ASYMPTOTIC BEHAVIOR OF ASSET MARKETS, I:

ASYMPTOTIC INEFFICIENCY

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ABSTRACT

This paper presents a model of an asset market with an infinite number of states of the world. Equilibria exist (under standard assumptions) provided that assets are denominated in a single numeraire commodity. For a given sequence of assets, necessary and sufficient conditions are established that the equilibria of the finite asset markets necessarily converge to an efficient allocation or to an equilibrium allocation of the underlying complete contingent claims market. The set of assets *failing* this condition is residual: it contains a countable intersection of dense, open sets.

1. INTRODUCTION

Underlying the Walrasian (Arrow-Debreu) model of economic activity are two assumptions: that agents act as price-takers, and that there is a market for every commodity. When there is uncertainty about the future, the latter assumption entails a complete set of contingent claims; i.e., claims to consumption streams dependent on the future state of the world.

Arrow (1953, 1964) presents a different model, with trading in futures markets for securities (assets) whose payoffs depend on the state of the world, and in spot markets for physical commodities.[1] Although the Walrasian model and the security market model are formally different, Arrow shows that, if security markets are complete (i.e., if every wealth pattern can be obtained from a portfolio of available securities), the two models are equivalent: they support the same equilibrium allocations (of physical commodities). In particular, if security markets are complete, equilibrium allocations are efficient (Pareto optimal).

If security markets are incomplete, however, the situation is quite different: the Walrasian model and the security market model are not equivalent. In particular, equilibrium allocations of security markets

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1. Arrow considers only nominal securities; i.e., securities denominated in units of account. Radner (1972) considers a model with real securities (i.e., securities denominated in physical commodities).

need not be Pareto optimal.[2] Indeed, equilibrium allocations need not
even be optimal within the set of allocations that can be obtained
through trades in the given securities. (See Hart (1974, 1975), Grossman
(1977), Newberry and J. Stiglitz (1982), Stiglitz (1982), and especially
Geanakoplos and Polemarchakis (1987).)

Security markets equilibria will be efficient if assets span all the
uncertainty; they may be inefficient if assets fail to span all the
uncertainty. Intuition might suggest (and mine did) that security
markets equilibria will be "nearly" optimal if assets span "most" of the
uncertainty, then . The results of this paper suggest that this intuition
may be wrong - badly wrong. In general, security markets equilibria
may be inefficient, and remain inefficient (i.e., bounded away from
Pareto optimal allocations) even when the set of assets expands to a set
which resolves all the uncertainty. Moreover, this "asymptotic
inefficiency" is robust: in fact, "asymptotic inefficiency" is a generic
property of asset sequences.

Our results suggest that, although a complete securities market is a
perfect substitute for a Walrasian (complete contingent claims) market,
a large securities market need not be a good approximation for a
Walrasian market.

To establish these results, we construct a model of a securities

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2. Of course, they may be optimal in certain circumstances; the capital asset pricing
   model provides a noteable example.

market with a countably infinite number of states of nature.[3] When the number of assets is finite, we are able to prove (under standard and natural assumptions about the returns on assets, and the preferences and endowments of consumers) that such a securities market always has a security market equilibrium (Theorem 1). To study the behavior of equilibrium allocations when the assets span "most" of the uncertainty, we fix an infinite sequence $\{\alpha_n\}$ of assets, and consider, for each N, the corresponding securities market. We say that such a sequence is *asymptotically efficient* if, for all consumer preferences and endowments (in a well-behaved class), the equilibrium allocations of the security markets involving only the assets $\{\alpha_n : 1 \le n \le N\}$ converge to Pareto optimal allocations of the underlying Walrasian markets. We identify a condition on an infinite sequence of assets which is necessary and sufficient that it be asymptotically efficient. Modulo a small technical caveat (that preferences be uniformly proper), sequences of assets which are asymptotically efficient are also *asymptotically complete* , in the sense that the equilibrium allocations of the security markets involving only the assets $\{\alpha_n : 1 \le n \le N\}$ converge to equilibrium allocations of the underlying Walrasian market (Theorem 2).[4] The requisite condition for asymptotic efficiency (and asymptotic completeness) is that every Arrow security can be *uniformly* approximated by the returns from a *finite* portfolio. This condition is extremely strong: the sequences of assets which *fail* to satisfy this

3. It would be natural to allow for continuous uncertainty, but this would give rise to serious technical difficulties that we wish to avoid; see Section 7 for further discussion.

4. This method of studying the assumption of complete markets seems analogous to the familiar method of studgying the assumption of price-taking behavior by consumers; see Anderson (1986) for ecxample

condition comprise a residual subset of the space of all asset sequences (Theorem 3).[5]

We find it convenient (for technical reasons) to work with *numeraire securities* ; i.e., securities denominated in a single commodity. Numeraire securities constitute a convenient halfway station between the purely financial securities of Arrow and the general securities of Radner (1972). As shown by Geanakoplos and Polemarchakis (1987), the problems of existence identified by Hart (1975) for general security models do not arise in the case of numeraire securities.[6]

Our results do not depend on pathologies in securities structures, endowments, or preferences. We assume the existence of a riskless asset, and that the returns on other assets are bounded; we could, without loss, assume that all assets have strictly positive payoffs. We assume that endowments are bounded away from zero; we could also assume that endowments are bounded above. Finally, our negative results are obtained with preferences representable by separable, strictly concave utility functions (with bounded marginal rates of substitution); similar constructions could be carried out with preferences representable by homogeneous utility functions.

The crucial idea underlying our negative results is that the requirement that consumption bundles be non-negative places severe

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5. Recall that residual sets are large, and their complements are small, so "asymptotic inefficiency" is a generic property.
6. This had already been established for models involving only purely financial securities by Cass (1964), Werner (1985) and Duffie (1985).

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constraints on the set of portfolios which can be traded. In other words, terminal wealth constraints matter.

If consumption bundles are not required to be non-negative (i.e., if we ignore terminal wealth constraints), the situation is quite different. On the one hand, security markets equilibria need not exist; moreover, equilibrium allocations of security markets may become unbounded as the set of available assets expands. On the other hand, if the infinite sequence $\{\alpha_n\}$ of assets spans all the uncertainty, then limits of equilibrium allocations of the finite securities markets will be equilibrium allocations of the underlying complete markets economy, *provided* that the finite security markets have equilibria and that these equilibrium allocations converge (Theorem 4).

The first research of which I am aware on the asymptotic behavior of security markets is due to Green and Spear (1987, 1988), and their thought-provoking work has provided some of the impetus for the present paper. However, the conclusions of the present paper are not entirely in harmony with those of Green and Spear. For further discussion, see Section 7.

The remainder of the paper is organized in the following way. We describe the model in Section 2. Section 3 provides the basic existence theorem and its proof. Section 4 discusses asymptotic efficiency and asymptotic completeness, Section 5 presents the generic analysis, and Section 6 discusses the case of unconstrained consumption. Finally, Section 7 concludes.

2. THE MODEL

The model we use is a variant of the model of Geanakoplos and Polemarchakis (1987), adapted to accommodate an infinite number of possible states of the world.

Transactions occur (at date 0) in assets (or securities) before the state of nature is known, and then (at date 1) in real commodities, after the state of nature is known. (There would be no difficulty in allowing for several periods, or for consumption before the state of nature is known.) The state of nature is described by an atomic probability space (S, σ), where $S = \{1, 2, 3, \ldots\}$ is the set of *states* of nature, and σ is a probability measure on S.[7] We assume that $\sigma(s) > 0$ for each $s \in S$ (this involves no loss of generality).

At each state there are available for consumption ℓ physical goods, $1, \ldots \ell$, of which the first is the *numeraire*. *Commodity bundles* (or *consumption patterns*) are elements of the *commodity space* $L_1(S, \sigma)^{\ell}$; i.e., functions $x : S \to \mathbb{R}^{\ell}$ for which the *norm*

$$\|x\| = \int |x(s)| \, d\sigma(s) = \int \sum_{i=1}^{\ell} |x_i(s)| \, d\sigma(s)$$

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7. The probability $\sigma(s)$ may be interpreted as the objective probability that state s will occur, or as the unanimous assessments of consumers, but neither of these interpretations is necessary. All that is necessary for our purposes is that assessments of consumers be consistent, in the minimal sense of allowing for the same consumption patterns.

is finite. Since σ is a probability measure, the norm of x is just the expectation of $|x| = \sum_i |x_i|$. We usually do not distinguish between functions $x : S \to \mathbb{R}^{\ell}$ and ℓ-tuples $(x_1, \ldots x_{\ell})$ of functions $x_i : S \to \mathbb{R}$. Since S is countable, a function $x_i : S \to \mathbb{R}$ may be identified with a sequence of real numbers, but it is convenient to use functional notation; we shall usually write $x(s,i)$ rather than $x_i(s)$. It is frequently convenient to identify a function $w \in L_1(S,\sigma)$ with the ℓ-tuple $(w, 0, \ldots 0) \in L_1(S,\sigma)^{\ell}$; we frequently call w a *numeraire pattern*. Given bundles $x, y \in L_1(S,\sigma)^{\ell}$, we write: $x \leq y$ to mean $x(s,i) \leq y(s,i)$ for each i, s; $x < y$ to mean $x(s,i) \leq y(s,i)$ for each i, s and $x \neq y$; $x \ll y$ to mean $x(s,i) < y(s,i)$ for each i, s. We write χ_{si} for the consumption pattern which is one unit of commodity i in state s; i.e., $\chi_{si}(s,i) = 1$, $\chi_{si}(r,j) = 0$ if $(r,j) \neq (s,i)$.

