OPTIMAL SEARCH WITH LEARNING

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Working Paper No. 580
Department of Economics
University of California, Los Angeles
March 1990
Revised September, 1990

We thank Ken Burdett, Thomas Ferguson, Nick Kiefer, Ed Leamer, John McCall and Michael Rothschild for helpful comments.

1 Introduction

Since the pioneering work of Stigler (1961, 1962) and McCall (1965), search theory has played an important role in microeconomic and macroeconomic modeling.[1] Most frameworks that use the simple sequential search model or one of its many variants assume that the searchers have complete knowledge about the distribution from which they are searching. This assumption, in most contexts, is unreasonable and represents an unjustified simplification. We investigate the sequential search problem without the assumption that the searcher knows the distribution he faces. Both search with recall and without recall are examined. Sufficient conditions for the existence of optimal stopping rules with the reservation property are provided. Comparative static results with respect to stochastic dominance (appropriately defined) are obtained.

The sequential search problem from a known distribution is as follows. Consider a buyer who wishes to purchase a good which is offered for sale by many different sellers. The price offered by each seller is an independent draw from a distribution which is known to the buyer. The buyer can elicit price quotations from a seller by paying a fee c. After each price quotation the buyer decides whether to stop and buy the good at the current price (or, if permitted, at the lowest price offered in the past) or to elicit one more price quotation from another seller after paying the fee, c. The buyer seeks to minimize his total costs, that is the price paid for the good plus the search costs.

If the variance of the distribution of price offers is finite then there exists an optimal stopping rule for the above problem (see DeGroot (1970)). (This is true even if the distribution is not known.) Moreover, there exists an optimal stopping rule for search, with or without recall, in an infinite horizon setting from a known distribution, which is *myopic*. That is, at each stage it is optimal to act as if exactly one more observation is allowed. When recall is allowed, the option is never exercised in this optimal stopping rule. Thus the optimal stopping rule for search with recall is feasible, and therefore also optimal, when recall is not allowed. Hence the minimum expected cost is the same for search with and without recall. Also, it turns out that the optimal stopping rule for search from a known distribution is a reservation price rule. That is, it is optimal to stop the first time one observes a price less than or equal to some cut-off price. Thus the optimal stopping rule for search from a known distribution is simple. More importantly, in markets where searchers follow reservation price strategies the demand for a seller's product is well-behaved in that if a seller increases price his expected sales will decrease. Hence it is of interest to know the structure of optimal policies because, besides determining the nature of the demand curve, prior distribution or in the cost of each sample (see Mortensen (1986), Kiefer and Neumann (1989)). We investigate comparative static results with respect to parameters related to the search environment when the distribution is random. First consider the case of a known distribution F. It is well-known (see Robbins (1970)) that the reservation price, r, of the optimal stopping rule is given by the equation

$$c = \int_0^r (r - x) dF(x) = \int_0^r F(x) dx \tag{1}$$

Moreover, the total expected cost from F when following the optimal rule is equal to the reservation price r. Thus if F dominates another known distribution G by first-order stochastic dominance, i.e., $F(x) \leq G(x)$, $\forall x$, then the expected cost under G is lower. As shown in Kohn and Shavell (1974), a similar result holds when F and G can be compared by second-order stochastic dominance. If F dominates G by second-order stochastic dominance, i.e., $\int_0^\infty x dF(x) = \int_0^\infty x dG(x)$ and $\int_0^y F(x)dx \leq \int_0^y G(x)dx$, $\forall y$, the expected cost under G is lower. Essentially if the distribution places more weight in the tails, the probability of getting a low price increases.

In our setting, where the true underlying distribution of prices is not known and the searcher has a prior on the set of possible distributions, it is not clear how to define stochastic dominance. Let F and G be prior distributions on the set of all possible distributions. That is, F and G are distributions on distributions. Let $E[F]$ and $E[G]$ denote the distributions obtained by multiplying probabilities in F and G, respectively. Since $E[F]$, $E[G]$ are distributions on R_+, one possible definition is to say that F dominates G by first-order (second-order) stochastic dominance if $E[F]$ dominates $E[G]$ by ordinary first-order (second-order) stochastic dominance. However, these definitions are inadequate for our purpose as the following examples show.

Example 1 *Let F_1, G_1, and G_2 be uniform distributions on $[0,1]$, $[0,\frac{1}{2}]$, and $[\frac{1}{2},1]$ respectively, and let G_0 be the degenerate distribution which gives the outcome 0 with probability one. The per period cost of search c is $\frac{1}{4}$. From (1) it is clear that when searching from the distribution F_1, the reservation price and the expected cost of the optimal stopping rule is 0.7071. Similarly, when searching from G_1, the reservation price and expected cost is $\frac{1}{2}$, and when searching from G_2 it is 1. The cost under G_0 is $\frac{1}{4}$. Let $F = (F_1, 1)$ and $G = (G_0, 0.02; G_1, 0.49; G_2, 0.49)$ be distributions on distributions. That is, F is degenerate at F_1, and under G, G_0 is chosen with probability 0.02, and G_1 and G_2 are each chosen with probability 0.49. Clearly, the minimum expected cost under F is 0.7071. After one observation from G the buyer knows with probability one whether he is sampling from G_0, G_1 or G_2. Further it is optimal to take exactly one observation from G and then stop and* the topology of weak convergence) as a posterior.

The rest of this paper is organized as follows. The next section contains some preliminaries and definitions. Section 3 deals with search with recall and Section 4 covers search without recall.

2 Preliminaries

We assume that the support of the random distribution of prices is contained in R_+. Let $\mathcal{B}$ be the $\sigma-$algebra of Borel subsets of R_+. Define $\mathcal{F}$ to be the set of countably additive probability measures on $(R_+, \mathcal{B})$ which have finite variance. $\mathcal{F}$ is endowed with the topology of weak convergence. Let $\mathcal{A}$ be the $\sigma-$algebra of Borel subsets of $\mathcal{F}$, and $\mathcal{P}$ be a probability distribution on $(\mathcal{F}, \mathcal{A})$. P denotes a *random probability measure* chosen according to $\mathcal{P}$, and $X_1, X_2, ..., X_n$ denotes a random sample chosen according to P. Sometimes, it will be useful to think of P as a stochastic process indexed by the subsets $B \in \mathcal{B}$. One can define a *random probability distribution*, F, corresponding to P

$$F(t) = P[0, t]$$

Let $E[\cdot]$ denote the expectation operator with respect to $\mathcal{P}$. It is easy to show that $E[P]$ is a countably additive probability measure on $(R_+, \mathcal{B})$. Let $E[F]$ be the distribution function corresponding to $E[P]$. If $h(\cdot)$ is any function defined on R_+ then

$$E[h(X_1)] = \int_0^\infty h(t) dE[F](t)$$

We will use the terms random probability distribution, and distribution on distributions interchangeably. Sometimes we shall refer to F as a random probability distribution on R_+. This means that, with probability one, F selects a distribution function with support in R_+. When we say that F has finite variance we mean that $E[F]$ has finite variance. The posterior distribution of F after observing $X_1 = x_1, ..., X_n = x_n$ is denoted by $F|x_1, ..., x_n$.

