EQUILIBRIUM MODELS DISPLAYING ENDOGENOUS FLUCTUATIONS

AND CHAOS: A SURVEY

Michele Boldrin

University of California, Los Angeles

and

Michael Woodford

University of Chicago

1. Introduction

The idea that market mechanisms are <u>inherently dynamically unstable</u> has not played a great role in studies of aggregate fluctuations over the past quarter century. Instead, the dominant strategy, both in equilibrium business cycle theory and in econometric modelling of aggregate fluctuations, has been to assume model specifications for which equilibrium is determinate and intrinsically stable, so that in the absence of continuing exogenous shocks the economy would tend toward a steady state growth path. The existence of a stationary pattern of fluctuations is then attributed to the existence of exogenous shocks of one kind or another -- most often either technology or taste shocks, or stochastic shifts in government policies.

Recent work, however, has seen a revival of interest in the hypothesis that aggregate fluctuations might represent an endogenous phenomenon that would persist even in the absence of stochastic "shocks" to the economy. Even without giving credence to the extreme (and surely implausible) view that macroeconomic fluctuations are purely deterministic, the possibility that exogenous shocks might play a relatively minor role in generating the size of aggregate fluctuations that we observed must be judged of no small importance.

A successful explanation of aggregate fluctuations in endogenous terms would have a dramatic impact on our view of the business cycles. It would, for example, imply that stabilization efforts should be directed more toward dampening the structural feedback loops through which the endogenous fluctuations are sustained, rather than the elimination of exogenous shocks either by making government actions more predictable or by using government

policies to offset temporary changes in technologies and preferences.[1] It would also, most likely, imply a greater degree of optimism regarding the extent to which undesired fluctuations could in principle be eliminated or reduced through appropriate structural interventions and/or reform of government policy responses, relative to what would be the case if one took the size of observed oscillations to indicate large stochastic shocks to the aggregate production possibilities or to tastes.

The endogenous cycle hypothesis is not new. Indeed, the earliest formal models of business cycles were largely of this type, including most notably the business cycle models proposed by Sir John Hicks, Nicholas Kaldor, and Richard Goodwin.[2] In all these models the stationary growth path for the economy is unstable, but deviations from it are eventually contained by "floors and ceilings," such as shortages of productive factors on the upside or technological limits to the amount by which investment can be made negative on the downside.

By the late 1950s, however, this way of attempting to model aggregate fluctuations had largely fallen out of favor, the dominant approach having become instead the Slutsky-Frisch-Tinbergen methodology of exogenous stochastic "impulses" that are transformed into a characteristic pattern of oscillations through the filtering properties of the economy's "propagation mechanism". There were probably two main reasons for the overwhelming popularity of the latter methodology, apart from whatever comfort may have been

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[1]The destabilizing feedback loops might, of course, involve the government's typical policy response to varying economic conditions, but the emphasis would be upon the nature of the government's reaction function rather than simply upon its predictability.

[2]For overviews of the early (non-optimizing, non-equilibrium) literature, see, e.g., Blatt [1983] or Lorenz [1988]. For recent extensions of Goodwin's model, see Goodwin and Punzo [1988].

provided by a vision of the market process as fundamentally self-stabilizing.

First of all, endogenous cycle models are essentially nonlinear; in linear models (for generic parameter values) the only possibilities are globally divergent dynamics -- oscillations that grow explosively without limit (not what we seem to observe) -- or globally convergent dynamics (so that continuing fluctuations require continual exogenous shocks). But the linear specifications that were possible in the case of the exogenous shock models were extremely convenient, both from the analytical point of view (the properties of nonlinear differential equations could be studied only in special cases, as when Goodwin was able to demonstrate a formal analogy between his model and the van der Pol oscillator) and from the point of view of empirical testing (multi-linear regression techniques were well understood, tractable ways of estimating nonlinear models much less so).

Second, the endogenous cycle hypothesis was thought to have been empirically refuted. Actually business cycles, it was easily shown, are far from being regular periodic motions.[3] The spectrum, for example, of an aggregate time series typically exhibits no pronounced peaks, let alone actual spikes as one would expect in the case of deterministic cycles. And econometric models were estimated that, when simulated with repeated exogenous stochastic shocks, produced data that looked like actual business cycles, but that when simulated without the exogenous shocks converged to a steady state (Adelman and Adelman [1959]). Demonstrations of this kind appeared to show that the true structural relations implied an intrinsically stable economy.

Both of these sorts of considerations have less force today than they must have seemed to around 1960. For one thing, advances in the mathematics

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[3]Sir John Hicks, in private communication, has indicated that this was the reason for his loss of interest in endogenous cycle models.

of dynamical systems, in the econometric theory of nonlinear models, and above all, in our computational capacity make both the analysis and the estimation of nonlinear models much more tractable. It is certainly still true that linear models are much simpler to handle, and if one wants to allow complicated lag structures not much constrained by any a priori theory, multilinear models alone easily provide one with a set of free parameters whose number is quite large compared to the size of one's data set. Nonetheless, it is clearly possible to imagine a demonstration that some kind of nonlinear specification allows for a reduction in squared error that more than justifies the inclusion of the additional free parameters, and it is certainly possible to test particular nonlinear specifications that might be suggested on theoretical grounds.

Furthermore, we have come to understand that the simple empirical "refutations" of the endogenous cycle hypothesis do not prove as much as they might have seemed to. It is now understood that deterministic dynamical systems can generate chaotic dynamics, that can look very irregular and that can have autocorrelation functions and spectra that exactly mimic those of a "stable" linear stochastic model, such as a stationary AR(1) model (Sakai and Tokumaru [1980]).[4] Furthermore, it is now recognized that the fact that a stable model gives the best fit within the class of models considered is no proof that the true (or more accurate) model may not be an unstable one that generates endogenous cycles. John Blatt [1978] showed that when a linear autoregression was fit to periodic data from a simulation

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[4]In fact, even before economists understood the possibility of chaotic dynamics, it was observed that some endogenous cycle models could produce cycles of irregular lengths. See Ando and Modigliani [1959]. Modern mathematical developments allow us to understand that this phenomenon can occur in robust examples.

of the Hicks cycle model, the parameter estimates implied a stable second-order autoregressive process for output, of the kind that is in fact obtained from autoregressions of actual GNP data. In fact, nonlinear deterministic models can generate chaotic data that, as far as linear econometric methods are concerned, look like perfect "white noise" -- not merely a stable model, but one with no persistence of fluctuations at all.

New techniques for the analysis of economic time series, of course, may eventually show more convincingly that observed behaviors are genuinely stochastic to a large extent. Techniques that can, in principle, distinguish stochastic fluctuations from purely deterministic chaotic data have been developed in the natural sciences, especially among physicists (see the excellent discussion in Eckmann and Ruelle [1985] and the more recent survey by Farmer and Sidorowich [1988]). They have been refined, improved and recently applied to economic data by Brock, Scheinkman and their coauthors. We will not review this literature here, but instead refer the reader to the updated report contained in Brock [1988]. None of the results yet obtained provide much evidence for the existence of deterministic chaos. In particular the application of these statistical instruments to the U.S. macro data, reported in Brock and Sayers [1988], finds little evidence of chaos. Some aggregate time series appear to involve significant nonlinearities, but the type of tests used against the null hypothesis of a linear specification give no indication as to whether the sort of nonlinearity that exists would give rise to endogenous fluctuations even in the absence of exogenous shocks. A type of test proposed by Hammour [1988b] is more promising in that it provides evidence (in the case of statistics for growth of U.S. industrial production) of a type of nonlinearity that would imply local instability of the steady state growth path in the absence of exogenous

shocks. But these results remain highly preliminary.

In general, the types of non-parametric tests for nonlinearity and endogenous instability that have been proposed seem to require quite large samples if reliable results are to be obtained, and this may ultimately mean that definitive conclusions will not be possible in the case of economic time series, proceeding in this fashion. The question of the relative empirical validity of the exogenous and endogenous cycle hypotheses is likely to be decided only by comparing the predictions of theoretical models from both classes, whose parameters are either econometrically estimated or "calibrated" after the fashion of Kydland and Prescott (1982). Thus far, no tests of this kind have been performed using endogenous cycle models (and admittedly, none of the available theoretical models appear likely to fare well under such a test -- some reasons for which are discussed below). The development of theoretical models that could be tested in this way should be a major object of further research.

Another reason for the decline from favor of the endogenous cycle hypothesis concerns the inadequate behavioral foundations of the early models of this kind.[5] The stability results obtained for many simple equilibrium models based upon optimizing behavior with perfect foresight -- in particular the celebrated "Turnpike Theorems" for optimal growth models (discussed further below in Section 3) doubtless led many economists to suppose that the endogenous cycle models were not only lacking in explicit foundations in terms of optimizing behavior, but depended upon behavioral assumptions that were necessarily inconsistent with optimization. This latter issue is the focus of the present paper.

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[5]We discuss this further in the context of a specific example in Section 2.

We survey the literature that shows that endogenous fluctuations (either periodic or chaotic) can persist in the absence of exogenous shocks, in rigorously formulated equilibrium models in which agents optimize with perfect foresight. We find it useful to divide the known examples into two categories. On the one hand (Sections 2, 3, and 4) are models with a unique perfect foresight equilibrium which involves perpetual fluctuations for most initial conditions. In such cases it is clear how the forces that bring about a competitive equilibrium also require the economy to exhibit endogenous fluctuations. On the other hand (Section 5 and 6) are models in which perfect foresight equilibrium is indeterminate and among the large set of possible equilibria are ones in which the state of the economy oscillates forever. In cases of this sort, the forces that bring about competitive equilibrium do not require that perpetual fluctuations occur (since a steady state position or paths converging to a steady state are also equally consistent with equilibrium, for generic initial conditions). While we regard the indeterminacy in cases of this sort to indicate a type of instability of the competitive process, it is of qualitatively a different sort than in the case of models of the first type, which are the primary focus of the present survey.

In particular there is no reason to regard models in which equilibrium is indeterminate as providing a theoretical reason for one to expect to observe something at all similar to chaotic deterministic dynamics in economic time series. For in these models, the deterministic equilibria involving perpetual oscillations are essentially just limit cases (of a very special sort) of stationary "sunspot" equilibria, in which the state of the economy follows a stationary stochastic process despite the absence of any stochastic shock to "fundamentals". And there is no reason to regard it as

more likely that one should observe nearly deterministic fluctuations rather than large stochastic fluctuations in response to "sunspots" events. Nevertheless, we include some discussion of models of the indeterminate kind in the present survey because of their historical importance as early examples of the possibility of complex equilibrium dynamics.

We begin (Section 2) with a quick discussion of an early nonoptimizing model of complex economic dynamics, because it allows us to raise some issues regarding the consistency of the postulated behavior with optimization that are then resolved in the more recent literature on optimizing models.

For reasons of brevity we have dispensed with mathematical definitions and theorems almost entirely. The reader who wishes to know more about the mathematics of endogenous cycles and chaos may wish to consult such standard references as Collet-Eckmann [1980], Devaney [1986], Guckenheimer-Holmes [1983], Iooss [1977] and Lasota-Mackey [1985]. Useful introductions to this branch of mathematics with a view to economic applications are provided by Baumol and Benhabib [1987] and Lorenz [1988].

2. __Non-optimizing Models of Economic Dynamics__[6]

Day [1982] considers a one-sector, neoclassical growth model in which the dynamics of capital accumulation has the form:

$$k_{t+1} = \frac{s(k_t) \cdot f(k_t)}{1+\lambda} - h(k_t) \tag{2.1}$$

where s is the saving function, f the production function and $\lambda > 0$ is the exogenous population's growth rate. This is a discrete-time version of

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[6]While we concentrate here only on the growth model of Day [1982] it is worth mentioning that also in recent times a very large literature has investigated the cyclic and chaotic properties of Keynes-Kaldor and Goodwin type models. The titles known to us are reported in the bibliography to this survey.

the famous Solow's growth model. The latter had used a continuous time specification to show that under neoclassical assumptions any capital accumulation path will converge to a steady-state position. In the discrete-time form (2.1) Solow's assumption of a constant, exogenous saving rate and of a neoclassical, concave production function give rise to a map $h(k_t)$ which is monotonically increasing and has one and only one interior steady state $k* = h(k*)$. A typical case is represented in Figure 1a. In order to "see" the dynamic paths one has only to pick any initial condition and iterate $h(k_t)$: we have done this for two initial conditions, $\bar{k}_0$ and $\tilde{k}_0$ respectively larger and smaller than $k*$. By following the arrows in Figure 1a, the reader can easily see that every accumulation path will converge, monotonically, to $k*$. Therefore not even damped oscillations are possible in this case.

