

**TWO-STAGE LOTTERIES WITHOUT THE REDUCTION AXIOM<sup>1</sup>**

by

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Abstract: This paper analyzes preference relations over two-stage lotteries, i.e., lotteries having as outcomes tickets for other, simple, lotteries. Empirical evidence indicates that decision makers do not always behave in accordance with the reduction of compound lotteries axiom, but it seems that they satisfy a compound independence axiom (also known as the certainty equivalent mechanism). It turns out that although the reduction and the compound independence axioms together with continuity imply expected utility theory, each of them by itself is compatible with all possible preference relations over simple lotteries. By using these axioms I analyze three different versions of expected utility for two-stage lotteries.

The second part of the paper is devoted to possible replacements of the reduction axiom. For this I suggest several different compound dominance axioms. These axioms compare two-stage lotteries by the probability they assign to the upper and lower sets of all simple lotteries  $X$ . (For a simple lottery  $X$ , its upper (lower) set is the set of lotteries that dominate (are dominated by)  $X$  by first order stochastic dominance.) It turns out that these axioms are all strictly weaker than the reduction of compound lotteries axiom. The main theoretical results of this part are: (1) an axiomatic basis for expected utility theory that does not require the reduction axiom and (2) a new axiomatization of the anticipated utility model (also known as expected utility with rank-dependent probabilities). These representation theorems indicate that to a certain extent the rank dependent probabilities model is a natural extension of expected utility theory. Finally, I show that some paradoxes in expected utility theory can be explained, provided one is willing to use the compound independence rather than the reduction axiom.

Keywords: Two-stage lotteries, compound independence, reduction of compound lotteries axiom, compound dominance, expected utility, anticipated utility.

## 1. INTRODUCTION

One of the common vindications of expected utility theory, besides its usefulness, is that it is based upon normatively appealing assumptions. Special attention was given to the independence axiom, which became almost synonymous with the theory itself. This axiom states that a lottery  $X = (x_1, p_1; \dots; x_n, p_n)$  is preferred to a lottery  $Y = (y_1, q_1; \dots; y_m, q_m)$  if and only if for every lottery  $Z = (z_1, r_1; \dots; z_\ell, r_\ell)$  and  $\alpha \in (0, 1]$ , the mixture  $\alpha X + (1-\alpha)Z = (x_1, \alpha p_1; \dots; x_n, \alpha p_n; z_1, (1-\alpha)r_1; \dots; z_\ell, (1-\alpha)r_\ell)$  is preferred to the mixture  $\alpha Y + (1-\alpha)Z = (y_1, \alpha q_1; \dots; y_m, \alpha q_m; z_1, (1-\alpha)r_1; \dots; z_\ell, (1-\alpha)r_\ell)$ . Essentially, this is the key axiom in Marschak (1950) and Herstein and Milnor (1953). Almost all writers in recent years have criticized and rejected this axiom. Some tried to weaken it (Quiggin, 1982; Chew, 1983; Dekel, 1986; Chew, Epstein, and Segal, 1988), to replace it with other axioms (Yaari, 1987; Röll, 1987), or to abandon it completely (Machina, 1982).

In this paper I try to consider nonexpected utility theory while keeping the spirit of the independence axiom. For this I use the richer setting of two-stage lotteries, their outcomes being tickets for other, simple, lotteries. When adapted to two-stage lotteries, the independence axiom states that the two-stage lottery  $A$ , yielding with probability  $\alpha$  a ticket for lottery  $X$  and with probability  $1-\alpha$  a ticket for lottery  $Z$ , is preferred to the two-stage lottery  $B$ , which is the same as  $A$  with  $Y$  instead of  $X$ , if and only if the one-stage lottery  $X$  is preferred to the one-stage lottery  $Y$ . Call this axiom compound independence, and call the independence axiom for simple lotteries mixture independence.

The compound independence axiom by itself does not imply expected utility theory, as it does not compare two-stage lotteries to one-stage

lotteries. For this, one has to add the reduction of compound lotteries axiom, stating that a two-stage lottery is equally as attractive as the one-stage lottery yielding the same prizes with the corresponding multiplied probabilities (see Samuelson, 1952). The compound independence axiom and the reduction of compound lotteries axiom together imply the mixture independence axiom. However, these two axioms are mutually independent, and each one of them by itself is compatible with all possible preference relations over simple lotteries.

The key question is therefore this: Suppose that in a richer setting one can distinguish between one- and two-stage lotteries, thus making it possible to assume compound independence in a form that is distinct from mixture independence. Will this richer setting and distinction imply a better understanding of decision making under uncertainty? I believe that the answer to this question is yes. This distinction can obtain more normatively acceptable axiomatizations of expected utility theory (see Section 3 below). It may also supply us with axiomatizations of alternative theories (Section 4). On the other hand, the compound independence axiom is sufficiently weak so that by itself it does not impose any restrictions on preference relations over simple lotteries (Section 2).

The distinction between one- and two-stage lotteries has some theoretical advantages. By assuming compound independence one can prove that Nash equilibrium always exists (Safra and Segal, 1988) even in a non-expected utility framework. However, if one assumes the reduction but not the compound independence axiom, Nash equilibrium exists only if preferences are quasi-concave (Crawford, 1987). Also, Green's (1987) claim that whenever his preferences fail to be quasi-convex, an individual can be manipulated to replace a lottery  $X$  with a lottery  $Y$  which is stochastically

dominated by  $X$ , depends on the reduction axiom. So does Border's (1987) defence of expected utility theory.

The compound independence axiom has a strong normative appeal. Wakker (1988) proved that violations of this axiom imply that decision makers may be better off rejecting information. Finally, assuming the compound independence but not the reduction axiom can explain some nonexpected utility behavioral patterns, as demonstrated in Section 5 below. (See also Segal, 1987b.) Consider, for example, the following decision problem from Kahneman and Tversky (1979):

Problem 1: Choose between  $X_1 = (3000, 1)$  and  $Y_1 = (0, 0.2; 4000, 0.8)$ .

Problem 2: Choose between  $X_2 = (0, 0.75; 3000, 0.25)$  and  $Y_2 = (0, 0.8; 4000, 0.2)$ .

Problem 3: Choose between  $A = (0, 0.75; X_1, 0.25)$  and  $B = (0, 0.75; Y_1, 0.25)$ .

The lotteries in problems 1 and 2 are simple lotteries. The lotteries in problem 3 are compound lotteries, depicted in Figure 1.

[Insert Figure 1 here.]

Kahneman and Tversky found that most subjects prefer  $X_1$  to  $Y_1$  but  $Y_2$  to  $X_2$  (a clear violation of expected utility theory and of the mixture independence axiom). Note that by the reduction axiom,  $A \sim X_2$  and  $B \sim Y_2$ , hence  $Y_2 > X_2$  implies  $B > A$ . By the compound independence axiom, on the other hand,  $A \succeq B$  if and only if  $X_1 \succeq Y_1$ , hence  $X_1 > Y_1$  implies  $A > B$ . Kahneman and Tversky found that most subjects prefer  $A$  to  $B$ , in agreement with the compound independence axiom, but in disagreement with the reduction axiom.

In the next section I formally define one- and two-stage lotteries and show the connection between the reduction of compound lotteries axiom,

compound independence, and mixture independence, as well as the connection between these axioms and different forms of expected utility for two-stage lotteries. As an alternative to the reduction axiom I suggest in Section 3 a compound dominance axiom that is a stronger version of the stochastic dominance axiom for two-stage lotteries but still weaker than the reduction axiom. It turns out that this axiom, together with the axioms of compound independence and strict first-order stochastic dominance for one-stage lotteries, implies the expected utility representation. In Section 4, I discuss the connection between the concept of compound dominance and Quiggin's (1982) anticipated utility theory (also known as expected utility with rank-dependent probabilities) and prove a representation theorem for this theory.

Section 5 discusses some empirical evidence and shows that a rejection of the reduction axiom while accepting the compound independence axiom may solve some nonexpected utility paradoxes, as well as some phenomena that do not contradict the expected utility hypothesis but seem to imply risk loving. Section 6 concludes with some remarks on the literature and some final comments.

## 2. DEFINITIONS

Let  $L_1 = \{(x_1, p_1; \dots; x_n, p_n) : x_1, \dots, x_n \in [0, M], x_1 \leq \dots \leq x_n, p_1, \dots, p_n \geq 0, \sum p_i = 1\}$ . Elements of  $L_1$ , denoted by  $X, Y$ , etc., represent simple lotteries, yielding  $x_i$  dollars with probability  $p_i$ ,  $i = 1, \dots, n$ . For  $X = (x_1, p_1; \dots; x_n, p_n) \in L_1$ , define the cumulative distribution function  $F_X$  by  $F_X(x) = \Pr(X \leq x)$ .

On  $L_1$  there exists a complete and transitive preference relation  $\succeq_1$ .  $X \sim_1 Y$  if and only if  $X \succeq_1 Y$  and  $Y \succeq_1 X$ , and  $X >_1 Y$  if and only if  $X \succeq_1 Y$  but not  $Y \succeq_1 X$ . Assume that the relation  $\succeq_1$  satisfies the

following continuity axiom:

Continuity:  $\succeq_1$  is continuous in the topology of weak convergence. That is, if  $X, Y, Y_1, Y_2, \dots \in L_1$  such that at each continuity point  $x$  of  $F_Y$ ,  $F_{Y_i}(x) \rightarrow F_Y(x)$ , and if for all  $i$ ,  $X \succeq_1 Y_i$ , then  $X \succeq_1 Y$ . Similarly, if for all  $i$ ,  $Y_i \succeq_1 X$ , then  $Y \succeq_1 X$ .

$V: L_1 \rightarrow \mathbb{R}$  represents the preference relation  $\succeq_1$  if for every  $X, Y \in L_1$ ,  $X \succeq_1 Y$  if and only if  $V(X) \geq V(Y)$ . The most celebrated representation is the expected utility functional

$$(1) \quad V(X) = \sum p_i u(x_i).$$

Preference relations represented by this functional satisfy the continuity axiom whenever  $u$  is continuous (and hence bounded). Of course, this axiom does not imply the expected utility functional. Further assumptions are required, either on  $\succeq_1$  itself or on its extension to two-stage lotteries.

The outline of the space  $L_1$  assumes that all the events in all the lotteries are ethically neutral in the sense that the decision maker cares about an event's probability, but not about the event itself (see Ramsey, 1931). This assumption is plausible when the prizes are measured in terms of money and the probabilities are determined by an objective mechanism such as roulette, coins, or dice. In particular, it implies that the decision maker does not care whether the winning event at the lottery  $(0, 0.75; 100, 0.25)$  is two heads on two coins or two heads from the same coin being tossed twice with no time passing between the two tosses. On the other hand, it is not necessarily true that the decision maker is indifferent between the lotteries  $Z^*$  - "flip two coins at time  $t_1$ , win \$100 if both fall heads up, \$0 otherwise" and  $W^*$ , which is the same as  $Z^*$  but with

the second coin to be tossed at time  $t_2 > t_1$ , especially when a sufficiently long time passes between  $t_1$  and  $t_2$ , or when for other reasons, the two stages are clearly distinct. This discussion leads to the construction of two-stage lotteries.

Let  $L_2 = \{(X_1, q_1; \dots; X_m, q_m) : X_1, \dots, X_m \in L_1, q_1, \dots, q_m \geq 0, \sum q_i = 1\}$ . Elements of  $L_2$ , called two-stage lotteries, are denoted by  $A, B$ , etc. A lottery  $A \in L_2$  yields a ticket to lottery  $X_i$  with probability  $q_i$ ,  $i = 1, \dots, m$ . More specifically, at time  $t_1$  the decision maker faces the lottery  $(1, q_1; \dots; m, q_m)$ . Upon winning the number  $i$ , he participates at time  $t_2 > t_1$  in the lottery  $X_i \in L_1$ . Assume throughout this paper that all prizes are delivered at time  $\bar{t} > t_2$ , and all decisions are made at time  $t_0 < t_1$ .

Natural isomorphisms exist between  $L_1$  and two subsets of  $L_2$ . The first subset,  $\Delta$ , consists of degenerate lotteries in  $L_2$ . The second subset,  $\Gamma$ , consists of lotteries in  $L_2$ , outcomes of which are degenerate in  $L_1$ . Formally,  $\Delta = \{(X, 1) : X \in L_1\}$ , and  $\Gamma = \{(x_1, 1), p_1; \dots; (x_n, 1), p_n) : X = (x_1, p_1; \dots; x_n, p_n) \in L_1\}$ . For  $X \in L_1$ , the elements of  $\Delta$  and  $\Gamma$  that correspond to  $X$  are denoted by  $\delta_X$  and  $\gamma_X$ , respectively.

On  $L_2$  there exists at time  $t_0$  a complete and transitive preference relation  $\succsim_2$ . Throughout this paper,  $U : L_2 \rightarrow \mathbb{R}$  denotes a representation function of  $\succsim_2$ . This preference relation induces by restriction preference relations  $\succsim_\Delta$  and  $\succsim_\Gamma$  on  $\Delta$  and  $\Gamma$ , respectively. These two are  $\succsim_1$ -type preferences in the sense that their domain is isomorphic to  $L_1$ .

The construction of the space  $L_2$  and the definition of two-stage lotteries assumed that decision makers do not find themselves obliged to multiply the probabilities of the two stages. If the above-mentioned two lotteries  $Z^*$  and  $W^*$ , which differ at the time at which the second coin



is tossed, ought to be considered equivalent by all decision makers regardless of their preferences, then both should be written as  $(0,0.75;100,0.25)$ . In other words, our setting assumes that decision makers do not find it necessary to follow the reduction of compound lotteries axiom, given below.

**Reduction of Compound Lotteries Axiom:** Let  $X_i = (x_1^i, p_1^i; \dots; x_{n_i}^i, p_{n_i}^i)$ ,  $i = 1, \dots, m$ , let  $A = (X_1, q_1; \dots; X_m, q_m)$ , and define

$$R(A) = (x_1^1, q_1 p_1^1; \dots; x_{n_1}^1, q_1 p_{n_1}^1; \dots; x_1^m, q_m p_1^m; \dots; x_{n_m}^m, q_m p_{n_m}^m)$$

The decision maker is indifferent between the two-stage lottery  $A$  and the one-stage lottery  $R(A)$ . That is,  $A \sim_2 \gamma_{R(A)}$ .