Define $X_0 \equiv \infty$ and let M_n represent the smallest available price at each stage n. For search with recall $M_n \equiv \min(X_0, X_1, ..., X_n)$, and for search without recall $M_n \equiv X_n$. The problem of finding a price from the random distribution function F at the least total cost is the same as finding an optimal stopping rule which minimizes

$$E[M_N + Nc]$$

In the interesting case, the search cost, c, is small enough so that it is not always optimal to stop after one price sample.

the reservation price at each stage is always greater than the reservation price for search from a known distribution equal to the expectation of the random distribution at that stage. The reservation prices do not depend on past observations and can be computed at the beginning of the search. In addition, we obtain conditions on the random distribution under which the optimal rule is bounded, that is, it always stops after a finite number of observations.

We assume throughout that the random probability distribution F has finite variance. In addition, consider the following assumptions.[9]

Assumption 1

For all $x_1, ..., x_n$

$$E[F|x_1,...,x_n](t) \leq E[F|x_1,...,x_{n-1}](t) \leq .... \leq E[F](t), \quad \forall t < \min(x_1,...,x_n)$$

Assumption 2

For all $x_1, ..., x_n$, $y_1, ..., y_n$

$$E[F|x_1,...,x_n](t) = E[F|y_1,...,y_n](t), \quad \forall t < \min(x_1,...,x_n,y_1,...,y_n)$$

Consider an individual with a prior (random) distribution, F. Suppose that he observes $n-1$ prices, all of which are greater than t. Then, the individual can calculate the probability that the next observed price is less than t. Assumption 1 states that if the nth observed price is also greater than t, then the probability that the next observed price is less than t will decrease. This, in a sense, formalizes the notion that the posterior probability of observing a low price, given that all the previous observations are high, decreases with the number of observations. Assumption 2 states that the posterior probability of observing a price less than t, given that all the previous observations are greater than t, depends only on the number of previous observations. Assumption 2 seems to be more stringent than Assumption 1.

It is useful to relate Assumptions 1 and 2 to the assumption that the posterior distribution of F is a convex combination of the prior and the empirical distribution. The empirical distribution function, $H_{x_1,...,x_n}(t)$, after observing $(x_1, ..., x_n)$ is given by

$$H_{x_1,...,x_n}(t) \equiv \frac{1}{n}\sum_{i=1}^{n} 1_{[x_i,\infty)}(t) \tag{2}$$

contains α_1 balls marked y_1, α_2 balls marked y_2, and so on. Each time a ball marked y_j, $j = 1$ or 2 or $...m$, is drawn from this urn, two balls marked y_j are returned to it. It is readily confirmed that the multinomial-Dirichlet distribution satisfies Assumption 3 with $a_n = \frac{n}{n+\sum \alpha_i}$.

The Dirichlet process is a generalization of the multinomial-Dirichlet distribution to the infinite case, that is the price observed can be from an infinite (possibly uncountably infinite) set. Essentially, after an observation x from a Dirichlet process, the posterior probability of any two sets B_i, $i = 1, 2$, such that $x \notin B_i$, decreases and the relative probability of B_1 and B_2 remains unchanged. The definition of Dirichlet process and some related results are provided in Appendix I. It turns out that the Dirichlet process satisfies Assumption 3 with a_n increasing (see (41), in Appendix I). Thus, under Bayesian learning, the Dirichlet process is a random probability distribution which satisfies Assumptions 1 and 2.

Recall that $\mathcal{F}$, the space of probability distributions on $(R_+, \mathcal{B})$, is endowed with the topology of weak convergence and $\mathcal{A}$ is the $\sigma-$algebra of Borel subsets of $\mathcal{F}$. Ferguson (1973) has shown that the support of a Dirichlet process is large in the sense that there exists a Dirichlet process with support $\mathcal{F}$ (see also Appendix I, Fact 2). Thus the following is true, and Assumptions 1 and 2 place no restriction on the set of permissible underlying distributions.

Lemma 2 *There exists a random probability distribution, F, on $(\mathcal{F}, \mathcal{A})$, which satisfies Assumptions 1 and 2 and has support $\mathcal{F}$.*

There exists a class of random probability distributions called neutral to the right distributions which satisfy Assumptions 1 and 2. In Appendix II we give a definition and summary of some well-known results on neutral to the right distributions. Essentially, after an observation x from a neutral to the right distribution, the posterior probability of any two sets B_i, $i = 1, 2$ such that $B_i \subset [0, x)$ decreases and the relative probability of B_1 and B_2 remains unchanged. Thus the Dirichlet process is neutral to the right. The class of neutral to the right distributions is strictly larger than the class of Dirichlet processes. (See Fact 6 and Remark 1 in Appendix II.) Both Dirichlet processes and the neutral to the right family of distributions are characterized by infinite dimensional parameters (see Remark 2 in Appendix II). In Appendix III, Lemma 13 we show that Assumptions 1 and 2 are satisfied by neutral to the right distributions.

We now show that if F satisfies Assumption 1 then the problem of search with recall from F is monotone in the sense of Chow, Robbins, and Siegmund (1971), and thus has which is equivalent to

$$\int_0^{M_n} (M_n - t) dE[F|x_1, x_2, ..., x_n](t) \le c \tag{8}$$

Thus to show that the sequence $M_{n+1} + c, M_{n+2} + 2c, ...$ is a submartingale it is enough to show that (8) implies

$$\int_0^{M_{n+1}} (M_{n+1} - t) dE[F|x_1, x_2, ..., x_n, x_{n+1}](t) \le c, \quad \forall x_{n+1} \tag{9}$$

Since $M_{n+1} \le M_n$, (8) implies that

$$\int_0^{M_{n+1}} (M_{n+1} - t) dE[F|x_1, x_2, ..., x_n](t) \le c \tag{10}$$

Let H_n, H_{n+1} be probability distributions on $[0, M_{n+1}]$ defined by

$$\begin{aligned} H_n(t) &= E[F|x_1, x_2, ..., x_n](t), \quad \forall t < M_{n+1} \\ H_n(M_{n+1}) &= 1 - H_n(M_{n+1}^-) \end{aligned}$$

and

$$\begin{aligned} H_{n+1}(t) &= E[F|x_1, x_2, ..., x_n, x_{n+1}](t), \quad \forall t < M_{n+1} \\ H_{n+1}(M_{n+1}) &= 1 - H_{n+1}(M_{n+1}^-) \end{aligned}$$

where $H(M_{n+1}^-) \equiv \lim_{t \uparrow M_{n+1}} H(t)$. Assumption 1 implies that $H_{n+1}(t) \le H_n(t)$, $\forall t \in [0, M_{n+1}]$ and therefore

$$\begin{aligned} \int_0^{M_{n+1}} (M_{n+1} - t) dE[F|x_1, x_2, ..., x_n](t) &= \int_0^{M_{n+1}} (M_{n+1} - t) dH_n(t) \\ &\ge \int_0^{M_{n+1}} (M_{n+1} - t) dH_{n+1}(t) \\ &= \int_0^{M_{n+1}} (M_{n+1} - t) dE[F|x_1, x_2, ..., x_n, x_{n+1}](t) \end{aligned}$$

which, combined with (10), implies (9).

