_δ is monotone on $[\gamma,1]$, with $r_\delta(\gamma) = 1$, there exists a preimage

of y in $[0,\gamma]$. Let p be the largest preimage of γ, i.e., $p = \sup\{x: \tau_\delta(x) = \gamma\}$. Then for $x \in (p,\gamma)$ the Euler equation holds:

$$(32) \qquad V_2(x,y) + \delta V_1(y,\tau_\delta(y)) = 0$$

This operation will yield an explicit functional form for τ_δ on $[p,\gamma]$, where $p = \tau_\delta^{-1}(\gamma)$ is the left preimage of γ. In fact, using (15) and the fact that $(1 - \tau_\delta(y))/(y-\gamma\tau_\delta(y)) = z*$ (proven in Proposition 9), we see that (32) may be rewritten as:

$$(33) \qquad V_2(x,y) + \delta(1-\alpha)(z*)^\alpha = 0$$

Thus we may use the implicit function theorem to derive an expression for $\tau_\delta'(x)$:

$$\tau_\delta'(x) = -V_{12}(x,y)/V_{22}(x,y) = (1-y)/(\gamma-x)$$

The differential equation $\tau_\delta'(x) = dy/dx = (1-y)/(\gamma-x)$ can be solved by separation of variables, to yield: $y = 1 - k(\gamma-x)$. Thus, the policy function is __linear__ on (p,γ), with a slope k that can be determined from (33). Now let $q = \inf\{x: \tau_\delta(x) = \gamma\}$. The monotonicity of $\tau_\delta(\cdot)$ implies that $[q,p]$ is an interval.[3] For $x \in (\tau_\delta^{-1}(p),q)$, the Euler equation holds again, and so:

$$(34) \qquad V_2(x,y) + \delta(1-\alpha)z^\alpha = 0$$

where $z = (1-\tau_\delta(y))/(y-\tau_\delta(y))$ is constant since $\tau_\delta(\cdot)$ is linear on (p,γ). Thus, $\tau_\delta(\cdot)$ is linear on $(\tau_\delta^{-1}(p),q)$ with a slope which can be calculated from (34). In fact, it is easily seen that for all $x < q$, both $(x,\tau_\delta(x))$ and $(\tau_\delta(x),\tau_\delta^2(x))$ are in the interior of D, so that the Euler equation must hold everywhere on $(0,q)$. Repeatedly taking preimages using

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[3]Observe that for $y \in [q,p]$, the Euler equation need not hold, and hence $\tau_\delta(\cdot)$ is only weakly monotone (see Fact 1).

the Euler equation (34) then yields the entire functional form for $r_\delta(\cdot)$.

Observe that when $\delta = (\alpha+\gamma)/(1-\alpha)$, Proposition 9 states that $r_\delta(\cdot)$ has a continuum of period-2 cycles! For

$$\delta \in [\frac{\alpha+\gamma}{1-\alpha} (\frac{\gamma}{1+\gamma})^\alpha, \frac{\alpha+\gamma}{1-\alpha}],$$

simulations indicate that a globally stable period-2 orbit exists which lives on the boundary of the set D, namely $r_\delta(\gamma) = 1$, $r_\delta(1) = \gamma$.

We may also prove that h_δ has period-2 cycles for δ-values in a neighborhood of $\gamma/(1-2\alpha)$:

<u>Proposition 10</u>: h_δ has a stable period-2 cycle when $\delta \in (\gamma/(1-2\alpha), \gamma/(1-2\alpha) + \epsilon)$, for some $\epsilon > 0$.

<u>Proof</u>: At $\delta = \gamma/(1-2\alpha)$, we have $h_\delta'(z*) = -1$, $h_\delta''(z*) \neq 0$, and $\partial[h_\delta^2]'(x*)/\partial\delta|_{\delta=\gamma/(1-2\alpha)} \neq 0$. We may thus apply the flip bifurcation theorem (see, e.g., Devaney [1986, p. 89]) to obtain existence of a cycle of period-2 in an open neighborhood of $\gamma/(1-2\alpha)$. To see that the cycle occurs for δ-values greater than $\gamma/(1-2\alpha)$, we observe that $h_\delta(z*)$ and $(\partial/\partial\delta)h_\delta(z*)$ have opposite signs at $\delta = \gamma/(1-2\alpha)$. Finally, stability of the cycle follows from the fact that $z*$ changes stability at $\delta = \gamma/(1-2\alpha)$.

∎

Obviously, the cycle we recover here for h_δ corresponds to the cycle we obtained for r_δ in Section 3, Proposition 4. The bifurcation theorem referred to in the proof above implies that the two-cycle will be close to $z*$ when δ is close to $\gamma/(1-2\alpha)$. Simulations reveal that as δ increases, the cycle moves away from $z*$, and that eventually one of its points lies on the upward sloping branch of h_δ. A necessary condition for this to occur is that the trapping region $I \times I$ contains part of the

upward sloping branch of h_δ, i.e., $h_\delta^2(1/\gamma) > 1/\gamma$. We saw above (in Proposition 8) that this is equivalent to

$$(1-\alpha) + \alpha[\frac{\delta(1-\alpha)}{\gamma}]^{1/\alpha} > [\frac{\delta(1-\alpha)}{\gamma}]^2.$$

5. The Model Without Instantaneous Depreciation

When μ is greater than zero the one-period return function becomes:

$$(35) \qquad V(x,y) = (1+\mu x-y)^\alpha (bx-\gamma y)^{1-\alpha},$$

where $b = 1 + \gamma\mu$ with a feasible set

$$(36) \qquad D = \{(x,y) \in [0,1/(1-\mu)] \times [0,1/(1-\mu)],$$

$$\text{s.t.,} \quad \mu x \le y \le \mu x + \min\{1,x/\gamma\}\}$$

Again, we may show that V is of class C^2 on $\bar{D} = \text{int}(D)\backslash\{x=y\}$, and strictly concave in each variable, everywhere on the interior of D, except for the point $x = y$ at which $V_{22} = 0$. As before, we also have $V_{11}V_{22} - V_{12}^2 = 0$ for all $(x,y) \in D$.