[Figure 1a about here.]

The trouble with Solow's model is that it is not an optimizing one, i.e., the aggregate saving function is not explicitly derived from considerations of intertemporal efficiency. One is therefore free to pick "reasonable" shapes for $s(k_t)$ (and $f(k_t)$ obviously) in order to prove his claim. A typical Solow-like pair would be a constant saving ratio σ and a Cobb-Douglas form for f, (2.1) then becomes:

$$k_{t+1} = \frac{\sigma B k_t^\beta}{1+\lambda} \qquad (2.2)$$

which is monotonic and therefore stable. The first modification Day suggests is to the production function. By introducing a "pollution effect" in it one obtains:

$$k_{t+1} = \frac{\sigma B k_t^{\beta}(m-k_t)^{\gamma}}{1+\lambda} \qquad (2.3)$$

which is unimodal and has period-three for certain ranges of parameter values. Returning to the Cobb-Douglas form and allowing instead for a variable saving rate, $s(k) = a(1-b/r)k/y$ Day obtains:

$$k_{t+1} = [\frac{\phi}{1+\lambda}] k_t[1 - \frac{b}{\beta B} k_t^{1-\beta}] \qquad (2.4)$$

using the fact that the rate of interest must be $r = \beta y/k$. This equation also displays topological chaos for feasible parameter values.

What is special in (2.3) and (2.4) that makes their dynamics so different from the ones originated by (2.2)? If one tries to picture the $h(k_t)$ functions implied by (2.3) and (2.4) a shape like the one in Figure 1b will typically obtain for reasonable parameter values. As the trajectory we have depicted there suggests, orbits need not converge to the steady state anymore. The function is not monotonic, it is hump-shaped, it maps some interval, say [0,1] into itself, it satisfies: $h(0) = 0$, $h(1) = 0$ and, denoting with $\bar{k}$ the critical point, one often has $h(\bar{k}) = 1$ for appropriate parameter values. Moreover the two stationary states (i.e., the origin and k*) are both unstable in the sense that $|h'(k)| > 1$ for $k = 0$ and $k = k*$.

[Figure 1b about here.]

Maps of the real line into itself that, more or less, satisfy these properties are called "unimodal". They have been extensively studied by mathematicians and natural scientists as they represent the simplest kind of dynamical system that displays "chaotic" behavior. A typical, and much studied, example is the quadratic map $h(x) = \mu x(1-x)$ for $\mu \in [1,4]$. As the parameter μ moves from 1 to 4 the underlying dynamical behavior

obtained by repeated iterations of h undergoes a series of "bifurcations", i.e., of qualitative changes. More exactly the structure of the "attracting sets" change: for $\mu < 3$ the steady state x* is globally attractive, at $\mu = 3$ a period-2 cycle becomes attractive, and subsequently periods 4, 8, 16 ... cycles emerge, become attractive for a while until a new cycle acquires the attracting property as μ keeps increasing. There is a precise order followed by the periodicity of these cycles (this is the famous Sarkovskij theorem) but we will not discuss it. What is more relevant is that when a period-3 cycle occurs then cycles of period N, for N <u>any natural</u> number, are simultaneously present (even if they need not, and will not in general, attract nearby orbits). Not only: there exists also a subset S of the domain of definitions of h (called "scrambled" set) on which "chaos" occurs. This is the celebrated "period-3 implies chaos" Theorem of Li-Yorke [1975]. It has been widely used by economists because of its simplicity as it requires only checking the existence of a period-3 orbit in order to deduce the existence of chaos.[7] Nevertheless this kind of chaos (to which we refer with the words "topological chaos"), may not be very interesting from the economists' viewpoint. The reason is that the scrambled set, S, sometimes, turns out to be of (Lebesgue) measure zero, i.e., the probability of starting in S is zero. Moreover, any initial condition outside S will produce orbits converging to a cycle of finite period. For the quadratic case this occurs, for example, at $\mu =$

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[7]Moreover there are a few simple, qualitative properties that guarantee the existence of a period-3. In fact let h: I → I be continuous, with I an interval, if there exist distinct, disjoint subintervals $I_1 \subset I$ and $I_2 \subset I$ such that $h(I_1) \supset I_2$ and $h(I_2) \supset (I_1 \cup I_2)$ then there is a period-3 for h. See Devaney [1986] for more details.

3.828427, where topological chaos exists but almost all initial conditions lead asymptotically to a period-three cycle. But there is also a more interesting type of chaos, that we call here "observable chaos" or "ergodic chaos". This is the case, for example when $\mu = 4$ for the quadratic map. Ergodic chaos (loosely speaking) means: (a) S has positive (Lebesgue) measure (it has full measure, for example, at $\mu = 4$) so that aperiodic trajectories are in fact observable and, (b) asymptotically the sequence $\{k_t\}_{t=0}^{\infty}$ obtained by iterating $h(k_t)$ approximates an ergodic and absolutely continuous distribution which is invariant under h and that summarize the limit statistical properties of the (deterministic) chaotic trajectories. Again, in the case of $\mu = 4$ the invariant distribution exists and it can easily be shown to be $f(x) = (\pi[x(1-x)]^{\frac{1}{2}})^{-1}$. Unfortunately to prove that "ergodic chaos" exists is not trivial at all. In particular no standard all-purpose theorems exist as of today. The one result that economists have found most useful goes, approximately, like this: if h: $I \to I$ is C^1 almost everywhere in I and such that $|h'(x)|$ > 1 for all $x \in I$ where it is defined (i.e., h is everywhere expansive) then there exists "ergodic chaos", (see Lasota-Mackey [1985]).

Day's examples (as well as many others) show that extremely simple behavioral hypotheses and model structures can produce very complicated dynamics. However, one may nevertheless question whether the sort of behavior assumed is in fact consistent with optimization within the assumed environment. For example, the assumption of a constant saving ratio was often used in the early "descriptive" growth models and can indeed be derived from intertemporal utility maximization under certain hypotheses but it becomes especially implausible when a production function of the type embodied in (2.3) is proposed. Why should a maximizing agent ever save up

to the point at which marginal returns to capital are negative if he can obtain the same output level with much less capital stock? This will clearly never happen; in turn this implies that (given the assumptions on the technology) a "policy function" of the type (2.3) would never occur in an "optimal growth model" of the Cass [1965] type. In fact, as it is well known, a "turnpike property" holds for that class of one sector models at every level of discounting.

Although it is less obvious, Day's case of a variable saving ratio and a monotonic production function (i.e., equation (2.4)) is equally inconsistent with intertemporal utility maximization. This was pointed out (in a general form) in Dechert [1984]. Dechert's argument goes as follows: pick, for example, Day's version of the Solow growth model and ask if the saving function he is using in his example could be determined, everything else equal, as the solution to a representative-agent infinite-horizon maximization problem. The answer is negative. More formally we have: let $y_t = f(k_t)$ be total output at time t, as a function of the existing stock of capital. The consumer-producer chooses how to split it between consumption and future capital in order to maximize: $\Sigma_{t=0}^{\infty} u(c_t)\delta^t$, where u is a concave utility function, δ is a time-discount factor, $\delta \in (0,1)$ and k_0 is given as an initial condition. It turns out that, even if the production function is not concave, the optimal program $\{k_0, k_1, k_2, \ldots\}$ can be expressed by a policy function $k_{t+1} = \tau(k_t)$ which is monotonic. The dynamical system induced in this way cannot therefore produce cycles or chaos. The economic prediction is that such a society will asymptotically converge to some stationary position. The latter is unique when f is concave (i.e., in this case τ looks like the h of Figure 1a). From this we have to conclude that the chaotic examples derived from a one-sector growth model

would not pass the rationality critique. Such a critique turns out to be rather special itself, as it holds true only for the special version of the one-sector growth model considered above. This will be illustrated in the next two sections.

3. Optimal Growth Models

An obvious class of models from which to begin thinking about equilibrium modeling of aggregate fluctuations is the class of optimal growth models, which we may interpret as describing perfectly competitive dynamic economies in which all agents are identical and optimize over the entire infinite horizon of the economy. A stochastic version of such a model is the basis for the influential class of "real business cycle models," studied by Fynn Kydland and Edward Prescott [1980] and [1982], the most fully worked out family of exogenous shock models of aggregate fluctuations. For our purposes it suffices to sketch here the basic ingredients of a very general model from which most of the adopted setups can be derived as special cases. In particular we will consider a world with a single (representative) agent that controls both consumption and production decisions and perfectly foresees even the more distant future (see Bewley [1982] and the literature therein for a reconciliation of this abstraction with the case of many independent consumers and producers). Also we will fully describe only the discrete time formalism even if, later on, we will have to use the continuous-time version of the same model: the translation should be immediate.[8]

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[8]The reader is referred to Cass and Shell [1976], Bewley [1982], Becker and Majumdar [1987] and especially McKenzie [1986] and [1987] for more complete treatments.

In every period $t = 0,1,2,\ldots$ an agent derives satisfaction from a "consumption" vector $c_t \in \mathbb{R}^m$, according to a utility function $u(c_t)$ which is taken increasing, concave and smooth as needed. Notice that c_t denotes a flow of goods that are consumed in period t. The state of the world is fully described by a vector $x_t \in \mathbb{R}^n_+$ of stocks and by a feasible set $F \subset \mathbb{R}^{2n}_+ \times \mathbb{R}^m$ composed of all the triples of today's stocks, today's consumptions and tomorrow's stocks that are technologically compatible, i.e., a point in F has the form (x_t, c_t, x_{t+1}). Now define:

$$V(x,y) = \max\ u(c) \quad \text{s.t.}\ (x,c,y) \in F \tag{3.1}$$

and let $D \subset \mathbb{R}^{2n}_+$ be the projection of F along the c's coordinates. Then V, which is called the short-run or instantaneous return function, will give the maximum utility achievable at time t if the state is x and we have chosen to go into state y by tomorrow. It should be easy to see that to maximize the discounted sum $\Sigma^\infty_{t=0}\ u(c_t)\delta^t$ s.t. $(x_t, c_t, x_{t+1}) \in F$ is equivalent to $\max \Sigma^\infty_{t=0}\ V(x_t, x_{t+1})\delta^t$ s.t. $(x_t, x_{t+1}) \in D$.

The parameter δ indicates the rate at which future utilities are discounted from today's standpoint (impatience): it takes values in $[0,1)$. For $\delta = 0$ the agent is infinitely impatient and there is a sense in which a repeated myopic optimization of this kind may represent the outcomes of an OLG model. In general δ will be greater than zero.

It is mathematically simpler to consider the problem in the latter (reduced) form. The following assumptions on V and D may be derived from more basic hypotheses on u and F:

(A.1) $V: D \to \mathbb{R}$ is strictly concave and smooth (if needed). $V(x,y)$ is

   increasing in x and decreasing in y.

(A.2) $D \subset X \times X \subset \mathbb{R}^{2n}_+$ is convex, compact and with non-empty interior. X

   is also convex, compact and with non-empty interior.

The initial state x_0 is given. Notice that the economy we are describing is essentially time-invariant: return function and feasible set do not change over time, the latter enters the picture only through discounting and the intrinsically intertemporal nature of the production process summarized by D.

The optimization problem we are facing can be equivalently described as one of Dynamic Programming:

$$W(x) = \text{Max}\{V(x,y) + \delta W(y), \quad s.t. (x,y) \in D\} \tag{3.2}$$

The latter is the Bellman equation and $W(x)$ is the value function for such a problem. A solution to (3.2) will be a map $\tau_\delta: X \to X$ describing the optimal sequence of states $\{x_0, x_1, x_2, \ldots\}$ as a dynamical system $x_{t+1} = \tau_\delta(x_t)$ on X. The time evolution described by τ_δ contains all the relevant information about the dynamic behavior of our model economy. In particular, the price vectors p_t of the stocks x_t that realize the optimal program as a competitive equilibrium over time follow a dynamic process that (when the solution $\{x_t\}$ is interior to X) is homeomorphic to the one for the stocks. In other words: $p_{t+1} = \theta(p_t)$ with $\theta = \delta DW \cdot \tau \cdot (DW\delta)^{-1}$, where D is the derivative operator.

The question that concerns us is: what are the predictions of the theory about the asymptotic behavior of the dynamical system τ_δ? Where should a stationary economy converge under Competitive Equilibrium and Perfect Foresight? A first, remarkable answer is given by the following:

TURNPIKE THEOREM (Discrete Time): Under Assumptions (A.1) and (A.2) plus smoothness of V there exists a level $\bar{\delta}$ of the discount factor such that for all the δ's in the non-empty interval $[\bar{\delta}, 1)$ the function τ_δ that solves (3.2) has a unique globally attractive fixed point $x^* = \tau_\delta(x^*)$.