An *asset* (or *security*) is a claim to a numeraire pattern at date 1. (Thus, securities are denominated in the numeraire commodity.) The *return* on asset α in state s is $\alpha(s)$, which may be positive, negative or zero. We frequently use the same notation for an asset and for its returns; it should always be clear from context what is intended. We assume that asset returns are *bounded*; i.e., for each α, there is a constant c such that $|\alpha(s)| \leq c$ for each state s.[8] We assume that the first asset α_1 is *riskless*; i.e., $\alpha_1(s) = 1$ for each $s \in S$.[9] (We usually write 1 for this element of $L_1(S,\sigma)$.)

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8. The requirement that asset returns be bounded is likely unnecessary, but it is very convenient.
9. This requirement too is likely unnecessary but very convenient.

If there are N assets $\alpha_1, \ldots \alpha_N$, a *portfolio* is a vector $\theta = (\theta_1, \ldots \theta_N) \in \mathbb{R}^N$; θ_k is the holding of the k-th asset, and may be positive, negative or zero. The *return* on the portfolio $\theta = (\theta_1, \ldots \theta_N)$ is the numeraire pattern:

$$\text{return}(\theta) = \sum \theta_k \alpha_k \in L_1(S, \sigma) .$$

Asset prices are vectors $q \in \mathbb{R}^N$, where q_k is the price of the k-th asset; if $\theta \in \mathbb{R}^N$ is a portfolio, then $q \cdot \theta = \sum q_k \theta_k$ is the *value* of the portfolio θ at the prices q . The asset prices q are *no-arbitrage prices* if $q \cdot \theta > 0$ for each portfolio θ such that $\text{return}(\theta) > 0$. *Commodity prices* are functions $p : S \to (\mathbb{R}^\ell)^+$; $p(s, i)$ is the price of commodity i in state s . We shall always normalize so that, for each state s , $\sum p(s, i) = 1$. (This is a free normalization, because there will be a different budget constraint in each state; see Geanakoplos and Polemarchakis (1987).) The *income pattern* $p \square x$ required to purchase a commodity bundle x at prices p is defined by:

$$p \square x(s) = \sum p(s, i) x(s, i) .$$

Consumers $h \in \{1, \ldots H\}$ are defined by *endowments* e^h and *preferences* $\prec^h$. We assume that aggregate endowments are strictly positive and that numeraire endowments are bounded away from 0 ; i.e., $\sum e^h \gg 0$, and there is a $\delta > 0$ such that $e^h(s, 1) \geq \delta$ for each consumer h and state s . *Consumption sets* for each consumer are the positive cone $[L_1(S, \sigma)^\ell]^+$; i.e., we require terminal consumption to

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be non-negative (except in Section 6, where we consider unconstrained consumption). *Preferences* are (norm) continuous, convex and strictly monotone; i.e., $x <^h y$ whenever $x < y$. (Such preferences are representable by continuous, quasi-concave, strictly monotonic utility functions $[L_1(S,\sigma)^{\ell}]^+ \to [0,\infty)$.) Finally, we assume that the numeraire is *desirable* in every state, in the sense that, for each commodity bundle $x \in [L^1(S,\sigma)^{\ell}]^+$ and state s, there is a $c > 0$ such that:

$$x <^h cX_{s1} + \sum_{i=2}^{\ell} e^h(s,i)X_{si}$$

(Note that the right hand side is the consumption pattern that yields c units of the numeraire and the endowment of every other commodity in state s and nothing in other states.)

A *securities* (or *asset*) *market* is a pair $\mathcal{E} = (\{\alpha_k\}, \{(e^h, <^h)\})$, where $\{\alpha_k : 1 \le k \le N\}$ is a finite set of assets and $\{(e^h, <^h) : 1 \le h \le H\}$ is a finite set of consumers. The assumptions above are understood to be in force at all times; in particular, α_1 is riskless.

Given securities prices q, commodity prices p, and endowments e^h, the *budget set* $B^h(q,p,e^h)$ for consumer h is the set of pairs (x^h, θ^h) consisting of a commodity bundle $x^h \in [L_1(S,\sigma)^{\ell}]^+$ and a portfolio $\theta^h \in \mathbb{R}^N$ with the properties:

(i) $q \cdot \theta^h \leq 0$ (θ^h is *affordable*);

(ii) $p \square (x^h - e^h - \text{returns}(\theta^h)) \leq 0$ (θ^h *finances* x^h)

(Note that (ii) is an infinite collection of budget constraints, one for each state, but that there is no overall budget constraint.)

A *(securities market) equilibrium* for the securities market $\mathcal{E}$ is a 4-tuple (q,p,x,θ) , where $q \in \mathbb{R}^N$ are asset prices, $p : S \to (\mathbb{R}^\ell)^+$ are commodity prices, $x = (x^1,\ldots x^H)$ is the *equilibrium allocation* (so that x^h is consumer h's equilibrium consumption bundle) and $\theta = (\theta^1,\ldots,\theta^H)$ is the profile of portfolios, satisfying:

(i) $\sum(x^h - e^h) = 0$ (commodity markets clear)

(ii) $\sum \theta^h = 0$ (assets are in zero net supply)

(iii) for all h , x^h is $\preceq^h$ - maximal in $B^h(q,p,e^h)$
                                (consumers optimize)

Underlying every securities market $\mathcal{E}$ is a Walrasian (Arrow-Debreu) economy $\mathcal{E}^{CM}$, with the same commodities and consumers, but with complete contingent claims to every state/commodity pair. *Commodity prices* for $\mathcal{E}^{CM}$ are non-negative linear functionals on the commodity space; i.e., elements of the space $[L_\infty(S,\sigma)^\ell]^+$ of non-negative, bounded functions, $\pi : S \to (\mathbb{R}^\ell)^+$. A *Walrasian (competitive)*

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equilibrium for $\mathcal{E}^{CM}$ is, as usual, a pair (π, x), where π are *commodity prices* and $x = (x^1, \ldots x^H)$ is the *equilibrium allocation* (so that x^h is consumer h's equilibrium consumption bundle), satisfying:

(i) $\sum (x^h - e^h) = 0$;

(ii) for each h, $\pi \cdot (x^h - e^h) \leq 0$;

(iii) for each h , if $x^h <^h y^h$ then $\pi \cdot (y^h - e^h) > 0$.

3. EXISTENCE OF SECURITIES MARKET EQUILIBRIUM

In this section, we show that securities market equilibria exist.

THEOREM 1: Every securities market satisfying the assumptions of Section 2 has a securities market equilibrium.

PROOF: Let $\mathcal{E} = (\{\alpha_k\},\{(e^h, \prec^h)\})$ be a securities market. We may assume, without loss of generality, that the assets have linearly independent returns, since redundant assets can be priced by arbitrage. We construct a securities market equilibrium for $\mathcal{E}$ as the limit of securities market equilibria for finite state securities markets that approximate $\mathcal{E}$.

The first step is to construct these finite state markets and their equilibria. To this end, fix a positive integer n, and write $S_n = \{1,\ldots n\}$, the first n states. Let L_n be the space of functions from S_n to $\mathbb{R}$, and let L_n^{ℓ} be the space of functions from S_n to $\mathbb{R}^{\ell}$. We identify L_n (respectively L_n^{ℓ}) with the subspace of $L_1(S,\sigma)$ (respectively $L_1(S,\sigma)^{\ell}$) consisting of functions which vanish off S_n. Let $P_n : L_1(S,\sigma) \to L_n$ and $Q_n : L_1(S,\sigma)^{\ell} \to L_n^{\ell}$ be the projections (so that $P_n(w)$ is the restriction of w to S_n, etc.).

For each n, we let $\mathcal{E}_n$ be the securities market with state space

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S_n , commodity space L_n^{ℓ} , assets ${}_n\alpha_k = P_n(\alpha_k)$, endowments ${}_n e^h = Q_n(e^h)$, and preferences the restrictions to L_n^{ℓ} of the given preferences. By a result of Geanakoplos and Polemarchakis (1987), $\mathcal{E}_n$ has a securities market equilibrium $(q_n, p_n, {}_n x, {}_n \theta)$. Without loss of generality, we may assume that the riskless asset α_1 has price 1 ; i.e., $q_n \cdot \alpha_1 = 1$. This completes the first step.

The second step is to extract a convergent subsequence of the sequence $\{(q_n, p_n, {}_n x, {}_n \theta)\}$ of securities market equilibria. Note first that the equilibrium bundles ${}_n x^h$ are non-negative and sum to the aggregate endowments $\sum e^h$. Passing to a subsequence if necessary, and keeping in mind that S is countable, we may assume that the equilibrium bundles converge; i.e., there are bundles $x^h \in L_1(S,\sigma)^{\ell}$ such that ${}_n x^h(s,i) \rightarrow x^h(s,i)$ for each s , i . Note that ${}_n x^h \rightarrow x^h$ in the norm topology of $L_1(S,\sigma)$ (again because S is countable and because all the bundles ${}_n x^h$ are bounded by $\sum e^h$). Write $x = (x^1, \ldots x^H)$.

Since we have normalized the state prices to sum to 1 , we may (again passing to a subsequence if necessary) also assume that the state prices $p_n(s,i)$ converge; say $p_n(s,i) \rightarrow p(s,i)$. If $p(s,i) = 0$ for any state s and commodity i , then, for n sufficiently large, the demands in state s would be unbounded. Hence, $p(s,i) \neq 0$ for each s , i .