The first and second partial derivatives of V are in fact:

$$V_1(x,y) = [\frac{1+\mu x-y}{bx-\gamma y}]^\alpha [b(1-\alpha) + \frac{\alpha\mu(bx-\gamma y)}{1+\mu x-y} > 0$$

$$V_2 = -[\frac{1+\mu x-y}{bx-\gamma y}] \gamma(1-\alpha) + \alpha\frac{bx-\gamma y}{1+\mu x-y} < 0$$

$$V_{11}(x,y) = -\alpha(1-\alpha) (\frac{1+\mu x-y}{bx-\gamma y})^\alpha \frac{(b-y)^2}{(bx-\gamma y)(1+\mu x-y)^2} < 0$$

$$V_{22}(x,y) = -\alpha(1-\alpha) (\frac{1+\mu x-y}{bx-\gamma y})^\alpha \frac{(\gamma-x)^2}{(bx-\gamma y)(1+\mu x-y)^2} < 0$$

$$V_{12}(x,y) = \frac{\alpha(1-\alpha)(\gamma-x)(b-y)}{(bx-\gamma y)(1+\mu x-y)^2} (\frac{1+\mu x-y}{bx-\gamma y})^\alpha \gtrless 0$$

The Euler equation along an interior path $\{k_t\}$ now becomes:

$$\delta\left(\frac{1+\mu k_t - k_{t+1}}{bk_t - \gamma k_{t+1}}\right)^\alpha \left[b(1-\alpha) + \alpha\mu \frac{bk_t - \gamma k_{t+1}}{1+\mu k_t - k_{t+1}}\right] =$$

$$= \left(\frac{1+\mu k_{t-1} - k_t}{bk_{t-1} - \mu k_t}\right)^\alpha \left[\gamma(1-\alpha) + \alpha \frac{bk_{t-1} - \gamma k_t}{1-\mu k_{t-1} - k_t}\right]$$

The OSS K* must therefore satisfy:

$$(\frac{1+\mu k^* - k^*}{(b-\gamma)k^*})^\alpha \left[(\delta b - \gamma)(1-\alpha) + \alpha(\delta\mu - 1)\frac{(b-\gamma)k^*}{1+\mu k^* - k^*}\right] = 0$$

which gives:

$$(37) \qquad k^* = \frac{(\delta b - \gamma)(1-\alpha)}{(\delta b - \gamma)(1-\alpha)(1-\mu) + \alpha(1-\delta\mu)(b-\gamma)}$$

Observe that (37) reduces to (21) when $\mu = 0$ (and $b = 1$). Observe also that $\delta b > \gamma$, i.e., $\delta > \gamma/(1+\gamma\mu)$ is now required to guarantee k^* to be bounded away from zero. Hence, an interior steady state will be present for δ values smaller than in the $\mu = 0$ case.

The formula for $V_{12}(x,y)$ given above shows that there are two sets of points at which $V_{12}(x,y) = 0$. The first is, as before, $x = \gamma$ and the second is $y = b - 1 + \gamma\mu$. Since $1 + \gamma\mu < \bar{k} = (1-\mu)^{-1}$ we need to check what happens at $y = 1 + \gamma\mu$. It is not difficult to see that, as long as τ_δ is interior, the value $y = 1 + \gamma\mu$ will never be crossed, so that $(x, \tau_\delta(x))$ in fact will always stay within $[0, 1+\gamma\mu) \times [0, 1+\gamma\mu)$. Thus, the critical point is again at $x = \gamma$, independently of δ. This is clearly coincidental. In general, when μ is positive, the critical point of τ_δ will depend on δ (see Boldrin [1988]). We also compute:

$$(38) \qquad k^* \gtreqless \gamma \quad \text{iff} \quad \delta \gtreqless \frac{\gamma}{1+\mu\gamma - \alpha}$$

Again, we see that the critical value of δ at which k^* moves on the

downward sloping branch of r_δ is smaller than when $\mu = 0$. The comparative statics of k* with respect to α, γ and δ is the same as before, and we also have, consistent with economic intuition, $\partial k*/\partial \mu > 0$.

Consider now equation (24) in Section 3. In this case we have:

$$[(V_{22}+\delta V_{11})^2 - 4\delta V_{12}^2]^{\frac{1}{2}} =$$

$$= [\alpha^2(1-\alpha)^2 (\frac{1+\mu k*-k*}{bk*-\gamma k*})^{2\alpha} \frac{1}{(bk*-\gamma k*)^2(1+\mu k*-k*)^4} \times$$

$$\times (\delta^2(b-k*)^4 + (\gamma-k*)^4 - 2\delta(b-k*)^2(\gamma-k*)^2)]^{\frac{1}{2}}$$

$$= \frac{\alpha(1-\alpha)}{(bk*-\gamma k*)(1+\mu k*-k*)^2} (\frac{1+\mu k*-k*}{bk*-\gamma k*}) (\delta*(b-k*)^2 - (\gamma-k*)^2)$$

Therefore the expression for the two roots reduces to:

$$\lambda_{1,2} = [(\gamma-k*)^2 + \delta(b-k*)^2 \pm (\delta(b-k*)^2 - (\gamma-k*)^2] [2\delta(\gamma-k*)(b-k*)]^{-1}$$

Recalling that $k* = (\delta b-\gamma)(1-\alpha) [(\delta b-\gamma)(1-\alpha)(1-\mu) + \alpha(1-\delta\mu)(b-\gamma)]^{-1}$, we get:

(39a) $$\lambda_1 = \frac{b-k*}{\gamma-k*} - \mu + \frac{\alpha(1-\delta\mu)(b-\gamma)}{\alpha\gamma(1-\delta\mu)(b-\gamma)-(\delta b-\gamma)(1-\alpha)[1-\gamma(1-\mu)]}$$

(39b) $$\lambda_2 = \frac{\gamma-k*}{b-k*} \delta^{-1} - \frac{\alpha\gamma(1-\delta\mu)(b-\gamma)-(\delta b-\gamma)(1-\alpha)[1-\gamma(1-\mu)]}{\alpha(1+\gamma\mu)(1-\delta\mu)(b-\gamma)-(\delta b-\gamma)(1-\alpha)[1-\gamma(1-\mu)]} \delta^{-1}$$

Some straightforward algebra shows that both eigenvalues are negative for $\delta > \gamma/(1-\alpha+\gamma\mu)$, and that their modulus behaves as in the case $\mu = 0$. In other words, there exists a pair $\gamma/(1-\alpha+\gamma\mu) < \delta^- < \delta^+ < 1$ such that:

(40a) $$\lambda_1 \in (-\infty,-1) \quad \text{for} \quad \delta \in (\gamma/(1-\alpha+\mu\gamma),\delta^+)$$

(40b) $$\lambda_2 \in (-\infty,-1) \quad \text{for} \quad \delta > \delta^-$$

and:

(41a) $$\lambda_1 \in (-1,0) \quad \text{for} \quad \delta > \delta^+$$

(41b) $\quad\quad\quad\quad \lambda_2 \in (-1,0) \quad$ for $\quad \delta \in (\gamma/(1-\alpha+\gamma\mu),\delta^-)$.