Such an x* is also interior to X under additional mild restrictions.

Not too bad indeed: under a set of hypothesis as general as (A.1) and (A.2) we are able to predict that if people are not "too impatient" relative to the given V and D then they should move toward a stationary state where history repeats itself indefinitely and no surprises ever arise. In the form given here the Turnpike Theorem is due to Scheinkman [1976], whereas McKenzie [1976] and Rockafeller [1976] proved it for the continuous time version (on this see also the discussion below), Bewley [1982] and Yano [1984] generalized it to the many-agents case (but see McKenzie [1986] for a more careful attribution of credits).

As remarkable as it is, the Turnpike property is also very sensitive to perturbations of its sufficient conditions. In particular, how close should δ be to one in order to obtain convergence and what happens when δ is smaller than $\bar\delta$? These are important questions. It is hard to rely heavily on a property that may depend critically on such a volatile and unobservable factor as "society's average degree of impatience".

The careful reader should have realized by now that the one-sector model we briefly introduced at the end of Section 2, and which was used by Dechert to prove that cycles and chaos are not optimal in that framework, is a special case of the general model we are considering here, with

$V(x_t, x_{t+1}) = u[f(x_t) - x_{t+1}]$ and $D = \{(x_t, x_{t+1})$ s.t. $0 \le x_{t+1} \le f(x_t)\}$.

For that model the Turnpike Theorem holds independently of the discount factor as τ_δ is always monotonic increasing. Unfortunately, such a nice feature does not persist even if the simplest generalization of the one-sector model is taken into account. This was proved by Benhabib and Nishimura [1985]. They considered a model with two goods -- consumption and capital -- which are produced by two different sectors by means of capital

and labor. Given the two production functions one can then define a
Production Possibility Frontier (PPF) $T(x_t, x_{t+1}) = c_t$, that gives the
producible amount of consumption when the aggregated capital stock is x_t
(a scalar), labor is efficiently and fully employed and the decision of
having an aggregated stock x_{t+1} tomorrow has been taken. The return func-
tion is now $V(x_t, x_{t+1}) = u[T(x_t, x_{t+1})]$ and $D = \{(x_t, x_{t+1})$ s.t. $0 \leq x_{t+1}$
$\leq F(x_t, 1)\}$ where F is the production function of the capital good sector
and labor has been normalized to one. In such a case τ_δ is not always
upward-sloping. If the consumption sector uses a capital/labor ratio higher
than the one used by the capital sector it will be downward-sloping. Let
$x*$ be the (unique) interior fixed point (i.e., $\tau_\delta(x*) = x*$). This is the
candidate for the Turnpike. Assume, for simplicity, that τ_δ is
differentiable in a neighborhood of $x*$. The derivative will be $\tau_\delta'(x*)$ at
the steady state, it is negative and it changes as δ moves in $(0,1)$,
everything else equal. Benhabib and Nishimura showed that it may take up
the value -1 for admissible δ's, in such a way that the conditions for a
flip (period-doubling) bifurcation are realized. In this case an optimal
cycle of period-2 will exist which can also be attractive: no more
Turnpike! One may provide examples of this phenomenon showing that such an
outcome is by no means due to "pathological" technologies and preferences.

Cycles are not a special feature of the discrete time version of our
model. This has been known since long ago. In a very early work Magill
[1979] had pointed out that cyclical (albeit converging to a steady state)
motions were possible for solutions to optimal undiscounted optimization
problems expressed in continuous time. He was able to show that the origin
of oscillations along optimal trajectories is directly related to the
existence of asymmetries in the Hessian function of the short run maximand

evaluated at the steady state. The use of a model without discounting prevented him from making these oscillatory motions persistent and to prove the existence of limit cycles. This was achieved in Benhabib and Nishimura [1979]. The two authors use the Hopf bifurcation theorem to prove that limit cycles can occur. Let us show very briefly how this can happen. In continuous time we face an optimal control problem of the form:

$$\max \int_0^\infty V(x,\dot{x}) \exp(-\rho t)dt \qquad s.t. \quad (x,\dot{x}) \in D, \quad x(0) \text{ given.} \qquad (3.3)$$

Here $x(t)$ is a vector depending on time, $\dot{x}$ is its time derivative, D again the convex feasible set, ρ the discount factor in $[0,\infty)$, ($\rho = 0$ is equivalent to $\delta = 1$ in discrete time). Using the Maximum Principle one defines a Hamiltonian:

$$H(x,q) = \max_{\dot{x}} \{V(x,\dot{x}) + <q,\dot{x}>, \quad s.t. \ (x,\dot{x}) \in D\} \qquad (3.4)$$

which can be interpreted as the current value of national income evaluated at the (shadow) prices q (on this point see Cass and Shell [1976]).

The dynamical system is then:

$$\dot{x} = \frac{\partial H(x,q)}{\partial q}$$

$$\dot{q} = \frac{-\partial H(x,q)}{\partial x} + \rho q \qquad (3.5)$$

Linearization of (3.5) around the steady state will yield, after some manipulations, a Jacobian matrix J that can be written as $J = \tilde{J} + (\rho/2)I$, where I is the $2n \times 2n$ identity matrix. As $\tilde{J}$ is a Hamiltonian matrix we may consider how its eigenvalues will change with the discount factor ρ and then add $\rho/2$ to obtain those of J. If $\rho = 0$ $\tilde{J}$ has the form:

$$\tilde{J} = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \qquad (3.6)$$

with $A = \partial^2 H(x,q)/\partial x \partial q$, $B = \partial^2 H(x,q)/\partial^2 q$, $C = -\partial^2 H(x,q)/\partial^2 x$. It is a well known result that under strict concavity in x and strict convexity in q of H the $2n$ eigenvalues of $\tilde{J}$ will split into n positive and n negative ones. The steady state will be a saddle point with a stable manifold of dimension n. As the latter is also the dimension of the control vector the optimal program will steer the system on the stable manifold thereby guaranteeing convergence to the Turnpike. For $\rho > 0$ this is not necessarily true: the saddle point property may be lost as some of the negative eigenvalues become positive. It remains true, and global convergence is assumed, in the special symmetric case $C = B^T$, this was proved in Magill-Scheinkman [1979]. As we noticed before when this symmetry condition does not hold there is room for oscillations that may destabilize the steady state and acquire asymptotic stability on their own for ρ large enough. The Turnpike Theorems give conditions under which the stability property of the saddle point is preserved for small ρ. For the purposes of this discussion a particularly useful form of Turnpike Theorem is the one proved by Rockafellar [1976] (see also the related Cass and Shell paper in the same issue of the Journal of Economic Theory). As we pointed out the Hamiltonian $H(x,q)$ is concave in x and convex in q; we say it is α-concave in x if $H(x,q) + \alpha/2 \|x\|^2$ is still concave on its domain of definition for all feasible q and $\alpha > 0$, it is β-convex in q if $H(x,q) - \beta/2 \|q\|^2$ is convex in q on its domain for all admissible x and $\beta > 0$. Then one has:

TURNPIKE THEOREM (Continuous Time): Suppose the Hamiltonian given in (3.4) is α-concave and β-convex in a convex neighborhood of $(\bar{x},\bar{q}) \in \mathbb{R}^n \times \mathbb{R}^n$, where $(\bar{x},\bar{q})$ is a rest point for (3.5) (i.e., $\partial H(\bar{x},\bar{q})/\partial q = 0$ and

$-\partial H(\bar{x},\bar{q})/\partial x + \rho\bar{q} = 0)$. Assume the discount rate satisfies: $\rho^2 < 4\alpha\beta$. Then (under a few additional technical conditions) for every initial condition (x_0,q_0) the unique solution $(x(t),q(t))$ to (3.5) that maximizes (3.3) converges to $(\bar{x},\bar{q})$ as $t \to +\infty$.

This form of the Turnpike property is useful because it relates the level of discounting to the "curvature" of the Hamiltonian which in turn depends (albeit in a very complicated way) on the curvatures of the technology and the preferences. The more concave-convex is H the higher is the level of impatience compatible with regular dynamic behaviors. But as Benhabib and Nishimura showed when ρ grows for fixed α and β a pair (or more than a pair) of eigenvalues may change the sign of their real part by crossing the imaginary axis. In such a case they proved that (care taken for the technical details) a Hopf bifurcation realizes. The limit cycle associated to it turns out to be an attractor for the system (3.5). Once again the Turnpike property is lost as people become "a bit more impatient".

Some characteristics of the oscillatory paths so obtained need to be stressed. First of all they are realized as "equilibrium paths", in the sense that all markets are continuously clearing at each point in time, prices adjust completely and no productive resource is left "involuntarily unemployed". Moreover, they are Pareto efficient in the sense that it is impossible to modify the allocation of resources they imply, in order to increase the welfare of some agent without making somebody else worse off. The economic policy implications of these facts are straightforward and we do not intend to elaborate further on them. In second place oscillations here are strictly market-driven: it is the existence of certain factor-intensity relations across sectors that make it profitable for the producers (and the consumers alike) to invest, produce (and consume) in an oscillatory

form. Even if all the prices are the "right ones" (i.e., no conditions for profitable arbitrage exist) still the pure seeking of individual profits will bring about cyclic behavior. All of this comes from sound economic theory and it is hard to rule it out on pure _a priori_ grounds. Not only, as we will see in a moment, this very same logic can be pursued further to allow for chaotic dynamics in the same class of model economies.

This theoretical possibility turns out to be one of the implications of a more general theorem according to which every dynamical behavior is consistent with these economies. This was proved in Boldrin and Montrucchio [1986b] (but see also Boldrin and Montrucchio [1984] and [1986a] and Montrucchio [1986] for additional results). The Theorem, formally speaking, has the following form: let $\theta: X \to X$ be a C^2-map describing a dynamical system on the compact, convex set $X \subset \mathbb{R}^n$, then there exist a technological set D, a return function V and a discount factor $\delta \in (0,1)$ satisfying (A.1), (A.2) and such that θ is the policy function τ_δ that solves (3.2) for the given D, V and δ. The proof is of a constructive type, so that one may effectively compute a fictitious economy for each given dynamics. This makes clear that any kind of strange dynamic behavior is fully compatible with competitive markets, perfect foresight, decreasing returns, etc. At about the same time and independently Deneckere and Pelikan [1986] also presented some one-dimensional examples of models satisfying our assumptions and having the quadratic map $4x(1-x)$ as their optimal policy function for appropriately selected values of δ.

All these results were given for the discrete time version of the model, but they are not specific to it. In Montrucchio [1987] it is in fact proved that exactly the same facts hold for the case in which time is continuous. The proof proceeds essentially as in the discrete case and

therefore permits the construction of fictitious economies that optimally

evolve according to any prescribed law of motion $\dot{x} = f(x)$.

The extension to the continuous time case is particularly useful in

clarifying the extent to which a high rate of time discount is not essential

for the existence of complex dynamics. In both Boldrin and Montrucchio

[1986a,b] and Deneckere-Pelikan [1986] all of the parametric examples of

chaotic optimal accumulation paths require very small discount factors, or

equivalently high rates of time preference -- e.g., a rate of time discount

of 10,000% per "period". This is patently absurd as a representation of the

attitudes toward future consumption in actual economies. Some have

therefore concluded that chaotic oscillations are mere mathematical curiosa

(at least in optimal growth models) and not a plausible line of research in

business cycle theory. But one must be careful about drawing such general

conclusions from a small number of examples. Further search may discover

more interesting cases. For example, Neumann et al. [1988] provide a

technical improvement on the Boldrin-Montrucchio Theorem that, after a

slight modification, enables one to derive the classical chaotic map $\tau_\delta(x)$

$= 4x(1-x)$ for $\delta = .25$, which is twenty five times larger than the initial

estimates (see Boldrin-Montrucchio [1989] Chapt. 4 for this and other

examples). Furthermore the parameterized example worked out in Boldrin-

Deneckere [1987] derives chaos for values of the discount factor in the

range (.3,.5) (see below).