The definition of securities markets equilibrium implies that the securities prices q_n are no-arbitrage prices for the assets $\{{}_n\alpha_k\}$, and hence for the assets $\{\alpha_k\}$. Since each of the assets α_k is bounded

and α_1 is a riskless bond, there is (for each k) a constant $c_k > 0$ such that $\alpha_1 + c_{k}(_n\alpha_k) \geq 0$ and $\alpha_1 - c_k(_n\alpha_k) \geq 0$; hence $_n\alpha_1 + c_i(_n\alpha_k) \geq 0$ and $\alpha_1 - c_k(_n\alpha_k) \geq 0$. Since $q_n \cdot \alpha_1 = 1$, it follows that $-c_k \leq q_n \cdot \alpha_k \leq c_k$. In particular, the sequence $\{q_n\}$ of securities prices lies in a bounded subset of $\mathbb{R}^N$. Passing to a subsequence if necessary, we may assume that the securities prices q_n also converge to a limit, say $q_n \to q$ for some $q \in \mathbb{R}^N$. Note that $q \cdot \alpha_1 = 1$ and that $-c_k \leq q \cdot \alpha_k \leq c_k$, for each k.

To show that the equilibrium portfolios $_n\theta$ lie in a bounded subset of $\mathbb{R}^N$, we show first that q is a no-arbitrage price for the securities $\{\alpha_k\}$. To see this, let γ be a portfolio of the securities $\{\alpha_k\}$ whose returns g are non-negative, and positive in some state r; without loss, we may assume that $g(r) = 1$. For each $n \geq r$, γ may be viewed as a portfolio of the securities $\{_n\alpha_k\}$, with returns $_ng = P_n(\zeta)$. In particular, $_ng(r) = g(r) = 1$. By the assumption of numeraire desirability, there is a number $C > 0$ such that
$$\sum e^h <^h C\chi_{r1} + \sum e^h(s,i)\chi_{si}$$ (summation over i) for each consumer h. By assumption, numeraire endowments are bounded away from 0; say $e^h(s,1) \geq \delta > 0$ for every h, s. Hence, for each n, the portfolio $(C+\delta)\gamma - \delta(_n\alpha_1)$ is feasible for each consumer. However, the returns on this portfolio, together with consumer h's endowment, are, by numeraire desirability, preferred to any feasible consumption bundle. Hence, this portfolio cannot be affordable at equilibrium, so its price must exceed that of the endowment. Thus, $q_n \cdot [(C+\delta)\gamma - \delta(_n\alpha_1)] > 0$, so that $q_n \cdot \gamma > [\delta/(C+\delta)]$. Passing to the limit yields

$q \cdot \gamma > [\delta/(C+\delta)]$. In particular, q is a no-arbitrage price, as asserted.

We can now show that, for each h , the equilibrium portfolios $_n\theta^h$ remain bounded (as $n \to \infty$). For, suppose not. Passing to a subsequence if necessary, we may assume that the portfolio directions $_n\theta^h/\|_n\theta^h\|$ have a limit $\tau \in \mathbb{R}^N$. Since $q_n \cdot _n\theta^h = 0$, it follows that $q_n \cdot (_n\theta^h/\|_n\theta^h\|) = 0$ and hence that $q \cdot \tau = 0$. Since q is a no-arbitrage price, it is impossible that τ has returns which are non-negative, and strictly positive in some state. We have assumed that the returns $\alpha_1,\dots\alpha_N$ are linearly independent, so that, for some m , the returns $P_m(\alpha_1),\dots P_m(\alpha_N)$ are also linearly independent. This means that no non-zero portfolio of $\alpha_1,\dots\alpha_N$ can have 0 returns in the states $1,\dots m$; in particular, τ cannot have 0 returns (because $\|\tau\| = 1$). Hence τ must have negative returns in some state s . We have assumed that the portfolios $_n\theta^h$ are unbounded, and the state prices $p_n(s,i)$ converge to non-zero limits; this implies that, for n sufficiently large, the returns on the portfolios $_n\theta^h$ create liabilities in state s which cannot be satisfied. Since this contradicts the equilibrium conditions, we conclude that the portfolios $_n\theta^h$ remain bounded, as asserted.

Passing once again to a subsequence if necessary, we may assume that the portfolios $_n\theta^h$ also converge; say $_n\theta^h \to \theta^h$ for each h . Write $\theta = (\theta^1,\dots\theta^H)$. This yields a limit 4-tuple (q,p,x,θ) , completing the second step.

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The third step is to show that (q,p,x,θ) is a securities market equilibrium for $\mathcal{E}$. It is routine to verify that assets are in zero net supply and that commodity markets clear, that all portfolios are feasible and affordable, and that each consumer's consumption bundle is in his budget set. It remains only to verify optimality of portfolios and consumption bundles.

To this end, suppose that consumer h's portfolio and consumption bundle are not optimal (in his budget set). Then there is a consumption bundle y which is strictly preferred to x^h and is financed by a feasible and affordable portfolio γ. We first construct a consumption bundle z which is strictly preferred to x^h and is financed by a feasible and affordable portfolio υ with the additional property that $q \cdot \upsilon < 0$. Let t be a real number (to be chosen later), with $0 < t < 1$, and let ζ be the returns on the portfolio γ. Since γ finances y, we have $p \square [y + \zeta - e^h] \leq 0$, so

$$p \square [ty + t\zeta - te^h] = t\{p \square [y + \zeta - e^h]\} \leq 0 \ .$$

Note that $ty + t\zeta - te^h = ty + (1-t)e^h + t\zeta - e^h$, so that the feasible portfolio $t\gamma$ finances the consumption bundle $ty + (1-t)e^h$. Since numeraire endowments are bounded away from 0, there is a real $\delta > 0$ such that $[ty + (1-t)e^h](s,1) \geq (1-t)\delta$. Hence the consumption bundle $z = ty + (1-t)e^h - (1/2)(1-t)\delta\alpha_1$ is non-negative in each state, and can be financed by the feasible portfolio $\upsilon = t\gamma - (1/2)(1-t)\delta\alpha_1$, with returns $\xi = t\zeta - (1/2)(1-t)\delta\alpha_1$. However, the value of the portfolio υ

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is $q \cdot \upsilon = t(q \cdot \gamma) - (1/2)(1-t)\delta < 0$. Finally, since $z \to y$ as $t \to 1$, continuity of preferences implies that $x^h <^h z$ for t sufficiently close to 1.

Now, consider the projections $Q_n(z)$ of the consumption bundle z and the projection $P_n(\xi)$ of the returns ξ (of the portfolio υ) into the first n states; write υ_n for υ, viewed as a portfolio of the assets ${}_n\alpha_1, \dots, {}_n\alpha_N$. By construction, asset prices and commodity prices converge; i.e., $q_n \to q$ and $p_n \to p$. It follows that $q_n \cdot \upsilon_n < 0$ for n sufficiently large. Moreover, the portfolio υ_n is feasible and finances the consumption bundle $Q_n(z)$, for n sufficiently large. On the other hand, continuity of preferences implies that $Q_n(z)$ is strictly preferred to ${}_nx^h$, for n sufficiently large, so this contradicts the fact that $(q_n, p_n, {}_nx, {}_n\theta)$ is an equilibrium for the securities market $\mathcal{E}_n$. We conclude that portfolios and consumption bundles are optimal (given prices), so that (q, p, x, θ) is an equilibrium for the securities market $\mathcal{E}$, as asserted. This completes the proof. ∎

4. ASYMPTOTICS

The result of the previous section guarantees that securities market equilibria exist; in this section we study the asymptotic behavior of such equilibria as the set of securities grows. We ask: When do equilibrium allocations of the securities markets converge to Pareto optimal allocations or to Walrasian (competitive equilibrium) allocations of the underlying Walrasian economy?

Before formalizing these questions, we address a small but important point. As Mas-Colell (1986) has pointed out, the usual assumptions on preferences and endowments which suffice to guarantee the existence of Walrasian equilibrilium in the finite dimensional setting do not suffice in infinite dimensional settings such as ours. The difficulty is that the consumption sets of consumers are assumed to be the positive cone, which has an empty interior. This leaves opens the possibility that individual preferred sets may not be supportable by prices; in such a case, competitive equilibria need not exist. To avoid this difficulty, Mas-Colell introduced a restriction on preferences which he called *uniform properness* ; in essence, uniform properness bounds marginal rates of substitution. In conjunction with the usual assumptions on preferences and endowments, uniform properness suffices to guarantee the existence of competitive equilibria. If we hope to show that equilibrium allocations of securities markets converge to equilibrium allocations of the underlying Walrasian economy, we must surely make

assumptions that are strong enough to guarantee the existence of Walrasian equilibria. The easiest way to do this is to assume that preferences are uniformly proper, and that is what we shall do.[10]

To formalize our questions about asymptotic behavior of securities market equilibria, we fix an infinite sequence $\{\alpha_k\}$ of assets (of which the first is riskless). Given a set $\{(e^h, \prec^h) : 1 \leq h \leq H\}$ of consumers, we call the tuple $S = (\{\alpha_k\}, \{(e^h, \prec^h)\})$ a *securities market structure* . For each N , let $\mathcal{E}_N = (\{\alpha_k : 1 \leq k \leq N\}, \{(e^h, \prec^h)\})$ be the corresponding securities market. We say that the securities market structure S is *asymptotically efficient* if, for each $\varepsilon > 0$, there is an integer N_0 such that: for $N \geq N_0$, every equilibrium allocation of the securities market $\mathcal{E}_N$ is within ε (in norm) of a Pareto optimal allocation of the underlying Walrasian economy $\mathcal{E}^{CM}$. (Since the space of states of nature is countable, all these allocations lie in a norm compact subset of the commodity space $L_1(S, \sigma)$, so this requirement is equivalent to the requirement that equilibrium allocations of $\mathcal{E}_N$ converge, as N tends to ∞ , to Pareto optimal allocations of $\mathcal{E}^{CM}$. Norm continuity of preferences implies that the utilities of equilibrium allocations of $\mathcal{E}_N$ also converge to the utilities of Pareto optimal allocations.) We say that the securities market structure S is *asymptotically complete* if, for each $\varepsilon > 0$, there is an integer N_0 such that, for $N \geq N_0$, every equilibrium allocation of the securities market $\mathcal{E}_N$ is within ε of an equilibrium allocation of the underlying Walrasian economy $\mathcal{E}^{CM}$. (Similarly, this is equivalent to the requirement that equilibrium allocations of $\mathcal{E}_N$ converge, as N tends to ∞ , to equilibrium

10. For more on the meaning of uniform properness, see Richard and Zame (1986).

allocations of $\mathcal{E}^{CM}$. Again, norm continuity of preferences implies that the utilities of equilibrium allocations of $\mathcal{E}_N$ also converge to the utilities of competitive equilibrium allocations of $\mathcal{E}^{CM}$.) If a securities market structure is not asymptotically efficient (respectively not asymptotically complete), we say it is *asymptotically inefficient* (respectively *asymptotically incomplete*).