The expression for δ^- and δ^+ in terms of α, γ, μ are, unfortunately, rather long. and not very informative; therefore, we have dropped them. The analogy between these results and those previously obtained for the simple model $\mu = 0$ should convince the reader that all of the analysis in Section 3 can be reproduced for the case $\mu > 0$ as well. Period-doubling bifurcations will be present for appropriate values of α, γ, δ and μ. Also, because $V_{12}(\gamma,y) = 0$ for all $y \in [0,\bar{k}\!]$, Fact 6 can be applied again.

Important differences with the case $\mu = 0$ nevertheless exist. In fact, simulations reveal that the chaotic solutions disappear rapidly as μ increases from the full depreciation value towards one. For example, with $\alpha = .03$, $\gamma = .02$ and $\delta = .025$ chaos exists for $\mu \in [0,0.09]$, but vanishes when $\mu \geq .1$. It is easy to understand why the chaos disappears in this example when capital does not depreciate too quickly. The high level of capital productivity means that the rising portion of the cycle is confined to a small region of very low levels of the capital stock; unless depreciation ratios are quite large, it will never be optimal to let the capital stock fall to such a low level. Interestingly enough, chaos re-appears as μ moves towards one. In fact, when μ is larger than .9 (i.e., depreciation ratios are on the order of 10 percent or less), we find period 3 orbits for $\alpha = .03$, $\gamma = .02$, and $\delta = .025$. The policy function for this case is illustrated in Figure 5. Observe that the mechanism generating chaos is now completely different from the one operative at $\mu = 0$. Indeed, the trapping region now lies completely to the right of the point of capital intensity reversal γ. This type of chaos, therefore, does not rely on the presence of capital intensity reversals. Rather, it exploits the interaction between the downward sloping portion of the policy function and

the depreciation constraint $k_t = \mu k_{t-1}$.

6. Chaotic Business Cycles?

We have proved in the previous two sections that, while for a wide range of parameter values our model produces dynamic accumulation paths that converge either to a stationary state or to a periodic orbit of finite periodicity, there exists a non-negligible (albeit rather extreme) set of values for α, γ and δ such that the capital stock moves erratically forever. In this section we examine some typical patterns of behavior for $(k_t, c_t, Y_t, r_t, q_t, w_t)$ under this chaotic regime and compare their basic correlation properties with those of the U.S. economy, following in this the methodology introduced by Kydland and Prescott [1982].

We present two sets of simulations, the first relative to the model with full depreciation and the second relative to the more realistic case where μ is in the range (.9,1) which also exhibits chaos as demonstrated in Section 5. Before reporting the results of our simulations we will briefly describe the correlation pattern which is theoretically predicted by our model (a similar analysis can be found in Benhabib-Nishimura [1987], Sec. III). To do this properly one should assume that the policy function r_δ is differentiable in order to take into account the effects that variations in k_t have on the values of $k_{t+1} = r_\delta(k_t)$. This is still a partially open problem even at the theoretical level. Moreover, our own simulations seem to suggest that for the model we are using r_δ may well fail to be of class C^1. We will therefore disregard these second order effects in what follows. In any case it seems to us that the qualitative results would not change in any precise direction once this effect is introduced.

We consider first the case where $\mu = 0$ (full depreciation). Total output in our economy is defined as $Y_t = c_t + q_t k_{t+1} - T(k_t, k_{t+1}) - T_2(k_t, k_{t+1}) k_{t+1}$. Therefore we have:

(42)
$$\frac{\partial Y_t}{\partial k_t} = T_1(k_t, k_{t+1}) - T_{12}(k_t, k_{t+1}) k_{t+1}$$

and

(43)
$$\frac{\partial Y_t}{\partial k_{t+1}} = -T_{22}(k_t, k_{t+1})$$

While the sign of (43) is always non-negative because of concavity (implying therefore that GNP and investments move together) the value of (42) varies along the cycle: output and capital stocks will move together in periods just preceding a recession (i.e., when k_t is such that $T_{12} < 0$) but may move in the opposite direction when k_t is going up. For our specific model the sign of (42) reduces to the sign of $(x - 2\gamma y + \gamma y^2)$ which is easily seen to be nonnegative for all values of x and y in D.

The correlation between c_t and k_t or k_{t+1} is obvious, c_t and k_t move together while consumption and investment move in opposite directions: sectors are out-of-phase in models of this type. For the three prices we have:

(44)
$$\frac{\partial r_t}{\partial k_t} = T_{11}(k_t, k_{t+1}); \quad \frac{\partial r_t}{\partial k_{t+1}} = T_{12}(k_t, k_{t+1})$$

(45)
$$\frac{\partial q_t}{\partial k_t} = -T_{12}(k_t, k_{t+1}); \quad \frac{\partial q_t}{\partial k_{t+1}} = -T_{22}(k_t, k_{t+1})$$

(46)
$$\frac{\partial w_t}{\partial k_t} = -T_{12}(k_t, k_{t+1}) k_{t+1} - T_{11}(k_t, k_{t+1}) k_t$$

$$(47) \qquad \frac{\partial w_t}{\partial k_{t+1}} = -T_{22}(k_t, k_{t+1})k_{t+1} - T_{12}(k_t \cdot l_{t+1})k_t$$

from which we can conclude that: the rate of return decreases with the capital stock while it moves procyclically with respect to the investment activity; the price of the capital stock moves in a direction opposite to k_t when the stock is increasing and together with k_t during the other periods of the cycle while it always goes up when investment goes up. Finally the wage rate moves together with both the capital stock and the investment activity when the consumption sector is more capital intensive but has no definite correlation in the other case.

