EFFICIENCY AND THE ROLE OF DEFAULT WHEN SECURITY MARKETS ARE INCOMPLETE

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ABSTRACT

Default appears to play an important role in the economy - but not in general equilibrium models. This paper provides a general equilibrium model of default that suggests that, when markets are incomplete, default is compatible with equilibrium, does not interfere with the orderly functioning of markets, and may promote - indeed even be necessary for - efficiency.

1. INTRODUCTION

Default appears to play an important role in the economy - but not in general equilibrium models. Analysis of default to date has largely been in the context of game-theoretic and/or partial equilibrium models. (Hart and Moore (1989) is a recent example.) From the perspective of general equilibrium theory, it would seem that the default which is actually observed in the economy is a sign of disequilibrium, and that this default interferes with the orderly and efficient functioning of markets.[1] The purpose of this paper is to provide (within a general equilibrium framework) a model of default that suggests, precisely to the contrary, that - in a world of uncertainty and incomplete security markets - default is not incompatible with equilibrium, does not interfere with the orderly functioning of markets, and may promote - indeed, even be necessary for - efficiency.

To explain this positive role for default, it is useful to look briefly at the usual general equilibrium models. Of course, default is not contemplated in the Walrasian (Arrow-Debreu) model. In particular, agents never promise deliveries of commodities that they do not personally own. Moreover, default could not promote efficiency, since the first welfare theorem guarantees that Walrasian (competitive) equilibria are already efficient (Pareto optimal).

This fundamental conclusion depends, of course, on the assumptions that underlie the Walrasian model, and in particular on the assumption that there is a market for every commodity. When there is uncertainty about the future, this assumption entails a complete set of contingent claims (to consumption patterns dependent on the future state of the world). Arrow (1953, 1964) has described an alternative model (a *security market*), in which trading takes place in spot markets for physical commodities and in futures markets for financial securities whose returns depend on the state of the world. Arrow assumes that the set of securities is *complete* (in the sense that every future wealth pattern can be realized as the returns on a portfolio of available securities), and shows that a complete security market is a perfect substitute for a market with complete contingent claims (a Walrasian market), at least in the sense that both structures support

1. The U.S. Bankruptcy Court reported 3.9 million filings for personal bankruptcy in the 10 year period ending June 30, 1989.

the same equilibrium allocations (of physical commodities). Default is not contemplated in Arrow's complete security market model, any more than in the Walrasian model. In the security market model, agents may promise deliveries of goods that they do not personally own, but they will be able to keep these promises because other agents make promised deliveries to them. Moreover, as in the Walrasian model, default could not promote efficiency; since the equilibrium allocations of a complete security market coincide with equilibrium allocations of the underlying Walrasian market, they are, in particular, Pareto optimal.[2]

If security markets are *incomplete* , however, the situation is quite different. If some wealth patterns cannot be realized as the returns on a portfolio of available securities, equilibrium allocations need not be (indeed, usually will not be) Pareto optimal, so there may be a gap between equilibrium and efficiency.[3] It seems natural to suppose, however, that this gap between equilibrium and efficiency would be small if the set of available securities were "nearly" complete, and would disappear entirely as the set of available securities expanded to complete the market.

This supposition is quite wrong, and it is here that default can play a positive role. To demonstrate this, we construct a model of a security market in which uncertainty is described by a countably infinite set of possible states of nature.[4,5] To examine the effect of expanding the set of available securities, we proceed by: fixing an infinite sequence $\{A_n\}$ of securities; examining, for each index N , the security market $\mathcal{E}^N$ in which only the securities $\{A_1,\ldots,A_N\}$ are available for trade; and asking about the behavior of equilibrium allocations of $\mathcal{E}^N$ as N tends to infinity.[6] (Note that, with an

2. Arrow considers only purely financial securities, denominated in units of account. Radner (1972) expands Arrow's model to allow for real securities, denominated in physical commodities. For further discussion of these models, see for example Diamond (1967), Hart (1975), Stiglitz (1982), Cass (1984), Duffie (1985), Werner (1985), Duffie and Shafer (1985), and Geanakoplos and Polemarchakis (1987).