In fact, it is clear that appropriate choice of the technology and the

single-period utility function can allow chaos to exist for as low a rate of

time preference as one likes. This is clearest in the case of the

continuous-time examples.[9] In such examples, arbitrary rescaling of one's time unit clearly allows one to obtain an example in which the rate of time preference (in, say, percent per year) is as low as one likes. It should be clear that for any given system of ordinary differential equations $\dot{x} = f(x)$ and constant $0 < \epsilon < 1$ one can obtain a perfectly equivalent dynamical behavior from $\dot{x} = g(x) = \epsilon f(x)$. Only the time parameterization has been changed. Nevertheless the C^2-norm of g can be made as small as one like for an appropriate choice of ϵ. As the size of the discount factor compatible with a given dynamics depends monotonically on that norm it is straightforward to conclude that any kind of dynamics can be made optimal at any level of discounting, no matter how small the latter is! A rigorous proof of this claim can be found in Boldrin and Montrucchio [1989, Ch. 3], here we content ourselves with one additional remark on the relations between this result and the Turnpike property.

In particular one may suspect that what we have just stated conflicts with the Turnpike Theorem of Rockafellar we quoted above. This is not the case because as ϵ goes to zero the degrees of concavity-convexity of the associated Hamiltonian also go to zero and the Turnpike property still holds for very small values of the discount rate ρ. We will not go through the algebra here, but the reader can easily check it by himself using the following method. Let $\dot{x} = f(x)$ and $\dot{q} = p(q)$ be the policy functions that solve (3.5). It is well known that the functions f and p have to satisfy the relation: $p(q) = Dh(h^{-1}(q)) \cdot f(h^{-1}(q))$ for some function h: $X \rightarrow \mathbb{R}^n$ which is a C^1 diffeomorphism with symmetric Jacobian (i.e., $Dh(x) = [Dh(x)]^T$), here D denotes the derivative operator. Let $H(x,q) =$

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[9]We are indebted to David Levine for clarification of this point.

$H(x,h(x))$ be the constructed Hamiltonian for which (3.5) holds with:

$$\dot{x} = \frac{\partial H(x,q)}{\partial q} = D_2 H(x,h(x)) = f(x) \tag{3.7}$$

$$\dot{q} = \frac{-\partial H(x,q)}{\partial x} + \rho q = -D_1 H(h^{-1}(q),q) + \rho q = p(q)$$

where D_i denotes differentiation with respect to the ith argument. By substituting $g(x) = \epsilon f(x)$ in (3.7) together with the associated $p_\epsilon(q)$ that one can calculate from the formula given above, it should be easy to see that the new Hamiltonian $\bar{H}(x,q)$ will become less and less concave-convex as $\epsilon \to 0$. This is, obviously, consistent with the basic intuition that relates a high degree of concavity in the objective function to dynamically regular behaviors. It also suggests an interesting area of research on the relation between the level of decreasing returns and the complexity of competitive accumulation paths.

The results of Boldrin and Montrucchio cited above show that an extremely wide range of types of dynamical behavior can arise within the optimal growth framework, but they provide relatively little insight into the economic logic behind these theoretical and mathematical results. What is it that makes it profitable for a competitive economy to oscillate erratically over time? As we noticed above the driving force seems to be the technological structure of the different sectors that induces profitability differentials across them at different points in time. Unfortunately even if we do not have a full-fledged analytical explanation for the multisectorial case (but see Magill [1977 and 1979], something can be said for the two-sector, two-good economy that is often used in macroeconomic applications. A theoretical analysis is provided in Boldrin [1986]. There are two goods -- consumption and capital -- produced by means of two factors

-- capital itself and labor, the resulting dynamics in the aggregate capital stock k_t is one-dimensional. It is proved that the policy function x_{t+1} = $\tau_\delta(k_t)$ is unimodal when (for example) factor-intensity reversal occurs between the two sectors (this is not strictly necessary). Remember that for the case in which the consumption sector always uses a capital-labor ratio higher than the capital sector, period-two cycles are possible. In the general case one is likely to find a level, say k*, of the aggregate capital stock such that when k_t is in [0,k*) the capital sector has a higher capital-labor ratio, whereas the opposite is true when k_t is in (k*,$\bar{k}$], where $\bar{k}$ is the maximum level of capital that the economy can sustain. This technological feature provides the unimodal shape for τ_δ (i.e., τ_δ is as in Figure 1b). Variations in the level of the discount factor δ then can produce a cascade of period-doubling bifurcations that (technicalities aside) leads to period-three orbits and chaos. The problem is taken up again in Boldrin and Deneckere [1987]. The authors consider the same two-sector economy as in Boldrin [1986]. The two production functions are respectively CES (consumption) and Leontief (capital). This implies that factor-intensity reversal occurs in this model. Once properly para-meterized the economy displays various types of dynamic behavior, from the simple convergence to a stationary state, to cycles of different finite periods, to "chaos". For levels of the elasticity of substitution in production that are not too extreme, aperiodic motions appear when the discount factor is in the range (.2,.3) (this still implies an interest rate around 400%). When the elasticity of substitution becomes extremely small, then chaos is present also for less unreasonable levels of discounting.

On the other hand Boldrin and Deneckere prove that chaotic orbits are still present when the elasticity of substitution is equal to one (i.e., the

Cobb-Douglas case), but in order to obtain them one has to select fairly unreasonable values for the parameters of the model.

4.   Models With Market Imperfections and Determinate Equilibrium Dynamics

In the event that markets are incomplete, imperfectly competitive, or otherwise less than fully efficient, the conditions under which endogenous equilibrium fluctuations can occur are less stringent. There is no general "turnpike" theorems for this case, even for an economy made up entirely of infinite lived consumers with stationary, additively separable preferences. In fact, it is possible to construct economies in which different borrowing constraints permit, for a given specification of technology and single-period utility functions, to make consumers' rate of time preference arbitrarily small while continuing to have endogenous cycles or even chaotic equilibrium dynamics.

Nor is the special type of inter-sectoral relations in the production technology considered earlier necessary in order for endogenous fluctuations to occur; for example, endogenous cycles and chaos can occur even in the case of a one-sector production technology, while this would not be possible with any rate of time preference in the case of complete, perfectly competitive markets.

A simple type of market imperfection that results in a greatly increased range of possible dynamics is an assumption that agents are unable to borrow against all types of future incomes. The first demonstration that borrowing constraints could make endogenous cycles possible even in the case of a finite number of infinite lived consumer types and a one-sector

production technology was due to Bewley [1986].[10] Bewley showed how

borrowing constraints could result in equilibrium dynamics in such a model

formally analogous to the capital accumulation paths that could occur in the

overlapping generations model of Diamond [1965]. Bewley's result depends

upon specifications that typically also imply indeterminacy of perfect fore-

sight equilibrium. Hence further discussion of this example is deferred to

Section 6.

Another model based on borrowing constraints is examined by Woodford

[1988b]. In this economy, there are two types of infinite-lived consumers

-- workers, who supply labor inputs to the production process, and entrepre-

neurs, who own the capital stock and organize production, and hence who make

the investment decisions. Workers are assumed to be unable to save by

accumulating physical capital and organizing production themselves (because

of a minimum efficient scale of operation, despite the existence of constant

returns to scale at the levels of production at which entrepreneurs choose

to produce). There is also a limitation upon the extent to which workers

can indirectly invest in productive capital by lending to entrepreneurs. In

the simplest case (though admittedly an extreme one), loan contracts are

assumed to be completely unenforceable. In this case workers must consume

each period exactly the wage bill, and entrepreneurs must finance investment

each period entirely out of retained earnings from that period's production.

The capital stock in each period will then be equal to the previous period's

gross returns to capital, times the fraction of their wealth that entrepre-

neurs do not wish to consume. In the case of a low rate of time discount on

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[10]A somewhat similar type of borrowing constraint is considered in
Scheinkman-Weiss [1986], but exogenous stochastic shocks play there a major
role in sustaining oscillations.

the part of entrepreneurs, this fraction will generally be close to one over

a wide range of anticipated rates of return on their investments, so that

the capital stock chosen for each period will tend to vary closely with the

level of gross returns to capital in the previous period. This can easily

result in a unimodal map $k_{t+1} = f(k_t)$ of the form shown in Figure 1b of

Section 2, since gross returns to capital will be a decreasing function of

the capital stock if labor supply is sufficiently inelastic at high levels

of labor supply, and capital is not too easily substituted for labor. For

example, if entrepreneurs seek to maximize

$$\sum_{t=0}^{\infty} \beta^t \log c_t^e \tag{4.1}$$

where c_t^e is the level of consumption by entrepreneurs in period t and

$0 < \beta < 1$, then entrepreneurs will consume a constant fraction $(1-\beta)$ of

their wealth in each period. If the preferences of workers are additively

separable between periods, equilibrium labor supply will depend only upon

the current real wage, so that it can be represented by a function $s(w_t)$.

If the production function $Y_t = F(K_t, L_t)$ exhibits constant returns to

scale, then the equilibrium real wage w_t will be determined by the current

capital stock K_t as the solution to the relation

$$F_L(K_t, s(w_t)) = w_t$$

This will have a unique solution $w(K_t)$ if $s(w)$ is monotonically increas-

ing, and as a result there will be a unique equilibrium solution for the

following period's capital stock, namely

$$K_{t+1} = f(K_t) = \beta K_t F_K(K_t, s(w(K_t))) \tag{4.2}$$

The function $f(K)$ in (4.2) can easily be unimodal. For example, if

workers' preference are of the form

$$\sum_{t=0}^{\infty} \gamma^t [(m \cdot c_t^w)^\alpha - \tfrac{1}{2} (L_t)^{2\alpha}]$$

where $m > 0$, $0 < \gamma < 1$, and $0 < \alpha \le 1$, and where c_t^w denotes consumption by workers, the labor supply function will be linear

$$s(w) = mw$$

If in addition the technology is Leontief, with $a > 0$ units of output being produced per unit of capital using $b > 0$ units of labor, then (4.1) becomes

$$K_{t+1} = \beta[aK_t - (b^2/m)K_t^2] \qquad (4.3)$$

This family of quadratic maps is well known to imply instability of the steady state equilibrium for parameter values with $\beta a^2 m/b^2 > 1$. (Note that if this condition holds, it continues to hold as β is made to approach 1. Hence there is no "Turnpike" property.) Setting $\mu = \beta a^2 m/b^2$ (4.3) becomes the quadratic map discussed earlier in Section 2 and chaos, both "topological" and "ergodic", will occur for $\mu \in [3.57,4]$.

The existence of strongly chaotic dynamics can be assured for open sets of parameter values by constructing an example in which the map $f(k)$ in (4.2) is everywhere expanding. This requires a kink in $f(k)$ at the peak, but this can easily come about if, for example, the elasticity of labor supply is discontinuous at this point. If the elasticity of labor supply falls sufficiently greatly at the kink, it is possible for $f(k)$ to be sharply increasing before the peak and sharply decreasing thereafter.

Endogenous cycles and chaos in this type of economy do not depend upon the assumption of a Leontief technology, but it is important that the substitutability between capital and labor not be too great. For the fall in k_{t+1} for large values of k_t obviously depends upon total gross

returns to capital being a decreasing function of the size of the current capital stock at that point, which is only possible if it is not possible to easily substitute capital for labor when the real wage rises. In fact, Hernandez [1988] proves a "turnpike" theorem for a class of economies in which consumers cannot borrow against future labor income, under the assumption that the production technology allows sufficient substitutability between factors for total returns to capital to be a monotonically increasing function of the capital stock. It is not clear how far this result can be generalized, but it suggests that low factor substitutability may be important for the existence of endogenous instability even in more complicated examples. See also the discussion by Becker and Foias [1987].

Similar dynamics are also possible in the case of a much weaker restriction upon financial intermediation, namely, in the case that workers can lend to entrepreneurs, but debt contracts contingent upon firm-specific technology shocks (not yet realized at the time of the investment and borrowing decision) are unenforceable. In such a case, if there is a continuum of firms with independent realizations of the technology shock, and the technology shock takes an appropriate form, the equilibrium real wage and aggregate production are deterministic functions of the aggregate capital stock despite the existence of a stochastic technology for each individual entrepreneur/firm. If entrepreneurs' preferences are homothetic (as in (4.1)), each entrepreneur will choose levels of consumption, investment, and borrowing that are proportional to the amount by which his gross returns in the current period exceed his debt commitment, so that aggregate entrepreneurial consumption, investment, and borrowing depend only upon the amount by which aggregate gross returns to capital exceed aggregate debt commitments. As a result the deterministic (perfect foresight) dynamics of

the aggregate capital stock are independent of the distribution of capital holdings across entrepreneurs with different histories of technology shocks. In this model, if the technology shock occurring in the worst state that cannot be insured against is sufficiently bad, then entrepreneurs will be unwilling to finance a large fraction of their investment by borrowing even when gross real returns to capital are expected (on average) to be well in excess of the gross real interest rate on debt. As a result, a low level of gross returns to capital (because real wages are currently high) will result in a low capital stock in the following period, even though total current income is high. Thus k_{t+1} can again be a decreasing function of k_t, for high values of k_t, and all of the phenomena discussed above can occur. The distribution of income continues to have a significant effect upon capital accumulation, despite the existence of a competitive market for (non-contingent) loans. Furthermore, the sort of market imperfection assumed in this model is of considerable empirical relevance, since firms are often unable to obtain external finance upon terms that make their repayment commitments contingent upon their level of revenues.