As mentioned in the Introduction, empirical experiments indicate that decision makers do not always obey this axiom. Recently, Schoemaker (1987) found new such evidence. Consider the lottery  $(0, 1-p; x, p)$ . The decision maker has to choose between the following two options: In  $A$ ,  $p = 0.5$ , and  $x$  has a uniform distribution over  $[0, 1]$ , while in  $B$ ,  $x = 0.5$ , and  $p$  has a uniform distribution over  $[0, 1]$ . Certainly, one can interpret these two options as two-stage lotteries (although this is not the only possible interpretation -- see Schoemaker, 1987). Using the reduction of compound lotteries axiom,  $A$  and  $B$  reduce to the lotteries  $X$  and  $Y$  in Figure 2. The lottery  $X$  is obtained from  $Y$  by a mean preserving increase in risk (note that  $\alpha$  and  $\beta$  are congruent triangles), hence a risk averse decision maker should prefer  $Y$  to  $X$  and  $B$  to  $A$ . As discovered by Schoemaker, most subjects prefer  $A$  to  $B$ . Other violations of the reduction axiom were found by Ronen (1971) and Snowball and Brown (1979), although, as reported by Keller (1985), these violations may depend on the way the problems are formed.

[Insert Figure 2 here.]

There may be several reasons why some decision makers do not use the reduction of compound lotteries axiom as a guideline for evaluating two-stage lotteries, even if one assumes that people accept the basic laws of probability theory. (For example, at  $t_1$  and  $t_2$  decision makers may use the rule that for independent events  $S_1$  and  $S_2$ ,  $P(S_1 \cap S_2) = P(S_1)P(S_2)$ .<sup>2</sup>) In this model, the reason is that some events are realized at time  $t_1$  while others are realized at time  $t_2$ . This may affect the desirability of a two-stage lottery (as compared to a similar one-stage lottery) in at least two ways. Firstly, the decision maker may have preferences for the number of lotteries in which he participates. This argument holds whenever the two stages are clearly distinct, even without the time element. Secondly, he may have preferences for early or later resolutions of uncertainty. That is, he is not indifferent between the lotteries  $\gamma_X$  and  $\delta_X$ , which are the same except for their timing; the uncertainty of  $\gamma_X$  is resolved at time  $t_1$ , while the uncertainty of  $\delta_X$  is resolved at time  $t_2$ .<sup>3</sup> This later reasoning is especially plausible if preferences are induced from more primitive decision problems such as consumption-saving problems. (See Mossin, 1969; Spence and Zeckhauser, 1972; Dreze and Modigliani, 1972; Kreps and Porteus, 1978, 1979; Epstein, 1980; Machina, 1984; and Chew and Epstein, 1987a).

Of course, if the decision maker does not care when the uncertainty is resolved, that is, if for every  $X \in L_1$ , he is indifferent between  $\gamma_X$  and  $\delta_X$ , then he will have the same preference relation over  $\Gamma$  and  $\Delta$ . Let

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<sup>2</sup>De Finetti (1937, 1977) proved that violations of this rule expose the decision maker to Dutch books. These arguments are relevant only when no real time is involved. See also Marschak (1975).

<sup>3</sup>This nonindifference may persist even, as assumed above, when the prizes of all lotteries are delivered at time  $\bar{t} > t_2$ .

$X, Y \in L_1$ . If  $\delta_X \sim_2 \gamma_X$  and  $\delta_Y \sim_2 \gamma_Y$ , then it follows from the transitivity assumption that  $\delta_X \succeq_\Delta \delta_Y$  if and only if  $\gamma_X \succeq_\Gamma \gamma_Y$ . Less evident is that the opposite holds true as well. That is, if the decision maker has the same preference relation over  $\Delta$  and  $\Gamma$ , then he is indifferent to the timing of the resolution of the uncertainty. Let  $X \in L_1$ . By continuity, there exists a number  $x$  such that  $\delta_X \sim_\Delta \delta_{(x,1)}$ . Because the decision maker's preferences over  $\Delta$  and  $\Gamma$  are the same, it is also true that  $\gamma_X \sim_\Gamma \gamma_{(x,1)}$ . The lotteries  $\delta_{(x,1)} \in \Delta$  and  $\gamma_{(x,1)} \in \Gamma$  represent the same lottery  $((x,1),1)$ , which is a sure gain of  $x$  dollars, paid at time  $\bar{t}$  (recall that all preferences are expressed at time  $t_0 < t_1$ ). It thus follows that  $\delta_X \sim_2 \gamma_X$ . This discussion is summarized in the following axiom and lemma:

**Time Neutrality Axiom:** For every lottery  $X \in L_1$ ,  $\delta_X \sim_2 \gamma_X$ .

**Lemma 1:** The preference relations  $\succeq_\Gamma$  and  $\succeq_\Delta$  are the same  $\succeq_1$ -type relation (i.e.,  $\delta_X \succeq_\Delta \delta_Y \iff \gamma_X \succeq_\Gamma \gamma_Y$ ) if and only if the decision maker satisfies the time neutrality axiom.

The implication of the timing of the resolution of the uncertainty on decision makers' behavior is especially important when it may affect current decisions such as consumption-saving problems (see references above). It is usually believed that this is the reason that decision makers are not indifferent between one- and two-stage lotteries. Although I believe that in general people care for the resolution timing of the uncertainty, I want to emphasize here the other factor, which is too often neglected. Consider again the lotteries  $Z^*$  and  $W^*$  of the above example. (These two lotteries differ in the time at which the second coin is flipped,  $t_1$  or  $t_2$ .) Lottery  $Z^*$  involves just one lottery at time  $t_1$ , but  $W^*$  involves two

lotteries, one at  $t_1$ , the other at  $t_2$ . It may well happen that, even with the same compound probabilities, the decision maker has preferences for more or less lotteries. I adopt this aspect of two-stage lotteries and will assume later on that the decision maker satisfies the time neutrality axiom, hence his preference relations on  $\Gamma$  and  $\Delta$  are the same.

Let  $X, Y \in L_1$  be such that  $\delta_X \succeq_{\Delta} \delta_Y$ . Originally, both  $\delta_X$  and  $\delta_Y$  are available, and once the decision maker announces his preferences, he participates in his preferred lottery. As mentioned above, his participation in  $X$  (or  $Y$ ) may be uncertain, because  $X$  and  $Y$  may themselves be possible outcomes of another, non-trivial, lottery. Formally, let  $A = (X, q; Z, 1-q)$  and  $B = (Y, q; Z, 1-q)$ . With probability  $1-q$ , both  $A$  and  $B$  yield a ticket for  $Z$ . The lotteries  $A$  and  $B$  yield different outcomes only if the  $q$ -probability event happens. In that case,  $A$  yields a ticket for  $X$  while  $B$  yields a ticket for  $Y$ . As the unconditional lottery  $\delta_X$  is preferred to the unconditional lottery  $\delta_Y$  (the uncertainty of these two lotteries is resolved at time  $t_2$ ), it is reasonable to assume that  $A \succ_2 B$ . Of course, this assumption does not follow from the assumptions made so far.

**Compound Independence Axiom:** Let  $X, Y \in L_1$ , and let  $A = (Z_1, q_1; \dots; X, q_1; \dots; Z_m, q_m)$  and  $B = (Z_1, q_1; \dots; Y, q_1; \dots; Z_m, q_m)$  be two lotteries in  $L_2$ .  $A \succeq_2 B$  if and only if  $\delta_X \succeq_{\Delta} \delta_Y$ .

Let  $CE_1(X)$  be the certainty equivalent of  $X$ , given implicitly by  $(CE_1(X), 1) \sim_1 X$ . Let  $CE_{\Gamma}(X)$  and  $CE_{\Delta}(X)$  be the certainty equivalents of  $X$  with respect to  $\succeq_{\Gamma}$  and  $\succeq_{\Delta}$ , respectively. That is,  $((CE_{\Gamma}(x), 1), 1) \sim_2 \gamma_X$  and  $((CE_{\Delta}(x), 1), 1) \sim_2 \delta_X$ . If  $\succeq_2$  satisfies the compound independence axiom, then

$$(X_1, q_1; \dots; X_m, q_m) \sim_2 ((CE_{\Delta}(X_1), 1), q_1; \dots; (CE_{\Delta}(X_m), 1), q_m)$$

The left-hand side of this last equivalence is a general two-stage lottery. The right-hand side is an element of  $\Gamma$ , the set of lotteries in  $L_2$  where all the uncertainty is resolved at time  $t_1$ .

The compound independence axiom and the reduction of compound lotteries axiom are compatible with all preference relations on  $L_1$ . Let the preference relation  $\succeq_1$  on  $L_1$  be represented by  $V$  and define two preference relations on  $L_2$  as follows:

- (a) Given  $A, B \in L_2$ , let  $A \succeq_2 B$  if and only if  $R(A) \succeq_1 R(B)$ . This preference relation on  $L_2$  is the only one to satisfy the reduction of compound lotteries axiom such that  $\succeq_\Gamma = \succeq_\Delta = \succeq_1$ . It can be represented by  $U(A) = V(R(A))$ .
- (b) Given  $A = (X_1, q_1; \dots; X_m, q_m)$  and  $B = (Y_1, r_1; \dots; Y_\ell, r_\ell)$ , let  $A \succeq_2 B$  if and only if  $(CE_1(X_1), q_1; \dots; CE_1(X_m), q_m) \succeq_1 (CE_1(Y_1), r_1; \dots; CE_1(Y_\ell), r_\ell)$ . This preference relation on  $L_2$  is the only one to satisfy the compound independence and time neutrality axioms such that  $\succeq_\Gamma = \succeq_\Delta = \succeq_1$ .<sup>4</sup> It can be represented by  $U(A) = V(CE_1(X_1), q_1; \dots; CE_1(X_m), q_m)$ .

Without the time neutrality axiom, the compound independence axiom is compatible with any two preference relations on  $L_1$ . Let  $\succeq_1^1$  and  $\succeq_1^2$  be two such preferences and let  $CE_1^2(X)$  be the certainty equivalent of  $X$  with respect to  $\succeq_1^2$ .

- (c) Given  $A$  and  $B$  as in (b), let  $A \succeq_2 B$  if and only if  $(CE_1^2(X_1), q_1; \dots; CE_1^2(X_m), q_m) \succeq_1^1 (CE_1^2(Y_1), r_1; \dots; CE_1^2(Y_\ell), r_\ell)$ . In this case,  $\succeq_\Gamma = \succeq_1^1$  and  $\succeq_\Delta = \succeq_1^2$ . If  $V^1$  represents  $\succeq_1^1$ , then  $\succeq_2$  can be represented by  $U(A) = V^1(CE_1^2(X_1), q_1; \dots; CE_1^2(X_m), q_m)$ .

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<sup>4</sup>This structure proves that time neutrality does not imply the reduction axiom.

To illustrate, consider the extensions of Quiggin's (1982) anticipated utility functional to two-stage lotteries via the reduction axiom and via the compound independence and the time neutrality axioms. Let  $X = (x_1, p_1; \dots; x_n, p_n)$  where  $x_1 \leq \dots \leq x_n$ . The anticipated utility of this lottery is given by

$$(2) \quad V(X) = \sum_{i=1}^{n-1} u(x_i) \left[ f\left(\sum_{j=i}^n p_j\right) - f\left(\sum_{j=i+1}^n p_j\right) \right] + u(x_n) f(p_n)$$

where  $u$  and  $f$  are continuous and strictly increasing,  $u(0) = 0$ ,  $f(0) = 0$ , and  $f(1) = 1$ . Let  $g(p) = 1 - f(1-p)$  and obtain from (2) that

$$V(X) = u(x_1)g(p_1) + \sum_{i=2}^n u(x_i) \left[ g\left(\sum_{j=1}^i p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right) \right].$$

Some writers use this version of the anticipated utility functional. However, for the discussion in Section 4, the expression in (2) is the more natural. The reader is left the straightforward but tedious task of extending (2) to two-stage lotteries as in cases (a) and (b) above (see Segal, 1987b). The reader can also easily verify that these two extensions coincide if and only if  $f$  is linear. Later on, I use a special case of these extensions. Let  $x > 0$  and let  $A = ((0, 1-p; x, p), q; (0, 1), 1-q)$ . By the reduction axiom,

$$(3) \quad V(A) = u(x)f(pq)$$

while by the compound independence and the time neutrality axioms,

$$(4) \quad V(A) = u(x)f(p)f(q).^5$$

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<sup>5</sup>The expression at (3) and (4) equal each other for all  $p$  and  $q$  if and only if  $f(p) = p^\alpha$  (Aczél, 1966).

Let  $u(x) = x$  and  $f(p) = (e^p - 1)/(e - 1)$  and get  $u(3000)f(1) > u(4000)f(0.8)$ , but  $u(4000)f(0.2) > u(3000)f(0.25) = u(3000)f(1)f(0.25) > u(4000)f(0.8)f(0.25)$ . These inequalities are in agreement with the reported common response to problems 1-3 of the Introduction.<sup>6</sup>

The reduction and the compound independence axioms, both on  $\succeq_2$ , together imply the following mixture independence axiom on  $\succeq_1$ :

**Mixture Independence:** Let  $X = (x_1, p_1; \dots; x_n, p_n)$ ,  $Y = (y_1, q_1; \dots; y_m, q_m)$ ,  $Z = (z_1, r_1; \dots; z_\ell, r_\ell) \in L_1$ , and let  $\alpha \in (0, 1]$ .  $X \succeq_1 Y$  if and only if  $\alpha X + (1-\alpha)Z = (x_1, \alpha p_1; \dots; x_n, \alpha p_n; z_1, (1-\alpha)r_1; \dots; z_\ell, (1-\alpha)r_\ell) \succeq_1 \alpha Y + (1-\alpha)Z = (y_1, \alpha q_1; \dots; y_m, \alpha q_m; z_1, (1-\alpha)r_1; \dots; z_\ell, (1-\alpha)r_\ell)$ . We say that  $\succeq_2$  satisfies this axiom if both  $\succeq_\Gamma$  and  $\succeq_\Delta$  satisfy it.

This is a slightly stronger version of Marschak's (1950) Postulate IV<sub>2</sub>. It is well known that this axiom, together with continuity, completeness, and transitivity, implies the expected utility representation (1). I now turn to a discussion of the connection between the mixture independence, compound independence, and reduction of compound lotteries axioms.

**Theorem 2:**

- (a) The three axioms, compound independence, reduction of compound lotteries, and mixture independence are pairwise independent -- no one implies another. Moreover, no one of them in conjunction with the time neutrality axiom implies any other.
- (b) The reduction axiom implies time neutrality, but mixture independence and compound independence, even together, do not.

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<sup>6</sup>For an explanation of this phenomenon, using disappointment theory with a similar compound lotteries analysis, see Loomes and Sugden (1986).

(c) The reduction and the compound independence axioms imply mixture independence, and the reduction and the mixture independence axioms imply compound independence. Mixture independence, compound independence, and time neutrality imply the reduction axiom, but no proper subset of these three axioms has this implication.

A natural question is, what preference relations are implied by these different axioms. For the next theorem consider the following three versions of expected utility for two-stage lotteries. In all cases and in Examples 1 and 2,  $X_i = (x_1, p_1^i; \dots; x_n, p_n^i)$  with  $x_1 \leq \dots \leq x_n$  and  $A = (X_1, q_1; \dots; X_m, q_m)$ .