Securities market structures are the appropriate objects of study if we view assets, preferences and endowments as fixed (or known). Alternatively, we might view assets as fixed (or known), but preferences and endowments as variable (or unknown). Taking the latter point of view, we shall say that a given sequence $\{\alpha_k\}$ of assets (of which the first is a riskless bond) is *asymptotically efficient* if, for all choices of consumers $\{(e^h, \prec^h)\}$, the securities market structure $S = (\{\alpha_k\}, \{(e^h, \prec^h)\})$ is asymptotically efficient. We say that $\{\alpha_k\}$ is *asymptotically complete* if, for all choices of consumers $\{(e^h, \prec^h)\}$ with uniformly proper preferences, the securities market structure $S = (\{\alpha_k\}, \{(e^h, \prec^h)\})$ is asymptotically complete. If an asset sequence is not asymptotically efficient (respectively asymptotically complete), we say it is *asymptotically inefficient* (respectively *asymptotically incomplete*). From this point of view, the basic questions are: When are asset sequences asymptotically efficient? aymptotically complete?

Theorem 2 provides complete answers to these questions about asset sequences (and its proof provides some information about the corresponding questions for securities market structures). Before giving

the formal statement, we collect some notation and terminology.

For $x, y \in L_1(S,\sigma)$, we define $d_\infty(x,y) = \sup_{s \in S} |x(s)-y(s)|$. Of course, this supremum will be infinite if $|x - y|$ is an unbounded function. Nevertheless, this "distance function" induces a well-defined (complete, metrizable) topology on $L_1(S,\sigma)$, which we call the *uniform topology*. If E is a subset of $L_1(S,\sigma)$, we denote its closure with respect to the uniform topology by $cl_\infty(E)$; $x \in cl_\infty(E)$ if and only if x can be *uniformly approximated* by elements of E. The distance from a point x to a set E is $d_\infty(x,E) = \inf \{d_\infty(x,y) : y \in E\}$. Note that $d_\infty(x,E) = 0$ exactly when $x \in cl_\infty(E)$. By an *Arrow security (for state s)* we mean the security χ_{s1} whose return is 1 in state s and 0 in every other state. (Note that this is in agreement with our previous useage.)

The following result completely characterizes asymptotic completeness and asymptotic efficiency for a sequence of assets.

THEOREM 2: Let $\{\alpha_k\}$ be a sequence of assets, of which the first is riskless. The following statements are equivalent:

(i)   the sequence $\{\alpha_k\}$ is asymptotically efficient;

(ii)   the sequence $\{\alpha_k\}$ is asymptotically complete;

(iii)   every Arrow security can be uniformly approximated by the returns on a finite portfolio of the securities α_k.

The last condition may be formulated equivalently as: for every state s and every $\varepsilon > 0$, there is a finite portfolio of the assets $\{\alpha_k\}$ whose returns differ from χ_{s1} by at most ε in every state.

PROOF: (iii) $\Rightarrow$ (i) and (ii): Fix H consumers $\{(e^h, \prec^h)\}$, and consider a sequence $\{\alpha_k\}$ of assets (of which the first is riskless) having the property that every Arrow security can be uniformly approximated by the returns on a finite portfolio. Assume we are given a subsequence $\mathcal{E}_{N(n)} = (\{\alpha_k : 1 \leq k \leq N(n)\}, \{(e^h, \prec^h)\})$ of $\{\mathcal{E}_N\}$, and equilibrium allocations $x_n = ({}_nx^1, {}_nx^2, \ldots {}_nx^H)$ for $\mathcal{E}_{N(n)}$, converging to $x = (x^1, x^2, \ldots x^H)$; let q_n be the corresponding asset prices and let p_n be the corresponding state prices. We proceed by showing that x is in the core of the underlying Walrasian economy $\mathcal{E}^{CM}$.

Suppose this were not so. Then there would be a set of consumers, which we may assume to be the consumers $M = \{1, 2, \ldots M\}$, and an allocation $y = (y^1, \ldots y^M)$ that is a redistribution of the endowments $(e^1, \ldots e^M)$ and is unanimously preferred to x by consumers in M. Continuity of preferences, together with the assumption that numeraire endowments are bounded away from zero, guarantees that we can find a state r, allocations $z = (z^1, \ldots z^M)$ and $\tilde{z} = (\tilde{z}^1, \ldots \tilde{z}^M)$, and a real number $\delta > 0$ such that $\tilde{z}$ is unanimously preferred to x by consumers in M and:

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$$\tilde{z}^m(s,i) = z^m(s,i) = 0 \qquad \text{for } s > r, \ 1 \le i \le \ell \ ;$$

$$\tilde{z}^m(s,i) = z^m(s,i) \qquad \text{for } s \le r, \ 2 \le i \le \ell \ ;$$

$$\tilde{z}^m(s,1) = z^m(s,1) - \delta \qquad \text{for } s \le r \ ;$$

$$e^m(s,1) \ge \delta \qquad \text{for all } s \ ;$$

$$\sum_m z^m(s,i) = \sum_m y^m(s,i) = \sum_m e^m(s,i) \qquad \text{for } s \le r, \ 1 \le i \le \ell \ .$$

Since we have normalized the state prices to sum to 1 , we may (passing to a subsequence if necessary) assume that the state prices $p_n(s,i)$ converge; say $p_n(s,i) \to p(s,i)$. As noted in the proof of Theorem 1, if $p(s,i) = 0$ for any state s and commodity i , then, for n sufficiently large, the state s demands in $\mathcal{E}_{N(n)}$ would exceed total endowments. Hence, $p(s,i) \ne 0$ for each s , i . (The limiting behavior of asset prices is irrelevant.)

For each price system p^* and each $m \in M$, define a numeraire pattern $w^m(p^*)$ by:

$$w^m(p^*,s) = [1/p^*(s,1)][p^* \,\square\, (z^m - e^m)](s) \quad \text{for } s \le r \text{ , and}$$

$$w^m(p^*,s) = 0 \qquad \text{for } s > r \ .$$

For $s \leq r$, $w^m(p*,s)$ is the amount of numeraire that must be transferred into state s in order to make the net purchase $(z^m - e^m)(s)$, at prices $p*$. Since z is a reallocation of endowments in states $s \leq r$, and $w^m(p*,s) = 0$ in states $s > r$, if follows that, for each price system $p*$ and each state s, $\sum w^m(p*,s) = 0$ (summation over $m \in M$).

For each m, $w^m(p)$ is a finite linear combination of Arrow securities, so (iii) enables us to find finite portfolios $\theta^1,...\theta^{M-1}$, such that $d_\infty(\text{returns}(\theta^m), w^m(p)) < \delta/M$ for $1 \leq m \leq M-1$. Set

$$\theta^M = - \sum_{m=1}^{M-1} \theta^m$$

so that θ^M is a finite portfolio, and $d_\infty(\text{returns}(\theta^M), w^M(p)) < \delta$. Since the state prices $p_n(s,i)$ converge to $p(s,i)$ for each s, i and $w^m(p*,s) = 0$ for states $s > r$, we conclude that $d_\infty(\text{returns}(\theta^m), w^m(p_n)) < \delta$. for each $m \in M$, provided that n is sufficiently large.

Our construction guarantees that, at all prices p_n sufficiently close to p, the portfolio θ^m is feasible for consumer m (i.e., it does not impose unsatisfiable liabilities). Since we have constructed θ^M so that $\sum \theta^m = 0$, at least one of the portfolios θ^m must have a non-positive price (at asset prices q_n). However, at prices p_n

sufficiently close to p , the returns ζ^m on the portfolio θ^m will finance purchase of the commodity bundle $\tilde{z}^m$. Since $_n x^m \to x^m$ and $x^m <^m \tilde{z}^m$, continuity of preferences implies that $_n x^m <^m \tilde{z}^m$ for n sufficiently large. Since $\tilde{z}^m$ belongs to the budget set of consumer m , this contradicts the equilibrium conditions in $\mathcal{E}_{N(N)}$. It follows that x is in the core of the underlying complete markets economy, as desired.

We have just proved that (iii) implies that every limit of equilibrium allocations of the finite securities markets is in the core of the underlying Walrasian economy. Since allocations in the core are Pareto optimal, this certainly yields asymptotic efficiency, and we obtain the implication (iii) $\Rightarrow$ (i) .

If we replicate the economy, and note that a securities market equilibrium for the original economy is necessarily a securities market equilibrium for the replicated economy, we conclude that every limit of equilibrium allocations of the finite securities markets is in the core of every replication of the underlying complete markets economy. We can now apply a result of Aliprantis, Brown and Burkinshaw (1987), which is the infinite dimensional version of the Debreu and Scarf (1963) core convergence theorem: assuming that preferences are uniformly proper, equilibrium allocations are precisely those in the core of every replication. This yields asymptotic completeness, and we obtain the implication (iii) $\Rightarrow$ (ii) .