When less than full depreciation is allowed the following derivatives can be computed (the arguments of the functions are omitted for brevity):

$$(48) \qquad \frac{\partial Y_t}{\partial k_t} = T_1 - [T_{12} - \mu T_{22}](k_{t+1} - \mu k_t) \gtrless 0$$

$$(49) \qquad \frac{\partial Y_t}{\partial k_{t+1}} = -T_{22}(k_{t+1} - \mu k_t) > 0$$

$$(50) \qquad \frac{\partial c_t}{\partial k_t} = T_1 - \mu T_2 > 0; \qquad \frac{\partial c_t}{\partial k_{t+1}} = T_2 < 0$$

$$(51) \qquad \frac{\partial r_t}{\partial k_t} = T_{11} - \mu T_{12} \gtrless 0; \qquad \frac{\partial r_t}{\partial k_{t+1}} = T_{12} \gtrless 0$$

$$(52) \qquad \frac{\partial q_t}{\partial k_t} = -T_{12} + \mu T_{22} \gtrless 0; \qquad \frac{\partial q_t}{\partial k_{t+1}} = -T_{22} > 0$$

$$(53) \qquad \frac{\partial w_t}{\partial k_t} = (k_{t+1} - \mu k_t)(\mu T_{22} - T_{12}) + (\mu T_{12} - T_{11})k_t \gtrless 0$$

(54)
$$\frac{\partial w_t}{\partial k_{t+1}} = -T_{22}(k_{t+1} - \mu k_t) - T_{12}k_t \gtreqless 0$$

From (48)-(54) conclusions similar to the one already reached for the case with full depreciation can be easily derived. We can therefore conclude that the general structure of the model is elastic enough to accommodate the most diverse patterns of correlation among the relevant macrovariables. This is reinforced by the fact that in the analysis above we have considered only partial derivatives, while along an optimal path both k_t and k_{t+1} are changing so that the observed correlations may turn out to have a different sign.

To obtain some numerical evidence we simulated our model economy for parameter values at which chaos is displayed. The practical method is as follows: a value function and a policy function were computed numerically by solving (8) over a grid of 1600 points on the unit interval and then interpolating these pairs $(x, r(x))$ by means of a piecewise linear approximation. The function $r: [0,1] \rightarrow [0,1]$ was then applied iteratively on a generic initial condition, which provided us with a sequence $\{k_t\}_{t=0}^{T}$. The latter was then used to obtain the sequences for the other relevant variables by means of equations (1)-(5), of the definition of GNP given above and of a measure of the real interest rate computed as:

(55)
$$R_t = r_t/q_t.$$

The parameter values we have chosen are such that the simulated policy function displays "ergodic chaos", i.e., for generic initial conditions the associated trajectory $\{k_t\}_{t=0}^{T}$ does not converge to any periodic orbit but tends to fill in the whole feasible set $[0,1]$. We report the results associated with $\alpha = .03$, $\gamma = .09$ and $\delta = .0997$, for the case $\mu = 0$,

and $\alpha = .03$, $\gamma = .02$, and $\delta = .025$ for the case $\mu = .9$ in Table 1, below. Even a casual glance at this table indicates that the Kydland-Prescott model is much better calibrated to the U.S. data than our own. Nevertheless, a correct interpretation of the results should proceed with some caution. First, the correlations reported in the first two columns of the table are not based upon raw data. Indeed, as explained in Hansen [1985], these correlations were computed on the basis of detrended data (using the Hodrick-Prescott filter), measured in logarithms. The correlations for our model are based upon unfiltered data, using levels rather than logs of the variables. It should also be pointed out that the negative correlation between investment and output is largely a consequence of the fact that the aggregate labor supply is fixed in our model.

7. Conclusion

In this paper we analyzed a simple general equilibrium model which produced unique, but sometimes cyclical and even chaotic paths for aggregate variables such as output, consumption, and investment. Despite the fact that an analytical expression for the policy function was often unavailable, we were able to characterize the dynamic behavior of our economy in terms of its basic parameters: α, the labor share of income in the consumption sector; γ, the capital/labor ratio in the investment sector; and δ, the discount factor. For many values of the parameters, the unique steady state was shown to be globally asymptotically stable. For other values of the parameters, we obtained a unique period-2 point, which was globally attractive. Successive bifurcations then led to a chaotic regime, but only for rather unrealistic values of the parameters. The statistical properties of the time series associated with this chaotic regime, while not orthogonal to

their realized counterparts for the postwar U.S. economy, performed much worse than those reported by Kydland and Prescott [1986].

This should not be too surprising. As the analysis above indicated, the analytical complexity of the nonlinear model greatly exceeds that associated with linear stochastic models. The primitive stage of our research technology forced us to work with a rather rudimentary and rigid model. The introduction of an elastic labor supply, a nonlinear utility function, and increasing returns to scale in production are all elements of realism that deserve attention. Nevertheless, our study casts some doubt on the notion that, in one-dimensional capital good models, chaos is a useful way to model the apparently self-sustained nature of the trade cycle. The highly nonlinear bell-shaped form for the policy function that is then necessary in order to produce complex dynamics forces one to resort to rather unrealistic higher dimensional state space is bound to be much more successful in this regard, since even slight departures from linearity may then produce strange attractors.

While still in its infancy, the study of nonlinearities in economic models is likely to provide insights into the forces behind observed economic fluctuations. In our model, we underlined the importance of _inter-sectoral substitution effects_ (induced by different degrees of profitability in different sectors) as well as intertemporal substitution effects in determining factor allocation decisions, investment activities, and so on. Bypassing the nonlinearities with first order approximations would have neglected the important contribution of these factors in amplifying and sustaining oscillations.