**EU1 - Expected Utility With Reduction:** A preference relation that can be represented by

$$(5) \quad U(A) = U(X_1, q_1; \dots; X_m, q_m) = \sum_{i=1}^m \sum_{j=1}^n q_i p_j^i(x_j)$$

**EU2 - Expected Utility With Time Neutrality:** A preference relation that induces the same expected utility representation (1) of  $\succeq_{\Gamma}$  and  $\succeq_{\Delta}$ .

**EU3 - Expected Utility Without Time Neutrality:** A preference relation that induces expected utility representations (1) of  $\succeq_{\Gamma}$  and  $\succeq_{\Delta}$ , but these two representations are not necessarily the same.

Obviously,  $EU1 \Rightarrow EU2 \Rightarrow EU3$ . To illustrate these definitions, consider the following two examples. The first demonstrates a preference relation which is EU2 but not EU1. The second provides an example for an EU3 preference relation that is not EU2.

**Example 1:** For  $X, Y \in L_1$ ,  $X \neq Y$ , let  $\alpha(X, Y) = \min(x: F_X(x) \neq F_Y(x))$ .

Define a relation  $R$  on  $L_1$  such that for  $X \neq Y$ ,  $XRY$  if and only if



either  $E(X) > E(Y)$  ( $E(X)$  is the expected value of  $X$ ), or  $E(X) = E(Y)$  and  $F_X(\alpha(X,Y)) < F_Y(\alpha(X,Y))$ . Let  $A = (X_1, q_1; \dots; X_m, q_m)$ , and assume, without loss of generality, that  $X_m R \dots R X_1$ . In the following example the representation functional depends on the order of the  $X_i$ 's, and the relation  $R$  is used to ensure that the lottery  $A$  has a unique exposition.

Let  $f: [0,1] \rightarrow [0,1]$  be onto, strictly increasing, but not linear. Let  $\bar{p}_j^i = f(\sum_{k=j}^n p_k^i) - f(\sum_{k=j+1}^n p_k^i)$ ,  $j = 1, \dots, n-1$ ,  $i = 1, \dots, m$ , and let  $\bar{p}_n^i = f(p_n^i)$ ,  $i = 1, \dots, m$ . Let  $\bar{q}_i = f(\sum_{\ell=i}^m q_\ell) - f(\sum_{\ell=i+1}^m q_\ell)$ ,  $i = 1, \dots, m-1$ , and let  $\bar{q}_m = f(q_m)$ . Let  $r_j = \sum_{i=1}^m \bar{q}_i \bar{p}_j^i$ ,  $j = 1, \dots, n$ , and let  $\bar{X} = (x_1, r_1; \dots; x_n, r_n)$ . In other words,  $\bar{X}$  is obtained from  $A$  by transforming the original distributions of  $X_1, \dots, X_n$  and of  $A$  by  $f$  and by using the reduction of compound lotteries axiom for the transformed distributions. We now transform this new distribution by using the inverse of  $f$ . Define recursively  $s_n = f^{-1}(r_n)$ , and  $s_j = f^{-1}[r_j + f(\sum_{k=j+1}^n s_k)] - \sum_{k=j+1}^n s_k$ ,  $j = n-1, \dots, 1$ . Let  $u: [0, M] \rightarrow \mathbb{R}$ , and define  $U(A) = \sum s_j u(x_j)$ . One can easily verify that this preference relation induces the same expected utility relation on  $\Gamma$  and  $\Delta$  (with the utility function  $u$ ), hence it satisfies the mixture independence and time neutrality axioms. It does not satisfy the reduction of compound lotteries or the compound independence axioms (hence, by Theorem 3-a, it is not EU1). For example, let  $f(p) = (e^p - 1)/(e - 1)$  and  $u(x) = x$ .  $U((0,1), 0.5; (0,0.5; 1,0.5), 0.5) = \ln[(\sqrt{e}-1)^2/(e-1) + 1] = 0.219 \neq 0.25 = U((0,0.75; 1,0.25), 1)$ , while by the reduction axiom these two lotteries are equally attractive. To obtain a violation of the compound independence axiom note that  $U((0,0.5; 1,0.5), 1) = U((0,0.75; 2,0.25), 1) = 0.5$ , but  $U((0,1), 0.5; (0,0.5; 1,0.5), 0.5) = 0.219 \neq 0.102 = U((0,1), 0.5; (0,0.75; 2,0.25), 0.5)$ .

**Example 2:** For continuous and increasing functions  $u$  and  $v$ , let  $\succeq_2$  be represented by  $U(A) = \sum_i q_i u(v^{-1}(\sum_j p_j^i v(x_j)))$ . This preference relation induces expected utility relations on  $\Gamma$  and  $\Delta$ , with  $u$  at  $\Gamma$  and  $v$  at  $\Delta$  (see Kreps and Porteus, 1978; and Selden, 1978). It satisfies mixture independence and compound independence, but not time neutrality or the reduction axiom unless  $v = au + b$ . For example, let  $u(x) = x^2$  and  $v(x) = x$ .  $U((0,1),0.5);(0,0.5;1,0.5),0.5) = 0.125 \neq 0.25 = U((0,1),0.75;(1,1),0.25)$ , while by the reduction axiom these two lotteries are equally attractive.

**Theorem 3:** Let  $\succeq_2$  induce continuous preferences  $\succeq_\Gamma$  and  $\succeq_\Delta$ .

- (a) It is EU1 if and only if it satisfies the mixture independence, time neutrality, and compound independence axioms (if and only if it satisfies the reduction of compound lotteries and the compound independence axioms).
- (b) It is EU2 if and only if it satisfies the mixture independence and the time neutrality axioms.
- (c) It is EU3 if and only if it satisfies the mixture independence axiom.

Of course, further results follow by combining Theorems 2 and 3.

Recently, Yaari (1987) suggested the following "dual independence" axiom for decision making under uncertainty: let  $X = (x_1, p_1; \dots; x_n, p_n)$ ,  $Y = (y_1, p_1; \dots; y_n, p_n)$ , and  $Z = (z_1, p_1; \dots; z_n, p_n)$ . Of course, there is no loss of generality in assuming the same probability vectors in all three lotteries. Yaari assumed that  $X \succeq_1 Y$  if and only if for every  $\alpha \in (0,1]$ ,  $(\alpha x_1 + (1-\alpha)z_1, p_1; \dots; \alpha x_n + (1-\alpha)z_n, p_n) \succeq_1 (\alpha y_1 + (1-\alpha)z_1, p_1; \dots; \alpha y_n + (1-\alpha)z_n, p_n)$ . (See also Röell, 1987.) The above discussion makes it evident that, in our richer setting, Yaari's dual theory concerns a duality with mixture independence. In fact, because his functional is a special case of (2), his dual

(mixture) independence theory can be consistent with compound independence.

### 3. COMPOUND DOMINANCE

This section discusses several possible extensions of the concept of stochastic dominance to two-stage lotteries. Let  $X$  and  $Y$  be two one-stage lotteries. We say that  $X$  stochastically dominates  $Y$  if for every  $x$ ,  $F_X(x) \leq F_Y(x)$ .  $X$  strictly stochastically dominates  $Y$  if  $X$  stochastically dominates  $Y$  and for some  $x$ ,  $F_X(x) < F_Y(x)$ . These definitions lead to the following two axioms.

**One-Stage (Strict) Stochastic Dominance Axiom:** If  $X$  (strictly) stochastically dominates  $Y$ , then  $X \succeq_1 Y$  ( $X \succ_1 Y$ ).

We say that the relation  $\succeq_2$  satisfies the one-stage stochastic dominance and the strict one-stage stochastic dominance axioms if the induced relations  $\succeq_\Gamma$  and  $\succeq_\Delta$  satisfy them. It is well known that  $X$  stochastically dominates  $Y$  if and only if for every increasing function  $u: [0, M] \rightarrow \mathbb{R}$ ,  $E[u(X)] \geq E[u(Y)]$  (Hanoch and Levy, 1969). The one-stage stochastic dominance axiom can thus be interpreted in two different ways. Firstly, if for every possible outcome  $x$ , the lottery  $X$  gives more than  $x$  with higher probability than the lottery  $Y$ , then  $X$  is preferred to  $Y$ . Secondly, if all expected utility maximizers with increasing utility functions prefer  $X$  to  $Y$ , then  $X$  is preferred to  $Y$ .

Each of these two interpretations has its own drawbacks. The first does not naturally extend to more general lotteries where there is no natural complete order over the prizes, for example, lotteries with prizes in  $\mathbb{R}^2$  (see Levhari, Paroush and Peleg, 1975). The second has little normative appeal in a nonexpected utility world. In this section I examine these two interpretations of stochastic dominance for  $L_2$ , together with

some possible extensions of this concept.

Let  $D$  be a set of outcomes with (possibly partial) order  $\succeq_D$ , and let  $L(D)$  be the space of lotteries with outcomes in  $D$ . The function  $u: D \rightarrow \mathbb{R}$  is increasing (with respect to  $\succeq_D$ ) if whenever  $a \succeq_D b$ ,  $u(a) \geq u(b)$ . Let  $u^*(D, \succeq_D)$  be the set of all the increasing (with respect to  $\succeq_D$ ) functions  $u: D \rightarrow \mathbb{R}$ .

**Definition:** Let  $A = (a_1, p_1; \dots; a_m, p_m)$ ,  $B = (b_1, q_1; \dots; b_\ell, q_\ell) \in L(D)$ . The lottery  $A$  stochastically dominates the lottery  $B$  with respect to  $\succeq_D$  if and only if for every  $u \in u^*(D, \succeq_D)$ ,  $\sum p_i u(a_i) \geq \sum q_i u(b_i)$ . A preference relation on  $L(D)$  is said to satisfy the  $\succeq_D$ -stochastic dominance axiom if  $A$  is preferred to  $B$  whenever  $A$  stochastically dominates  $B$  with respect to  $\succeq_D$ .<sup>7</sup>

**Lemma 4:** Let  $A, B \in L(D)$ . The lottery  $A$  stochastically dominates the lottery  $B$  with respect to  $\succeq_D$  if and only if  $A = (a_1, q_1; \dots; a_m, q_m)$  and  $B = (b_1, q_1; \dots; b_m, q_m)$ , where  $a_i \succeq_D b_i$ ,  $i = 1, \dots, m$  (Kamae, Krengel, and O'Brien, 1977).<sup>8</sup>

Consider now the case  $D = L_1$ , with  $X \succeq_{L_1} Y$  if and only if  $X$  stochastically dominates  $Y$ . To simplify terminology, I adopt the term two-stage stochastic dominance. Let  $A = (X_1, p_1; \dots; X_m, p_m)$  and  $B = (Y_1, q_1; \dots; Y_\ell, q_\ell)$  be two two-stage lotteries. The lottery  $A$  dominates the lottery  $B$  by two-stage stochastic dominance if and only if for every  $V$ :

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<sup>7</sup>See Levhari, Paroush, and Peleg (1975), Fishburn and Vickson (1978), and Hansen, Holt, and Peled (1978) for the case  $D = \mathbb{R}^n$ .

<sup>8</sup>It is of course assumed that  $(a_1, q_1; a_1, q_2; a_3, q_3; \dots) \sim_D (a_1, q_1 + q_2; a_3, q_3, \dots)$ .

$L_1 \rightarrow \mathbb{R}$  which is increasing with respect to one-stage stochastic dominance,<sup>9</sup>  
 $\sum p_i V(X_i) \geq \sum q_i V(Y_i)$ .

**Two-Stage Stochastic Dominance Axiom:** If the lottery  $A$  dominates the lottery  $B$  by two-stage stochastic dominance, then  $A \succeq_2 B$ .

Let  $A, B \in L_2$  such that  $A$  dominates  $B$  by two-stage stochastic dominance. It follows from Lemma 4 that  $R(A)$  stochastically dominates  $R(B)$ . (The one-stage lottery  $R(A)$  is obtained from  $A$  by the reduction of compound lotteries axiom.) Assume, by Lemma 4, that  $A = (X_1, q_1; \dots; X_m, q_m)$  and  $B = (Y_1, q_1; \dots; Y_m, q_m)$  with  $X_i$  stochastically dominating  $Y_i$ ,  $i = 1, \dots, m$ . For every  $x$ , the probability that  $R(A)$  yields  $x$  or less is  $\sum q_i F_{X_i}(x) \leq \sum q_i F_{Y_i}(x)$ , which is the corresponding probability for  $R(B)$ . The opposite, however, is not true. That is, it may happen that  $R(A)$  stochastically dominates  $R(B)$ , but  $A$  does not dominate  $B$  by two-stage stochastic dominance. Let  $A = ((0, 0.8; 1, 0.2), 0.5; (0, 0.1; 1, 0.9), 0.5)$  and let  $B = ((0, 0.6; 1, 0.4), 0.5; (0, 0.4; 1, 0.6), 0.5)$ , and obtain  $R(A) = (0, 0.45; 1, 0.55)$  and  $R(B) = (0, 0.5; 1, 0.5)$ . Obviously,  $R(A)$  stochastically dominates  $R(B)$ , but it follows immediately from Lemma 4 that  $A$  does not dominate  $B$  by two-stage stochastic dominance.

Stronger results hold for the one-stage and the two-stage stochastic dominance axioms, provided  $\succeq_2$  satisfies the reduction axiom or the compound independence and the time neutrality axioms. Formally:

**Theorem 5:** If the relation  $\succeq_2$  satisfies the reduction of compound lotteries axiom, or if it satisfies the compound independence and the time neutrality axioms, then it satisfies the two-stage stochastic dominance

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<sup>9</sup>The functional  $V$  is not necessarily an expected utility functional.

axiom if and only if it satisfies the one-stage stochastic dominance axiom.

Consider again the one-stage stochastic dominance axiom. This axiom implies that if for every outcome  $x$  the probability of winning more than  $x$  under the lottery  $X$  is at least as large as the corresponding probability under the lottery  $Y$ , then  $X$  should be preferred to  $Y$ . The major problem in adapting this idea to two-stage lotteries is the lack of an objective complete order on  $L_1$ . Instead, one can try to use an objective partial order on this space, namely, the partial one-stage stochastic dominance order. Formally, for  $X \in L_1$ , let  $X^* = \{Y: Y \text{ stochastically dominates } X\}$ . For each  $A = (X_1, q_1; \dots; X_m, q_m)$  and  $Q \subseteq L_1$ , let  $P_A(Q) = \sum_{i: X_i \in Q} q_i$  be the probability that  $A$  yields an element of  $Q$ . The above discussion suggests that if for every  $X$ ,  $P_A(X^*) \geq P_B(X^*)$ , then  $A \succeq_2 B$ . I call this axiom upper compound dominance. This is, however, not the only possible extension. The one-stage stochastic dominance axiom for simple lotteries also says that if for every  $x$  the probability of winning less than  $x$  under  $X$  is less than the corresponding probability under  $Y$ , then  $X$  is preferred to  $Y$ . Let  $X_* = \{Y: X \text{ stochastically dominates } Y\}$ . This last observation leads to the assumption that if for all  $X$ ,  $P_A(X_*) \leq P_B(X_*)$ , then  $A \succeq_2 B$ . I call this axiom lower compound dominance.