Finally, we need to establish (7). Since $E[X_1] < \infty$, and for any $\tau \in \Delta$, $P\{\tau > k\} \to 0$, as $k \to \infty$, we have

$$\liminf_k \int_{\{\tau > k\}} X_1 = 0$$

$$
\begin{aligned}
&= \int_0^{M_n(y)} (M_n(x) - t) dE[F|x_1, ..., x_{n-1}, x](t) \\
&\quad + \int_{M_n(y)}^{M_n(x)} (M_n(x) - t) dE[F|x_1, ..., x_{n-1}, x](t) \\
&\geq \int_0^{M_n(y)} (M_n(y) - t) dE[F|x_1, ..., x_{n-1}, x](t) \\
&= \int_0^{M_n(y)} (M_n(y) - t) dE[F|x_1, ..., x_{n-1}, y](t) \\
&= J_n(y)
\end{aligned}
$$

∎

The main result of this section is[10]

Theorem 1 *If F is a random probability distribution which satisfies Assumptions 1 and 2 then the following hold for search with recall:*

(i) There exists an optimal stopping rule which is myopic and has the reservation price property.

(ii) The reservation price functions, r_n, $n \geq 1$, corresponding to this optimal stopping rule do not depend on the prices observed.

(iii) Let r_0 be the optimal reservation price when searching from a known distribution equal to the expected prior distribution, $E[F]$. Then $r_0 \leq r_1 \leq ....r_n \leq ....$

Proof: (i) Follows directly from Lemmas 3 and 4.

(ii) From (i) and (11) we know that it is optimal to stop after observing price samples $x_1, ..., x_n$ if and only if

$$
\begin{aligned}
c &\geq \int_0^{M_n} (M_n - t) dE[F|x_1, ..., x_n](t) \\
&= \int_0^{M_n} E[F|x_1, ..., x_n](t) dt \qquad (12)
\end{aligned}
$$

where the equality follows from integration by parts.

For all r in the support of $E[F]$ define

$$Q_1(r) \equiv c - \int_0^r E[F|r](t) dt \qquad (13)$$

By Assumption 2, for any $x > r$ in the support of $E[F]$,

$$Q_1(r) = c - \int_0^r E[F|x](t) dt \qquad (14)$$

where $M_n(x) = \min(x, M_{n-1})$. The inequality follows from Assumption 2 and the fact that if $x > r_n$ then $M_n(x) > r_n$, and the equality from Assumption 2 and (16). Thus (12) implies that it is optimal to continue if $X_n > r_n$. Therefore, when $M_{n-1} > r_n$, r_n is an optimal reservation price. A similar proof establishes that even when $M_{n-1} = r_n$ the optimal reservation price is r_n.

If instead $M_{n-1} < r_n < \infty$ then

$$\begin{aligned} c &> \int_0^{M_{n-1}} E[F|\overbrace{M_{n-1}, M_{n-1}, \ldots, M_{n-1}}^{\text{n } M'_{n-1}\text{s}}](t)dt \\ &\geq \int_0^{M_n(x)} E[F|x_1, \ldots, x_{n-1}, x](t)dt, \quad \forall x \end{aligned}$$

where the first inequality follows from (16) and the second from Assumption 2, and the fact that $M_n(x) \leq M_{n-1} < r_n$, $\forall x$. Thus if $M_{n-1} < r_n$ then regardless of the value of X_n it is optimal to stop after observing X_n. Since $M_n(X_n) \leq M_{n-1} < r_n$, r_n is an optimal reservation price.

If $r_n = \infty$ then (12) is always satisfied regardless of the observations $x_1, \ldots, x_n$, and thus r_n is an optimal reservation price. This completes the proof of (ii).

(iii) Assumption 1 implies that

$$E[F|\overbrace{r, r, \ldots, r}^{\text{n r's}}](t) \leq E[F|\overbrace{r, r, \ldots, r}^{\text{n}-1\text{ r's}}](t), \quad \forall t < r, \ \forall n \geq 1$$

Therefore $Q_n(r) \geq Q_{n-1}(r)$. Since $Q_n(r)$ is decreasing in r for all n, the definition of r_i, $i = 0, 1, 2, \ldots, n$ implies $r_n \geq r_{n-1} \geq \ldots \geq r_0$. ∎

The sequence of optimal reservation prices is increasing.[11] Essentially, a higher price realization from a random distribution makes the searcher more pessimistic about finding a low price in the future. This intuition also explains the fact that the reservation price at each stage is always greater than the reservation price for search from a known distribution equal to the expectation of the random distribution at that stage. Theorem 1 implies that the optimal reservation prices can be determined before beginning the search. Also, if in period n the lowest available price, M_n, is such that $r_{n+k-1} \geq M_n \geq r_{n+k}$ for some k, then at most k additional observations will be taken. Since the optimal reservation prices, r_i, $i \geq 1$ are greater than the optimal reservation price when searching from a known distribution equal to the expected prior distribution, the expected number of samples is lower when sampling from the random distribution F. In Example 8, Appendix III, we provide a neutral to the right distribution and its optimal reservation price policy.

3.2 Comparative Statics

We turn to some comparative static results for sampling with recall. Our main result is Theorem 2. We show that if, after any set of observations, the posterior distribution of F dominates the posterior distribution of G by first-order stochastic dominance, and a mild assumption is satisfied, then the minimum expected search cost is lower under G. In Theorem 2 it is not assumed that an optimal reservation price policy exists. In Lemma 7 we show that when each of two random distributions F and G has an optimal reservation policy, and the hypotheses of Theorem 2 are satisfied, then the reservation price under F is greater than the reservation price under G. The analogous result for second-order stochastic dominance seems to be less generally true. The comparative static result with respect to the cost of search generalizes easily.

For the rest of this section we do not assume that F satisfies Assumptions 1, 2 or 3, unless otherwise stated. Before proceeding, it is useful to consider a truncated problem in which the buyer is allowed to observe at most T price samples from F. As mentioned in Section 2, an optimal stopping rule exists in the untruncated problem. Optimal stopping rules in the truncated problem are obtained by backward induction (see DeGroot (1970), section 12.4). Let $\hat{V}_T(F,m)$ and $\hat{V}(F,m)$ be the minimum expected cost in the truncated and untruncated problems respectively, where m is the minimum price available from previous observations, if any. Thus

$$\begin{aligned}\hat{V}_1(F,m) &= \min\{m, \int_0^\infty \min(m,t)dE[F](t) + c\},\\ \hat{V}_T(F,m) &= \min\{m, \int_0^\infty \hat{V}_{T-1}(F|t,\min(m,t))dE[F](t) + c\}, \quad \forall T \geq 2 \end{aligned} \tag{20}$$

The following lemma, which is based on results in DeGroot (1970), shows that $\hat{V}_T(F,m)$ converges to $\hat{V}(F,m)$. Thus results which are true for the truncated problem can be extended to the untruncated problem.