3. Indeed, when there is more than one consumption good, equilibrium allocations need not be optimal even within the set of allocations that can be obtained through trades in the given securities; see for instance Hart (1975), Stiglitz (1982), or Geanakoplos and Polemarchakis (1987). This "constrained suboptimality" is not relevant for us here, however, since we work in the context of a single consumption good - a setting in which security market equilibria are constrained optimal; see Diamond (1967).

4. It seems difficult to address these questions in a model in which the set of possible states of the world is finite: in such a setting, an expanding set of securities will - at some finite stage - already comprise a complete set.

5. Green and Spear (1987, 1989) and Zame (1988) have described similar models.

infinite state space, none of the security markets $\mathcal{E}^N$ can be complete.) If the sequence $\{A_n\}$ is complete (in an appropriate approximate sense) we might expect that, as N tends to infinity, equilibrium allocations should converge to Walrasian equilibrium allocations (and *a fortiori*, to Pareto optimal allocations) of the underlying complete markets economy (and hence, assuming continuity of utility functions, that the corresponding utilities also converge).

In fact, however, equilibrium allocations of $\mathcal{E}^N$ may remain bounded away from Pareto optimal allocations as N tends to infinity; indeed, *all* feasible allocations of $\mathcal{E}^N$ may remain bounded away from Pareto optimal allocations (and utilities of feasible allocations remain bounded away from Pareto optimal utilities). Moreover, far from representing an extreme or pathological situation, this *asymptotic inefficiency* represents (in a sense we make precise) the typical situation.[7]

Surprising though this may be, it has a simple explanation. The requirement that future consumption patterns be non-negative places constraints on the set of portfolios which can be traded; in some cases, these constraints will bind. After all, securities provide a method of shifting wealth between states; a typical portfolio will yield positive returns in some states and impose liabilities in others. If these liabilities are so great that they exceed endowments, satisfying them would violate the requirement that consumption be non-negative; such portfolios cannot be traded. Hence, if such portfolios are required in order to implement - or to approximate - Pareto optimal trades, feasible security market allocations may be inefficient - and remain inefficient even as the set of available securities expands.

To explore the role played by default, we adapt a model introduced by Dubey,

6. This method of analysis is in the same spirit as the familiar method of studying perfect competition through competitive sequences of economies; see Hildenbrand (1974) for instance.
7. The first examples of asymptotic inefficiency were given by Green and Spear (1987). More examples were given by Zame (1988), who identifies a condition on a sequence of securities that is necessary and sufficient for asymptotic efficiency. What is required is that every Arrow security (a security returning one unit of the numeraire commodity in one state, and nothing in every other state) be uniformly approximable by the returns on a finite portfolio of the given securities. This is an extremely stringent condition, and most sequences of securities *fail* this condition. These conclusions are completely consistent with those reached here. Green and Spear (1987) appears to reach different conclusions; however, there are some difficulties with this work. The conclusions of Green and Spear (1989) are not at variance with those of Zame (1988) and the present paper.

Geanakoplos and Shubik (1988). In the present context, a security is simply a promise to pay; default means that (some) agents can not or do not keep (some of) these promises. In the real world, such default might entail various consequences: creditors might be able to seize assets and be awarded judgments against future earnings, defaulters might be barred from future credit markets, etc. Rather than attempt to model such institutional details, we assume here that the only consequences of default are penalties assessed against the defaulters, and that these penalties are assessed directly in terms of utility. Such penalties might be interpreted as extra-economic (debtor's prison, flogging, indentured servitude, public humiliation, etc.), but we prefer to interpret them simply as proxies for economic penalties.[8,9]

In the same spirit of simplification, we also assume that the default penalty is independent of the security and of the state of nature, is the same for all consumers, and is proportional to the amount of default; we write λ for the constant of proportionality.[10] Since we work in a general equilibrium, perfect foresight framework, we assume that all default is perfectly anticipated, but anonymous, and that default is spread equally among all creditors.[11] In particular, each agent observes - and is affected by - only the aggregate default on each security.