Financial constraints are not the only kind of market imperfections that can give rise to endogenous fluctuations even with an arbitrarily low rate of time discount. Deneckere and Judd [1986] demonstrate the possibility of endogenous fluctuations in the rate of introduction of new products in an economy in which the creation of a new product involves a one-time fixed cost, and allows the innovator a one-period monopoly of production of the new product. (After the first period of production, imitation is possible, so that the market for the product becomes perfectly competitive.) The model is a simplified discrete-time variant of that of Judd [1985]. Consumers are assumed to have a "taste for variety" of the Dixit-Stiglitz

sort. Each period, new products are introduced to the point where the monopoly rents from the production of each new product are no greater than the fixed cost of creation of a new product. Since monopoly rents are earned only for a single period, expectations regarding the number of products to be produced in the future have no effect on the decision. As a result, the total number of products produced in a given period (N_t), is a determinate function of the number of old (non-monopolized) products in existence. Deneckere and Judd show that for a certain parametric class of preferences, N_{t+1} is a function of N_t of the form shown in Figure 2.

[Insert Figure 2 Here.]

For $N_t \leq \hat{N}$, N_{t+1} is a decreasing function, because a larger number of old products being produced reduces the level of monopoly rents from production of new products to such an extent that the equilibrium number of new products falls by an amount even greater than the increase in the number of old products. For sufficiently large N_t (i.e., $N_t > N^*$), N_{t+1} is actually less than N_t, because the number of old products produced in period t+1 is assumed to be only δN_t (where δ is a fraction less than 1), due to a constant rate of obsolescence of products. If $N_t = \hat{N}$, the level of monopoly rents is equal to the fixed cost of innovation, even if no new products are created. For larger values of N_t, the level of monopoly rents that would exist in the case of innovation is even lower, but the rate of innovation cannot fall below zero. Hence for $N_t \geq \hat{N}$, N_{t+1} is simply equal to δN_t. This results in a kinked function of the kind shown in the figure. If the slope of the deceasing segment (which is linear in the case analyzed by Deneckere and Judd) is steeper than -1, the steady state equilibrium at N^* is unstable. On the other hand, the dynamics converge eventually to a bounded interval $[\underline{N}, \tilde{N}]$, and fluctuate within this interval

forever after. Depending upon parameter values, the dynamics are either asymptotically convergent to a stable cycle of period 2 for most initial conditions, or are chaotic for most initial conditions.

Like the Woodford example, this one in no way relies upon high rates of time preference. Producers' decisions about whether to introduce new products do not depend upon the rate at which profits are discounted, because profits are obtained only for a single period and the fixed cost is paid in that same period. The level of monopoly rents depends only upon consumers' elasticity of substitution between different products within a given period, not upon their preferences regarding present as opposed to future consumption.

Neither of these examples can easily be parameterized for comparison with actual data, since each makes a number of very special assumptions in order to obtain equilibrium dynamics that can be described by a first-order nonlinear difference equation for a single state variable. This is, as of now, the only case in which a reasonably thorough analytical characterization of the types of possible asymptotic dynamics is available. Further progress in evaluation of the empirical relevance of the endogenous cycle hypothesis will doubtless depend on the use of numerical simulations to determine whether chaotic fluctuations occur for realistic parameter values in more complicated models.

5. Indeterminacy and Endogenous Fluctuations in Overlapping Generations Models

Endogenous cycles and chaos can also occur as equilibrium phenomena even in the presence of complete and perfectly competitive intertemporal markets, in the case of an economy made up of overlapping generations of finite lived consumers. In this case, as noted earlier, the equilibria involving perpetual deterministic fluctuations are only some members of a

large set of rational expectations equilibria, which also includes equilibria converging to a stationary state. Nonetheless, this class of examples has been crucial for the development of modern interest in the endogenous cycle hypothesis, since it provided the first general equilibrium examples of the possibility of chaotic economic dynamics, through the work of Benhabib and Day [1982].

Consider the simple overlapping generations model treated by Gale [1973]. The economy consists of a sequence of generations of two period lived consumers, each identical in number. There is a single perishable consumption good each period of which each consumer has an endowment w_1 in the first period of life and w_2 in the second period of life. Each consumer born in period t seeks to maximize $U(c_{1t}, c_{2t+1})$, where c_{jt} is consumption in period t by consumers in the jth period of life, and where U is a concave function increasing in both arguments. Finally, there exists a single asset, fiat money, in constant supply $M > 0$, all of which is initially held by a group of consumers who are already in their final period of life in the first period of the model.

Let p_t be the price of the consumption good in terms of money in period t. Then a young consumer who chooses to hold money in the quantity M_t at the end of period t must expect a lifetime consumption pattern $(c_{1t}, c_{2t+1}) = (w_1 - M_t/p_t, w_2 + M_t/p_{t+1})$. Optimal choice of M_t, given the price level p_t that he faces and the price level p_{t+1} expected for the following period, must satisfy the first-order condition

$$U_1(w_1 - M_t/p_t, w_2 + M_t/p_{t+1}) = U_2\left(w_1 - \frac{M_t}{p_t} w_2 + \frac{M_t}{p_{t+1}}\right) \frac{p_t}{p_{t+1}} \qquad (5.1)$$

(assuming boundary conditions on the utility function that rule out

existence of a corner solution). A perfect foresight equilibrium price
sequence $\{p_t\}$ is a sequence of positive prices, for $t = 0,1,2,\ldots,$ that
when substituted into (5.1) result in a money demand by the young of $M_t = M$
in every period $(t = 0,1,2,\ldots)$. Thus the price sequence must solve the
difference equation

$$U_1\left(w_1 - \frac{M}{p_t}, \; w_2 + \frac{M}{p_{t+1}}\right) = U_2\left(w_1 - \frac{M}{p_t}, \; w_2 + \frac{M}{p_{t+1}}\right)\frac{p_t}{p_{t+1}} \qquad (5.2)$$

The relation between M/p_t and M/p_{t+1} required by equation (5.2) can
be graphed as in Figure 3.

[Insert Figure 3 here.]

It can easily be shown that if both first and second period consumption are
normal goods, the graph must be such that there is a unique positive solu-
tion for M/p_t for each positive value for M/p_{t+1}, that we may write

$$\frac{M}{p_t} = f\left(\frac{M}{p_{t+1}}\right)$$

However, as indicated by the figure, the function f need not be invert-
ible. The sort of unimodal function shown occurs if preferences are such
that desired saving in youth is a decreasing function of the expected real
return on money, for high enough levels of that return. The equilibrium
dynamics are graphed for the case of a given expectation regarding the price
level at some date T far in the future; the figure shows what price levels
must be expected to occur in each of several periods prior to period T.

It is apparent that, when the unimodal map is steep enough (which
Grandmont [1985] shows to require simply that the marginal utility of second
period consumption fall sharply enough with increases in second period
consumption, near the level of consumption that occurs in the stationary

monetary equilibrium), the (backwards) perfect foresight trajectories so traced can involve complex oscillations. As was first noted by Gale [1973], deterministic cycles are possible. Cass, Okuno and Zilcha [1979] showed that the deterministic cycles could be of arbitrary period, and Grandmont discusses in detail the order in which cycles of various periods occur as the map is made progressively steeper, applying the theory of unimodal maps set out in Collet and Eckmann [1980]. All such cycles obviously represent possible equilibria of the forward perfect foresight dynamics as well; i.e., they can be extended indefinitely into the future as well as indefinitely into the past.

Grandmont also discusses conditions under which the backward perfect foresight dynamics would exhibit topological chaos. This result is of less obvious significance for the forward perfect foresight dynamics, since the "chaotic" property of a trajectory can be defined only in terms of its asymptotic behavior as it is continued indefinitely, and the existence of such a property for trajectories extended backward indefinitely from a date T does not necessarily imply anything about the kind of trajectories that exist going _forward_ from a date T. Nonetheless, it is clear that not just chaotic but _genuinely random_ perfect foresight trajectories do exist for the forward dynamics, albeit for reasons that do not require the use of the theory of nonlinear maps of the interval. For it is evident from Figure 3 that for many values of p_t, there would be two different values of p_{t+1}, expectation of either of which would result in a market clearing price of p_t. So we can construct a (forward) perfect foresight equilibrium trajectory by starting at some arbitrary $p_0 \geq \underline{P}$ (where $M/\underline{P} = \sup_{p>0} f(M/p)$), and proceeding recursively, for each value of t, choosing p_{t+1}, given p_t, so that (5.3) is satisfied, and also so that $p_{t+1} \geq \underline{P}$. It is clear

that this iteration can be continued forever.

Let us suppose furthermore that the map f is so steep that $f(M/\underline{P}) < f^2(M/\underline{P}) < f^3(M/\underline{P}) < M/\underline{P}$, as shown in Figure 4. Then for any choice of p_0 so that M/p_0 lies within the open interval $I - (f(M/\underline{P}),M/\underline{P})$, it is also possible to continue the iteration forever, always choosing p_t so that M/p_t lies within I. Furthermore, each time M/p_t lies within the interval $I_2 - (f^2(M/f),M/f)$, there are two possible choices of p_{t+1} which will continue the series. One may use a randomizing device (e.g., a coin toss) to chose between them. And each time M/p_t lies within the interval $I_1 - (f(M/f), f^2(M/f)]$, the unique available choice for p_{t+1} places M/p_{t+1} within I_2. Hence there is never more than one consecutive period in which one does not get to use the randomizing device. The resulting time series for the price level is accordingly quite random. In order for it to represent a perfect foresight equilibrium, we must suppose that the outcomes of the coin tosses are known to consumers in advance, so that the price level p_{t+1} is known with certainty at least as of period t. But since consumer behavior will never in any way reveal the outcome of the coin toss until period $t+1$'s price level is reclined, an econometrician observing the time series would have to model it as one in which random events continually occur that determine the evolution of the price level.

It is also known from the work of Shell [1977], Azariadis [1983], Azariadis-Guesnerie [1986] and many subsequent authors that genuinely stochastic ("sunspot") equilibria exist in this model in which consumers do not know the following periods price level with certainty. These equilibria correspond to stochastic processes for $\{p_t\}$ that satisfy the generalization of (5.2)

$$E_t\left[U_1\left(w_1 - \frac{M}{P_t}, \; w_2 + \frac{M}{P_{t+1}}\right) - U_2\left(w_1 - \frac{M}{P_t}, \; w_2 + \frac{P_t}{P_{t+1}}\right)\right] = 0$$

Because such equilibria exist in a model like Grandmont's, there is no need of the concept of chaotic deterministic dynamics to explain how erratic, "random-looking" time series could be generated, even in the absence of stochastic variations in endowments or preferences. (The existence of such stochastic equilibria is closely related to the indeterminacy of perfect foresight equilibrium, as is discussed further in Woodford [1984].)