(i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) : We establish the contrapositives. Suppose that some Arrow security, say χ_{11} (without loss), cannot be uniformly approximated by returns on a finite portofolio. Note that the set of returns on finite portfolios constitutes a linear subspace of $L_1(S,\sigma)$ coinciding with the linear span span$\{\alpha_k\}$ of the securities. Set $4\rho = d_\infty(\chi_{11}, \text{span}\{\alpha_k\}) > 0$. We find two consumers so that the equilibrium allocations of the corresponding securities markets are bounded away from Pareto optimal allocations of the underlying Walrasian economy. It is convenient to give the construction first for the case of one commodity (the numeraire) in each state; the general case requires only a simple adaptation.

For the one commodity case, let $u : [0,\infty) \to [0,\infty)$ be any continuously differentiable, strictly concave function such that $u'(0) < \infty$ and $u'(\infty) > 0$. The two consumers will have identical utility functions $U^1 = U^2 = U$, where $U(x) = \sum a_k u(x(k))\sigma(k)$, and $\{a_k\}$ is a bounded sequence of strictly positive numbers, to be chosen later.[11] Endowments are: $e^1 = (3, \rho, \rho, \rho, \dots)$, $e^2 = (1, 3\rho, \rho, \rho, \dots)$. Write $e = e^1 + e^2$ for the aggregate endowment. Since consumers have identical, separable, strictly concave utility functions, it is easy to see that every Pareto optimal allocation is of the form $(\lambda e, (1-\lambda)e)$ for some real number λ with $0 \le \lambda \le 1$. To obtain the Pareto optimal allocation $(\lambda e, (1-a)\lambda)$ requires the net trade (for consumer 1):

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11. Note that any choice of the sequence (a_k) will lead to a well-defined utility function U on $L_1(S,\sigma)$; U will be uniformly proper if we choose the sequence (a_k) bounded away from 0 .

$$\lambda e - e^1 = (4\lambda - 3, (4\lambda - 1)p, (2\lambda - 1)p, (2\lambda - 1)p, \ldots)$$

Equilibrium allocations of securities markets (and hence their limits) are individually rational, so any Pareto optimal allocation which is the limit of equilibrium allocations of the securities markets must also satisfy the individual rationality requirements $U(\lambda e) \geq U(e^1)$ and $U((1-\lambda)e) \geq U(e^2)$. By choosing the coefficient sequence $\{a_k\}$ appropriately, we can guarantee that these inequalities are satisfiable only if λ is very close to $1/2$. (We leave the details to the reader.)

Now consider a securities market $\mathcal{E}_N = (\{\alpha_k : 1 \leq k \leq N\}, \{(e^h, \prec^h)\})$, and an equilibrium allocation (x^1, x^2) of $\mathcal{E}_N$; suppose that (x^1, x^2) is close (in the L_1 norm) to a Pareto optimal allocation. If we choose $\{a_k\}$ so that λ is close to $1/2$, then x^1 must be close (in the L_1 norm) to λe, for some λ close to $1/2$, and the net trade of consumer 1, which is $x^1 - e^1$, must be close (in the L_1 norm) to $\lambda e - e^1$. If λ is close to $1/2$, the net trade of consumer 1 must be close to 1 in the first state, close to p in the second state, and be bounded by p in every other state (since neither consumer 1 nor consumer 2 can incur liabilities greater than endowments). Hence, we can obtain (by choosing $\{a_k\}$ so that λ is sufficiently close to $1/2$) that $d_\infty(x^1 - e^1, \chi_{11}) \leq 2p$. Since there is only one commodity in each state, equilibrium trades in the securities market $\mathcal{E}_N$ must be effected entirely through transactions in available securities. In particular, we can find a finite portfolio whose returns are precisely $x^1 - e^1$, and hence differ from χ_{11} by at most $2p$. This contradicts our supposi-

tion that $d_\infty(\chi_{11}, \text{span}\{\alpha_k\}) = 4\rho$. We conclude that, (for appropriately chosen $\{a_k\}$), no equilibrium allocation of the securities market $\mathcal{E}_N = (\{\alpha_k : 1 \le k \le N\}, \{(e^h, \prec^h)\})$ can be close to a Pareto optimal allocation of the underlying Walrasian economy; since Walrasian allocations are Pareto optimal, it follows *a fortiori* that no equilibrium allocation of the securities market $\mathcal{E}_N = (\{\alpha_k : 1 \le k \le N\}, \{(e^h, \prec^h)\})$ can be close to a Walrasian allocation of the underlying Walrasian economy. This completes the proofs of (iii) $\Rightarrow$ (i) and (ii) for the case of one commodity in each state.

To obtain the case of ℓ commodities from the case of one commodity, we simply choosae utility functions so that, in each state, all ℓ commodities are perfect substitutes. (Of course, this is incompatible with strict concavity of utility functions, but strict concavity can be restored by making a tiny perturbation. Again, we leave the details to the reader.) ∎

The argument of Theorem 2 actually proves a bit more. The assumption that Arrow securities can be uniformly approximated by the returns on finite portfolios entails that, for every numeraire pattern x that is non-zero in only a finite number of states, and every $\varepsilon > 0$, there is a finite portfolio of the assets $\{\alpha_k\}$ whose returns differ from x by at most ε in every state. However, the argument that equilibrium allocations of $\mathcal{E}_N$ converge to a core allocation (and in particular, to a Pareto optimal allocation) of the underlying Walrasian economy, uses a

bit less. To be precise, this argument uses only that for every numeraire pattern x that is non-zero in only a finite number of states, every $\varepsilon > 0$, and every state s_0, there is a finite portfolio of the assets $\{\alpha_k\}$ whose returns differ from x by at most ε in states $s \leq s_0$, and are bounded (in absolute value) in every other state by numeraire endowments divided by the number of consumers. This is equivalent to the assumption that every numeraire pattern x that is non-zero in only a finite number of states $1, 2, \ldots, s_0$ is the limit, in the L_1 norm, of portfolio returns that are bounded in states $s > s_0$ by numeraire endowments divided by the number of consumers. Informally: if the assets span a norm dense subspace of $L_1(S, \sigma)$, and endowments are "large enough" then securities markets equilibria will converge to Pareto optima.[12] However, this is a stringent requirement; see Example 4 below and Section 5, particularly the concluding discussion.

We now give a number of examples. (See also Green and Spear (1988).) It is convenient to describe an asset sequence by an infinite matrix, the j-th column of which represents returns on the j-th asset (so that the entry in the i-th row and j-th column is the return paid by the j-th asset in the i-th state, etc.).

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12. The reader should *not* infer that asymptotic efficiency is a generic property when endowments are uniformly large. However, analysis of that case seems to require different methods.

EXAMPLE 1:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 1 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 1 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix}$$

A is asymptotically efficient; of course, it is simply a riskless securi-
ty, together with a complete set of Arrow securities.

EXAMPLE 2:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 1 & 0 & 0 & . & . & . \\ 1 & 1 & 1 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

B is asymptotically efficient. This is merely to illustrate the point
that asymptotic efficiency depends only on the space spanned by the as-
sets; note that the space spanned the first n columns of B is pre-
cisely the same as that spanned by the first n columns of A .

EXAMPLE 3:

$$C \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 1 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

C is asymptotically inefficient. Indeed, the distance from χ_{11} to the span of the columns of C is $1/2$.

EXAMPLE 4:

$$D \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & 0 & . & . & . \\ 1 & -1 & 1 & 0 & 0 & . & . & . \\ 1 & 0 & -1 & 1 & 0 & . & . & . \\ . & 0 & 0 & -1 & 1 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

D is asymptotically inefficient; as in Example 3, the distance from χ_{11} to the span of the columns of D is $1/2$. Note that any sequence in the span of the columns of D that converges to χ_{11} in the $L_1(S,\sigma)$

norm is necessarily unbounded. (See the comments following the proof of Theorem 2 above.)

We have required that endowments be bounded away from 0 ; the final example shows the kind of difficulties that could arise if we did not make this restriction.

EXAMPLE 5: Consider the sequence of assets $\{\alpha_k\}$ whose returns are $\alpha_k(s) = k^{-s}$. If endowments are $e^h(s) = \exp(-s)$, then *no finite non-zero portfolio* (except for the 0 portfolio) is dominated by endowments. Hence *no finite portfolio can be traded* . We surely cannot expect to say much about asymptotic efficiency for such sequences of assets.

5. GENERICITY

Roughly speaking, the results of Section 4 tell us that a sequence of assets leads asymptotically to efficiency whenever it is possible to approximate any given numeraire pattern which is non-zero in only a finite number of states by portfolios which do not involve "large trades" in other states. As we have noted by simple examples, this is not an innocuous restriction. Here we show that it is in fact a very strong restriction. We do this by showing that "most" asset sequences cannot approximate *any* numeraire pattern which is non-zero in only a finite number of states without simultaneously requiring "large trades" in other states. The meaning of "large trades" should become clear in what follows; to give precise meaning to "most" we need a topology on the set of asset sequences.

Let $\alpha = \{\alpha_k\}$ be a sequence of assets, of which the first is riskless. Since we are interested only in the span of the assets, there is no loss of generality in assuming that $\|\alpha_k\| = 1$ for each k; such a sequence *normalized*. Note that a normalized sequence of assets gives rise to a bounded operator (continuous linear transformation) R^α from ℓ_1 (the space of all summable real sequences) into $L_1(S,\sigma)$ defined by

$$R^\alpha(c) = \sum c_k \alpha_k ,$$

for $c = (c_1, c_2, \dots) \in \ell_1$. If we identify normalized asset sequences

with the operators they induce, it seems natural to define the *operator distance* between asset sequences α, β as

$$d^{op}(\alpha,\beta) = \|R^\alpha - R^\beta\|_{op} = \sup\left\{\sum \sigma(s) | R^\alpha(c)(s) - R^\beta(c)(s)|\right\}$$

where the sum extends over all states s, and the supremum extends over all sequences $c \in \ell_1$ with $\|c\|_{\ell_1} = \sum |c_i| \leq 1$. In the topology induced by this distance function, two asset sequences α, β are close together if the same portfolios yield nearly the same (expected) returns, uniformly over all portfolios representing total trades of at most 1 share. (See Dunford and Schwartz (1958) for further discussion.)