For each default penalty λ (with $0 \leq \lambda \leq \infty$), a default equilibrium exists. If $\lambda = 0$, default goes unpunished; in such a situation, no optimizing agent will ever keep promises to pay, and there will be no trade at the default equilibrium. At the other extreme, if $\lambda = \infty$, no optimizing agent will ever default, and a default equilibrium coincides with a security market equilibrium in the sense discussed previously. (So the security market model might be viewed as a special case of the default model.) However,

8. It might be kept in mind, however, that extra-economic penalties have played an important role in the past.

9. Kehoe and Levine (1989) have constructed a model in which the penalty for default is denial of access to future credit markets. However, in their model, markets are complete and there is no equilibrium default. In work in progress, Geanakoplos and Zame construct a model in which collateral may be seized and there is equilibrium default,

10. As in Dubey, Geanakoplos and Shubik (1988), state-specific, security-specific, consumer-specific, and non-proportional default penalties could all be easily accommodated. The essential requirements are that penalties be concave in the amount of default and become sufficiently severe for large default.

11. Of course, the identity of defaulters must be known to some central authority, responsible for imposition of penalties.

for all intermediate values of λ, there will generally be equilibrium default (although the probability and expected magnitude of default will both be small if λ is large).[12]

As we have said, when default is not possible, the requirement that liabilities be satisfied may severely restrict the portfolios that can actually be traded. When default is possible, however, some liabilities may be left unsatisfied; this will enlarge the set of portfolios that can actually be traded. Moreover, since the decision to default is endogenous, this enlargement seems likely to be in precisely the right directions to lead to efficiency. Indeed this is the case: so long as the sequence of securities (A_n) is complete (in an appropriate approximate sense) and the default penalty is sufficiently large, default equilibria will be close to Pareto optima (indeed, to Walrasian equilibria). A bit more precisely, let $\mathcal{E}^{N,\lambda}$ be the security market in which the securities $(A_1,\dots,A_N)$ are available for trade and the default penalty is λ. If the sequence (A_n) is complete, and N and λ are sufficiently large, then default equilibrium allocations of $\mathcal{E}^{N,\lambda}$ are close to Walrasian equilibrium allocations of the underlying complete markets economy (and the corresponding utilities are close, taking default penalties into account); moreover, all Walrasian equilibrium allocations are in turn close to approximate default equilibrium allocations of $\mathcal{E}^{N,\lambda}$.

We emphasize that the positive role of default depends crucially on *incompleteness* of security markets. If security markets are complete (or if a complete set of contingent claims is available), equilibrium allocations will already be Pareto optimal, and default - whenever it takes place - will necessarily have a Pareto worsening effect.

We have nothing to say here about why security markets are incomplete, although this is certainly an important question. It does seem useful, however, to analyze the consequences of incompleteness, even without addressing the causes. Similarly, we have nothing to say about the origin of securities (a question that is certainly connected to the reasons for market incompleteness), or moral hazard, or adverse selection.[13] Other issues we hope to address in future work include the role of collateral, and the possible magnitude of default.[14]

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12. If the security market is complete, there will be no equilibrium default if the default penalty is sufficiently high.

13. Adverse selection could certainly be addressed in a version of the present model, sufficiently enriched to allow for informational asymmetries.