Lemma 6 *If F is a random probability distribution with finite variance then*

$$\lim_{T\to\infty} \hat{V}_T(F,m) = \hat{V}(F,m).$$

Proof: Theorem 12.10.1 in DeGroot (1970) gives sufficient conditions for $\hat{V}_T(F,m) \to \hat{V}(F,m)$ as $T \to \infty$. As shown in Christensen (1983), these sufficient conditions are satisfied if F has finite variance. ∎

The induction hypothesis is: if $F \succeq_f G$ then $\hat{V}_{T-1}(F,m) \geq \hat{V}_{T-1}(G,m)$. This implies

$$\begin{aligned}
\hat{V}_T(F,m) &= \min\{m, \int_0^\infty \hat{V}_{T-1}(F|t, \min(m,t))dE[F](t) + c\} \\
&\geq \min\{m, \int_0^\infty \hat{V}_{T-1}(F|t, \min(m,t))dE[G](t) + c\} \\
&\geq \min\{m, \int_0^\infty \hat{V}_{T-1}(G|t, \min(m,t))dE[G](t) + c\} \\
&= \hat{V}_T(G,m)
\end{aligned}$$

Since F is increasing the induction hypothesis implies that $\hat{V}_{T-1}(F|t,m)$ and therefore the integrand are increasing in t. Thus the fact that $E[F]$ dominates $E[G]$ by ordinary first-order stochastic dominance implies the first inequality. The second inequality follows since $F|t \succeq_f G|t$. Therefore, $\hat{V}_T(F,m) \geq \hat{V}_T(G,m)$, $\forall T$.

By Lemma 6,

$$\hat{V}(F,m) = \lim_{T\to\infty} \hat{V}_T(F,m) \geq \lim_{T\to\infty} \hat{V}_T(G,m) = \hat{V}(G,m)$$

If, instead, G is increasing the proof is symmetric. ∎

From the proof of Theorem 2 it is clear that if $F \succ_f G$ then $\hat{V}(F,m) > \hat{V}(G,m)$. The following lemma obtains a comparative static result for the optimal policy when it is a reservation price rule.

Lemma 7 *Let F and G be random probability distributions on R_+ such that $F \succ_f G$ and either F or G is increasing. Suppose that F and G have an optimal stopping policy which is reservation price. Then after any history whenever it is optimal to stop under G, it is optimal to stop under F.*

Proof: Let $F' \equiv F|x_1, x_2, ..., x_n$ and $G' \equiv G|x_1, x_2, ..., x_n$ be the posterior distributions of F and G, respectively, after $x_1, x_2, ..., x_n$ have been observed. Then it is clear from Definitions 1 and 2 that $F' \succ_f G'$. Thus Theorem 2 implies that $\hat{V}(F',m) > \hat{V}(G',m)$. Let r_F and r_G be the optimal reservation prices of F' and G' respectively, and let $m \equiv (x_1, x_2, ..., x_n)$ be the minimum available price.

Suppose that $r_G \leq m$. Then the definition of r_F and r_G implies that

$$\begin{aligned}
r_G &= \hat{V}((G'|r_G), \min(m, r_G)) + c \\
&< \hat{V}((F'|r_G), \min(m, r_G)) + c
\end{aligned}$$

Since $r \leq \hat{V}((F'|r), \min(m,r)) + c$, $\forall r \leq r_F$ we have $r_F \geq r_G$.

Thus the expected prior distribution of the prices is

$$\begin{aligned} E[F](t) &= \left(\frac{a}{a+1}\right)\frac{t}{w_0}, \quad \text{if } 0 < t < \left(\frac{a+1}{a}\right) w_0 \\ &= 1, \quad \text{if } t \geq \left(\frac{a+1}{a}\right) w_0 \end{aligned}$$

Suppose that price samples $x_1, x_2, ..., x_n$ are observed. Let $w_n \equiv \max(w_0, x_1, x_2, ...x_n)$. The posterior distribution of w given samples $x_1, x_2, ..., x_n$ is a Pareto distribution with parameters w_n, $a+n$ (see DeGroot (1970), pg. 172). The expected posterior distribution is

$$\begin{aligned} E[F|x_1, x_2, ..., x_n](t) &= \left(\frac{a+n}{a+n+1}\right)\frac{t}{w_n}, \quad \text{if } 0 < t < \left(\frac{a+n+1}{a+n}\right) w_n \\ &= 1, \quad \text{if } t \geq \left(\frac{a+n+1}{a+n}\right) w_n \end{aligned}$$

Since $\max(w_n, x) \geq \max(w_n, y)$ for $x \geq y$ it follows that

$$E[F|x_1, x_2, ..., x_n, x](t) \leq E[F|x_1, x_2, ..., x_n, y](t)$$

Thus F is increasing.

Consider another random distribution G where the prices are uniformly distributed on $[0, w]$ and w has a Pareto distribution with parameters $\hat{w}_0$, $\hat{a}$, which satisfy $\hat{w}_0 \leq w_0$, $\hat{a} \geq a$. It is easily verified that $F \succeq_f G$ and thus that $\hat{V}(F, m) \geq \hat{V}(G, m)$.

Example 6 *The underlying set of distributions is exponential with unknown parameter λ. The prior distribution of λ is a gamma distribution with parameters a, b. Thus the conditional density of the prices given λ is*

$$\begin{aligned} f(t|\lambda) &= \lambda \exp^{-\lambda t}, \quad \text{if } t > 0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

and the prior density of λ is

$$\begin{aligned} h(\lambda|a, b) &= \frac{b^a}{\Gamma(a)}\lambda^{a-1}\exp^{-b\lambda} \quad \text{if } \lambda > 0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

Thus the expected prior distribution of the prices is

$$E[F](t) = 1 - \left(\frac{b}{b+t}\right)^a, \quad \text{if } t > 0$$

$$\hat{V}_{T-1}((G|t),m) = \hat{V}_{T-1}((G|t'),m), \quad \forall t \geq t' \geq m \tag{27}$$

$$\hat{V}_{T-1}((F|t),t) \geq \hat{V}_{T-1}((F|t'),t'), \quad \forall t \geq t' \tag{28}$$

We first show that the induction hypothesis is true for $T = 1$.
Since $E[F](t) \leq E[G](t)$, $\forall t < m$,

$$\begin{aligned}
\hat{V}_1(F,m) &= \min\{m, \int_0^\infty \min(m,t)dE[F](t) + c\} \\
&\geq \min\{m, \int_0^\infty \min(m,t)dE[G](t) + c\} \\
&= \hat{V}_1(G,m)
\end{aligned}$$

Since F satisfies Assumption 2, if $t \geq t' \geq m$ then

$$\begin{aligned}
\hat{V}_1((F|t),m) &= \min\{m, \int_0^\infty \min(m,s)dE[F|t](s) + c\} \\
&= \min\{m, \int_0^m sdE[F|t](s) + m(1 - E[F|t](m)) + c\} \\
&= \min\{m, \int_0^m sdE[F|t'](s) + m(1 - E[F|t'](m)) + c\} \\
&= \min\{m, \int_0^\infty \min(m,s)dE[F|t'](s) + c\} \\
&= \hat{V}_1((F|t'),m)
\end{aligned} \tag{29}$$