These two interpretations of dominance coincide on  $\mathbb{R}$ , but not on  $L_1$ . (See the proof of Theorem 6 for counterexamples.) The following axiom therefore seems a possible combination of those two axioms:

**Weak Compound Dominance Axiom:** If for every  $X$ ,  $P_A(X^*) \geq P_B(X^*)$ , and if for every  $X$ ,  $P_A(X_*) \leq P_B(X_*)$ , then  $A \succeq_2 B$ .

Alternatively, one could suggest the following axiom:

**Strong Compound Dominance Axiom:** If for every  $X$ ,  $P_A(X^*) \geq P_B(X^*)$ , or if for every  $X$ ,  $P_A(X_*) \leq P_B(X_*)$ , then  $A \succeq_2 B$ .

The following theorem discusses the connection between these axioms, the reduction of compound lotteries axiom, and the compound independence axiom:

**Theorem 6:** Let the preference relation  $\succeq_2$  satisfy the one-stage stochastic dominance axiom.

- (a) The reduction of compound lotteries axiom implies the strong compound dominance axiom, but the strong compound dominance axiom does not imply the reduction axiom.
- (b) The strong compound dominance axiom implies both the upper and the lower compound dominance axioms, but none of these two implies the strong compound dominance axiom.
- (c) Each of the upper and the lower compound dominance axioms implies the weak compound dominance axiom, but it implies neither of them.
- (d) The weak compound dominance axiom implies, but is not implied by, the two-stage stochastic dominance axiom.

Let the monotonic (with respect to one-stage stochastic dominance) preference relation  $\succeq_2$  induce continuous preferences  $\succeq_\Gamma$  and  $\succeq_\Delta$ . We know that if  $\succeq_2$  satisfies the reduction and the compound independence axioms, then it can be represented by the expected utility functional (5). However, as argued above, in an intertemporal framework the reduction axiom may not be supportable on normative grounds, and, descriptively, some decision makers violate it. On the other hand, it follows from Theorem 5 that if  $\succeq_2$  satisfies the compound independence and the time neutrality axioms, then it also satisfies the two-stage stochastic dominance axiom, hence all continuous and monotonic preference relations on  $L_1$  can be extended to  $L_2$  through

compound independence and time neutrality to satisfy the two-stage stochastic dominance axiom. As the strong, the upper and lower, and the weak compound dominance axioms are successive (strict) weakenings of the reduction axiom and, moreover, as they all have some normative appeal over  $L_2$ , the question naturally arises as to what preference relations are consistent with compound independence and these axioms. Partial answers to this question are given by Theorems 7 and 9. For these we need the following definitions.

Let  $X \in L_1$  and define  $X^\circ = \text{Cl}(\{(x,p) \in [0,M] \times [0,1] : p > F_X(x)\})$  to be the epigraph of  $F_X$ . Let  $L_1^\circ = \{X^\circ : X \in L_1\}$  be the set of these epigraphs. Let  $H = \{[x,y] \times [p,q] \subset [0,M] \times [0,1] : x < y, p < q\}$  and let  $\Psi = \{(X^\circ, h) \in L_1^\circ \times H : \text{Int } X^\circ \cap \text{Int } h = \emptyset, X^\circ \cup h \in L_1^\circ\}$ . Finally, for  $S \in L_1^\circ$ ,  $S^+$  is the lottery in  $L_1$  such that  $S = (S^+)^\circ$ . In Figure 3,  $X^\circ \in L_1^\circ$ ,  $h_1, h_2, h_3 \in H$ ,  $(X^\circ, h_2) \in \Psi$ , but  $(X^\circ, h_1), (X^\circ, h_3) \notin \Psi$ .

[Insert Figure 3 here.]

**Theorem 7:** Let  $\succeq_2$  induce continuous preference relations  $\succeq_\Gamma$  and  $\succeq_\Delta$ . It satisfies the one-stage strict stochastic dominance, compound independence, time neutrality, and strong compound dominance axioms if and only if it can be represented by the expected utility functional (5) (that is, if and only if it is an EU1 relation) with a strictly increasing utility function  $u$ .

Note that this theorem assumes the strict one-stage stochastic dominance axiom. The following example shows that this is indeed a necessary condition, as the one-stage stochastic dominance, compound independence, time neutrality, and strong compound independence axioms do not imply expected utility.



Example 3: The preference relation  $\succeq_1$  can be represented by  $V(X) = \sup\{x: 1-x \geq F_X(x)\}$ . This relation is continuous and satisfies the one-stage stochastic dominance axiom. Its extension to two-stage lotteries via compound independence and time neutrality satisfies the strong compound dominance axiom. (This occurs because the preference relation is isomorphic to a preference relation on lotteries over the  $[(0,1),(1,0)]$  segment.) Obviously,  $\succeq_1$  does not satisfy the one-stage strict stochastic dominance axiom. This preference relation cannot be represented by an expected utility functional. Indeed, by expected utility theory,  $X = (0, 1/2; 1/3, 1/2) \succeq_1 Y = (0, 3/4; 1/2, 1/4)$  if and only if  $Z = (0, 1/4; 1/3, 3/4) \succeq_1 W = (0, 1/2; 1/3, 1/4; 1/2, 1/4)$ . However,  $V(X) = 1/3 > 1/4 = V(Y)$ , but  $V(Z) = 1/3 = V(W)$ .

#### 4. ANTICIPATED UTILITY<sup>10</sup>

In the last few years, several authors have suggested alternatives to expected utility theory. One of the most promising of these new theories is anticipated utility theory (also known as "expected utility with rank-dependent probabilities"), first suggested by Quiggin (1982). It helps in solving several paradoxes, including the Allais paradox (Quiggin, 1982; Segal, 1987a; Allais, 1988), the preference reversal phenomenon (Karni and Safra, 1987), and the Ellsberg paradox (Segal, 1987b).

According to this theory, the value of the lottery  $(x_1, p_1; \dots; x_n, p_n)$  with  $x_1 \leq \dots \leq x_n$  is given by (2), where  $f(0) = 0$ ,  $f(1) = 1$ , and  $u(0) = 0$ . When  $f$  is linear, this functional reduces to the expected utility representation (1). One can easily verify that for continuous  $u$  and  $f$ , this functional satisfies the continuity axiom and for increasing  $u$  and

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<sup>10</sup>I am especially grateful to Bill Zame for extensive discussions of this section.

$f$ , it satisfies the one-stage stochastic dominance axiom as well. From Chew, Karni, and Safra (1987), Yaari (1986), Rölll (1987), and Segal (1987a), we know that in this theory risk aversion, in the sense of aversion to a mean-preserving spread of the distribution, holds if and only if  $u$  is concave and  $f$  is convex.

Several authors have axiomatized this theory. Quiggin himself suggested weakening the mixture independence axiom, but an essential part of his axiomatic basis leads to the conclusion that  $f(0.5) = 0.5$ . However, as risk aversion is associated with a convex  $f$ , assuming that  $f(0.5) = 0.5$  takes a lot of power out of this theory.

Yaari (1987) suggested another axiomatic basis, necessarily leading to the conclusion that the utility function  $u$  is linear. An attempt to obtain the general form of this theory is found in Segal (1984, 1987c), but the approach taken there lacks normative appeal. Recently, Chew and Epstein (1987b) gave a unifying axiomatic approach to anticipated utility and Chew's (1983) weighted utility theory and Luce (1988) analyzed this model with finite gambles and subjective, rather than objective probabilities. In this section I suggest what I believe to be a normatively appealing set of axioms implying (2) with a general utility function  $u$  (thus avoiding the linearity of Yaari's functional) that allows  $f$  to be either concave or convex (thus letting in the concept of risk aversion). This axiomatic basis includes the compound independence axiom and extended concepts of the compound dominance axioms. One advantage of this set of axioms is that it makes anticipated utility a natural extension of expected utility theory.

Consider again the one-stage stochastic dominance axiom. One possible interpretation of it is that if for every  $x$ ,  $\Pr(X > x) \geq \Pr(Y > x)$  (or if for every  $x$ ,  $\Pr(X \leq x) \leq \Pr(Y \leq x)$ ), then  $X \succeq_1 Y$ . According to this

interpretation, the decision maker is interested in the probability of receiving more (or less) than every possible outcome  $x$ . It is therefore a natural extension of this axiom to assume that whenever he compares  $X$  and  $Y$ , the decision maker ignores similar tails. Formally:

**Ordinal Independence Axiom** (Green and Jullien, 1988): Let  $X, Y, X', Y' \in L_1$ , and let  $x^* \in (0, M)$ . If for every  $x \geq x^*$ ,  $F_X(x) = F_Y(x)$ ,  $F_{X'}(x) = F_{Y'}(x)$ , and for every  $x < x^*$ ,  $F_X(x) = F_{X'}(x)$ ,  $F_Y(x) = F_{Y'}(x)$ , then  $X \succeq_1 Y$  if and only if  $X' \succeq_1 Y'$  (see Figure 4).<sup>11</sup> We say that  $\succeq_2$  on  $L_2$  satisfies this axiom if it is satisfied by  $\succeq_\Gamma$  and  $\succeq_\Delta$ .

[Insert Figure 4 here.]

**Lemma 8** (Green and Jullien, 1988. See also Segal, 1984, 1987c; and Chew and Epstein, 1987b): The following two conditions are equivalent:

- (a) The complete and transitive preference relation  $\succeq_1$  satisfies the continuity, one-stage strict stochastic dominance, and ordinal independence axioms.
- (b) There is a finitely additive measure  $\nu$  on  $[0, M] \times [0, 1]$  such that  $V(X) = \nu(X^0)$  represents the preference relation  $\succeq_1$ .

Let  $h(x, p) = \nu([0, x] \times [1-p, 1]) = V(0, 1-p; x, p)$ . Obviously,  $\nu$  and  $V$  can be reconstructed from  $h$ , as  $\nu([x, y] \times [1-p, 1-q]) = h(y, p) - h(x, p) - h(y, q) + h(x, q)$ , and  $X^0$  can be represented as the union of a finite set of rectangles  $\{Q_i\}$  where  $i \neq j \Rightarrow \text{Int } Q_i \cap \text{Int } Q_j = \emptyset$ . Different  $h$  functions thus define different representation functions. Consider the following four well-known examples:

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<sup>11</sup>This axiom is similar to but slightly weaker than the cancellation axiom in Segal (1984) where Lemma 8 is proved by assuming the later one.

- (a) Expected value:  $h(x,p) = px$ ,  $V(X) = \sum p_i x_i$ , and  $\nu([x,y] \times [1-p,1-q]) = [y-x][p-q]$ .
- (b) Expected utility:  $h(x,p) = pu(x)$ ,  $V(X) = \sum p_i u(x_i)$ , and  $\nu([x,y] \times [1-p,1-q]) = [u(y)-u(x)][p-q]$ .
- (c) Dual theory (Yaari, 1987):  $h(x,p) = xf(p)$ ,  $V(X) = \sum_{i=1}^{n-1} x_i [f(\sum_{j=i}^n p_j) - f(\sum_{j=i+1}^n p_j)] + x_n f(p_n)$ , and  $\nu([x,y] \times [1-p,1-q]) = [y-x][f(p)-f(q)]$ .
- (d) Anticipated utility (Quiggin, 1982):  $h(x,p) = f(p)u(x)$ ,  $V(X) = \sum_{i=1}^{n-1} u(x_i)[f(\sum_{j=i}^n p_j) - f(\sum_{j=i+1}^n p_j)] + u(x_n)f(p_n)$ , and  $\nu([x,y] \times [1-p,1-q]) = [u(y)-u(x)][f(p)-f(q)]$ .

In all four examples,  $h$  is multiplicatively separable, and the corresponding  $\nu$  are product measures. Anticipated utility is the most general form of a product measure and my next aim is to guarantee that  $\nu$  is indeed such a measure.<sup>12</sup>

Let  $A = (X_1, p_1; \dots; (X_i^o \cap X_j^o)^+, p; \dots; (X_i^o \cup X_j^o)^+, p; \dots; X_m, p_m)$  and  $B = (X_1, p_1; \dots; X_i, p; \dots; X_j, p; \dots; X_m, p_m)$ . As is clear from their definitions, the upper compound dominance axiom implies that  $A \succeq_2 B$  while by the lower compound dominance axiom,  $B \succeq_2 A$ . I do not know whether these conditions are equivalent to the upper and the lower compound dominance axioms, but they are certainly not stronger. I will therefore replace the upper and the lower compound dominance axioms by these weaker conditions.

**Weak Upper Compound Dominance Axiom:**  $A = (X_1, p_1; \dots; (X_i^o \cap X_j^o)^+, p; \dots; (X_i^o \cup X_j^o)^+, p; \dots; X_m, p_m) \succeq_2 B = (X_1, p_1; \dots; X_i, p; \dots; X_j, p; \dots; X_m, p_m)$

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<sup>12</sup> For a different approach to this separability issue see Rubinstein (1988).

**Weak Lower Compound Dominance Axiom:**  $B = (X_1, p_1; \dots; X_i, p; \dots; X_j, p; \dots; X_m, p_m) \succeq_2 A = (X_1, p_1; \dots; (X_i^o \cap X_j^o)^+, p; \dots; (X_i^o \cup X_j^o)^+, p; \dots; X_m, p_m)$ .

The main result of this section is presented in the following theorem:  
**Theorem 9:** Let  $\succeq_2$  induce continuous preference relations  $\succeq_\Gamma$  and  $\succeq_\Delta$ , and assume that it satisfies the one-stage strict stochastic dominance, compound independence, time neutrality, and ordinal independence axioms. The relations  $\succeq_\Gamma$  and  $\succeq_\Delta$  can be represented by the anticipated utility functional (2) with a strictly increasing utility function  $u$  and a strictly increasing and concave (convex) distribution transformation function  $f$  if and only if it satisfies the weak upper (lower) compound dominance axiom.

Given the ordinal independence axiom, Theorem 7 follows immediately from Theorem 9. By theorem 6, the strong compound dominance axiom implies upper compound dominance, which in turn implies weak upper compound dominance. Similarly, the strong compound dominance axiom implies weak lower compound dominance, hence, by Theorem 9, the strong compound dominance and the ordinal independence axioms imply that  $f$  is both convex and concave, that is, linear. Linear  $f$  means expected utility, which is the result of Theorem 7. Theorem 7 is of course much stronger, because it does not assume ordinal independence. I do not know whether Theorem 9 can be proved without this axiom.

In the anticipated utility model, risk aversion (loving), in the sense of aversion to a mean-preserving spread of the distribution, implies that  $f$  is convex (concave). Theorem 9 thus indicates a connection between the concept of risk aversion and the compound dominance axioms.