14. As we shall show, if the number of securities and the default penalty are sufficiently large,

The remainder of the paper is organized in the following way. Section 2 describes the basic security market model, and establishes the existence of equilibrium. Asymptotic inefficiency of security markets is discussed in Section 3. Section 4 presents the default model, and establishes the existence of default equilibrium. Section 5 discusses the asymptotic properties of default equilibrium. Proofs are collected in Section 6.

then the probability of default and the expected magnitude of default are both small. However, we have nothing to say about the magnitude (or fraction) of default, *conditional* on the event that default actually occurs.

2. THE SECURITY MARKET MODEL

In this Section we describe a model of a security market, adapted to accommodate an infinite set of states of nature. We use what seems to be the simplest possible model because it is adequate for our purposes and avoids the technical difficulties that would arise in more general models.

We consider a model with two dates, 0 and 1, and uncertainty about the state of nature at date 1. A single good is available for consumption at date 0 and in each state at date 1. At date 0, trade takes place in the single consumption good and in each of a finite number of securities, whose date 1 payoffs depend on the state of nature. At date 1, the state of nature is revealed, securities pay their returns, and consumption takes place. Since we consider only a single consumption good, there will of course be no trade at date 1.[15]

Formally, we describe uncertainty by the set Ω of *states of nature*, which we assume to be a countably infinite set; for convenience, we write $\Omega = \{1,2,....\}$. A *consumption plan* (or *consumption pattern*) specifies consumption at date 0 and in each state at date 1. Thus, a consumption plan is a pair $x = (x(0), x(\cdot))$, where $x(0) \in \mathbb{R}$ is consumption at date 0, and $x(\cdot): \Omega \to \mathbb{R}$ is a date 1 consumption plan; $x(\omega)$ is consumption if state ω occurs. For simplicity, we shall assume throughout that all conceivable consumption plans are bounded.[16] We write C^1 for the space of date 1 consumption plans, and $C = \mathbb{R} \times C^1$ for the space of all consumption plans.

For notational convenience, we sometimes write $\Omega^* = \{0\} \cup \Omega$, and view consumption plans as functions on Ω^*. Thus $x(0)$ represents consumption in date 0, in conformity with our previous notation. We identify a date 1 consumption pattern $x \in C^1$ with $(0,x) \in C$. Given plans $x, y \in C = \mathbb{R} \times C^1$, we write: $x \geq y$ to mean $x(\omega) \geq y(\omega)$ for all $\omega \in \Omega^*$; $x > y$ to mean $x(\omega) > y(\omega)$ for all $\omega \in \Omega^*$, and $x >> y$ to mean that there is

15. There would be no difficulty in allowing for multiple consumption goods, and opportunities for trade at date 1, so long as securities are all denominated in the same good; see Zame (1988, 1989) for details.

16. Since Ω is countable, a bounded function on Ω may be identified with a bounded sequence of real numbers; for our purposes, however, it is convenient to use functional notation. For other formalizations, see Green and Spear (1987, 1989) and Zame (1988).

a positive number $\varepsilon > 0$ such that $x(\omega) > y(\omega) + \varepsilon$ for all $\omega \in \Omega^*$. In particular, $x >> 0$ means that x is bounded away from 0.

Securities are claims to consumption patterns at date 1, and thus are elements of C^1. The *return* on a security A in state ω is $A(\omega)$. We shall assume, for the sake of parallelism with the default model to follow, that security returns are non-negative; i.e., $0 \le A(\omega)$ for each ω. If there are N securities $A_1, \ldots A_N$, a *portfolio* is a vector $\theta = (\theta_1, \ldots \theta_N) \in \mathbb{R}^N$; θ_n is the holding of the n-th security. The *returns* on the portfolio $\theta = (\theta_1, \ldots \theta_N)$ are:

$$\text{returns}(\theta) = \sum \theta_n A_n \in C^1 .$$

Security prices are vectors $q \in \mathbb{R}^N$; if $\theta \in \mathbb{R}^N$ is a portfolio, then $q \cdot \theta = \sum q_n \theta_n$ is the *value* of the portfolio θ at the prices q. We take date 0 consumption as numeraire, so that security prices are denominated in date 0 consumption.[17]