Similarly $\hat{V}_1((G|t),m) = \hat{V}_1((G|t'),m)$, $\forall t \geq t' \geq m$. Also if $t \geq t'$ then

$$\begin{aligned}
\hat{V}_1((F|t),t) &= \min\{t, \int_0^\infty \min(t,s)dE[F|t](s) + c\} \\
&\geq \min\{t', \int_0^\infty \min(t',s)dE[F|t](s) + c\} \\
&= \hat{V}_1((F|t),t') \\
&= \hat{V}_1((F|t'),t')
\end{aligned} \tag{30}$$

where the last equality follows from (29). Thus the induction hypothesis is satisfied for $T = 1$. Now suppose that the induction hypothesis is satisfied for $T - 1$. Let F and G satisfy Assumption 2, (i), (23) and (24). Then

$$\begin{aligned}
\hat{V}_T(F,m) &= \min\{m, \int_0^m \hat{V}_{T-1}((F|t),t)dE[F](t) + \int_m^\infty \hat{V}_{T-1}((F|t),m)dE[F](t) + c\} \\
&= \min\{m, \int_0^m \hat{V}_{T-1}((F|t),t)dE[F](t) + \hat{V}_{T-1}((F|m),m)(1 - E[F](m)) + c\} \\
&\geq \min\{m, \int_0^m \hat{V}_{T-1}((F|t),t)dE[G](t) + \hat{V}_{T-1}((F|m),m)(1 - E[G](m)) + c\}
\end{aligned}$$

points of discontinuity. Let N_t^f and N_t^g be the Lévy measures corresponding to F and G respectively. Suppose that

$$\begin{aligned} N_t^f(z) &= \gamma_f(t)N(z) \\ N_t^g(z) &= \gamma_g(t)N(z) \end{aligned} \tag{33}$$

where $N(z)$ is a measure such that $\int_0^\infty \frac{z}{1+z}dN(z) < \infty$ and $\gamma_i(t), i = f, g$ are nondecreasing, absolutely continuous functions with $\gamma_i(0) = 0$, and $\lim_{t\to\infty}\gamma_i(t) = \infty$. Then, if

$$\gamma_f(t) \geq \gamma_g(t),\ \forall t \tag{34}$$

then F and G satisfy (ii) of Theorem 3. To see this, let $\hat{Z}_t^f$ and $\hat{Z}_t^g$ denote the independent increments processes corresponding to the posterior distributions $F|x_1, ..., x_n$ and $G|x_1, ..., x_n$ respectively. From (33), (34), and from (44), (45) of Appendix II we see that $\forall t < \min(x_1, ..., x_n)$

$$\begin{aligned} \log \mathcal{M}_{\hat{Z}_t^f}(-1) &= \gamma^f(t)\int_0^\infty (\exp^{-z} - 1)\exp^{-nz} dN(z) \\ &\geq \gamma^g(t)\int_0^\infty (\exp^{-z} - 1)\exp^{-nz} dN(z) \\ &= \log \mathcal{M}_{\hat{Z}_t^g}(-1) \end{aligned}$$

It is readily confirmed that (56) of Appendix II implies (ii) of Theorem 3. Hence the neutral to the right random distribution functions F and G satisfy the assumptions of Theorem 3.

We now turn to comparative statics with respect to second-order stochastic dominance. The following is analogous to Definition 1.

Definition 3 *Let F and G be random probability distributions. Then F dominates G by second-order stochastic dominance under learning, denoted by $F \succeq_s G$, if $\forall x_1, x_2, ..., x_n$*

$$\begin{aligned} \int_0^\infty t dE[F|x_1, x_2, ..., x_n](t) &= \int_0^\infty t dE[G|x_1, x_2, ..., x_n](t) \\ \int_0^t E[F|x_1, x_2, ..., x_n](s)ds &\leq \int_0^t E[G|x_1, x_2, ..., x_n](s)ds, \quad \forall t \end{aligned}$$

Let F and G be Dirichlet processes, $F \in \mathcal{D}(\alpha)$ and $G \in \mathcal{D}(\beta)$ with $\alpha(R_+) = \beta(R_+)$. If $E[F]$ dominates $E[G]$ by second-order stochastic dominance then $F \succeq_s G$. More generally, we have the following lemma. The proof is straighforward and is omitted.

Lemma 8 *Suppose that F and G are random probability distributions which satisfy Assumption 3, and that for all $\mathrm{x}^n = (x_1, x_2, ..., x_n)$, $E[F|\mathrm{x}^n]$ and $E[G|\mathrm{x}^n]$ attach the same*

$$
\begin{aligned}
&= \min\{\min(m,\mathrm{x}^{\mathrm{n}}), \int_0^\infty \min(m,\mathrm{x}^{\mathrm{n}},t)dE[F|\mathrm{x}^{\mathrm{n}}](t) + c\} \\
&= \min\{\min(m,\mathrm{x}^{\mathrm{n}}), (1-a_n)\int_0^\infty \min(m,\mathrm{x}^{\mathrm{n}},t)dE[F](t)) + a_n \min(m,\mathrm{x}^{\mathrm{n}}) + c\} \\
&\geq \min\{\min(m,\mathrm{x}^{\mathrm{n}}), (1-a_n)\int_0^\infty \min(m,\mathrm{x}^{\mathrm{n}},t)dE[G](t)) + a_n \min(m,\mathrm{x}^{\mathrm{n}}) + c\} \\
&= \hat{V}_1((G|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}}))
\end{aligned}
$$

where the inequality follows from the fact that $\min(m, \mathrm{x}^{\mathrm{n}}, t)$ is an increasing, concave function of t, and $E[F]$ dominates $E[G]$ by second-order stochastic dominance. Next suppose that $\hat{V}_{T-1}((F|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}})) \geq \hat{V}_{T-1}((G|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}}))$. A similar argument shows that

$$
\begin{aligned}
&\hat{V}_T((F|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}})) \\
&= \min\{\min(m,\mathrm{x}^{\mathrm{n}}), \int_0^\infty \hat{V}_{T-1}((F|\mathrm{x}^{\mathrm{n}},t), \min(m,\mathrm{x}^{\mathrm{n}},t))dE[F|\mathrm{x}^{\mathrm{n}}](t) + c\} \\
&= \min\{\min(m,\mathrm{x}^{\mathrm{n}}), (1-a_n)\int_0^\infty \hat{V}_{T-1}((F|\mathrm{x}^{\mathrm{n}},t), \min(m,\mathrm{x}^{\mathrm{n}},t))dE[F](t) \\
&\qquad + \frac{a_n}{n}\sum_{i=1}^n \hat{V}_{T-1}((F|\mathrm{x}^{\mathrm{n}},x_i), \min(m,\mathrm{x}^{\mathrm{n}})) + c\} \\
&\geq \min\{\min(m,\mathrm{x}^{\mathrm{n}}), (1-a_n)\int_0^\infty \hat{V}_{T-1}((G|\mathrm{x}^{\mathrm{n}},t), \min(m,\mathrm{x}^{\mathrm{n}},t))dE[G](t) \\
&\qquad + \frac{a_n}{n}\sum_{i=1}^n \hat{V}_{T-1}((G|\mathrm{x}^{\mathrm{n}},x_i), \min(m,\mathrm{x}^{\mathrm{n}})) + c\} \\
&= \hat{V}_T((G|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}}))
\end{aligned}
$$

Thus $\hat{V}_T((F|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}})) \geq \hat{V}_T((G|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}}))$, $\forall T$. Taking limits as $T \to \infty$ establishes that $\hat{V}((F|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}})) \geq \hat{V}((G|\mathrm{x}^{\mathrm{n}}), \min(m,\mathrm{x}^{\mathrm{n}}))$, $\forall \mathrm{x}^{\mathrm{n}}$.