Alternatively, we can define another distance between asset sequences α, β by:

$$d^1(\alpha,\beta) = \sum \sigma(s) | \alpha_k(s) - \beta_k(s)|$$

the sum extending over all indices k and all states s. (Strictly speaking, this is not a distance function, since it might be infinite; nonetheless, it induces a well-defined topology.[13])

Since we have required that the returns on assets be bounded, we should take account of this. There are several ways in which we might

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13. This is the topology arising from the *trace norm*; again, we refer to Dunford and Schwarz (1958) for further details. One point to note: if $d^1(\alpha,\beta)$ is finite, then the difference $R^\alpha - R^\beta$ is a compact operator from ℓ_1 to $L_1(S,\sigma)$; i.e., it maps bounded subsets of ℓ_1 to relatively compact subsets of $L_1(S,\sigma)$.

do so; the simplest is to restrict our attention to asset sequences for which the k-th asset is bounded by a preassigned number. To this end, fix, once and for all, a sequence $b = \{b_k\}$ of positive numbers, with $b_k > 1$ for each k, and let $\mathcal{A}$ be the set of normalized asset sequences $\{\alpha_k\}$ for which $|\alpha_k(s)| \leq b_k$ for each s, k. It is not hard to see that the topology on $\mathcal{A}$ induced by d^1 is stronger than that induced by d^{op}, and that they are both complete, metric topologies. (Note that $\mathcal{A}$ is certainly not empty, since it contains the asset sequence each term of which is the riskless asset.)

The identification of asset sequences as operators $\ell_1 \to L_1(S,\sigma)$ suggests at least one way to answer the question of what it should mean for a given infinite set of assets to be *complete*. Informally, completeness of a set of assets should mean that any prescribed wealth stream can be obtained as the returns of a suitable portfolio. Since the set of states and available assets is infinite, we should allow for infinite portfolios. However, it is not entirely clear *which* infinite portfolios we should admit; questions of convergence must be addressed. For normalized asset sequences, it seems clear that, as a minimum, we should certainly allow any portfolio in ℓ_1 (since, for $\theta \in \ell_1$, the series $\sum \theta_j \alpha_j$ converges absolutely). If the returns of such portfolios exhaust all possible wealth streams (i.e., if the returns operator $R^\alpha : \ell_1 \to L_1(S,\sigma)$ is onto, and *a fortiori* if the returns operator is invertible) we should certainly say that the sequence α is complete. The set of invertible operators $\ell_1 \to L_1(S,\sigma)$ and the set of onto operators $\ell_1 \to L_1(S,\sigma)$ are both open in the topology induced by

d^{op} , and *a fortiori* in the topology induced by d^1 (since it is stronger). Countability of S implies that these sets of operators are not empty. Hence the set of complete, normalized asset sequences is a relatively open, non-empty subset of $\mathcal{A}$. It follows that, so long as $b_k \geq 1/\sigma(k)$ for each k , the set of *complete* , normalized asset sequences is certainly a non-empty open subset of $\mathcal{A}$, since it contains asset sequences that induce invertible operators $\ell_1 \to L_1(S,\sigma)$. (See Example 2 of Section 4.)

Recall that a subset of a complete metric space is *residual* if it contains the countable intersection of dense, open sets. (Recall that residual sets are large and that their complements are small. In particular, the Baire category theorem implies that a residual subset of a complete metric space is dense.) Since we have identified two different complete metric topologies on $\mathcal{A}$, we have two possible interpretations of "residual" for subsets of $\mathcal{A}$; fortunately, the sets of interest to us are residual in *both* these topologies.

THEOREM 3: The set of asymptotically inefficient asset sequences is a residual subset of $\mathcal{A}$ in both the d^1 and d^{op} topologies.

To motivate the proof, write $C_0(S)$ for the set of functions $v \in L_1(S,\sigma)$ with the property that $|v(s)| \to 0$ as $s \to \infty$. Note that $C_0(S)$ is exactly the uniform closure of the subspace of $L_1(S,\sigma)$

spanned by Arrow securities. Theorem 2 tells us that the asset sequence $\{\alpha_k\}$ is asymptotically efficient exactly when every Arrow security, and hence every numeraire pattern in $C_0(S)$, lies in the d_∞ closure of $\text{span}\{\alpha_k\}$. Equivalently, for every vector $v \in C_0(S)$, we have $d_\infty(v, \text{span}\{\alpha_k\}) = 0$. Of course, 0 is the closest a numeraire pattern can be to $\text{span}\{\alpha_k\}$; since $\alpha_1 = 1$, the furthest a numeraire pattern can be from $\text{span}\{\alpha_k\}$ is its distance from the one-dimensional subspace spanned by the asset 1. We show that, for a residual set of asset sequences, all vectors in $C_0(S)$ are at least half as far from $\text{span}\{\alpha_k\}$ as they could possibly be.

Before beginning the proof proper, it is convenient to isolate the most technical part in a Lemma.

LEMMA: For each $v \in L_1(S,\sigma)$, the set $Q(v)$ of normalized asset sequences $\{\alpha_k\}$ with the property:

$$d_\infty(v, \text{span}\{\alpha_k\}) \geq (1/2)d_\infty(v, \text{span}\{1\})$$

is a residual subset of $\mathcal{A}$ in both the d^1 and d^∞ topologies.

PROOF: For each $s \in S$, and $w \in L_1(S,\sigma)$, write $w^{(s)}$ for the numeraire pattern which coincides with w in states $1,2,\ldots s$ and is 0 elsewhere. Fix positive integers m, n; write $p = (1/2) - 2^{-m}$, and let $Q(v,m,n)$ be the set of asset sequences $\{\alpha_k\}$ in $\mathcal{A}$ such that:

(i) $\alpha_1^{(n)}, \ldots \alpha_n^{(n)}$ are linearly independent;

(ii) $d_\infty(v, \text{span}\{\alpha_1, \ldots \alpha_n\}) > pd_\infty(v, \text{span}\{1\})$.

It is easily seen that $Q(v)$ contains the intersection (taken over all integers n , m) of the sets $Q(v,m,n)$, so it suffices to show that each $Q(v,m,n)$ is an open subset of $\mathcal{A}$ with respect to d^{op} and is dense with respect to d^1 .

To see that $Q(v,m,n)$ is an open subset of $\mathcal{A}$ with respect to d^{op} , note first that the linear independence of $\alpha_1^{(n)}, \ldots \alpha_n^{(n)}$ is certainly preserved by any perturbation of the sequence $\{\alpha_k\}$ which is small in the first n states and for the first n terms of the sequence. Hence the set of sequences $\{\alpha_k\}$ satisfying the linear independence condition (i) is open. Note too that, if

$$d_\infty(v, \text{span}\{\alpha_1, \ldots \alpha_n\}) > pd_\infty(v, \text{span}\{1\}) ,$$

then there is a state s such that

$$d_\infty(v^{(s)}, \text{span}\{\alpha_1^{(s)}, \ldots \alpha_n^{(s)}\}) > pd_\infty(v, \text{span}\{1\}) .$$

Since the vectors $v^{(s)}, \alpha_1^{(s)}, \ldots \alpha_n^{(s)}$ all lie in the finite dimensional space $\mathbb{R}^s$, a simple continuity argument shows that the last inequality is also satisfied by any perturbation of $\{\alpha_k\}$ which is sufficiently

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small in the first s states and for the first n terms of the sequence. Hence, the set of sequences $\{\alpha_k\}$ satisfying the span condition (ii) is open, as desired.

To see that $Q(v,m,n)$ is a dense subset of $\mathcal{A}$ with respect to d^1, fix a sequence $\beta = \{\beta_k\} \in \mathcal{A}$; we construct a perturbed sequence $\tilde{\beta} = \{\tilde{\beta}_k\} \in Q(v,m,n)$ such that $\sum \sigma(r)|\tilde{\beta}_k(r) - \beta_k(r)|$ is arbitrarily small (the sum extending over all states r, and all indices $k \leq n$). Since $\|\beta_i - \beta_i^{(s)}\| \to 0$ as s tends to ∞, we may (choosing s sufficiently large), assume that $\beta_j^{(s)} = \beta_i$ for $i = 2,\ldots n$. Moreover, since linear independence of $\beta_1^{(n)},\ldots\beta_n^{(n)}$ is equivalent to the vanishing of an $n\times n$ determinant, this condition can be achieved by an arbitrarily small perturbation, which we assume to have been already carried out.