Benhabib and Day [1982] present a variant of the model in which the existence of chaotic perfect foresight equilibria is of greater interest. Their model is the same, except that instead of assuming the existence of a fixed positive supply of fiat money, they assume the existence of government lending to the finite lived private consumers. If the initial old consumers owe to the government real indebtedness of d_0, and if the government's lending policy each period is to lend (at a market clearing real interest rate) a quantity exactly equal to the amount currently being repaid by the old consumers, then each period the government will lend d_{t+1}/r_t to the young at a gross real interest rate of r_t, where d_t and r_t satisfy

$$U_1\left(w_1 + \frac{d_{t+1}}{r_t}, \; w_2 - d_{t+1}\right) = U_2\left(w_1 + \frac{d_{t+1}}{r_t}, \; w_2 - d_{t+1}\right)r_t$$

$$\frac{d_{t+1}}{r_t} = d_t$$

These equations in turn imply that a sequence $\{d_t\}$ corresponds to a perfect foresight equilibrium if and only if it satisfies

$$I_1(w_1 + d_t, \; w_2 - d_{t+1}) = U_2(w_1 + d_t, \; w_2 - d_{t+1})\frac{d_{t+1}}{d_t} \tag{5.4}$$

for all $t \geq 0$, given the initial obligation d_0. Now equation (5.4) is identical to (5.2) except that d_t replaces $-M/p_t$, and so the equilibrium dynamics can be studied using the same offer curve diagram as in Figure 3, except that now (assuming the government is a net creditor) we are interested in the lower left rather than the upper right quadrant. Rotating the diagram by 180° we get Figure 5. If consumption is a normal good in both periods of life, (5.4) can be solved for d_{t+1} as a unique function of d_t, which we may write

$$d_{t+1} = f(d_t) \tag{5.5}$$

Furthermore, the function f in (5.5) is exactly the same as the function f in (5.3), if one constructs a Benhabib-Day economy by reversing the time pattern of endowments and preferences in the Grandmont economy (i.e., one replaces (w_1, w_2) by (w_2, w_1), and $U(c_1, c_2)$ by $U(c_2, c_1)$). Hence Benhabib and Day's results on the conditions under which endogenous cycles and chaos are possible are parallel to Grandmonts. The map f can be sharply downward-bending, as shown in Figure 5, if desired borrowing does not increase very much as the real rate of interest on loans falls, which in turn occurs if the marginal utility of first period consumption falls rapidly with increases in first period consumption;. Again, equilibrium cycles and chaotic dynamics are possible. In Benhabib and Day's model, the chaotic paths relate to the __forward__ perfect foresight equilibrium dynamics, so they do indicate the possibility of chaotic equilibrium trajectories starting from a given initial condition. Furthermore, in the Benhabib and Day model, perfect foresight equilibrium is unique, so that for certain initial conditions, a cyclic or a drastic trajectory will be the __only__ possible equilibrium (assuming that the initial private sector obligation

d_0 is fixed in real rather than nominal terms).[11] It can however be objected that Benhabib and Day only give sufficient conditions for the existence of "topological chaos", i.e., for a map f such that a chaotic equilibrium occurs for some initial values d_0, but perhaps only for d_0 in a set of zero measure. Hence their example does not clearly provide an explanation for the observation of apparently stochastic fluctuations. It should be pointed out, on the other hand, that "ergodic chaos" can occur in their model, for some particular choices of preferences and endowments, but it is not clear how robust their examples are.

Grandmont's cycle example has been criticized on other grounds as well. Some have argued that the sharply backward-bending supply of savings as a function of the real return is not empirically plausible. Kehoe et al. [1986] show that deterministic cycles and chaotic equilibria are both impossible in a general stationary overlapping generations model (allowing for an arbitrary finite number of goods per period, an arbitrary number of consumer types per operation, and an arbitrary finite number of periods of life per consumer), if all goods are gross substitutes, in the sense that the excess demand for each good is a decreasing function of its own price and an increasing function of the prices of all other goods (including all goods in other periods).

However, the Kehoe et al. result applies only to endowment economies, i.e., economies in which there is no production. Reichlin [1986] shows that endogenous cycles are possible in an overlapping generations economy with

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[11] One could reduce the degree of indeterminacy in the Grandmont model as well by treating it as a model in which outstanding government debt is rolled over forever, assuming that the initial outstanding government debt is given in real terms. Then the initial value of $-d_0$ ($= M/p_0$) would be predetermined rather than arbitrary. But there would still exist stochastic equilibria because of the non-invertibility of the map f in (5.3).

production, even when consumers demands for all goods satisfy the gross
substitutes condition (savings are an increasing function of the real rate
of return, labor supply is an increasing function of the real wage).
Riechlin [1987] also shows that chaotic equilibrium dynamics are possible in
an overlapping generations economy with production. Other results relating
to endogenous cycles and chaos in overlapping generations models with
production are presented by Farmer [1986], Benhabib and Laroque [1988], and
Jullien [1988].

   Sims [1986] has also criticized the relevance of the Grandmont example
for business cycle theory on the ground that, since the deterministic cycles
must last for two or more periods, and since in Grandmont's model the
lifetime of consumers is only two periods, the example shows only that
endogenous cycles are possible with periods equal to or greater than a human
lifetime. Sims also presents an informal argument suggesting that such a
result might more generally hold, in which case the sort of mechanisms
illustrated by Grandmont's example would not be relevant for the explanation
of actual business cycles, that occur over periods of only a few years.

   The conditions under which endogenous cycles might exist in overlapping
generations models with long-lived consumers have not yet been much studied.
Aiyajan [1987] considers a family of overlapping generations models in which
the number of periods that each generation lives is made progressively
longer, while the rate of time preference and the elasticity of substitution
of consumption between periods remain constant, and while each period's
endowment continues to fall between the same upper and lower bounds. He
shows that for any integer k, k-period deterministic cycles eventually
cease to exist, within such a family of economies, once the lifetime T is
made large enough. This provides some support for the view that short-

period cycles are not likely to occur in economies with long-lived consumers. However, it should be noted that Aiyagan's result does not actually show that the cycles that are possible must always be long compared to the lifetime of consumers; he does not even show that within one of his families of economies, there exists any positive lower bound G such that one must have k/T > t in order for a cycle to be possible. Furthermore, the result is obtained only for a relatively special class of preferences and lifetime endowment patterns, and only for endowment economies (i.e., there is no production).

Whitesell [1986] shows that relatively short period cycles are possible in overlapping generations models with relatively long lived consumers. He considers a continuous-time model with stochastic lifetimes for individuals, but a deterministic rate of death for each generation in aggregate, of the kind first examined by Blanchard [1985]. Whitesell's variant allows for endogenous labor supply and a one-sector production technology. He exhibits a number of sets of parameter values for which endogenous equilibrium cycles will exist; for example, he shows that cycles with a period of about 4 years can exist in an economy in which the average consumer lifetime is 50 years, labor endowments decline with age at a rate of 4 percent per year, and consumers' rate of time preference is 10 percent per year.[12]

The objection raised earlier to the Grandmont example of endogenous cycles, that there exist many other equilibrium as well (including an equilibrium with a constant price level), can be answered if one asserts that perfect foresight equilibrium is a relevant equilibrium concept only when

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[12]The periods of the Whitesell cycles, which he does not report, can be calculated as approximately $2\pi(trJ/|J|)^{\frac{1}{2}}$, where trJ and $|J|$ are evaluated for the parameter values at which the Hopf bifurcation occurs, i.e., at which Z changes sign in Whitesell's table.

viewed as the eventual limit of a disequilibrium "learning" process. In this case, it is possible that the stationary equilibrium would be <u>unstable</u> under the learning dynamics, which would instead converge, from most initial conditions, to one of the equilibrium cycles. Grandmont [1985] and Grandmont and Laroque [1986] give conditions upon the "learning" process for this to occur.

For example, if expectations regarding the following period's price level are always formed by looking at past prices according to the simple rule

$$p^e_{t+1} = p_{t+1} \tag{5.6}$$

then the temporary competitive equilibrium dynamics[13] (in which p_t is determined by (5.2) each period, but where p_{t+1} in (5.2) is replaced by p^e_{t+1}) will take the form

$$\frac{M}{p_t} = f(\frac{M}{p_{t+1}})$$

where f is the same function as in (5.3). In this case, the temporary competitive equilibrium dynamics are uniquely determined by the initial condition p_{-1}. Furthermore, if the map f is such that the stationary monetary equilibrium is unstable in the backward perfect foresight dynamics, and there exists a two-period cycle that is globally stable under the backward perfect foresight dynamics then the (forward) temporary competitive equilibrium dynamics converge asymptotically to the two-period cycle for most initial conditions p_{-1}.

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[13]For further discussion of the concept of temporary competitive equilibrium, see Grandmont [1977, 1983].

Grandmont and Laroque show that this result holds not only in the case of the extremely simple forecasting rule (5.6), but also for a wide class of forecast functions giving p_{t+1}^e as a function of the history of past price levels, and similar results are given in the case of equilibrium cycles with periods longer than two as well. These results provide some reason to suppose that the deterministic cycles treated by Grandmont could represent observable phenomena. But no such results exist in the case of the chaotic perfect foresight dynamics shown to be possible in the Grandmont model. It is obviously true that some specifications of preferences, endowments, and forecast function would imply chaotic temporary competitive equilibrium dynamics for most initial conditions; for example, forecast function (5.6) would imply this in the case of backward perfect foresight dynamics that exhibit "ergodic chaos". But these chaotic temporary competitive equilibrium dynamics would involve a systematic failure of expectations to be fulfilled, that would not improve with time. Under such circumstances, it is implausible that consumers would continue to simply mechanically apply a rule so crude as (5.6).

6. **Indeterminacy and Endogenous Fluctuations in Models With Infinite Lived Consumers and Market Imperfections**

The overlapping generations examples are also relevant as an indication of what can happen in economies with long-lived consumers, facing various sorts of financial constraints. As noted in Section 4, Bewley [1986] has shown that borrowing constraints in an economy with a finite number of infinite lived consumer types can result in dynamics formally analogous to those in an overlapping generations economy. An economy with equilibrium dynamics like those of the Grandmont [1985] example is discussed by Bewley [1980]. Suppose that the economy is made up of equal numbers of two types

of infinite lived consumers, each of whom has preferences of the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \cdot$$

where $0 < \beta < 1$ and u is an increasing, concave function. Type A consumers have an endowment of w_1 in all even periods and w_2 in all odd periods (where $w_1 > w_2$) while the endowment pattern of type B consumers is exactly the reverse. Suppose furthermore that debt contracts are unenforceable, so that neither consumer type is able to borrow against future income. Consumers can save only by holding fiat money, which exists in a fixed positive supply $M > 0$. An equilibrium is possible in which the entire money supply is held at the end of each period by consumers of the type with endowment w_1 if the price level sequence $\{p_t\}$ satisfies the following sequence of conditions:

$$u'\left[w_1 - \frac{M}{p_t}\right] = \beta u'\left[w_2 + \frac{p_t}{p_{t+1}}\right] \frac{p_t}{p_{t+1}} \tag{6.1}$$

$$u'\left[w_2 + \frac{M}{p_t}\right] \geq \beta u'\left[w_1 - \frac{p_t}{p_{t+1}}\right] \frac{p_t}{p_{t+1}} \tag{6.2}$$

Conditions (6.1) shows how p_t is determined in period t as a function of expectations regarding the value of p_{t+1}; and the relation is exactly the same as in the case of an overlapping generations model with consumers whose first period endowment is w_1, whose second period endowment is w_2, and whose preferences over lifetime consumption patterns (c_1,c_2) are given by

$$U(c_1,c_2) = u(c_1) + \beta u(c_2)$$

There exists solutions to (6.1) if and only if

$$u'(w_1) < \beta u'(w_2) \tag{6.3}$$

which is to say, if endowments are sufficiently variable. Condition (6.2)
necessarily holds in all periods if (B.1) does, and p_t always satisfies

$$p_t - \underline{P} = \frac{2M}{w_1 - w_2} \tag{6.4}$$

This condition is necessarily satisfied by the stationary solution to (6.1),
i.e., the constant price p^* satisfying

$$u'(w_1 - \frac{M}{p^*}) = \beta u'(w_2 + \frac{M}{p^*})$$

since $\beta < 1$ and strict concavity of u implies that $p^* > \underline{P}$.

Now just as in the case of the overlapping generations model it is
possible for the dynamics consistent with (6.1) to involve endogenous cycles
or even chaos. This requires simply that the function u exhibit suffici-
ent curvature near the point $c = w_2 + M/p^*$.) If the fluctuations in
question remain in a sufficiently small neighborhood of the stationary
equilibrium price level p^*, (6.4) will necessarily be satisfied as well,
and these will be possible equilibria of the model with borrowing
constraints. It should be noted, however, that in order for examples of
large fluctuations to be constructed in this manner (or for that matter, in
order for examples of chaotic dynamics to be constructed without resort to
extreme curvature assumptions, since chaos cannot occur unless the price
level fluctuates over a large enough range to allow the demand for real
balances to be sometimes an increasing and sometimes a decreasing function
of the expected return on money balances), the bound $\underline{P}$ must be much lower
than p^*. Since $G(\underline{P})/G(p^*) = \beta$, where $G(p)$ is the increasing function

$$G(p) = \frac{u'(w_2 + \frac{M}{p})}{u'(w_1 - \frac{M}{p})}$$

this will in general require β to be much less than 1.

If consumers smooth their consumption path by accumulating capital (rather than fiat money) in their high-endowment periods, and one interprets the endowments as being of labor rather than of the consumption good, a similar model of infinite lived consumers subject to borrowing constraints can mimic the dynamics of capital accumulation in the Diamond [1965] over-lapping generations model. In this model, endogenous equilibrium cycles are possible in the case that aggregate savings are a backward-bending function of the expected real return. In this way Bewley [1986] shows that endogenous cycles are possible. This example again depends upon a high rate of time discount $(\beta = .5)$.