(ii) By Lemma 1, F [G] satisfies Assumptions 1 and 2. Therefore Theorem 1 implies that there exists an optimal stopping rule which is reservation price. Let $r_n(F)$ [$r_n(G)$] be the reservation price at stage n when searching from F [G]. The fact that for all x^{n}, $E[F|\mathrm{x}^{\mathrm{n}}]$ dominates $E[G|\mathrm{x}^{\mathrm{n}}]$ by second-order stochastic dominance implies that

$$
\int_0^r E[F|\overbrace{r,r,\ldots,r}^{\mathrm{n}\ r\text{'s}}](t)dt \leq \int_0^r E[G|\overbrace{r,r,\ldots,r}^{\mathrm{n}\ r\text{'s}}](t)dt, \quad \forall r
$$

The definition of the reservation price in the proof of Theorem 1 implies that $r_n(F) \geq r_n(G)$. ∎

Finally, we show that $\hat{V}(F,m)$ increases as the cost of sampling, c, increases. In the following lemma we use the notation $\hat{V}_T(F,m,c)$ and $\hat{V}(F,m,c)$ instead of $\hat{V}_T(F,m)$ and $\hat{V}(F,m)$ to indicate explicitly the dependence on c.

Lemma 10 *If F is a random probability distribution with finite variance then*

$$\lim_{T\to\infty} V_T(F) = V(F).$$

The main result of this section is

Theorem 5 *Let F be a random probability distribution on R_+ which satisfies Assumption 3. The problem of search without recall from F has an optimal stopping rule with the reservation price property.*

Proof: Let $\mathbf{x}^n \equiv (x_1, x_2, \ldots, x_n)$. We need to show for all $\mathbf{x}^n$

$$V(F|\mathbf{x}^n, y_1) - V(F|\mathbf{x}^n, y_2) \leq y_1 - y_2, \quad \forall y_1 \geq y_2 \tag{36}$$

To see that (36) implies the reservation price property note that it is optimal to stop after observing $(\mathbf{x}^n, y_1)$ if and only if

$$V(F|\mathbf{x}^n, y_1) + c \geq y_1 \tag{37}$$

The above inequality, together with (36), implies that $V(F|\mathbf{x}^n, y_2) + c \geq y_2$ for all $y_2 \leq y_1$.

Let $\mathbf{ky} \equiv (y, y, \ldots, y)$ denote a k-vector of y's. First we establish that for all T, for all $\mathbf{x}^n$ and k, $V_T(F|\mathbf{x}^n, \mathbf{ky}_1) - V_T(F|\mathbf{x}^n, \mathbf{ky}_2) \leq y_1 - y_2$, $\forall y_1 \geq y_2$. For $T = 1$ we have

$$\begin{aligned} V_1(F|\mathbf{x}^n, \mathbf{ky}_1) - V_1(F|\mathbf{x}^n, \mathbf{ky}_2) &= k\frac{a_{n+k}}{n+k}(y_1 - y_2) \\ &\leq y_1 - y_2 \end{aligned}$$

Suppose that for all $\mathbf{x}^n$, for all k, for all $y_1 \geq y_2$, $V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_1) - V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_2) \leq y_1 - y_2$. Then

$$\begin{aligned} & V_T(F|\mathbf{x}^n, \mathbf{ky}_1) - V_T(F|\mathbf{x}^n, \mathbf{ky}_2) \\ = \; & (1 - a_{n+k}) \int_0^\infty \{\min(t, V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_1, t) + c) - \min(t, V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_2, t) + c)\} dE[F](t) \\ & + \frac{a_{n+k}}{n+k} \sum_{i=1}^n \{\min(x_i, V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_1, x_i) + c) - \min(x_i, V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_2, x_i) + c)\} \\ & + k\frac{a_{n+k}}{n+k} \{\min(x_i, V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_1, y_1) + c) - \min(x_i, V_{T-1}(F|\mathbf{x}^n, \mathbf{ky}_2, y_2) + c)\} \\ \leq \; & (1 - a_{n+k})(y_1 - y_2) + a_{n+k}\left(\frac{n}{n+k} + \frac{k}{n+k}\right)(y_1 - y_2) \\ = \; & (y_1 - y_2) \end{aligned}$$

to the empirical distribution, and $E[F]$ dominates $E[G]$ by ordinary second-order stochastic dominance. Then

(i) $V(F) \geq V(G)$;

(ii) after any history the optimal reservation price under F is greater than that under G.

Christensen (1983) generalizes a result in Rothschild (1974) to show that if F and G are Dirichlet processes $F \in \mathcal{D}(\alpha)$, $G \in \mathcal{D}(\beta)$, $\alpha(R) = \beta(R)$ and $E[F]$ dominates $E[G]$ by second-order stochastic dominance, then $V(F) \geq V(G)$. Thus Theorem 7 is a generalization of these results.

Finally, we state the following lemma, the proof of which is easy.

Lemma 12 *Let F be a random probability distribution on R_+. If $c_1 \geq c_2$ then $V(F, c_1) \geq V(F, c_2)$.*

5 Concluding Remarks

We examine the sequential search model under learning. We adopt a non-parametric approach in the sense that we do not assume that the true underlying distribution belongs to any finite dimensional parametric family. Sufficient conditions are obtained under which the optimal policy, for search with and without recall, has the reservation property. These conditions do not place any restrictions on the space of permissible prior or posterior distributions but imply that learning is "local"

For search with recall it is shown that under local learning the optimal policy is myopic and has the reservation property. The sequence of reservation prices is increasing and can be computed before beginning search. The local learning assumption is satisfied by well-known classes of random distributions with large support in the space of distributions on $(R_+, \mathcal{B})$ (eg., Dirichlet processes and neutral-to-the-right random distributions.)

It appears that the sufficient conditions for obtaining optimal reservation policies are stronger for search without recall than for search with recall. It is shown that if the posterior distribution is a convex combination of the (expected) prior distribution and the empirical distribution, then for search without recall a (history dependent) reservation rule is optimal. Since the Dirichlet process satisfies this condition our result generalizes Rothschild (1974) and Christensen (1983). Also, comparative static results with respect to first-order and second-order stochastic dominance for random distributions are provided.[14]

Appendix I
The Dirichlet Process

The Dirichlet process is analyzed in Ferguson (1973). In this appendix results of Ferguson (1973) that are relevant for our paper are summarized. Ferguson (1974) is an excellent survey of distributions on distributions.