TABLE 1

Correlations with Output for the U.S. and Artificial Economies

	U.S. Time Series	Kydland and Prescott Model	Our Model With Complete Depreciation	Our Model With Less Than Full Depreciation
Y_t	1.00	1.00	1.00	1.00
c_t	.85	.94	.99	.99
I_t	.92	.80	-.57	-.79
k_t	.04	-.07	.99	.99
R_t	-	.47	-.98	-.98
r_t			-.84	-.98
q_t			.94	.98
w_t			.99	-.64

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APPENDIX

Proof of Proposition 5

The line of reasoning we will follow here is not new. In fact, it is an adaptation of the main argument in Benhabib and Nishimura [1985]. Consider the Euler equation (20): if $k_t \neq \gamma$ it induces a map $F: \tilde{D} \rightarrow X$, defined as: $k_{t+1} = F(k_t, k_{t-1})$. Following Benhabib and Nishimura [1985] we may derive a dynamical system $H: \tilde{D} \rightarrow D$:

$$\begin{bmatrix} k_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} F(k_t, y_t) \\ k_t \end{bmatrix} = H(k_t, y_t).$$

Note that:

$$DH(k_t, y_t) = \begin{bmatrix} \dfrac{\partial F}{\partial k_t} & \dfrac{\partial F}{\partial y_t} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{V_{22}(k_{t-1}, k_t) + \delta V_{11}(k_t, k_{t+1})}{\delta V_{12}(k_t, k_{t+1})} & -\dfrac{V_{21}(k_{t-1}, k_t)}{\delta V_{12}(k_t, k_{t+1})} \\ 1 & 0 \end{bmatrix}$$

If $x(\delta), y(\delta)$ satisfies: $r_\delta(x(\delta)) = y(\delta)$, $r_\delta(y(\delta)) = x(\delta)$, it must also satisfy (deleting the dependence on δ):

$$V_2(x,y) + \delta V_1(y,x) = 0$$

$$V_2(y,x) + \delta V_1(x,y) = 0.$$

The latter implies: $x = F(y,x)$, $y = F(x,y)$, or $[x,y] = H(y,x)$.

Now set $z_t = (k_t, y_t)$. We have:

$$z_{t+2} = H(z_{t+1}) = H(H(z_t)) = H^2(z_t) = \begin{bmatrix} F[F(k_t, y_t), k_t] \\ F(k_t, y_t) \end{bmatrix}$$

and: $[x,y] = H^2(x,y)$, so that the period-2 is a fixed point for H^2. Let

us compute the eigenvalues of DH^2 at (x,y). $DH^2(z_t) =$

$$DH(H(z_t))DH(z_t) = \begin{bmatrix} \dfrac{\partial F(k_{t+1},y_{t+1})}{\partial k_{t+1}} & \dfrac{\partial F(k_{t+1},y_{t+1})}{\partial y_{t+1}} \\[2ex] 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial F(k_t,y_t)}{\partial k_t} & \dfrac{\partial F(k_t,y_t)}{\partial y_t} \\[2ex] 1 & 0 \end{bmatrix}$$

Now set: $k_t = y_{t+1} = x$ and $k_{t+1} = y_t = y$ and

$$DH^2(x,y) = \begin{bmatrix} a_{11} & a_{12} \\[1ex] a_{21} & a_{22} \end{bmatrix}$$

to yield:

$$a_{11} = F_1(y,x)F_1(x,y) + F_2(y,x)$$

$$a_{12} = F_1(y,x)F_2(x,y)$$

$$a_{21} = F_1(x,y)$$

$$a_{22} = F_2(x,y)$$

The characteristic polynomial corresponding to the linearization of H^2 around the two cycle is

$$\lambda^2 - (a_{11}+a_{22})\lambda - a_{12}a_{21} + a_{11}a_{22} = 0.$$

Recall from above that

$$F_1(x,y) = -\frac{V_{22}(y,x) + \delta V_{11}(x,y)}{\delta V_{12}(x,y)}; \quad F_1(y,x) = -\frac{V_{22}(x,y) + \delta V_{11}(y,x)}{\delta V_{12}(y,x)}$$

$$F_2(x,y) = -\frac{V_{21}(y,x)}{\delta V_{12}(x,y)}; \quad F_2(y,x) = -\frac{V_{21}(x,y)}{\delta V_{12}(y,x)}.$$

Thus $\lambda_1\lambda_2 = \mathrm{Det}(DH^2(x,y)) = \delta^{-2}$ and $\lambda_1 + \lambda_2 = a_{11} + a_{22}$.

From now on we use the notation: $V^*_{ij} = V_{ij}(x,y)$ and $V_{ij} = V_{ij}(y,x)$. We need to compute the sign of

$$\mathrm{tr}(DH^2(x,y)) = a_{11} + a_{22} = \frac{V^*_{22}V_{22} + \delta^2 V^*_{11}V_{11} + \delta V^*_{11}V^*_{22} + \delta V_{11}V_{22} - \delta V^{*2}_{12} - \delta V^2_{12}}{\delta^2 V^*_{12}V_{12}}$$

Some tedious, but very simple, algebra show that the concavity of V implies that the numerator of the above expression is strictly positive. Then, sign $(a_{11}+a_{22}) = \mathrm{sign}(V^*_{12}V_{12}) = \mathrm{sign}(\lambda_1+\lambda_2)$. Because the two roots have the same sign, and we want one of them to cross -1 at $\delta = \delta^0$, it is clear that V^*_{12} and V_{12} must be of different signs. This, in turn, is possible in our model if and only if $x > \gamma$ and $y < \gamma$ or vice versa.

Notice also that $\lambda_{1,2} = (a_{11} + a_{12} \pm \sqrt{D})/2$, where $D = (a_{11}+a_{22}) - 4\delta^{-2}$. D may be shown to be nonnegative again by concavity of V. Both roots are therefore real.

Without loss of generality, let us assume $|\lambda_1| \leq |\lambda_2|$. From $\lambda_1\lambda_2 = \delta^{-2}$ we see that $\lambda_1 \in (-\infty,-1)$ for all δ in the relevant interval. In particular, if λ_2 happens to be equal to -1 for some δ, then $\lambda_1 = -\delta^{-2}$. This is possible only if

$$-(1+\delta^{-2}) = [V^*_{22}V_{22} + \delta^2 V^*_{11}V_{11} + \delta(V^*_{11}V^*_{22} - V^{*2}_{12})$$
$$+ \delta(V_{11}V_{22} - V^2_{12})] [\delta^2 V^*_{12}V_{12}]^{-1}$$

holds for some $\delta \in (\delta^{--},\delta^{++})$. By using the fact that $\lambda_2 = (a_{11} + a_{22} + \sqrt{D})/2$, and the definitions above some additional algebra yields

(i)   if: $(1+\delta^2)V^*_{12}V_{12} + [V^*_{22}V_{22} + \delta^2 V^*_{11}V_{11} + \delta(V^*_{11}V^*_{22} - V^{*2}_{12})$

  $+ \delta(V_{11}V_{22} - V^2_{12})] = 0$, then $\lambda_1 = -\delta^{-2}$ and $\lambda_2 = -1$.