Woodford [1988a] shows that cash-in-advance constraints can result in dynamics similar to those of an overlapping generations model with short lifetimes even when all agents are infinite lived. Consider, for example, the cash-in-advance model studied by Wilson [1979], in the case of prefer-ences that are additively separable between consumption and leisure. All consumers are infinite lived and seek to maximize

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - v(n_t)] \tag{6.5}$$

where c_t is consumption in period t and n_t is labor supply, $u', v^2, v^n > 0$, $u'' < 0$, and $0 < \beta < 1$, subject to the sequence of budget constraints

$$p_t c_t \leq M_t \tag{6.6a}$$

$$M_{t+1} = M_t + p_t(n_t - c_t) \tag{6.6b}$$

where M_t is money balances carried into period t. (One can also intro-duce a competitive loan market, as Wilson does, but as long as bonds are in

zero net supply this has no effect upon equilibrium dynamics for the price level and output). The technology allows one unit of labor to be converted into one unit of the same period's consumption, and hence the price of consumption goods p_t must also be the nominal wage. In equilibrium, the representative consumer's desired money holdings M_t must in every period equal the constant money supply M_t, and his labor supply n_t must equal his consumption demand c_t. In equilibria in which the cash-in-advance constant (6.6a) always binds, consumption demand c_t always equals the level of real balances M/p_t. In this latter case, the price level sequence $\{p_t\}$ must satisfy

$$v'(M/p_t) = \beta u'(M/p_{t+1}) \frac{p_t}{p_{t+1}} \tag{6.7}$$

$$u'(M/p_t) \geq v'(M/p_t) \tag{6.8}$$

in all periods. Condition (6.7) indicates how the equilibrium price level p_t is determined by expectations regarding p_{t+1}. This condition has the same form as that arising in the case of an overlapping generations model in which consumers supply output n_t in the first period of life and consumer c_{t+1} in the second so as to maximize

$$U(n_t, c_{t+1}) = \beta u(c_{t+1}) - v(n_t)$$

There exist price level sequences satisfying (6.7) as long as $v'(0) < u''(0)$. Any price sequence also satisfies (6.8) as long as

$$p_t \geq p = \arg \max_p \ [u(\frac{M}{p}) - v(\frac{M}{p})] \tag{6.9}$$

Condition (6.9) is necessarily satisfied by the stationary solution to (6.7), i.e., the constant price level p^* satisfying

$$v'(\frac{M}{p^*}) = \beta u'(\frac{M}{p^*})$$

since $\beta < 1$.

Endogenous cycles and chaos characterize some solutions of (6.7) in the case of certain kinds of preferences (in particular, if u exhibits sufficient curvature near the level of consumption c = M/p*). Again, such solutions must satisfy (6.9) if the fluctuations round the stationary equilibrium price level p* are small enough. Again, such a construction works in the case of large fluctuations only if β is much less than 1, for again $G(\underline{p})/G(p*) = \beta$, where in this case G(p) is the increasing function

$$G(p) = \frac{u'(\frac{M}{p})}{v'(\frac{M}{p})} \quad .$$

Similar results are possible even in the case of exogenous labor supply if, as in Lucas and Stokey [1987], one allows consumers to substitute between "cash goods" (purchases of which are subject to the cash-in-advance constraint) and "credit goods" (purchases of which are not). Suppose that the representative consumer seeks to maximize

$$\sum_{t=0}^{\infty} \beta^t [u(c_{1t}) + v(c_{2t})]$$

where c_{1t} represents consumption of "credit goods" in period t, c_{2t} represents consumption of "cash goods", u and v are both increasing concave functions, and $0 < \beta < 1$, and suppose that the fixed labor endowment (that can be used to produce either "cash goods" or "credit goods" at equal cost) is n > 0 per period. Then equilibria in which the cash-in-advance constant always binds involve price level dynamics identical to that in an overlapping generations model with two period lived consumers who seek to maximize

$$U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \beta v(c_{2t+1})$$

and who have an endowment of n in the first period of life and zero in the second period. Again endogenous fluctuations analogous to those in the Grandmont example are possible (for low enough β) if v exhibits enough curvature near the stationary equilibrium level of consumption of "cash goods".

Woodford [1986] also shows that equilibrium dynamics similar to those occurring in an overlapping generations model with production can result in a model with infinite lived workers and entrepreneurs, from the existence of a cash-in-advance constraint upon workers' consumption purchases, together with a constraint of the sort discussed in Section 4, according to which entrepreneurs must finance investment entirely out of the returns to capital. In this case the equilibrium dynamics are very close to those characteristic of the Reichlin [1986] model, and as a result endogenous cycles are possible under circumstances similar to those discussed by Reichlin. In particular, in this case (unlike the Bewley example of a production economy) endogenous cycles can occur even when savings are an increasing function of the expected return.

These examples show that it is not entirely fair to criticize the examples of endogenous fluctuations in economies with overlapping generations of two-period lived consumers as involving mechanisms that could generate cycles only at very low frequencies (i.e., cycles as long or longer than the lifetimes of consumers), and hence mechanisms that must be irrelevant for the explanation of actual business cycles. The same sort of intertemporal substitution can also generate endogenous fluctuations in models where consumers are represented as infinite lived, and in which the period of the fluctuations bears no relation to any time scale relating to human biology. Indeed, the "periods" in the models with financial

constraints just described must surely be interpreted as relatively short, so that if anything these models should be criticized for only generating cycles of frequency too <u>high</u> to correspond to actual "business cycles". For example, in all of the models that yield dynamics analogous to those of the Grandmont model, perfect foresight equilibrium fluctuations must have the property that the level of real balances must never increase for more than one consecutive period; the observed pattern must consist of single periods in which real balances increase, each separated from the others by some number of periods in which real balances decrease. (This is a property of the sort of unimodal map shown in Figure 1b. Similarly, real balances cannot be higher than their stationary equilibrium level for more than one consecutive period. Hence, although (as shown by Grandmont) it is possible for equilibrium cycles to have arbitrarily long periodicity, either cycles or chaotic fluctuations will usually exhibit a strong degree of negative serial correlation, so that the period of a typical cycle (measured, say, from peak to peak) would be not much longer than two "periods" on average. In the case of the cash-in-advance interpretation of this model, a "period" should represent only the length of time for which the money received from the sale of labor must be held before it can be spent upon "cash goods", presumably a very short period, so that cycles of this kind would be very short. In the case of the borrowing constraint interpretation, the "periods" could be longer but not in any event longer than the reciprocal of the velocity of money (so only a few months), so that the cycles are still too short to be business cycles. Reichlin's overlapping generations model yields cycles that are approximately six "periods" in length (again, from peak to peak, since the fluctuations need not repeat exactly in any finite time), or three times the lifetime of consumers, in the life cycle

interpretation of his model. This is a time scale much longer than that on which "business cycles" occur, it is even much too long to be a theory of "Kondraftieff waves". But in the Woodford reinterpretation of this model, six "periods" is still too short for a "business cycle", under the most reasonable interpretation of the length of a "period" in a cash-in-advance model (a week or at most a month). (From this point of view, the restriction of attention to purely deterministic endogenous fluctuations in these models makes them look particularly unrealistic. For example, Woodford [1988a] shows that if one considers stationary "sunspot" equilibria rather than only the limiting case of purely deterministic cycles, it is possible to obtain cycles that last for several quarters even if "periods" are identified as a week or less.)

Perfect foresight equilibrium is also indeterminate in many other kinds of models with infinite lived agents in the case of various market imperfections that need not involve financial constraints. In many such cases endogenous cycles or chaotic dynamics are among the possible types of equilibrium dynamics. For example, Shleifer [1986] demonstrates the existence of a large number of types of equilibrium cycles in the case of an economy in which the firm that implements a cost-saving technological innovation obtains a one-period monopoly of its market (before imitation restores perfect competition, as in the Deneckere-Judd model discussed in Section 4), and in which firms can choose to deter implementation of such discoveries so as to obtain the one period of monopoly rents in a higher-demand period. Chaotic dynamics are also consistent with perfect foresight equilibrium, since Shleifer shows that the equilibrium conditions place only very weak restrictions upon the possible pattern of time lags between periods of implementation of new technologies. Murphy et al. [1988]

similarly demonstrate the indeterminacy of equilibrium and the possibility of equilibrium cycles of many different periods in the case of a model of durable goods production with increasing returns. In this model equilibrium must be cyclic (no steady state equilibrium exists), although many different cycles are possible. Diamond and Fudenberg [1984] demonstrate the possible existence of endogenous equilibrium cycles in an economy with search externalities of the kind proposed by Diamond [1982]. Drazen [1988] also makes a similar point. Indeterminacy and endogenous cycles are shown to be possible for similar reasons in other models with externalities by Howitt and McAfee [1988] and Hammour [1988a].

As in the previous section, an objection to all of these models (except that of Murphy et al.), is that the conditions for perfect foresight equilibrium in themselves provide no reason why a fluctuating equilibrium should occur, rather than a stationary equilibrium or an equilibrium converging asymptotically to a stationary state. Again, however, one may well be able to show that reasonable sorts of disequilibrium learning dynamics would under some circumstances diverge from the stationary state and converge to perpetual fluctuations. In the case of each of the examples just discussed that result in perfect foresight equilibrium dynamics analogous to those of the Grandmont [1985] model, the analysis of learning dynamics by Grandmont and Laroque [1986] is directly applicable. Similar results are not available for other examples, and such analyses would be a useful topic for further research.