Let $S = \{(y_1, y_2, ..., y_m) : y_i \geq 0, \sum_1^m y_i = 1\}$ be the $(m-1)$-dimensional simplex. Suppose that $(Y_1, Y_2, ..., Y_m)$ has Dirichlet distribution with parameter $(\alpha_1, \alpha_2, ..., \alpha_m)$, $\alpha_i \geq 0$ and $\sum_1^m \alpha_i > 0$. Then the density of $(Y_1, Y_2, ..., Y_m)$ is

$$d(y_1, y_2, ..., y_m | \alpha_1, \alpha_2, ..., \alpha_m) = \frac{\Gamma(\alpha_1 + \alpha_2 + ... + \alpha_m)}{\Gamma(\alpha_1)...\Gamma(\alpha_m)} \prod_1^m y_k^{(\alpha_k - 1)} 1_S(y_1, y_2, ..., y_m)$$

where $\Gamma(\cdot)$ denotes the gamma function and $1_S(y_1, y_2, ..., y_m)$ represents the indicator function of the simplex S. If $(Y_1, Y_2, ..., Y_m)$ has Dirichlet distribution with parameter $(\alpha_1, \alpha_2, ..., \alpha_m)$, then the marginal distribution of Y_i is beta with parameters $(\alpha_i, \sum_{j \neq i} \alpha_j)$. The Dirichlet process is a generalization of the Dirichlet distribution to a stochastic process.

Definition 4 *Let $\alpha(\cdot)$ be a finite non-null measure on $(R_+, \mathcal{B})$ and let $P(\cdot)$ be a stochastic process indexed by elements of $\mathcal{B}$. P is a Dirichlet process with parameter α if for every finite measurable partition $\{I_1, I_2, ..., I_m\}$ of R_+, the random vector $(P(I_1), P(I_2), ..., P(I_m))$ has a Dirichlet distribution with parameter $(\alpha(I_1), \alpha(I_2), ..., \alpha(I_m))$.*

When the parameter of the Dirichlet process α has finite support, $\{y_1, y_2, ..., y_m\}$, we obtain a Dirichlet distribution. This is the case considered by Rothschild (1974), where it is assumed that the price can take one of m values, $y_1, y_2, ..., y_m$. The prior distribution of the prices is multinomial with parameters $\alpha(\{y_1\}), \alpha(\{y_2\}), ..., \alpha(\{y_m\})$. After observing a price y_j, the posterior distribution is multinomial with parameters $\alpha(\{y_1\}), \alpha(\{y_2\}), ..., \alpha(\{y_j\}) + 1, ..., \alpha(\{y_m\})$. This follows from the rule for updating a Dirichlet process specified below in Fact 1.

Let F be the random distribution function corresponding to P. We write $P \in \mathcal{D}(\alpha)$ (or $F \in \mathcal{D}(\alpha)$) if P is a Dirichlet process with parameter α. Since the one-dimensional marginal distributions of the Dirichlet are beta, if $P \in \mathcal{D}(\alpha)$ then for every $B \in \mathcal{B}$, $P(B)$ has a beta distribution with parameters $(\alpha(B), \alpha(R_+) - \alpha(B))$. Also $F(t)$ has a beta distribution with parameters $(\alpha(t), \alpha(R_+) - \alpha(t))$, where $\alpha(t) \equiv \alpha([0, t])$, for all $t \in R_+$. Thus

$$E[F](t) = \alpha(t)/\alpha(R_+) \tag{39}$$

Appendix II
Neutral Random Probability

In this appendix we give the main properties of neutral to the right random probabilities. This is a large class of distributions which includes the Dirichlet process. Our interest in them follows from the fact that random probability distributions in this class satisfy Assumptions 1 and 2 of Section 3.1. The concept of neutrality to the right was introduced by Doksum (1974). Here we summarize results from that paper and from Ferguson (1974). In Appendix III we show that neutral to the right distributions satisfy the assumptions made in Section 3.1.

Definition 5 *F, P and $\mathcal{P}$ are said to be neutral to the right if for every $k > 1$ and $t_1 < t_2 < \ldots < t_k$ there exist independent random variables $V_1, V_2, \ldots, V_k$ such that $(1 - F(t_1), 1 - F(t_2), \ldots, 1 - F(t_k))$ has the same distribution as $(V_1, V_1V_2, \ldots, \Pi_1^k V_k)$.*

The definition implies that $(1 - F(t_i))/(1 - F(t_{i-1}))$ has the same distribution as V_i. Thus F is neutral to the right means that

$$1 - F(t_1), (1 - F(t_2))/(1 - F(t_1)), \ldots, (1 - F(t_k))/(1 - F(t_{k-1}))$$

are independent for all $0 \le t_1 < t_2 < \ldots < t_k \le \infty$. Because of the possibility of dividing by zero, Definiton 5 is preferred. Neutral to the right distribution functions are completely characterized as follows.

Fact 3 *[Doksum (1974)] A random distribution function, F, is neutral to the right if and only if it has the same probability distribution as*

$$1 - \exp[-Y_t]$$

for some a.s. nondecreasing, a.s. right-continuous, independent increment stochastic process Y_t with $\lim_{t\to 0} Y_t = 0$ a.s. and $\lim_{t\to\infty} Y_t = \infty$ a.s.

It is easy to see why Fact 3 is true. If F is neutral to the right then $Y_t \equiv -\log(1 - F(t))$ is as in Fact 3. And if Y_t is as in Fact 3, then $F(t) \equiv 1 - \exp[-Y_t]$ is neutral to the right, since $(1 - F(t_i))/(1 - F(t_{i-1})) = \exp[-(Y_t - Y_{t-1})]$. A Dirichlet process is neutral to the right.[15] Therefore, Fact 4 follows from Fact 2.

Fact 4 *[Ferguson (1973), Doksum (1974)] There exists a random probability on $(\mathcal{F}, \mathcal{A})$ which is neutral to the right and has support $\mathcal{F}$.*

where c_i is a normalizing constant. If x is a prior fixed discontinuity, that is $x = t_k$ for some k, then the posterior density of the jump at $x = t_k$ is

$$g_{t_k}(s|x = t_k) = c_k(1 - \exp^{-s})g_{t_k}(s) \tag{47}$$

If x is not a prior fixed discontinuity then it becomes one in the posterior distribution of Y_t. For the general case the expression for the posterior density of the jump at x is complicated, but it can be simplified if we make the following assumption:

$$N_t(z) \equiv \gamma(t)N(z) \tag{48}$$

where $N(z)$ is a measure such that $\int_0^\infty \frac{z}{1+z}dN(z) < \infty$ and $\gamma(t)$ is a nondecreasing, continuous function with $\gamma(0) = 0$, $\lim_{t\to\infty}\gamma(t) = \infty$. If (48) holds then the distribution of the jump at x is given by

$$G_x(s|x) = \frac{\int_0^s(1 - \exp^{-z})dN(z)}{\int_0^\infty(1 - \exp^{-z})dN(z)} \tag{49}$$

Note that $g_x(s|x)$ is independent of x. In Appendix III, we do not require the posterior density of the jump at x. Therefore, we will not assume (48).