## 5. SOME EMPIRICAL EVIDENCE

This section discusses some empirical evidence in a nonexpected utility framework. For this, I use as an example the anticipated utility model. My

first aim here is to show that these experimental data support the claim that decision makers tend to accept the compound independence axiom while rejecting the reduction axiom. Secondly, I show that within the anticipated utility model, these data agree with some other nonexpected utility behavior patterns. Finally, I show that what seems to be a risk-loving attitude may actually be compatible with risk aversion, provided one is willing to forgo the reduction axiom.

Section 2 discussed the extensions of (2) to two-stage lotteries via the reduction of compound lotteries axiom or via the compound independence and the time neutrality axioms. Similarly to (4), it follows that if  $\succsim_2$  satisfies the compound independence and the time neutrality axioms, then the value of the two-stage lottery  $((y,1),1-p;(y,1-q;x,q),p)$  where  $y \leq x$  is  $u(y) + [u(x)-u(y)]f(p)f(q)$ . It thus follows that  $((y,1),1-p;(y,1-q;x,q),p) \succsim_2 ((y,1),1-p';(y,1-q';x,q'),p')$  if and only if  $f(p)/f(p') > f(q')/f(q)$ . Let  $p > p' \geq q' > q$  such that  $pq = p'q'$ . Let  $\alpha = p/p' = q'/q$ . A sufficient condition for  $f(p)/f(p') > f(q')/f(q)$  is that for every  $\alpha > 1$ ,  $f(\alpha p)/f(p)$  is increasing with  $p$ . This occurs if and only if

$$\alpha f'(\alpha p)f(p) > f(\alpha p)f'(p) \Leftrightarrow$$

$$\frac{\alpha p f'(\alpha p)}{f(\alpha p)} > \frac{p f'(p)}{f(p)}$$

The elasticity of a function  $f$  is defined as  $xf'(x)/f(x)$ . Thus, if the elasticity of  $f$  is increasing, then the desirability of a two-stage lottery decreases as the two stages become less degenerate. Given a compound probability  $r$ , this last discussion asserts that the least-preferred combination of  $p$  and  $q$  is when  $p = q = \sqrt{r}$ . These results agree with the empirical findings of Ronen (1971), where most of his subjects preferred  $((-50000,1),0.1;(-50000,0.5;70000,0.5),0.9)$  to  $((-50000,1),0.4;$

$(-50000, 0.25; 70000, 0.75), 0.6)$ . Moreover, this analysis shows that decision makers' attitude towards two-stage lotteries are highly correlated to their responses to the common ratio effect. Let  $x > y$  and  $p < q$  such that  $(0, 1-p; x, p) \sim_1 (0, 1-q; y, q)$ . Let  $\alpha < 1$ . By the common ratio effect,  $(0, 1-\alpha p; x, \alpha p) >_1 (0, 1-\alpha q; y, \alpha q)$ . (See MacCrimmon and Larsson, 1979; and Kahneman and Tversky, 1979.) Note that such a behavior violates the mixture independence axiom, but not the compound independence axiom, because it does not involve two-stage lotteries. It is proved in Segal (1987a) that anticipated utility can handle this phenomenon provided the elasticity of  $f$  is increasing. (See also the numerical example in Section 2, dealing with problems 1-3 of the Introduction.)

The extension of the anticipated utility model to two-stage lotteries through compound independence and time neutrality helps in analyzing several paradoxes in expected utility theory, where what seems to be consistent with risk aversion violates the assumption that the utility function  $u$  is concave. Elsewhere I showed that the extension of anticipated utility to two-stage lotteries via the compound independence and the time neutrality axioms can solve the probabilistic insurance phenomenon (Segal, 1988) and the Ellsberg paradox (Segal, 1987b). I now show that it can also explain Schoemaker's (1987) findings, described above in Section 2.

The functionals at (1) and (2) can easily be extended to continuous, rather than discrete, bounded random variables. Let  $X$  be a random variable with outcomes in  $[0, M]$ . Let  $F_X$  be its cumulative distribution function, where  $F_X(x) = \Pr(X \leq x)$ . The expected utility of  $X$  is given by

$$\int_0^M u(x) dF_X(x)$$

and the anticipated utility of  $X$  is

$$(6) \quad - \int_0^M u(x) df(1-F_X(x)) = \int_0^M f(1-F_X(x)) du(x).$$

Consider now the anticipated utility model with the compound independence and the time neutrality axioms. By (2), the certainty equivalent of  $(0, 1-p; x, p)$  is  $y = u^{-1}[u(x)f(p)]$ . Consider the two-stage lottery A where  $p = 0.5$  and  $X$  is uniformly distributed over  $[0, 1]$ . Define the random variable  $Y = u^{-1}[u(X)f(0.5)]$  with the distribution function  $F_Y$ , given by  $F_Y(y) = \Pr(Y \leq y) = \Pr(u^{-1}[u(X)f(0.5)] \leq y) = \Pr(X \leq u^{-1}[u(y)/f(0.5)]) = u^{-1}[u(y)/f(0.5)]$ . The smallest possible value of  $Y$  is  $u^{-1}[u(0)f(0.5)] = u^{-1}(0) = 0$ , its larger possible value is  $u^{-1}[u(1)f(0.5)]$ , and by (6) it follows that

$$U(A) = \int_0^{u^{-1}[u(1)f(0.5)]} u'(y) f(1-F_Y(y)) dy.$$

Substitute  $y = u^{-1}[u(x)f(0.5)]$  and obtain

$$(7) \quad U(A) = f(0.5) \int_0^1 u'(x) f(1-x) dx = f(0.5) \int_0^1 u(x) f'(1-x) dx.$$

Consider now the two-stage lottery B where  $x = 0.5$ , and  $P$  is uniformly distributed over  $[0, 1]$ . Define the random variable  $Q = u^{-1}[u(0.5)f(P)]$  with the distribution function  $F_Q$ , given by  $F_Q(q) = \Pr(Q \leq q) = \Pr(u^{-1}[u(0.5)f(P)] \leq q) = \Pr(P \leq f^{-1}[u(q)/u(0.5)]) = f^{-1}[u(q)/u(0.5)]$ . The smallest and largest possible values of  $Q$  are 0 and  $u^{-1}[u(0.5)f(1)] = 0.5$ , respectively, hence

$$U(B) = \int_0^{0.5} u'(q) f(1-F_Q(q)) dq.$$

Substitute  $q = u^{-1}[u(0.5)f(p)]$  and obtain

$$(8) \quad U(B) = u(0.5) \int_0^1 f'(p) f(1-p) dp.$$



Note that when  $f(p) = p$ , that is, when (6) is reduced to the expected utility functional,  $U(A) = 0.5 \int_0^1 u(x)dx$  and  $U(B) = 0.5 u(0.5)$ . These are indeed the values of these lotteries when the reduction axiom is employed together with the expected utility functional.

There are concave utility functions  $u$  and convex distribution transformation functions  $f$  for which  $U(A) > U(B)$ . For example, let  $u(x) = \ln(x+1)$  and let  $f(p) = p^3$ . It follows from (7) and (8) that  $U(A) = 0.026 > 0.020 = U(B)$ .

## 6. SOME REMARKS ON THE INDEPENDENCE AXIOM

The best known evidence against the expected utility hypothesis is the Allais paradox. Allais (1953) found that most people prefer  $X_1 = (0, 0.9; 5 \text{ million}, 0.1)$  to  $Y_1 = (0, 0.89; 1 \text{ million}, 0.11)$ , but  $Y_2 = (1 \text{ million}, 1)$  to  $X_2 = (0, 0.01; 1 \text{ million}, 0.89; 5 \text{ million}, 0.1)$ , while by expected utility theory,  $X_1 \succeq_1 Y_1$  if and only if  $X_2 \succeq_1 Y_2$ . Such behavior certainly contradicts the mixture independence axiom (see Machina, 1982, p. 287). Let  $X = (0, 1/11; 5 \text{ million}, 10/11)$ ,  $Y = (1 \text{ million}, 1)$ , and  $Z = (0, 1)$ . By the mixture independence axiom,  $X_1 = 0.11X + 0.89Z \succeq_1 Y_1 = 0.11Y + 0.89Z$  if and only if  $X_2 = 0.11X + 0.89Y \succeq_1 Y_2 = 0.11Y + 0.89Y$ , while by the Allais paradox  $X_1 \succ_1 Y_1$  but  $Y_2 \succ_1 X_2$ . Beyond doubt, however, this argument does not prove a behavioral violation of the compound independence axiom unless one assumes the reduction of compound lotteries axiom. Indeed, nonexpected utility theories like Chew's (1983) weighted utility or Quiggin's (1982) anticipated utility, which may be consistent with the compound independence axiom, are not contradicted by the Allais paradox.

Some might argue that the mixture and the compound independence axioms have the same normative justification. This, in my view, is false. The

rationale for the compound independence axiom is that if  $X$  is preferred to  $Y$ , then it should be preferred to  $Y$  even when receiving  $X$  or  $Y$  becomes uncertain and other prizes are possible. This argument cannot justify the mixture independence axiom, as there is no initial preference relation between half lotteries like  $(0, 0.01; 5 \text{ million}, 0.1; -)$  and  $(1 \text{ million}, 0.11; -)$ . Similarly, we usually assume that  $(x_1, x_2, \dots, x_n) \succeq (x'_1, x_2, \dots, x_n)$  if and only if  $x_1 \geq x'_1$ , because there is a well-defined natural order on quantities of commodities. However, we do not necessarily assume that  $(x_1, x_2, x_3, \dots, x_n) \succeq (x'_1, x'_2, x_3, \dots, x_n)$  if and only if  $(x_1, x_2, y_3, \dots, y_n) \succeq (x'_1, x'_2, y_3, \dots, y_n)$ , because there is no initial natural order on the half bundles  $(x_1, x_2, -)$ .

In this paper I interpret the compound independence axiom as a mechanism that transforms two-stage lotteries into one-stage lotteries. This results from using the certainty equivalents of the possible outcomes in the compound lotteries.<sup>13</sup> According to this approach, the compound independence axiom and the reduction of compound lotteries axiom should not be used together. Indeed, if the decision maker uses the reduction axiom, then the compound independence axiom becomes meaningless, because he never really considers two-stage lotteries as such. However, using the compound dominance axioms does not rule out the compound independence axiom, because they do not change the structure of a compound lottery. (Recall that these compound dominance axioms become redundant in the presence of the reduction axiom, as follows from Theorem 6.) I therefore believe that Theorem 7 gives a better normative basis for expected utility theory than the standard one. Moreover, the compound dominance axioms prove, as demonstrated by Theorems 7 and 9, that anticipated utility is a natural extension of expected utility theory.

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<sup>13</sup> For a nonaxiomatic approach using this mechanism see Kahneman and Tversky (1979) and Loomes and Sugden (1986).

APPENDIXProof of Theorem 2:

(a) The extension of the anticipated utility functional (2) to two-stage lotteries via the reduction of compound lotteries axiom proves that the reduction and time neutrality axioms together do not imply the compound independence or the mixture independence axioms. The extension of (2) to two-stage lotteries via the compound independence and the time neutrality axioms proves that these axioms, even together, do not imply the reduction axiom, nor do they imply the mixture independence axiom. Example 1 in Section 2 proves that the mixture independence and the time neutrality axioms do not imply the reduction or the compound independence axioms.

(b) Obviously, the reduction axiom implies time neutrality. Example 2 in Section 2 proves that mixture independence and compound independence together do not imply time neutrality.

(c) The reduction and the compound independence axioms obviously imply the mixture independence axiom. To prove that the reduction and the mixture independence axioms imply the compound independence axiom, let  $\delta_X \succeq_{\Delta} \delta_Y$ , and let  $A = (Z_1, q_1; \dots; X, q_1; \dots; Z_m, q_m)$  and  $B = (Z_1, q_1; \dots; Y, q_1; \dots; Z_m, q_m)$  be two two-stage lotteries. By time neutrality,  $\delta_X \succeq_{\Delta} \delta_Y \Leftrightarrow \gamma_X \succeq_{\Gamma} \gamma_Y$ . By the reduction and by the mixture independence axioms,  $A \succeq_2 B \Leftrightarrow \gamma_{R(A)} \succeq_{\Gamma} \gamma_{R(B)} \Leftrightarrow \gamma_X \succeq_{\Gamma} \gamma_Y$ .

Assume next that  $\succeq_2$  satisfies the mixture independence, compound independence, and the time neutrality axioms. Let  $A = (X_1, q_1; \dots; X_m, q_m) \in L_2$  where  $X_i = (x_{1i}, p_{1i}^i; \dots; x_{ni}, p_{ni}^i)$ ,  $i = 1, \dots, m$ . There is no loss of generality in assuming the same prizes in all the lotteries  $X_i$ , as some of the probabilities may equal zero.  $A \sim_2 ((CE_{\Delta}(X_1), 1), q_1; \dots; (CE_{\Delta}(X_m), 1), q_m) \sim_2$

$((CE_{\Delta}(X_1), q_1; \dots; CE_{\Delta}(X_m), q_m), 1) \sim_2 ((x_1, q_1 p_1^1; \dots; x_n, q_1 p_n^1; \dots; x_1, q_m p_1^m; \dots; x_n, q_m p_n^m), 1) = ((x_1, \Sigma q_i p_1^i; \dots; x_n, \Sigma q_i p_n^i), 1) \sim_2 ((x_1, 1), \Sigma q_i p_1^i; \dots; (x_n, 1), \Sigma q_i p_n^i)$ . It thus follows that  $\succeq_2$  satisfies the reduction axiom.

Finally, Example 2 in Section 2 proves that the compound independence and the mixture independence axioms do not imply the reduction axiom.

Q.E.D.

**Proof of Theorem 3:** For the proof that mixture independence implies the expected utility representation (1) see, for example, Fishburn (1982). The rest of the proof is trivial.

**Proof of Theorem 5:** Let  $\succeq_2$  satisfy the compound independence and the one-stage stochastic dominance axioms and let  $A, B \in L_2$  such that  $A$  dominates  $B$  by two-stage stochastic dominance. By Lemma 4,  $A = (X_1, q_1; \dots; X_m, q_m)$  and  $B = (Y_1, q_1; \dots; Y_m, q_m)$ , such that for every  $i$ ,  $X_i$  stochastically dominates  $Y_i$ . As  $\succeq_2$  satisfies the one-stage stochastic dominance axiom, it follows that for every  $i$ ,  $X_i \succeq_{\Delta} Y_i$ , hence  $A \succeq_2 B$ .

Let  $\succeq_2$  satisfy the time neutrality and the two-stage stochastic dominance axioms and let  $X, Y \in L_1$  such that  $X$  stochastically dominates  $Y$ . The lottery  $A = (X, 1)$  dominates the lottery  $B = (Y, 1)$  by two-stage stochastic dominance, hence  $A \succeq_2 B$ . In other words,  $X \succeq_{\Delta} Y$ , and by time neutrality,  $X \succeq_{\Gamma} Y$ .