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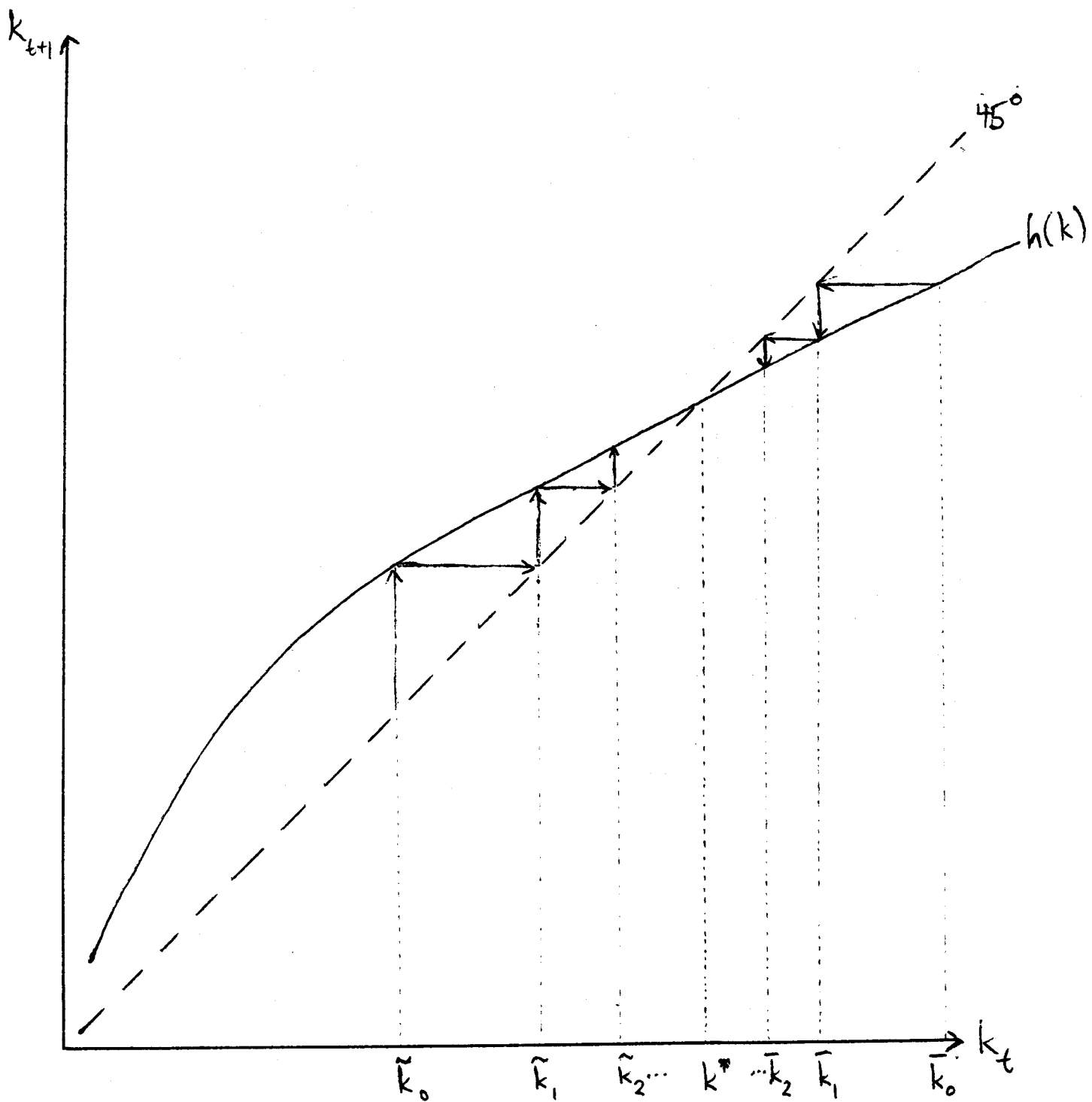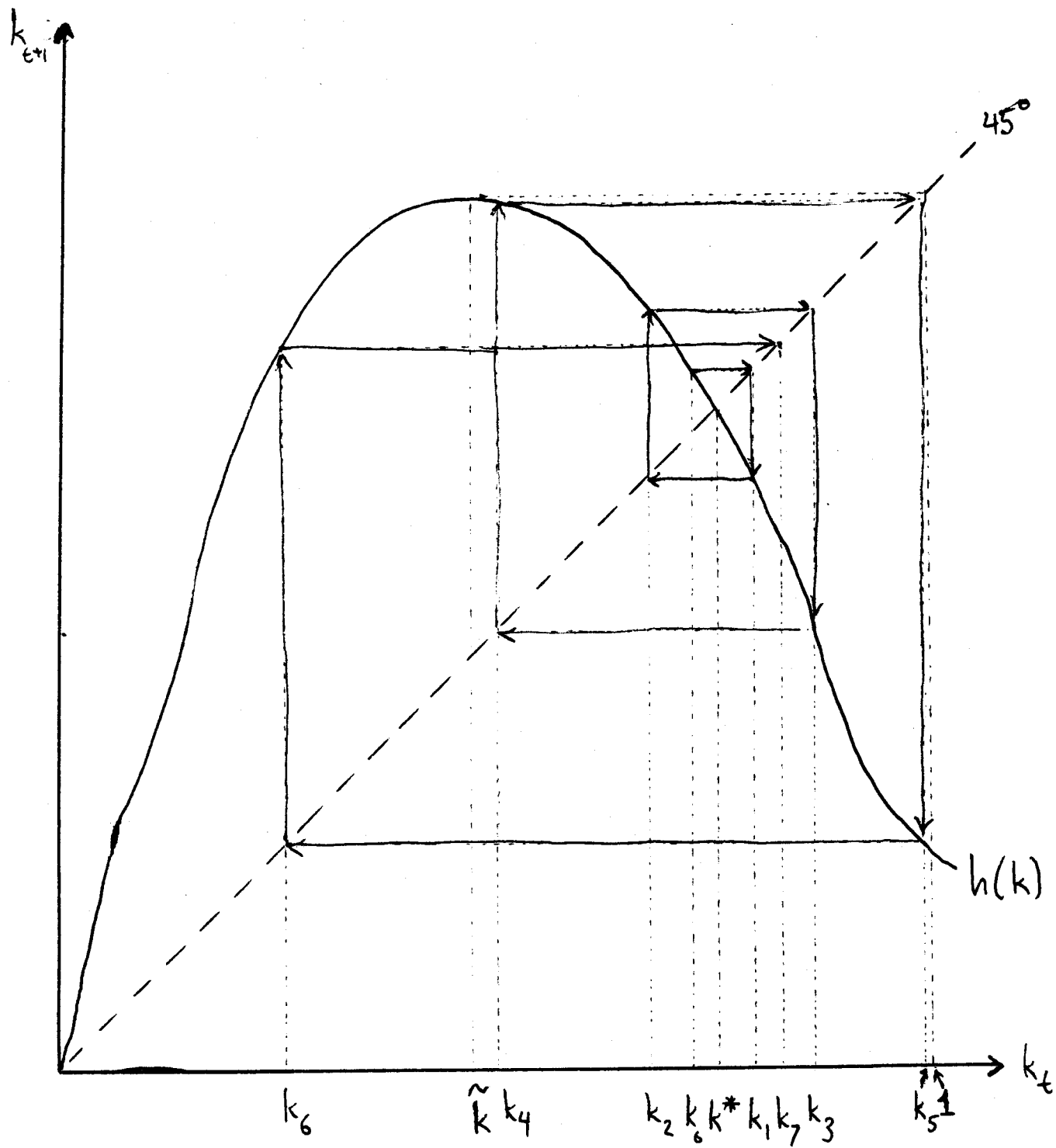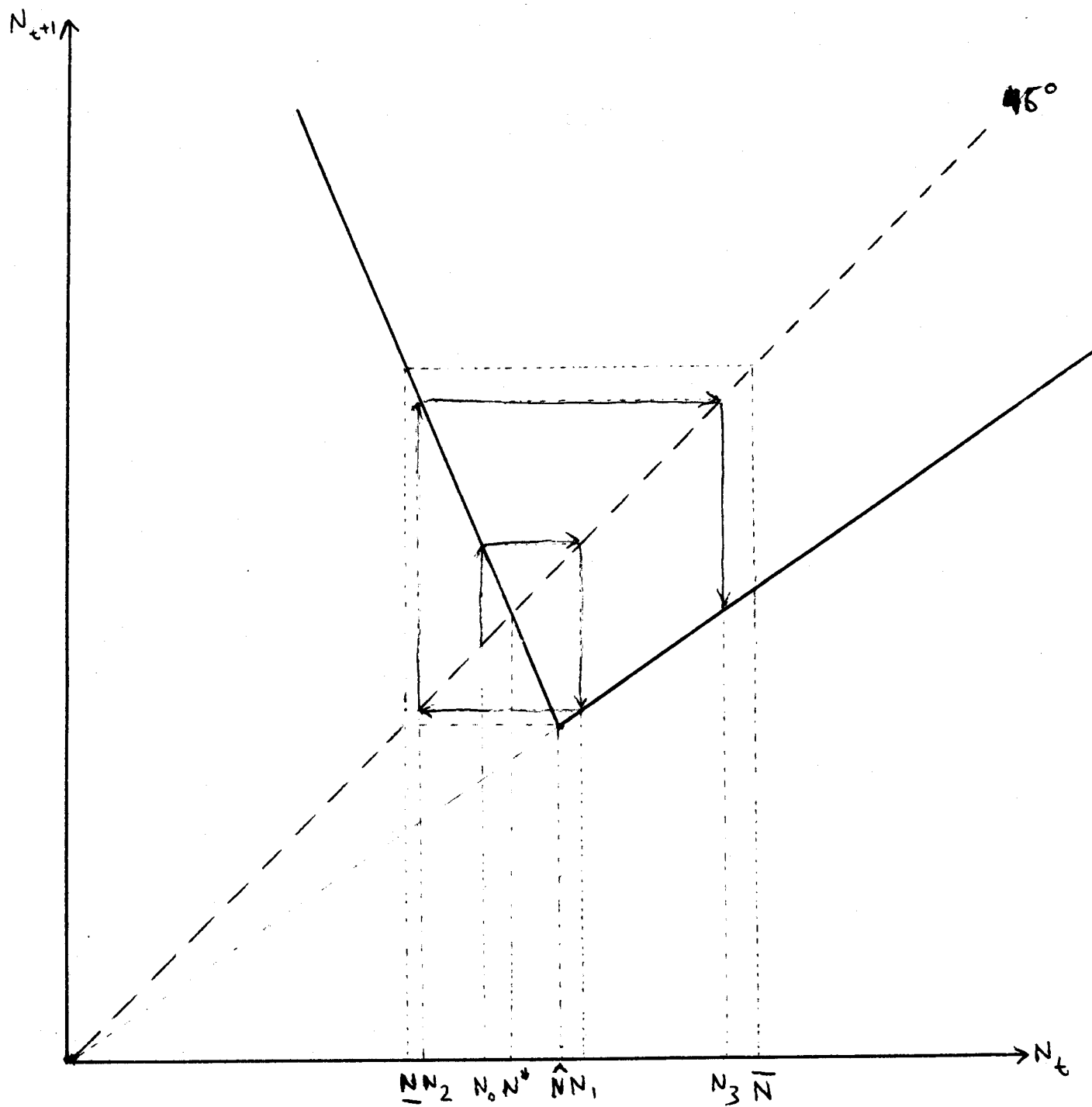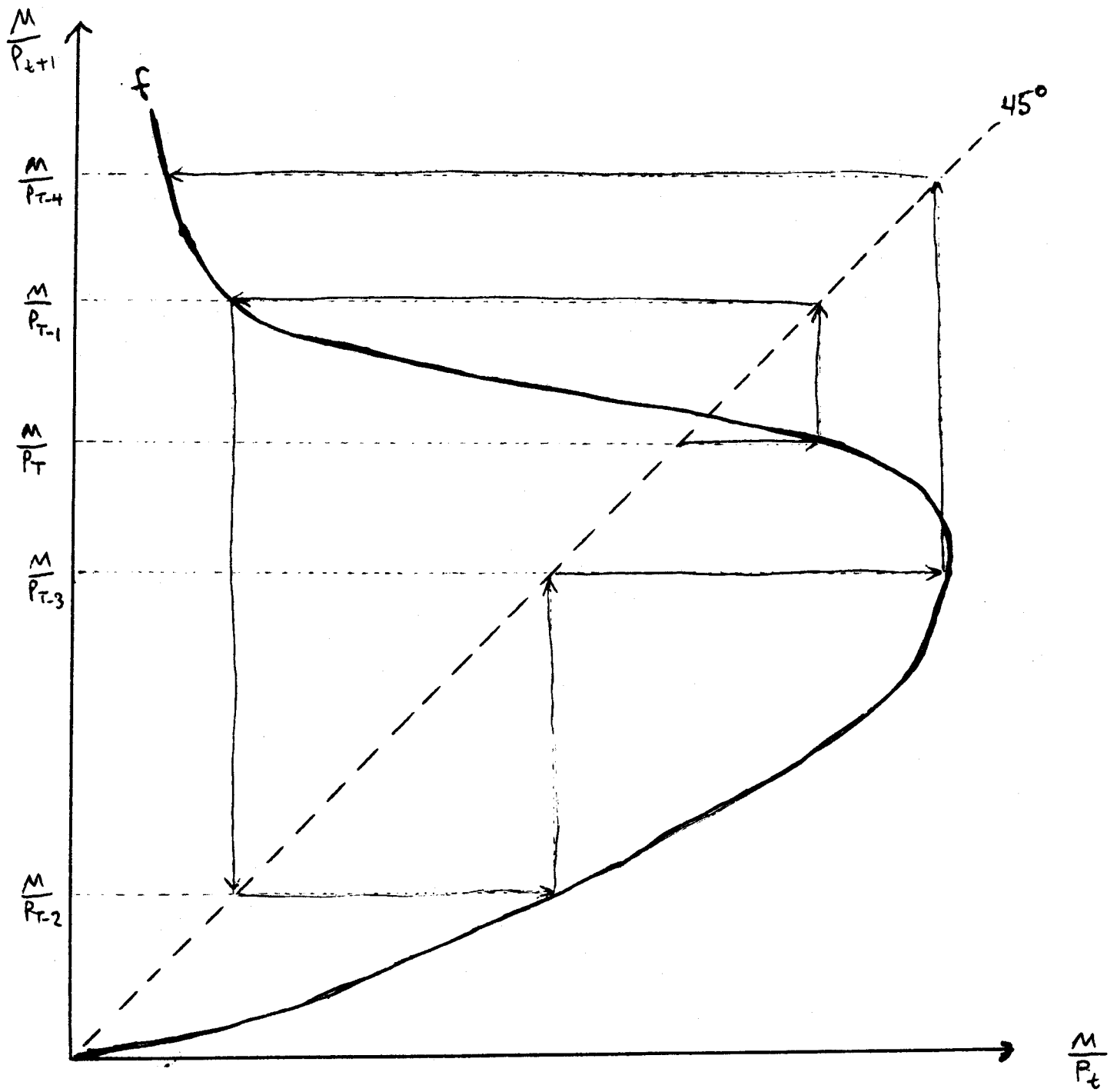Insensitivity to Initial Conditions," Journal of Mathematical Economics
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Figure 1a.

Figure 1b.

Figure 2.

Figure 3.

Figure 4.

Figure 5.