From (45) and (46) it is clear that the distribution of increments $Y_t - Y_{t-\epsilon}$ for $t < x$ and $\epsilon > 0$ are changed by multiplying the density by $\exp^{-s}$ and renormalizing, while the distribution of increments $Y_{t+\epsilon} - Y_t$ for $t > x$ and $\epsilon > 0$ remain unchanged. Consequently,

$$\frac{F(t+\epsilon) - F(t)}{1 - F(t)} = \frac{F|x(t+\epsilon) - F|x(t)}{1 - F|x(t)} \text{ a.s.}, \quad \forall t > x$$

where $F|x$ is the random posterior distribution given $X = x$. Also, (45) and (46) imply that

$$F|x(t) = F|y(t), \text{ a.s.} \quad \forall t < \min(x, y) \tag{50}$$

This property is similar to Assumption 2 of Section 3.1.

The following is an example of a neutral to the right random probability distribution.

Example 7 *In this example, the independent increments process Y_t corresponding to the prior distribution, F, is a Poisson process with parameter λ. Since the distribution of Y_t is Poisson with parameter λt, the moment generating function of Y_t is*

$$\mathcal{M}_{Y_t}(u) = E\exp^{uY_t} = \exp[\lambda t(\exp^u - 1)]$$

and its Lévy formula is

$$\log E\exp^{uY_t} = \int_0^\infty(\exp^{uz} - 1)dN_t(z)$$

random variables $F(t_k)$, $\frac{F(t_k)-F(t_{k-1})}{F(t_k)}, \ldots, \frac{F(t_2)-F(t_1)}{F(t_2)}$ are independent. Thus F is neutral to the left if and only if it has the same probability distribution as $\exp[Y_t]$ for some a.s. nondecreasing, a.s. right-continuous, independent increment stochastic process Y_t with $\lim_{t\to 0} Y_t = -\infty$ a.s. and $\lim_{t\to\infty} Y_t = 0$ a.s.

Fact 6 *[Doksum (1974)] The Dirichlet process is both neutral to the left and neutral to the right.*

Let F be a neutral to the right distribution and $Y_t = Z_t + \sum_j S_j 1_{[t_j,\infty)}(t)$ the corresponding independent increment process. Let $\mathcal{M}_R(u)$ represent the moment generating function of a random variable R, i.e.,

$$\mathcal{M}_R(u) = E[\exp^{uR}] \tag{54}$$

Then

$$\begin{aligned} E[F](t) &= 1 - E[\exp\{-Y_t\}] \\ &= 1 - E[\exp\{-Z_t - \textstyle\sum_{t_j \le t} S_j\}] \\ &= 1 - \mathcal{M}_{Z_t}(-1) \textstyle\prod_{t_j \le t} \mathcal{M}_{S_j}(-1) \end{aligned} \tag{55}$$

If F^i, $i = 1, 2$ are two neutral to the right distributions, then from (55) it follows that

$$E[F^1](t) \le E[F^2](t) \text{ iff } \log \mathcal{M}_{Z_t^1}(-1) + \sum_{t_j^1 \le t} \log \mathcal{M}_{S_j^1}(-1) \ge \log \mathcal{M}_{Z_t^2}(-1) + \sum_{t_j^2 \le t} \log \mathcal{M}_{S_j^2}(-1) \tag{56}$$

We shall use these results in Appendix III.

Lemma 14 *Suppose that F is neutral to the right with support $[0, K]$, $K < \infty$, and that its independent increments process, Z_t, has no prior fixed points of discontinuity. Then*

$$\lim_{n\to\infty} E[F|\overbrace{K, K, ..., K}^{\text{n K's}}](t) = 0, \quad \forall t < K$$

Proof: Define $F^n(t) \equiv E[F|\overbrace{K, K, ..., K}^{\text{n K's}}(t)]$. Since $F(t) = 1 - \exp^{-Z_t}$ and Z_t has no prior fixed points of discontinuity, the Lévy formula for Z_t, as specified in (44) is

$$\log \mathcal{M}_{Z_t}(u) = \int_0^\infty (\exp^{uz} - 1) dN_t(z)$$

Let Z_t^n be the independent increments process corresponding to $F^n(t)$. From the posterior distribution specified in Appendix II it is clear that Z_t^n has no fixed discontinuities for $t < K$ and that

$$\log \mathcal{M}_{Z_t^n}(-1) = \int_0^\infty (\exp^{-z} - 1) \exp^{-nz} dN_t(z) \tag{59}$$

Since $\lim_n (\exp^{-z} - 1) \exp^{-nz} = 0$, and $0 \geq (\exp^{-z} - 1) \exp^{-nz} \geq -1$, the dominated convergence theorem implies that $\log \mathcal{M}_{Z_t^n}(-1) \to 0$ as $n \to \infty$. Therefore, (55) implies that $\lim_n F^n(t) = 0$, $\forall t < K$. ∎

We end the appendix by computing the optimal stopping rule for search with recall for the neutral to the right distribution of Example 7 of Appendix II.

Example 8 *Consider a neutral to the right random distribution, $F(t)$, generated by a Poisson process Y_t with rate λ. That is, $F(t) = 1 - \exp(-Y_t)$. The Lévy formula of Y_t is*

$$\log E \exp^{uY_t} = \int_0^\infty (\exp^{uz} - 1) dN_t(z)$$

where the Lévy measure $N_t(z) = tN(z)$ with

$$\begin{aligned} N(z) &= 0, \quad \text{if } z < 1 \\ &= \lambda, \quad \text{if } z \geq 1 \end{aligned}$$

Thus the moment generating function of Y_t is

$$\mathcal{M}_{Y_t}(u) = E \exp^{uY_t} = \exp[\lambda t(\exp^u - 1)]$$

and

$$E[F](t) = 1 - \mathcal{M}_{Y_t}(-1) = 1 - \exp[-\lambda t(1 - \exp^{-1})]$$

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Footnotes

[1] See Lippman and McCall (1976, 1981), McMillan and Rothschild (1989), Mortensen (1986), Sargent (1987), and Stokey, Lucas and Prescott (1989), and references cited therein.

[2] When the distribution is not known, the probability mass assigned to any interval $[-\infty, x)$ is a random variable.

[3] The Dirichlet process was first studied by Ferguson (1973). It is often used for deriving Bayesian decision rules in a variety of nonparametric problems. We provide a summary of its properties in Appendix I.

[4] Burdett and Vishwanath (1988) show that in a job search model with learning, under certain conditions, if the optimal strategy has the reservation property then the reservation wages are declining.

[5] Neutral to the right distributions are analyzed in Doksum (1974). We provide an introduction to such processes in Appendix II.

[6] It is assumed that $x_1, x_2, ..., x_n$ are in the support of $E[F]$ and $E[G]$.

[7] See Bikhchandani, Segal, and Sharma (1989) for necessary and sufficient conditions for first-order stochastic dominance under Bayesian learning.

[8] This differs from the definition of the reservation property in Kohn and Shavell (1974) where the reservation price function in any stage n may depend on $(x_1, x_2, ..., x_n)$.

[9] Clearly if F satisfies Assumptions 1 and 2, then so does $F|y$ for all y.

[10] It is too much to ask that every optimal stopping rule be a reservation price rule. Suppose that when searching from some known distribution, it is always optimal to stop if and only if the price observed is less than 10. Then, as long as the probability of observing 5 is zero (that is, the distribution does not have an atom at 5), then the rule which stops if and only if the price observed is less than 10 but not equal to 5, is also optimal.