(ii)   if: $(1+\delta^2)V^*_{12}V_{12} + [V^*_{22}V_{22} + \delta^2 V^*_{11}V_{11} + \delta(V^*_{11}V^*_{22} - V^{*2}_{12})$

  $+ \delta(V_{11}V_{22} - V^2_{12})] > 0$, then $\lambda_1 \in (-\infty,-1)$ and $\lambda_2 \in (-1,0)$

(iii) if: $(1+\delta^2)V^*_{12}V_{12} + [V^*_{22}V_{22} + \delta^2 V^*_{11}V_{11} + \delta(V^*_{11}V^*_{22} - V^{*2}_{12})$

$\qquad + \delta(V_{11}V_{22} - V^2_{12})] < 0$

then λ_1 and λ_2 both belong to $(-\infty,-1)$.

From here on, the proof proceeds analogously to Theorem 1 in Benhabib and Nishimura [1985]. Define $\tilde{H}: \tilde{D} \to D$ as $\tilde{H}(z_t) = H^2(z_t)$. Obviously, $\text{Fix}(\tilde{H}) \subset \text{Fix}(\tilde{H}^2)$, but not vice-versa. In fact, a cycle of period-4 of τ_δ is a fixed point of $\tilde{H}^2$ which is not a fixed point of $\tilde{H}$.

As we are only considering interior solutions, we may examine the map $M(z)$ on the interior of D:

$$M(z) = z - \tilde{H}^2(z).$$

A zero for M is a fixed point for $\tilde{H}^2$. Furthermore, if we denote G the Jacobian of $\tilde{H}$ evaluated at a fixed point for $\tilde{H}$, then G^2 is the Jacobian of $\tilde{H}^2$ evaluated at the same point. Thus, if λ_1 and λ_2 are the eigenvalues of G, then the eigenvalues of the Jacobian of M evaluated at $(x(\delta), y(\delta))$, the fixed point of $\tilde{H}$, will be $(1-\lambda_1^2)$ and $(1-\lambda_2^2)$. Consider now the homotopy $M(x(\delta), y(\delta))$ from $[\delta^{--},\delta^{++}]$ on $\text{int}(D)$. The rest of the proof follows points (i) and (ii) of Theorem 1 in Benhabib and Nishimura, verbatim. ∎

Proof of the Corollary to Proposition 5

$G(\delta)$, for our model, may be written as:

$$\alpha^2(1-\alpha)^2 (\frac{1-y}{x-\gamma y})^\alpha (\frac{1-x}{x-\gamma y})^\alpha (x-\delta y)^{-1} (y-\gamma x)^{-1} \cdot [(\frac{x-\gamma}{1-x})^2 (\frac{y-\gamma}{1-y})^2 + \delta^2 +$$

$$+ (1+\delta)^2 (\frac{x-\gamma}{1-x}) (\frac{y-\gamma}{1-y})] = G(\delta)$$

We only have to consider the portion within the last square brackets as the

other part is always positive on int(D). Simple calculations will show that $G(\delta) = 0$ when either condition (i) or (ii) of the Corollary are satisfied. Moreover, as the sign of $G(\delta)$ changes when δ goes through either δ^{--} or δ^{++} the argument of Proposition 4 can be applied. ∎

Proof of Proposition 6: We need to compute the first three derivatives of h:

$$h'(z) = [a+bz^{-1}]^{1/\alpha} \left[\frac{\gamma z - 1}{\gamma z + \alpha/(1-\alpha)} \right]$$

$$h''(z) = \frac{[a+bz^{-1}]^{1/\alpha} (1-\alpha)}{z[\alpha-(1-\alpha)\gamma z]^2} > 0$$

$$h'''(z) = \frac{-(1-\alpha)[a+bz^{-1}]^{1\alpha} [1+\alpha+3\gamma(1-\alpha)z]}{z^2[\alpha+\gamma(1-\alpha)z]^3} < 0$$

Observe that $h'(z) > 0$ for $z > 1/\gamma$ and $h'(z) < 0$ for $z < 1/\gamma$. Each of these derivatives is continuous on $(0,\infty)$, and thus on every compact subset of it as well.

Now recall that the Schwartzian derivative of a function f at a point z is defined as (Devaney, 1986, p. 68):

$$Sf(z) = (f'''(z)/f'(z)) - 3/2(f''(z)/f'(z))^2.$$

Thus:

$$Sh(z) = [(1-\gamma z)z(\alpha+\gamma(1-\alpha)z)]^{-2}$$

$$[(1-\gamma z)(1+\alpha+3\gamma(1-\alpha)z)-3/2]$$

which is negative whenever the following quadratic is negative:

$$L(z) = -3\gamma^2(1-\alpha)z^2 + 2\gamma(1-\alpha)z + \alpha - 1/2.$$

But $L(z)$ attains a negative maximum at $z = (1-2\alpha)/(3\gamma(1-\alpha))$, for $\alpha < \frac{1}{2}$, proving the desired result. ∎
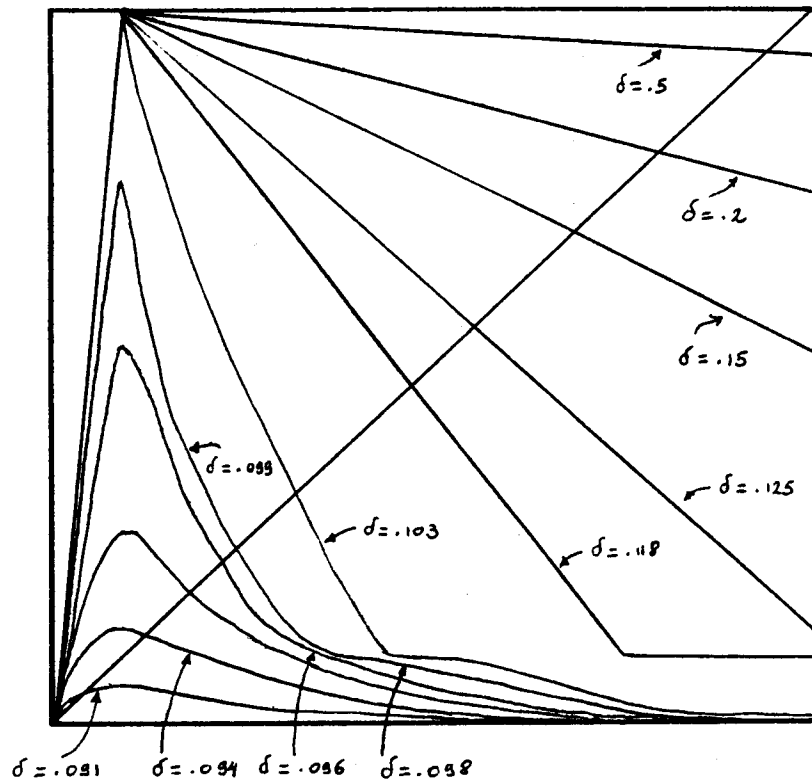
Figure 1: Evolution of the policy function τ_δ as a function of the discount parameter δ (α = .03, γ = .09).
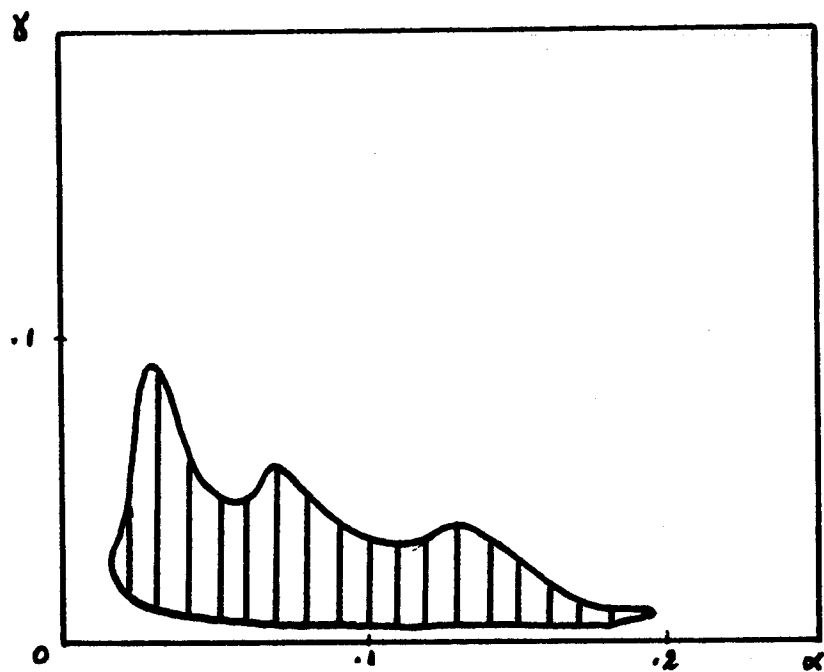
Figure 2: Values of the parameters α and γ for which chaos is present for some δ in $(0,1)$.