Let  $\succeq_2$  satisfy the reduction and the one-stage stochastic dominance axioms. Obviously, if  $A$  dominates  $B$  by two-stage stochastic dominance, then  $A \succeq_2 B$  (see Section 3 in the text). Let  $\succeq_2$  satisfy the reduction and the two-stage stochastic dominance axioms. Since it satisfies the time neutrality axiom (Theorem 2), it also satisfies the one-stage stochastic dominance axiom.

Q.E.D.

**Proof of Theorem 6:**

(a) The reduction of compound lotteries axiom implies the strong compound dominance axiom: I will first prove that if  $\succ_2$  satisfies the one-stage stochastic dominance and the reduction axioms, then  $[\forall X P_A(X^*) \geq P_B(X^*)] \Rightarrow A \succ_2 B$ . Let  $A = (X_1, q_1; \dots; X_m, q_m)$  and  $B = (Y_1, q_1; \dots; Y_m, q_m)$  (there is no loss of generality in assuming the same probability vectors), such that for all  $Z \in L_1$ ,  $P_A(Z^*) \geq P_B(Z^*)$ . The preference relation  $\succ_2$  satisfies the reduction axiom, hence  $A$  and  $B$  can be replaced by  $\gamma_X$  and  $\gamma_Y$ , where  $F_X = \sum q_i F_{X_i}$  and  $F_Y = \sum q_i F_{Y_i}$ . As  $\succ_2$  satisfies the one-stage stochastic dominance axiom, it is sufficient to prove that for all  $x$ ,  $\sum q_i F_{X_i}(x) \leq \sum q_i F_{Y_i}(x)$ . Let  $Z_p = (0, 1-p; x, p)$ . For every  $p$ ,  $P_A(Z_p^*) \geq P_B(Z_p^*)$ , hence

$$\sum_{i: F_{X_i}(x) \leq 1-p} q_i \geq \sum_{i: F_{Y_i}(x) \leq 1-p} q_i$$

Let  $G_X = (1-F_{X_1}(x), q_1; \dots; 1-F_{X_m}(x), q_m)$  and  $G_Y = (1-F_{Y_1}(x), q_1; \dots; 1-F_{Y_m}(x), q_m)$ . For every  $p$ ,

$$\begin{aligned} \Pr(G_X \geq p) &= \sum_{i: 1-F_{X_i}(x) \geq p} q_i = \sum_{i: F_{X_i}(x) \leq 1-p} q_i \geq \\ &\sum_{i: F_{Y_i}(x) \leq 1-p} q_i = \sum_{i: 1-F_{Y_i}(x) \geq p} q_i = \Pr(G_Y \geq p). \end{aligned}$$

In other words,  $G_X$  stochastically dominates  $G_Y$ . As stated in Section 3, this happens if and only if for every increasing function  $u$ ,  $E[u(G_X)] \geq E[u(G_Y)]$ . In particular, for  $u(x) = x$ , it follows that  $\sum q_i (1-F_{X_i}(x)) \geq \sum q_i (1-F_{Y_i}(x))$ , hence  $\sum q_i F_{X_i}(x) \leq \sum q_i F_{Y_i}(x)$ .

The proof that if for every  $X$ ,  $P_A(X_*) \leq P_B(X_*)$ , then  $A \succ_2 B$ , is similar. Let  $Z^P = (x, 1-p; M, p)$ . The proof follows from the assumption that for every  $p$ ,  $P_A(Z^P_*) \leq P_B(Z^P_*)$ .

The strong compound dominance axiom does not imply the reduction axiom:

let  $Z = (0,0.5;1,0.5)$  and define  $V: L_1 \rightarrow \mathbb{R}$  by

$$V(X) = \begin{cases} 1 & X \in Z^* \\ 0 & X \notin Z^* \end{cases}$$

The preference relation  $\succeq_1$  on  $L_1$  is represented by  $V$ , and  $A = (X_1, p_1; \dots; X_m, p_m) \succeq_2 B = (Y_1, q_1; \dots; Y_\ell, q_\ell)$  if and only if  $V(V(X_1), p_1; \dots; V(X_m), p_m) \geq V(V(Y_1), q_1; \dots; V(Y_\ell), q_\ell)$ . Obviously,  $\succeq_2$  satisfies the compound independence and the one-stage stochastic dominance axioms. It also satisfies the strong compound dominance axiom. Indeed, if  $\forall X P_A(X^*) \geq P_B(X^*)$ , then in particular  $P_A(Z^*) \geq P_B(Z^*)$  and by the one-stage stochastic dominance axiom,  $A \succeq_2 B$ . Suppose that  $\forall X P_A(X_*) \leq P_B(X_*)$  and let

$$F_W(x) = \min(\min\{F_{X_i}(x) : X_i \notin Z^*\}, \min\{F_{Y_j}(x) : Y_j \notin Z^*\}).$$

It follows that

$$\sum_{i: X_i \in Z^*} P_i = 1 - \sum_{i: X_i \in W_*} P_i \geq 1 - \sum_{j: Y_j \in W_*} q_j = \sum_{j: Y_j \in Z^*} q_j$$

hence  $A \succeq_2 B$ .

The preference relation  $\succeq_2$  does not satisfy the reduction of compound lotteries axiom. For example,  $V(V(0,1), 1/3; V(0, 1/3; 1, 2/3), 2/3) = V(0, 1/3; 1, 2/3) = 1$ , but  $V(V(0, 5/9; 1, 4/9), 1) = V(0, 1) = 0$ , although these two lotteries are equivalent by the reduction axiom.

(b)-(c) Obviously, the strong compound dominance axiom implies the upper and the lower compound dominance axioms, and each of them implies the weak compound dominance axiom. To prove that the opposite does not hold true, construct counterexamples based on the observation that by the lower compound dominance axiom,  $A = ((0, 1/3; 1, 2/3), 1/2; (0, 2/3; 2, 1/3), 1/2) \succeq_2 B = ((0, 2/3; 1, 1/3), 1/2; (0, 1/3; 1, 1/3; 2, 1/3), 1/2)$ , by the upper compound

dominance axiom,  $B \succeq_2 A$ , by the strong compound dominance axiom,  $A \sim_2 B$ , while the weak compound dominance axiom does not compare these two lotteries.

(d) It is easy to verify, by Lemma 4, that the weak compound dominance axiom implies the two-stage stochastic dominance axiom. To see that the opposite is false, construct a counterexample based on the observation that by weak compound dominance

$$A = ((0, \frac{1}{3}; 2, \frac{2}{3}), \frac{1}{3}; (0, \frac{1}{3}; 1, \frac{1}{3}; 3, \frac{1}{3}), \frac{1}{3}; (1, \frac{1}{3}; 2, \frac{1}{3}; 3, \frac{1}{3}), \frac{1}{3}) \succeq_2$$

$$B = ((0, \frac{1}{3}; 1, \frac{2}{3}), \frac{1}{3}; (1, 1), \frac{1}{3}; (0, \frac{1}{3}; 2, \frac{1}{3}; 3, \frac{1}{3}), \frac{1}{3})$$

while the two-stage stochastic dominance axiom does not compare these two lotteries. Q.E.D.

**Proof of Theorem 7:** Obviously, if  $\succeq_2$  can be represented by the expected utility functional (5) with a strictly increasing utility function  $u$ , then it satisfies the one-stage strict stochastic dominance, compound independence, time neutrality and strong compound independence axioms. (Recall that EU1 satisfies the reduction of compound lotteries axiom, hence, by Theorem 6, it also satisfies the strong compound independence axiom.) To prove the "only if" part of the theorem, let  $\succeq_1 = \succeq_\Gamma = \succeq_\Delta$ . I first show that  $\succeq_1$  can be represented by a measure of the epigraph of  $F_X$  and then, that this measure is actually the expected utility functional. Because this implies the mixture independence axiom, the theorem will follow from Theorem 3(a).

**Lemma 7.1:** Let  $X, Y \in L_1$  such that  $X^\circ \subseteq Y^\circ$ , and let  $h \in H$  such that  $(X^\circ, h), (Y^\circ, h) \in \Psi$ . Then  $A = ((X^\circ \cup h)^+, 0.5; Y, 0.5) \sim_2 B = (X, 0.5; (Y^\circ \cup h)^+, 0.5)$ .

**Proof:** Let  $Z \in L_1$ . If  $P_A(Z^*) = 0$ , then obviously  $P_B(Z^*) \geq P_A(Z^*)$ . If  $P_A(Z^*) = 0.5$ , then either  $(X^0Uh)^+$  stochastically dominates  $Z$ , but  $Y$  does not, or  $Y$  stochastically dominates  $Z$ , but  $(X^0Uh)^+$  does not. In both bases,  $(Y^0Uh)^+ \in Z^*$ , hence  $P_B(Z^*) \geq P_A(Z^*)$ . If  $P_A(Z^*) = 1$ , then  $X = ((X^0Uh) \cap Y^0)^+ \in Z^*$ , and  $P_B(Z^*) = P_A(Z^*)$ . By the strong compound dominance axiom,  $B \succeq_2 A$ . Similarly, for each  $Z \in L_1$ ,  $P_A(Z_*) \leq P_B(Z_*)$ , hence  $A \succeq_2 B$ . It thus follows that  $A \sim_2 B$ . ■

**Lemma 7.2:** Let  $(X^0, h), (Y^0, h) \in \Psi$ . Then  $(X^0Uh)^+ \succeq_1 (Y^0Uh)^+$  if and only if  $X \succeq_1 Y$ .

**Proof:** By the compound independence and time neutrality axioms and by Lemma 7.1,  $(X^0Uh)^+ \succeq_1 (Y^0Uh)^+ \Leftrightarrow ((X^0UY^0)^+, 0.5; (X^0Uh)^+, 0.5) \succeq_2 ((X^0UY^0)^+, 0.5; (Y^0Uh)^+, 0.5) \Leftrightarrow ((X^0UY^0Uh)^+, 0.5; X, 0.5) \succeq_2 ((X^0UY^0Uh)^+, 0.5; Y, 0.5) \Leftrightarrow X \succeq_1 Y$ . ■

Define on  $H$  partial orders  $R_X$  by  $h_1 R_X h_2$  if and only if  $(X^0, h_1), (X^0, h_2) \in \Psi$ , and  $(X^0Uh_1)^+ \succeq_1 (X^0Uh_2)^+$ .

**Lemma 7.3:** For every  $X$  and  $Y$ ,  $R_X$  and  $R_Y$  do not contradict each other. In other words, if  $h_1$  and  $h_2$  can be compared by both  $R_X$  and  $R_Y$ , then  $h_1 R_X h_2$  if and only if  $h_1 R_Y h_2$ .

**Proof:** Let  $X, Y \in L_1$  such that  $(X^0, h_i), (Y^0, h_i) \in \Psi$ ,  $i = 1, 2$ , and let  $Z^0 = X^0 \cap Y^0$ . Obviously,  $Z^0 \in L_1^0$ , and  $(Z^0, h_i) \in \Psi$ ,  $i = 1, 2$ . There exist  $h_1^1, \dots, h_1^s$ , and  $h_2^1, \dots, h_2^t$  such that  $\forall j (Z^0 \cup \cup_{k=1}^{j-1} h_i^k, h_i^j) \in \Psi$ ,  $i = 1, 2$ ,  $X^0 = Z^0 \cup \cup_{k=1}^s h_1^k$ , and  $Y^0 = Z^0 \cup \cup_{k=1}^t h_2^k$ . By Lemma 7.2,  $h_1 R_X h_2 \Leftrightarrow (Z^0Uh_1^1 \cup \dots \cup h_1^sUh_1^1)^+ \succeq_1 (Z^0Uh_1^1 \cup \dots \cup h_1^sUh_2^1)^+ \Leftrightarrow \dots \Leftrightarrow (Z^0Uh_1^1)^+ \succeq_1 (Z^0Uh_2^1)^+ \Leftrightarrow \dots \Leftrightarrow (Z^0Uh_2^1 \cup \dots \cup h_2^tUh_2^1)^+ \succeq_1 (Z^0Uh_2^1 \cup \dots \cup h_2^tUh_2^1)^+ \Leftrightarrow h_1 R_Y h_2$ . ■



Let  $R = \bigcup_{X \in L_1} R_X$ . That is,  $h_1 R h_2$  if and only if there exists  $X \in L_1$  such that  $h_1 R_X h_2$ . It can be proved that  $R$  is acyclic. That is,  $h_1 R h_2 R \dots R h_t R h_1$  implies  $h_1 R h_2 R \dots R h_t R h_1$  (see Segal, 1987c). Let  $\succeq^*$  be the transitive closure of  $R$ :  $h_1 \succeq^* h_2$  if and only if there are  $h_3, \dots, h_t$  such that  $h_1 R h_3 R \dots R h_t R h_2$ .

**Lemma 7.4:** There exist  $V: L_1 \rightarrow \mathbb{R}$  and  $W: H \rightarrow \mathbb{R}$  such that

- (a)  $V$  represents the relation  $\succeq_1$ .
- (b)  $W$  is finitely additive. That is, if  $h_1 \cup h_2 \in H$ , then  $W(h_1 \cup h_2) = W(h_1) + W(h_2) - W(h_1 \cap h_2)$ .
- (c) If  $X^0 = \bigcup_{k=1}^t h_k$  where  $\forall j (\bigcup_{k=1}^{j-1} h_k, h_j) \in \Psi$ , then  $V(X) = \sum_{k=1}^t W(h_k)$ .

**Proof:** Let  $[0, x] \times [0, p] \sim^* [x, y_1] \times [p, 1]$  (see Figure 5) and let  $W([0, x] \times [0, p]) = W([x, y_1] \times [p, 1]) = 1$ . By the continuity assumption

[Insert Figure 5 here.]

there exist  $z$  and  $w$  such that  $[x, w] \times [p, 1] \sim^* [0, z] \times [0, p] \sim^* [w, y_1] \times [p, 1]$ . Define  $W([x, w] \times [p, 1]) = W([w, y_1] \times [p, 1]) = 0.5$ . This can be repeated again and again for the  $x$  as well as for the  $p$  axes. By the one-stage strict stochastic dominance axiom, the areas of all these rectangles will become smaller and smaller. The function  $W$  can thus be defined as an atomless, continuous, finitely additive measure on  $[0, x] \times [0, p]$  and  $[x, y_1] \times [p, 1]$ . Similarly, it can be defined for the rectangles  $[y_i, y_{i+1}] \times [p, 1] \sim^* [0, x] \times [0, p]$ ,  $i = 1, \dots$ . By the one-stage strict stochastic dominance and the continuity axioms, the strictly increasing sequence  $\{y_i\}$  is finite. Indeed, let  $\lim y_i = y \leq M$ . For all  $i$ ,  $([0, y] \times [p, 1])^+ >_1 ([0, y_i] \times [p, 1])^+ -_1 (([0, y_{i-1}] \times [p, 1]) \cup ([0, x] \times [0, p]))^+$ , in contradiction to the continuity and the strict one-stage stochastic dominance axioms. This process defines a finitely additive measure  $W$  on  $[x, M] \times [p, 1]$ ,

which can be extended to  $[0, M] \times [0, p]$  and to  $[0, x] \times [p, 1]$  and thus to  $[0, M] \times [0, 1]$ . Define  $V$  as in part (c) of Lemma 7.4. Because  $W$  is finitely additive,  $V$  does not depend on the choice of  $h_1, \dots, h_t$ .