To achieve the span condition, it is convenient to work with the linear transformation $R^\beta : \mathbb{R}^n \to L_1(S,\sigma)$. If the span condition (ii) is not satisfied, let C be the set of vectors $c \in \mathbb{R}^n$ such that

$$\|R^\beta(c) - v\|_\infty \leq \rho d_\infty(v, \text{span}(1)) .$$

Since $\{\beta_1^{(n)},\ldots\beta_n^{(n)}\}$ is a linearly independent set, the linear transformation R^β is an isomorphism of $\mathbb{R}^n$ with a finite dimensional subspace of $L_1(S,\sigma)$, so that C is compact. For each $c \in C$, the above inequality, the triangle inequality, and the facts that $\beta_1 = 1$ and that $\rho < 1/2$ imply that:

$$\sum_{i=2}^{n} |c_i| \|\beta_i\|_\infty \geq \|R^\beta(c) - c_1 1\|_\infty > \rho d_\infty(v, \text{span}\{1\})$$

Choose *any* state $r > s$. For $1 = 2,\ldots,n$, write $\delta_i = 0$ if $c_i = 0$, $\delta_i = -1$ if $c_i \neq 0$ and $v(r)/c_i > 0$, and $\delta_i = +1$ otherwise. Define $\tilde{\beta}_i$ and β^*_i by:

$$\tilde{\beta}_i(t) = \beta_i(t) \text{ for } t \neq r,$$

$$\tilde{\beta}_i(r) = \delta_i \|\beta_i\|_\infty$$

$$\beta^*_i = \tilde{\beta}_i / \|\tilde{\beta}_i\|$$

This yields a perturbation $\{\beta^*_i\}$ of $\{\beta_i\}$ such that

$$\|R^{\beta^*}(c) - v\|_\infty > \rho d_\infty(v, \text{span}\{1\})$$

Since R^{β^*} is continuous, we conclude that

$$\|R^{\beta^*}(c') - v\|_\infty > \rho d_\infty(v, \text{span}\{1\})$$

for all c' in some neighborhood W_c of c. Since C is compact, we can cover C with a finite number of these neighborhoods. Since we can make these perturbations in different states r, we conclude that there is a single perturbation $\{\tilde{\beta}_i\}$ (a normalized asset sequence) such that $\|R^{\tilde{\beta}}(c) - v\|_\infty > \rho d_\infty(v, \text{span}\{1\})$ for every $c \in C$. Since we have made

these perturbations in states where the β_i vanished, we conclude that $\|R^{\tilde{\beta}}(c'') - v\|_\infty > \rho d_\infty(v, \text{span}\{1\})$ for every $c'' \in \mathbb{R}^n \setminus C$. Finally, since we can make these perturbations only in states of arbitrarily low probability, we can guarantee that $d^1(\beta, \tilde{\beta})$ is as small as we like. Hence the perturbation $\tilde{\beta}$ has all the required properties. We conclude that $Q(v,m,n)$ is dense with respect to d^1, as desired. This completes the proof. ∎

With this technical result in hand, we turn to the proof of Theorem 3.

PROOF OF THEOREM 3: Write Q for the set of asset sequences $\{\alpha_k\} \in \mathcal{A}$ having the property that, for each $v \in C_0(S)$,

$$d_\infty(v, \text{span}\{\alpha_k\}) \geq (1/2)d_\infty(v, \text{span}\{1\})$$

We claim that Q is a residual subset of $\mathcal{A}$ in both the topologies d^1 and d^∞. To see this, note that for each $v \in C_0(S, \sigma)$, the Lemma provides a residual set $Q(v)$ of asset sequences $\{\alpha_k\}$ such that

$$d_\infty(v, \text{span}\{\alpha_k\}) \geq (1/2)d_\infty(v, \text{span}\{1\})$$

If $\{\alpha_k\} \in Q(v)$ and $d_\infty(v, v') < (1/4)d_\infty(v, \text{span}\{1\})$, the triangle inequality implies that

$$d_\infty(v', \text{span}\{\alpha_k\}) \geq (1/4) d_\infty(v', \text{span}\{1\})$$

Since $C_0(S,\sigma)$ is d_∞-separable, we may choose a countable dense subset $\{v_i\}$. Set $Q = \cap Q(v_i)$; as the countable intersection of residual sets, Q is also a residual set. For each non-zero vector $w \in C_0(S)$, we can find a v_i such that

$$d_\infty(w, v_i) < (1/2) d_\infty(w, \text{span}\{1\})$$

It follows that $d_\infty(w, \text{span}\{\alpha_k\}) > (1/2) d_\infty(w, \text{span}\{1\})$ for all $\{\alpha_k\} \in Q(v_i)$, and *a fortiori* for all $\{\alpha_k\} \in Q$. Thus, Q has the properties required.

In view of Theorem 2, each asset sequence in Q is asymptotically inefficient; this completes the proof of Theorem 3. ∎

An observation about Q yields a strong negative implication about the asymptotic inefficiency of securities market structures, and may serve to explicate further the sense in which "large trades" are important. Consider any two consumers $(e^1, \prec^1)$, $(e^2, \prec^2)$, for which:

(i) there is an $s*$ such that $e^1(s) + e^2(s)\} \leq 1$
     for $s \geq s*$

(ii) if z is an individually rational, Pareto optimal

net trade (for consumer 1), then $z(1) \geq +4$ and $z(2) \leq -4$

(Such pairs of consumers are easy to find.). Then, for any asset sequence $\{\alpha_k\} \in Q$ (in particular, for a residual set of asset sequences), the securities market structure $\mathcal{S} = \{\{\alpha_k\}, ((e^1, \prec^1))\}$ is asymptotically inefficient. To see this, let w be an equilibrium net trade in the securities market $\mathcal{S}_N = \{\{\alpha_k : 1 \leq k \leq N\}, ((e^h, \prec^h))\}$. Feasibility of w implies that $|w(s)| \leq 1$ for $s \geq s*$. On the other hand, if z is any Pareto optimal net trade then (ii) entails $d_\infty(z^{(s*)}, \mathrm{span}\{1\}) > 4$, and the definition of Q therefore entails $d_\infty(z^{(s*)}, \mathrm{span}\{\alpha_k\}) > 2$. In particular, $d_\infty(z^{(s*)}, w) > 2$. Since $|w(s)| \leq 1$ for $s \geq s*$, we conclude that $d_\infty(z^{(s*)}, w^{(s*)}) > 1$, and hence that

$$\|z - w\| \; > \; \sum_{s \leq s*} \sigma(s)$$

That is, no security market equilibrium net trade is close to any Pareto optimal net trade, so no security market equilibrium allocation is close to a Pareto optimal allocation.

Finally, we note that we could obtain similar genericity results if we restrict ourselves to sequences of assets with positive returns.[14] The only substantive change required is in the Lemma. Given a sequence β of assets, we must construct a perturbed sequence $\tilde{\beta}$ which belongs to $Q(v, n, m)$. In the argument given, the assets we construct may not have

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14. At least if $b_k \geq 2$ for each k.

positive returns, even if the original assets do. However, the argument can be modified in the following way. Given a sequence $\{\beta_k\}$ of assets with positive returns, and with $\beta_1 = 1$, we consider the sequence $\{\gamma_k\}$ defined by $\gamma_1 = 1$ and $\gamma_k = \beta_k - 1$ for $k \geq 2$. Arguing as before, we construct a small perturbation $\{\tilde{\gamma}_k\}$ which belongs to $Q(v,n,m)$, and which consists entirely of assets whose returns are bounded below by -1. Then $\{\tilde{\gamma}_k + 1\}$ is a sequence of assets with positive returns which is a small perturbation of $\{\tilde{\beta}_k\}$ and belongs to $Q(v,n,m)$. The remainder of the argument is as before.

6. UNCONSTRAINED CONSUMPTION

As we have noted earlier, the asymptotic inefficiency we have demonstrated may be traced directly to the requirement that consumption bundles be non-negative in each state. In this section, we briefly discuss the case of *unconstrained consumption* ; i.e., *we assume in what follows that the consumption set of each consumer is the entire commodity space* $L_1(S,\sigma)^{\Omega}$. (The other assumptions of Section 2 are understood to remain in force. In particular, preferences are monotone, norm continuous and convex, and endowments are positive.) Unconstrained consumption seems not unnatural if the commodities are themselves (capital) assets, and the discussion that follows might be viewed in the light of asset pricing models.

The first comment that needs to be made is that, when consumption is unconstrained, the set of feasible consumption bundles is not compact; as a consequence, equilibria need not exist. This can, of course, be the case even for complete market economies with one commodity and two states of nature. (Werner (1987) and Nielsen (1986) have given elegant treatments of the existence problem in the finite dimensional case, but the infinite dimensional case does not appear susceptible to such an elegant treatment.) Unboundedness of feasible consumption bundles also opens the possibility that the securities markets corresponding to each finite set of assets might have equilibria, but that these equilibrium allocations might not converge (or have a

convergent subsequence) as the set of assets expands. In short, without consumption constraints, existence and convergence of equilibria are problematical; as we shall see, however, *efficiency* of limits of equilibria is not at all problematical. If assets span a dense subspace of the set of all wealth streams (a condition that seems like a natural formalization of the idea that assets span all the uncertainty), and the securities markets corresponding to a sequence of assets have equilibria, and the equilibrium allocations converge, then the limit is an equilibrium allocation of the underlying complete markets economy (and in particular is Pareto optimal).

A small point should be addressed here. Positivity constraints imply that all feasible allocations (and thus all equilibrium allocations of finite securities markets) lie in a norm compact set, so norm convergence of equilibrium allocations of finite securities markets is the relevant notion. Moreover, since preferences are norm continuous, norm convergence of allocations implies convergence of the corresponding utilities. In the absence of consumption constraints, it seems useful to consider *weak* convergence of equilibrium allocations. However, weak convergence of allocations does not imply convergence of the corresponding utilities. This motivates us to incorporate convergence of utilities in the definition below.