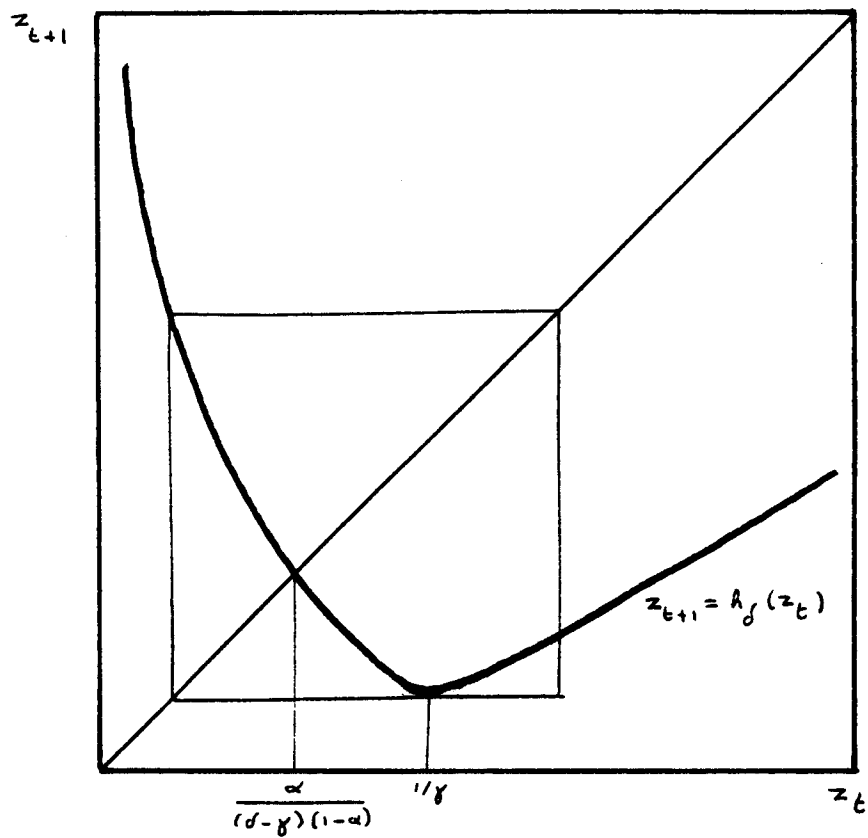
Figure 3: Typical graph of the function h_δ (with $\delta > \gamma(1 - \alpha)$)
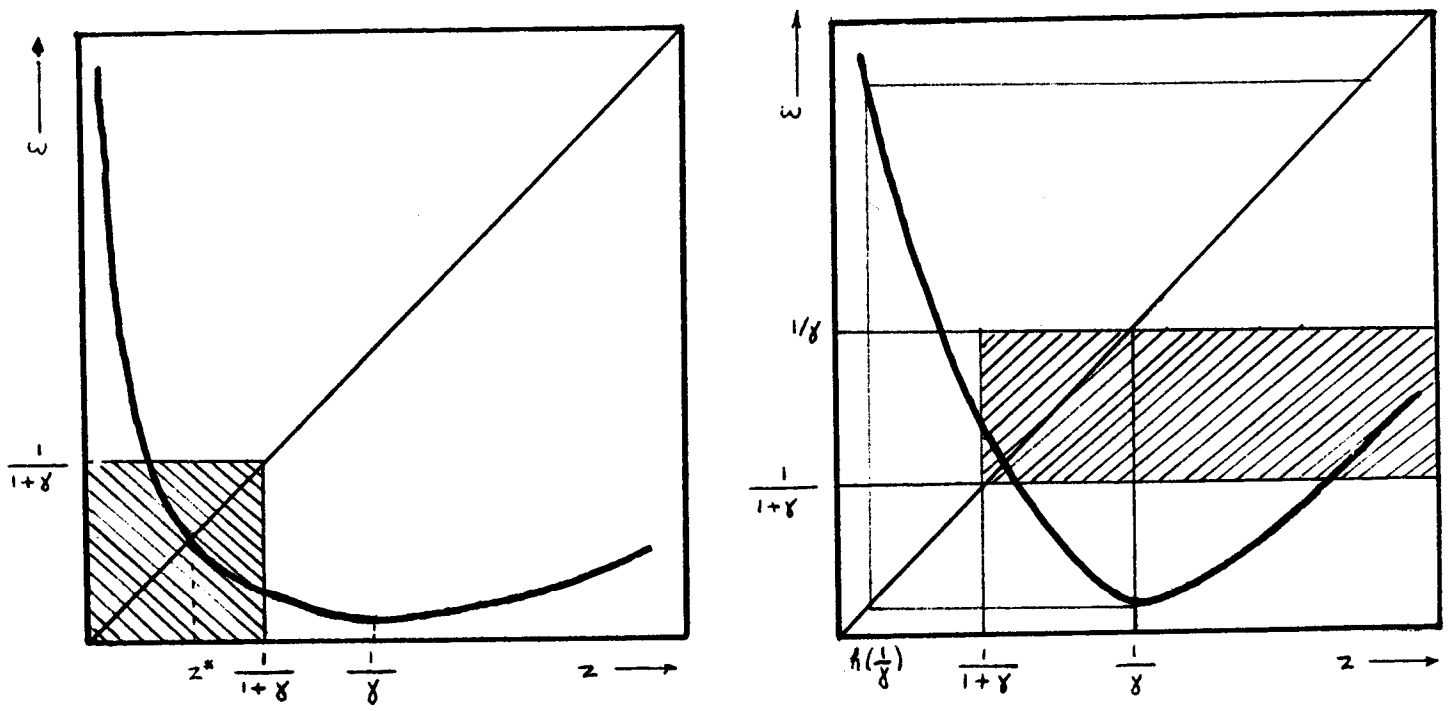
Figure 4: Illustration for two possible configurations for period two points of h_δ.
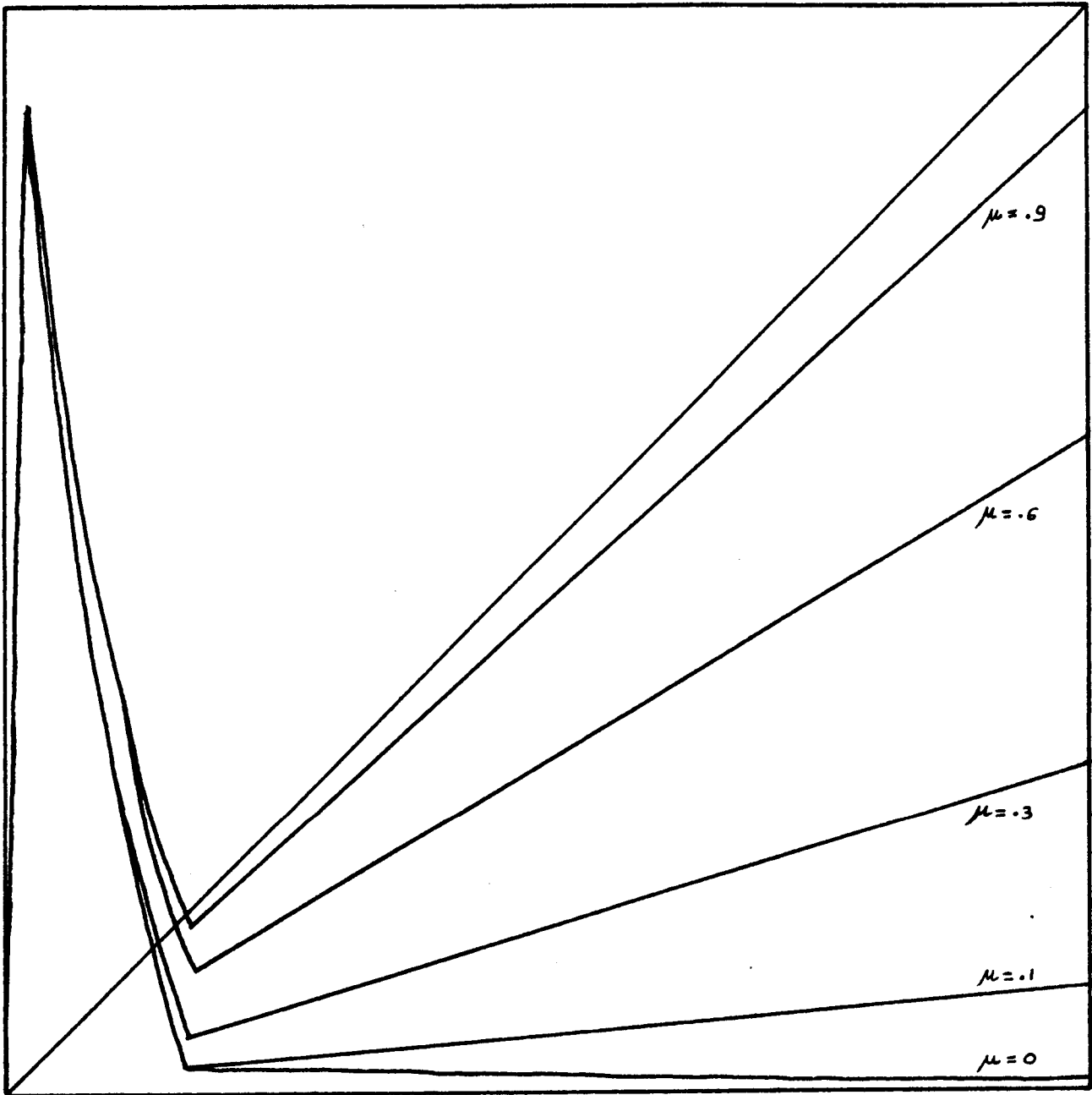
Figure 5:   Evolution of the policy function τ_δ as a function of the depreciation rate $(1 - \mu)$ for $\alpha = .03$, $\gamma = .02$, and $\delta = .025$