Let  $X^0 = \cup_{k=1}^t h_k$  and  $Y^0 = \cup_{\ell=1}^s g_\ell$  where  $\forall j (\cup_{k=1}^{j-1} h_k, h_j), (\cup_{\ell=1}^{j-1} g_\ell, g_j) \in \Psi$ , such that  $X \geq_1 Y$ . We want to construct two sequences  $(h'_k)_{k=1}^{t'}$  and  $(g'_\ell)_{\ell=1}^{s'}$  such that  $X^0 = \cup_{k=1}^{t'} h'_k$ ,  $Y^0 = \cup_{\ell=1}^{s'} g'_\ell$ ,  $\forall j (\cup_{k=1}^{j-1} h'_k, h'_j), (\cup_{\ell=1}^{j-1} g'_\ell, g'_j) \in \Psi$ , and for every  $j \leq s'$ ,  $h'_j \sim^* g'_j$ . We will do it by finite induction. If  $h_1 \sim^* g_1$ , then let  $h'_1 = h_1$  and  $g'_1 = g_1$ . If  $h_1 >^* g_1$ , construct  $h'_1, h'_2 \in H$  such that  $h'_1 \sim^* g_1$ ,  $h'_2 = Cl(h_1 \setminus h'_1) \in H$ ,  $(h'_1, h'_2) \in \Psi$ , and let  $g'_1 = g_1$ . If  $g_1 >^* h_1$ , construct  $h'_1, g'_1$ , and  $g'_2$  similarly. It thus follows that in each step we can reduce the number of nonequivalent elements either in  $X$  or in  $Y$  (or in both) by one. The desired representation will thus be constructed in a finite number of steps. As  $X \geq_1 Y$ , it follows that  $t' \geq s'$ . Obviously,  $V((\cup_{k=1}^{s'} h'_k)^+) = V((\cup_{\ell=1}^{s'} g'_\ell)^+)$ . As  $t' \geq s'$ , it follows that  $V(X) \geq V(Y)$ , which completes the proof of the lemma. ■

I now turn to the proof of the theorem. Let  $0 < x_1 < \dots < x_n$ ,  $0 < p < 1$ , and  $0 < \epsilon < p$ . Let  $\bar{X}_i = (0, p; x_i, 1-p)$ , and  $Y_i = (0, p-\epsilon; x_i, \epsilon; x_i, 1-p)$ ,  $i = 1, \dots, n$ . It follows from the proof of Lemma 7.1 that for every  $i$  and  $j$ ,

$$(\bar{X}_1, \frac{1}{n}; \dots; \bar{X}_{i-1}, \frac{1}{n}; Y_i, \frac{1}{n}; \bar{X}_{i+1}, \frac{1}{n}; \dots; \bar{X}_n, \frac{1}{n}) \sim_2$$

$$(\bar{X}_1, \frac{1}{n}; \dots; \bar{X}_{j-1}, \frac{1}{n}; Y_j, \frac{1}{n}; \bar{X}_{j+1}, \frac{1}{n}; \dots; \bar{X}_n, \frac{1}{n}).$$

The compound independence and the time neutrality axioms together imply that on  $L_2$ , the relation  $\geq_2$  can be represented by  $U(X_1, q_1; \dots; X_m, q_m) = \phi(V(X_1), q_1; \dots; V(X_m), q_m)$ . Assume, without loss of generality, that  $\phi(a, 1)$

- a. We want to show that  $\phi = \sum q_i V(X_i)$  is a possible representation. By Lemma 7.4 there exists  $\beta > 0$  such that  $V(Y_i) = V(\bar{X}_i) + \beta$ ,  $i = 1, \dots, n$ . Moreover, for every sufficiently small  $\beta$  there exists an appropriate  $\epsilon$ . Let  $0 < y^* \leq y_1 < \dots < y_n$ <sup>14</sup> be in the interior of the range of  $V$ . Let  $p \in (0, 1)$  and  $\bar{X}_i = (0, 1-p; x_i, p)$  such that  $V(\bar{X}_i) = y_i$ ,  $i = 1, \dots, n$ . There is  $\beta^* > 0$  such that for every  $0 < \beta < \beta^*$  and for every  $i$  and  $j$ ,

$$(A.1) \quad \phi(y_1, \frac{1}{n}; \dots; y_{i-1}, \frac{1}{n}; y_i + \beta, \frac{1}{n}; y_{i+1}, \frac{1}{n}; \dots; y_n, \frac{1}{n}) = \\ \phi(y_1, \frac{1}{n}; \dots; y_{j-1}, \frac{1}{n}; y_j + \beta, \frac{1}{n}; y_{j+1}, \frac{1}{n}; \dots; y_n, \frac{1}{n}).$$

Let  $y_1, \dots, y_n > 0$  be in the interior of the range of  $V$ . Let  $y^* = \min(y_1, \dots, y_n)$  and let  $\beta^* \leq y^*/2$  be appropriate for  $y^*/2$ . Let  $z_1, \dots, z_n$  be in the interior of the range of  $V$  such that  $\sum z_i = \sum y_i$  and  $\max |y_i - z_i| \leq \beta^*/n$ . By (A.1),

$$\phi(y_1, \frac{1}{n}; \dots; y_n, \frac{1}{n}) = \phi(y_1, \frac{1}{n}; \dots; y_{n-1} + y_n - z_n, \frac{1}{n}; z_n, \frac{1}{n}) = \\ \phi(y_1, \frac{1}{n}; \dots; y_{n-2} + y_{n-1} - z_{n-1} + y_n - z_n, \frac{1}{n}; z_{n-1}, \frac{1}{n}; z_n, \frac{1}{n}) = \\ \dots = \phi(z_1, \frac{1}{n}; \dots; z_n, \frac{1}{n})$$

hence

$$\phi(V(\bar{X}_1), \frac{1}{n}; \dots; V(\bar{X}_n), \frac{1}{n}) = f(\sum V(\bar{X}_i)).$$

Let  $x \geq 0$ .  $V(x, 1) = \phi(V(x, 1), 1) = \phi(V(x, 1), 1/n; \dots; V(x, 1), 1/n) = f(nV(x, 1))$ , hence  $f(\alpha) = \alpha/n$ . It thus follows that  $U(X_1, 1/n; \dots; X_n, 1/n) = \phi(V(X_1), 1/n; \dots; V(X_n), 1/n) = \sum V(X_i)/n$ , and by the continuity assumption

<sup>14</sup>The assumption that  $y_1 \leq \dots \leq y_n$  is not essential, because the value of a lottery depends on its prizes and not on their order.

it follows that  $\phi(V(X_1), p_1; \dots; V(X_n), p_n) = \sum p_i V(X_i)$ . Let  $u(x) = V(x, 1)$ . It follows that on  $\Gamma$ ,  $\succeq_\Gamma$  can be represented by  $\sum p_i u(x_i)$ . We assumed  $\succeq_1 = \succeq_\Gamma$ , hence the theorem. Q.E.D.

**Proof of Theorem 9:** Let  $\succeq_1 = \succeq_\Gamma = \succeq_\Delta$ . By Lemma 8,  $\succeq_1$  can be represented by a measure  $\nu$ . I first prove that if  $\succeq_2$  satisfies the weak upper compound dominance axiom then  $\succeq_1$  can be represented by (2) with a concave distribution transformation function  $f$  (Propositions 9.1-9.3). Then I show that if  $\succeq_1$  can be represented by (2) with concave  $f$ , then  $\succeq_2$  satisfies the weak upper compound dominance axiom (Proposition 9.4). The proof for the weak lower compound dominance axiom - convex  $f$  case is similar.

**Proposition 9.1:** Assume the weak upper compound dominance axiom and let  $x < y \leq x' < y'$  such that  $\nu([x, y] \times [0, 1]) = \nu([x', y'] \times [0, 1])$ . For every  $p, q$ , and  $\beta$  such that  $0 \leq p < p + \beta \leq q < q + \beta \leq 1$  there is  $\epsilon > 0$  such that if  $y - x \leq \epsilon$ , then  $\nu([x, y] \times [p, p + \beta]) \leq \nu([x', y'] \times [q, q + \beta])$ .

**Proof:** Let  $X, Y \in L_1$  and  $h \in H$  such that  $X^0 \subset Y^0$ ,  $(X^0, h), (Y^0, h) \in \Psi$ ,  $X \sim_1(x, 1)$ ,  $(X^0 \cup h)^+ \sim_1(y, 1)$ , and  $Y \sim_1(x', 1)$ . The existence of  $X, Y$ , and  $h$  follows from the assumption that  $x$  and  $y$  are sufficiently close. Since  $\nu$  is a measure,  $\nu([x', y'] \times [0, 1])$ , hence  $(Y^0 \cup h)^+ \sim_1(y', 1)$ .

Let  $Z, W \in L_1$  such that  $W \succeq_1(y', 1) \succ_1(x', 1) \succeq_1 Z \succeq_1(y, 1)$  and let  $z$  and  $w$  be such that  $Z \sim_1(z, 1)$  and  $W \sim_1(w, 1)$ . By the weak upper compound dominance axiom it follows that  $((0, 1), p; X, \beta; Z, q-p-\beta; (Y^0 \cup h)^+, \beta; W, 1-q-\beta) \succeq_2 ((0, 1), p; (X^0 \cup h)^+, \beta; Z, q-p-\beta; Y, \beta; W, 1-q-\beta) \Rightarrow (0, p; x, \beta; z, q-p-\beta; y', \beta; w, 1-q-\beta) \succeq_1 (0, p; y, \beta; z, q-p-\beta; x', \beta; w, 1-q-\beta) \Rightarrow \nu([x', y'] \times [q, q + \beta]) \geq \nu([x, y] \times [p, p + \beta])$ . ■

**Proposition 9.2:** Let  $x, y, x',$  and  $y'$  be as in Proposition 9.1. For every  $0 < p < q \leq 1$ ,  $\nu([x,y] \times [p,q]) = \nu([x',y'] \times [p,q])$ .

**Proof:** By Proposition 9.1 it follows that if  $x$  and  $y$  are sufficiently close to each other then for every  $n$  and  $i \leq n-2$ ,  $\nu([x,y] \times [p + i(q-p)/n, p + (i+1)(q-p)/n]) \leq \nu([x',y'] \times [p + (i+1)(q-p)/n, p + (i+2)(q-p)/n])$ , hence for every  $n$ ,  $\nu([x,y] \times [p, q - (q-p)/n]) \leq \nu([x',y'] \times [p + (q-p)/n, q])$ , and by the continuity of  $\nu_1$  it follows that  $\nu([x,y] \times [p,q]) \leq \nu([x',y'] \times [p,q])$ .

Similarly,  $\nu([x,y] \times [0,p]) \leq \nu([x',y'] \times [0,p])$  and  $\nu([x,y] \times [q,1]) \leq \nu([x',y'] \times [q,1])$ . Since  $\nu([x,y] \times [0,1]) = \nu([x',y'] \times [0,1])$ , it follows that  $\nu([x,y] \times [p,q]) = \nu([x',y'] \times [p,q])$ . The lemma now follows by the additivity of the measure  $\nu$ . ■

Define  $u(x) = \nu([0,x] \times [0,1])$ . By the one-stage strict stochastic dominance axiom, the function  $u$  is strictly increasing.

**Proposition 9.3:** There is strictly increasing and concave function  $f: [0,1] \rightarrow [0,1]$  such that  $\nu([x,y] \times [p,q]) = [u(y)-u(x)][f(1-p)-f(1-q)]$ .

**Proof:** By the definition of  $u$ ,  $\nu([x,y] \times [0,1]) = u(y) - u(x)$ . By Proposition 9.2, if  $u(y) - u(x) = m[u(y')-u(x')]/n$ , then  $\nu([x,y] \times [p,q]) = m\nu([x',y'] \times [p,q])/n$ . Hence, by the continuity assumption,  $\nu([x,y] \times [p,q]) = \theta(p,q)[u(y)-u(x)]$ . Define  $f(p) = \theta(1-p,1)$ . Because  $\nu$  is a measure,  $\nu([x,y] \times [p,q]) = [u(y)-u(x)][f(1-p)-f(1-q)]$ .

It follows from Propositions 9.1 and 9.2 that for every  $n$ ,  $\nu([x,y] \times [p, p+1/n]) \leq \nu([x,y] \times [p+1/n, p+2/n])$ . Hence  $f(1-p) - f(1-p-1/n) \leq f(1-p-1/n) - f(1-p-2/n)$ , and  $f$  is concave. The one-stage strict stochastic dominance axiom implies that  $f$  is strictly increasing. ■

**Proposition 9.4:** If  $\succ_2$  satisfies the compound independence and the time neutrality axioms and the induced preference relations  $\succ_\Gamma$  and  $\succ_\Delta$  can be represented by the same anticipated utility functional (2) with concave  $f$ , then  $\succ_2$  also satisfies the weak upper compound dominance axiom.