Let $S = ((\alpha_k), ((e^h, \prec^h)))$ be a securities market structure, and let $u^1, \ldots u^H$ be norm continuous, quasi-concave utility functions representing the preferences $\prec^1, \ldots \prec^H$. We say that S is

conditionally asymptotically complete if every allocation

$x = (x^1, \ldots x^H)$ that is the weak limit of equilibrium allocations $\{_n x\}$

of a subsequence $\{\mathcal{E}_{N(n)}\}$ of the securities markets

$\mathcal{E}_{N(n)} = (\{\alpha_k : 1 \leq k \leq N(n)\}, \{(e^h, \prec^h)\})$ is an equilibrium allocation for the

underlying complete markets economy, *and* has the additional property

that $u^h(_n x^h) \to u^h(x^h)$ for every consumer h . (The latter property is

independent of the choice of utility functions.) We say that an asset

sequence $\{\alpha_k\}$ is *conditionally asymptotically complete* if, for all

consumers $\{(e^h, \prec^h)\}$, the securities market structure $(\{\alpha_k\}, \{(e^h, \prec^h)\})$

is conditionally asymptotically complete.[15]

THEOREM 4: If $\{\alpha_k\}$ is a sequence of assets, and $\text{span}\{\alpha_k\}$ is a
(norm) dense subspace of $L_1(S, \sigma)$, then $\{\alpha_k\}$ is conditionally
asymptotically complete .

PROOF: The argument is very similar to the argument of Theorem 2.
Fix H consumers $(e^h, \prec^h)$, and utility functions u^h representing the

preferences $\prec^h$. Assume we are given a subsequence

$\mathcal{E}_{N(n)} = (\{\alpha_k : 1 \leq k \leq N(n)\}, \{(e^h, \prec^h)\})$ of $\{\mathcal{E}_N\}$, and equilibrium

allocations $_n x = (_n x^1, _n x^2, \ldots _n x^H)$ for $\mathcal{E}_{N(n)}$, converging weakly to

$x = (x^1, x^2, \ldots x^H)$; let q_n be the corresponding asset prices and let

p_n be the corresponding state prices.

We first establish the following:

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15. We could define conditional asymptotic efficiency in a similar way.

CLAIM: there does not exist a set C of consumers and an allocation y which is feasible for consumers in C (i.e., the restriction of y to C is a reallocation of the endowments of consumers in C), such that $u^c(y^c) > \limsup_n u^c(_n x^c)$ for every $c \in C$.

If this is not so, then there is a set of consumers, which we may assume to be the consumers $M = \{1,2,...M\}$, and an allocation $y = (y^1,...y^M)$ that is a redistribution of the endowments $(e^1,...e^M)$ and has the property that $u^m(y^m) > \limsup_n u^m(_n x^m)$ for $m \in M$. Continuity of preferences, together with the assumption that numeraire endowments are bounded away from zero, guarantees that we can find a state r , allocations $z = (z^1,...z^M)$ and $\tilde{z} = (\tilde{z}^1,...\tilde{z}^M)$, and a real number $\delta > 0$ such that $u^m(\tilde{z}^m) > \limsup_n u^m(_n x^m)$ for $m \in M$ and:

$$\tilde{z}^m(s,i) = z^m(s,i) = 0 \qquad \text{for } s > r, 1 \le i \le \ell ;$$

$$\tilde{z}^m(s,i) = z^m(s,i) \qquad \text{for } s \le r, 2 \le i \le \ell ;$$

$$\tilde{z}^m(s,1) = z^m(s,1) - \delta e^m(s,1) \qquad \text{for } s \le r ;$$

$$\sum_m z^m(s,i) = \sum_m y^m(s,i) = \sum_m e^m(s,i) \qquad \text{for } s \le r, 1 \le i \le \ell .$$

Since we have normalized the state prices to sum to 1 , we may (passing to a subsequence if necessary) assume that the state prices

$p_n(s,i)$ converge; say $p_n(s,i) \to p(s,i)$. If $p(s,i) = 0$ for any state s and commodity i, then, for n sufficiently large, the demands in state s would be unbounded. Hence, $p(s,i) \neq 0$ for each s, i. (The limiting behavior of asset prices is irrelevant.)

For each price system $p*$ and each $m \in M$, define a wealth stream $w^m(p*)$ by:

$$w^m(p*,s) = [1/p*(s,1)][p* \square (z^m - e^m)](s) \quad \text{for } s \leq r, \text{ and}$$

$$w^m(p*,s) = 0 \qquad\qquad\qquad\qquad\qquad \text{for } s > r.$$

For $s \leq r$, $w^m(p*,s)$ is the amount of numeraire that must be transferred into state s in order to make the net purchase $(z^m - e^m)(s)$, at prices $p*$. Since z is a reallocation of endowments in states $s \leq r$, and $w^m(p*,s) = 0$ in states $s > r$, if follows that, for each price system $p*$ and each state s, $\sum w^m(p*,s) = 0$ (summation over $m \in M$).

We now use the density of $\text{span}\{\alpha_k\}$ in $L_1(S,\sigma)$ to find portfolios $\theta^1,\dots\theta^{M-1}$ whose returns $\zeta^1,\dots\zeta^{M-1}$ are within δ/M of $w^m(p)$ in the L_1 norm; i.e., $\|\zeta^m - w^m(p)\| < \delta/M$ for $1 \leq m \leq M-1$. If we set

$$\theta^M = - \sum_{m=1}^{M-1} \theta^m$$

we obtain a portfolio whose returns ζ^M are also within δ of $w^M(p)$

in the L_1 norm; i.e., $\|\zeta^M - w^M(p)\| < \delta$. Since the state prices $p_n(s,i)$ converge to $p(s,i)$ for each s, i and $w^m(p^*,s) = 0$ for states $s > r$, we conclude that $\|\zeta^m - w^m(p_n)\| < \delta$ for each $m \in M$, provided that n is sufficiently large.

Our construction guarantees that, at all prices p_n sufficiently close to p, the portfolio θ^m is feasible for consumer m. Since we have constructed θ^M so that $\sum \theta^m = 0$, at least one of the portfolios θ^m must have a non-positive price (at asset prices q_n). However, at prices p_n sufficiently close to p, the returns ζ^m on the portfolio θ^m will finance purchase of the commodity bundle $\tilde{z}^m$. By construction, $u^m(\tilde{z}^m) > \lim \sup u^m(_n x^m)$ for each m. Since $\tilde{z}^m$ belongs to the budget set of consumer m, this contradicts the equilibrium conditions in $\mathcal{E}_{N(n)}$. This establishes the CLAIM.

if x is not in the core of the underlying complete markets economy, there is a set C of consumers and an allocation y which is feasible for consumers in C such that $u^c(y^c) > u^c(x^c)$ for $c \in C$. However, norm continuity and quasi-concavity of utility functions imply that $\lim \sup u^h(_n x^h) \le u^h(x^h)$ for every h (i.e., utility functions are weakly upper semi-continuous), so this would contradict the CLAIM. We conclude that x is in the core of the underlying complete markets economy; that it is a competitive equilibrium allocation follows, as before, by replication.

It remains only to establish that utilities converge. If not there

would be a consumer, say consumer 1, for whom $\{u^1(_n x^1)\}$ does not converge to $u^1(x^1)$. Passing to a subsequence if necessary, and keeping in mind that utility functions are weakly upper semi-continuous, we obtain $\limsup u^1(_n x^1) < u^1(x^1)$. Continuity allows us to choose a number $r < 1$, sufficiently close to 1 so that $\limsup u^1(_n x^1) < u^1(rx^1)$. Write $y^h 1 = rx^1$ and $y^h = x^h + (1/H)(1 - r)x^1$ for $h \neq 1$. Then $y = (y^1,...y^H)$ is a redistribution of endowments, and $u^j(y^j) > \limsup u^j(_n x^j)$ for every j, so this again contradicts the CLAIM. This completes the proof. ∎

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7. CONCLUDING REMARKS

In this paper, we have examined the asymptotic behavior of securities markets. Our results show that a large, but incomplete, securities market need not be a good approximation to a Walrasian market. Green and Spear (1987, 1988) appear to reach different conclusions (in a somewhat different model). However, the earlier version (1987) contains errors;[16] the later version is not at variance with the results here.[17]

We have have found it convenient to work in the commodity space $L_1(S,\sigma)$, but other choices could be made. So long as the set of states of nature is countable, all our results results and arguments over to all of the spaces $L_p(S,\sigma)$, $1 \leq p < \infty$, provided we change the definitions of the distance functions d^1 and d^{0p} (and hence the notion of a residual set) in the obvious way. For the space $L_\infty(S,\sigma)$, it seems natural to substitute the Mackey topology for the norm topology and the weak star topology for the weak topology (see Bewley (1972)); with these changes, the arguments given for Theorems 1, 2 and 4 work equally well in $L_\infty(S,\sigma)$. It is not clear, however, what the appropriate distance function(s) on asset sequences should be (and hence, what notion(s) of genericity we should use) in this case, so it is not entirely clear what

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16. In particular, the proof of existence of equilibrium is not correct, and the efficiency assertion is not true.

17. In the later version, Green and Spear give a proof of the existence of equilibrium which is an adaptation of the one given here. They also identify a condition on asset sequences and endowments that guarantees asymptotic efficiency. This condition is very strong; in particular, it is generically not satisfied.

form Theorem 3 should take.

As noted earlier, it would be natural to model the set of states of nature by a continuum (rather than by a countable set, as we have done). However, the existence proof given here does not generalize to this setting; with a continuum of states, it is not at all clear that the limit portfolios represent feasible trading plans at the limit prices (because the mapping $(p,x) \to p \square x$ will not be jointly continuous in the relevant topologies). In the setting of complete markets, Bewley (1972) cleverly finesses a similar issue, but it seems that Bewley's method does not work in the setting of incomplete markets.[18] In the special case of a single good, this difficulty does not arise, and it is possible to obtain results similar to those given here; see Zame (1989).

The methods employed here could easily be adapted to the context of multiple trading dates, but they do not seem easily adaptable to continuous time models of trading (for precisely the reason above); see Duffie and Huang (1985), Duffie and Zame (1987) for example. For some subsequent work in that area, we refer the reader to Zame (1989).

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18. Hellwig, Mas-Colell and Zame have given arguments for the case of separable preferences and two periods, but the general case seems quite difficult.

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