**Proof:** Let  $A = (X_1, 1/m; \dots; X_m, 1/m)$  and  $B = (X_1, 1/m; \dots; (X_i^o \cap X_j^o)^+, 1/m; \dots; (X_i^o \cup X_j^o)^+, 1/m; \dots; X_m, 1/m)$ . Assume that the lotteries in  $A$  and  $B$  are ordered from the worst one to the best one by  $\succ_1$ . The explicit form of  $B$  is

$$\begin{aligned}
 B = & \begin{matrix} (1) & & (s) & (s+1) & (s+2) & & (i) & (i+1) & & (j-1) \end{matrix} \\
 & (X_1, \frac{1}{m}; \dots; X_s, \frac{1}{m}; (X_i^o \cap X_j^o)^+, \frac{1}{m}; X_{s+1}, \frac{1}{m}; \dots; X_{i-1}, \frac{1}{m}; X_{i+1}, \frac{1}{m}; \dots; X_{j-1}, \frac{1}{m}; \\
 & \begin{matrix} (j) & & (t-1) & (t) & (t+1) & & (m) \end{matrix} \\
 & X_{j+1}, \frac{1}{m}; \dots; X_t, \frac{1}{m}; (X_i^o \cup X_j^o)^+, \frac{1}{m}; X_{t+1}, \frac{1}{m}; \dots; X_m, \frac{1}{m}).
 \end{aligned}$$

By continuity, it is sufficient to prove that the value of  $B$  is not less than that of  $A$ . From compound independence, time neutrality, and (2) it follows that the value of  $A$  is given by

$$U(A) = V(X_1) + \sum_{k=2}^m [V(X_k) - V(X_{k-1})] f\left(\frac{m-k+1}{m}\right)$$

Hence,

$$\begin{aligned}
 U(B) - U(A) = & \sum_{k=j+1}^t [V(X_k) - V(X_{k-1})] \times \left[ f\left(\frac{m-k+2}{m}\right) - f\left(\frac{m-k+1}{m}\right) \right] + \\
 & [V((X_i^o \cup X_j^o)^+) - V(X_t)] \times \left[ f\left(\frac{m-t+1}{m}\right) - f\left(\frac{m-t}{m}\right) \right] - \\
 & \sum_{k=s+2}^i [V(X_k) - V(X_{k-1})] \times \left[ f\left(\frac{m-k+1}{m}\right) - f\left(\frac{m-k}{m}\right) \right]
 \end{aligned}$$

$$\begin{aligned}
& \left[ v(X_{s+1}) - v((X_i^o \cap X_j^o)^+) \right] \times \left[ f\left(\frac{m-s}{m}\right) - f\left(\frac{m-s-1}{m}\right) \right] > \\
& \sum_{k=j+1}^t \left[ v(X_k) - v(X_{k-1}) \right] \times \left[ f\left(\frac{m-j+1}{m}\right) - f\left(\frac{m-j}{m}\right) \right] + \\
& \left[ v((X_i^o \cup X_j^o)^+) - v(X_t) \right] \times \left[ f\left(\frac{m-j+1}{m}\right) - f\left(\frac{m-j}{m}\right) \right] - \\
& \sum_{k=s+2}^i \left[ v(X_k) - v(X_{k-1}) \right] \times \left[ f\left(\frac{m-i+1}{m}\right) - f\left(\frac{m-i}{m}\right) \right] \\
& \left[ v(X_{s+1}) - v((X_i^o \cap X_j^o)^+) \right] \times \left[ f\left(\frac{m-i+1}{m}\right) - f\left(\frac{m-i}{m}\right) \right] - \\
& \left[ v((X_i^o \cup X_j^o)^+) - v(X_j) \right] \times \left[ f\left(\frac{m-j+1}{m}\right) - f\left(\frac{m-j}{m}\right) \right] - \\
& \left[ v(X_i) - v((X_i^o \cap X_j^o)^+) \right] \times \left[ f\left(\frac{m-i+1}{m}\right) - f\left(\frac{m-i}{m}\right) \right] - \\
& v(X_i^o \setminus X_j^o) \left\{ \left[ f\left(\frac{m-j+1}{m}\right) - f\left(\frac{m-j}{m}\right) \right] - \left[ f\left(\frac{m-i+1}{m}\right) - f\left(\frac{m-i}{m}\right) \right] \right\} > 0 \quad \blacksquare
\end{aligned}$$

The proof for convex  $f$  and the weak lower compound dominance axiom is similar.

Q.E.D.

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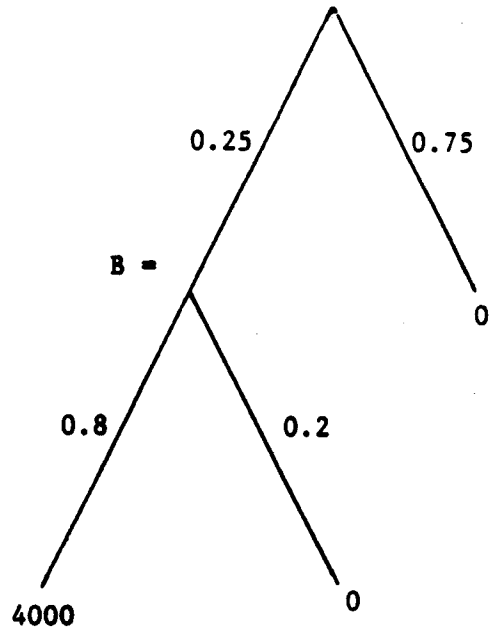
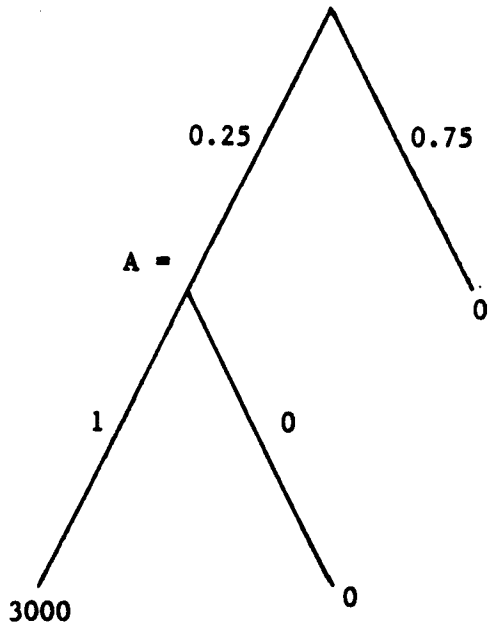


FIGURE 1

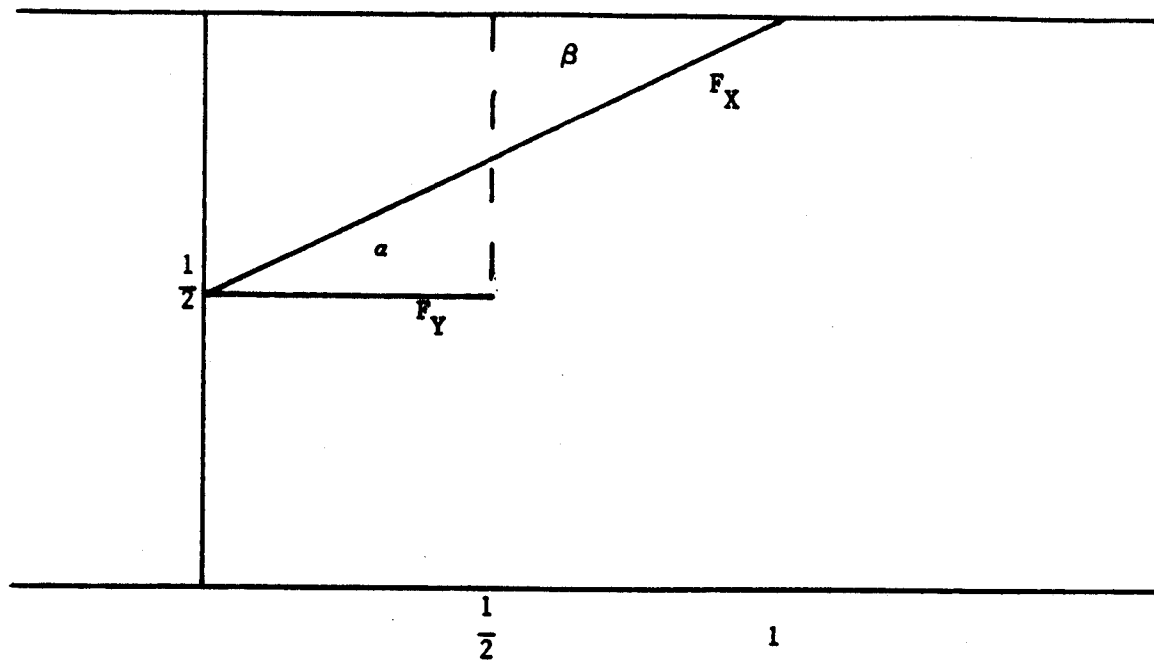


FIGURE 2

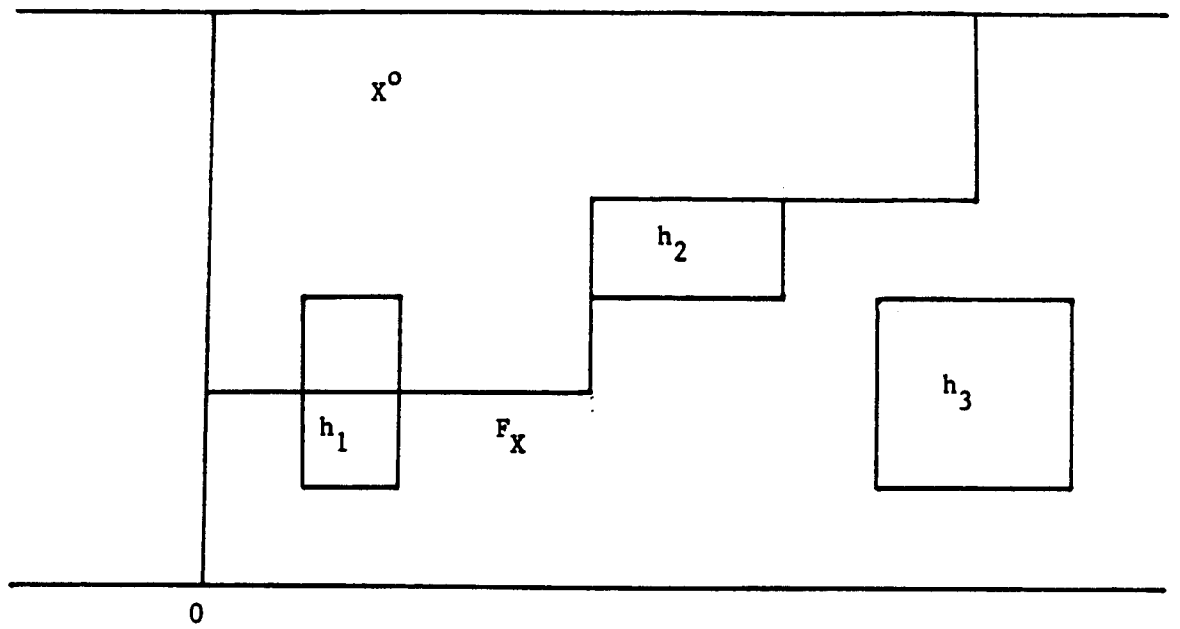


FIGURE 3

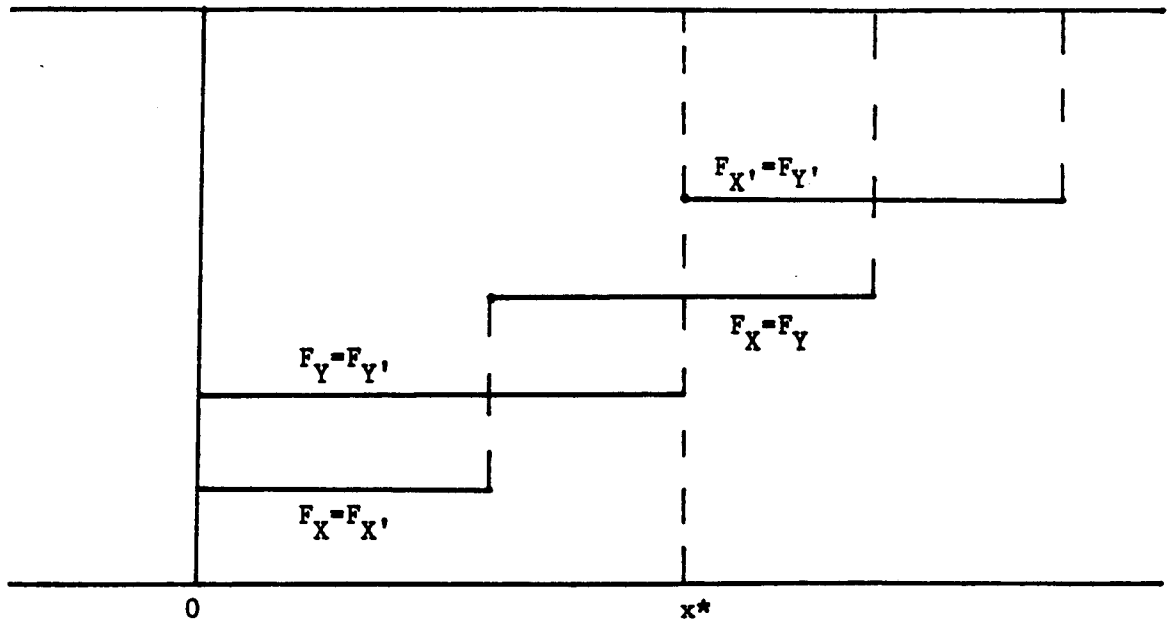


FIGURE 4



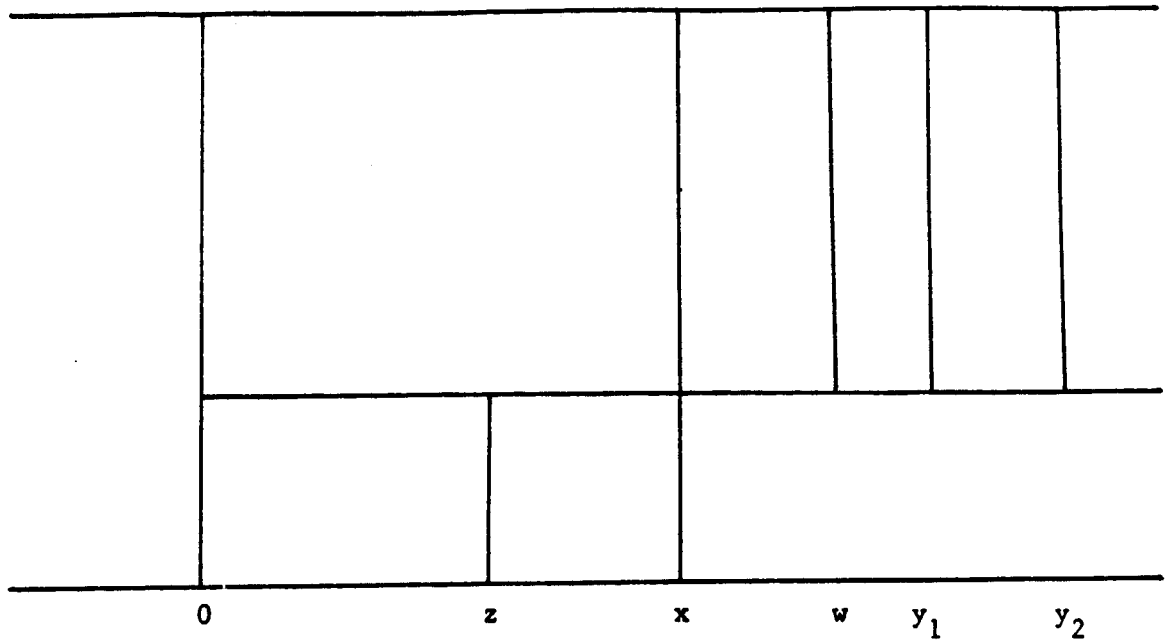


FIGURE 5