Consumers $h \in \{1, \ldots H\}$ are defined by *endowments* e^h and *utility functions* U^h; *consumption sets* for each consumer are the positive cone $C^+ = \mathbb{R}^+ \times (C^1)^+$. For the sake of simplicity we assume that each consumer maximizes the sum of utility for consumption at date 0 and expected utility for consumption at date 1 according to some probability distribution μ^h on Ω. We allow for the possibility that different consumers have different probability assessments μ^h, but we assume that assessments are consistent, in the weak sense that consumers find each of the possible states of nature to have positive probability; $\mu^h(\{\omega\}) > 0$ for each h, ω. Our assumptions on utility functions mean that there are functions v_h, u_h such that:

$$\begin{aligned} U^h(x) &= v_h(x(0)) + \int u_h(x(\omega))\, d\mu^h \\ &= v_h(x(0)) + \sum u_h(x(\omega))\, \mu^h(\{\omega\}) \end{aligned}$$

We shall assume that the functions $v_h, u_h : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, strictly concave, and strictly monotone, and that the (right hand) derivatives $v_h'(0)$, $u_h'(0)$ are finite. Finally, we assume that endowments are bounded away from 0; i.e., $e^h >> 0$ for all h.

17. Since we take date 0 consumption as numeraire and there is no trading in date 1, spot prices for commodities are irrelevant.

We equip each bounded subset of C with the topology of *simple convergence*: $x^n \to x$ if $x^n(0) \to x(0)$ and $x^n(\omega) \to x(\omega)$ for every state $\omega \in \Omega$.[18] Note that (on bounded sets) simple convergence implies convergence at date 0 and convergence in expectation with respect to every probability measure μ on Ω (just convergence in expectation, for short). Conversely, if $\mu(\omega) > 0$ for each state ω, then convergence in expectation with respect to μ implies simple convergence (and hence convergence in expectation with respect to every other probability measure). If we fix consumers' probability assessments $\mu^1, \ldots, \mu^H$, a particularly convenient metric for convergence in expectation suggests itself. Let $\|\cdot\|_E$ be the expectation norm (with respect to $\sum\mu^h$)

$$\|x\|_E = |x(0)| + \sum \int |x(\omega)|\, d\mu^h(\omega)$$

and define the metric d_E by

$$d_E(x,y) = \|x-y\|_E$$

for $x, y \in C$. Of course, this metric defines a topology on all of C, and not just on bounded sets. We caution the reader that convergence in expectation and simple convergence coincide only on *bounded* sets.

It is easily seen that utility functions are continuous on bounded sets, and hence on the set of feasible consumptions (which is bounded), and that the set of feasible consumptions is compact.[19]

The data of a *security market* is a tuple $\mathcal{E} = (\{(e^h, U^h)\}, \{A_n\})$, where $\{(e^h, U^h): 1 \le h \le H\}$ is a finite set of consumers and $\{A_n : 1 \le n \le N\}$ is a finite set of securities. The assumptions above are understood to be in force at all times.

Because there is a single physical commodity, there will be no trading in commodities after the state of nature is revealed. Hence, given security prices q, we define the *budget set* $B^h(q, e^h, U^h)$ for a consumer with endowment e^h and utility function U^h to be the set of triples (x^h, φ^h, ψ^h), where x^h is a feasible (i.e., non-negative) consumption plan and φ^h, ψ^h are non-negative portfolios - of security *purchases* and security *sales*,

18. A subset $B \subset C$ is *bounded* if there is a constant $M > 0$ such that $\|x\| \le M$ for each $x \in B$.
19. For a related discussion, see Bewley (1972).

respectively - such that:

(i) $q \cdot (\varphi^h - \psi^h) = e^h(0) - x^h(0)$

(ii) $x^h(\omega) = e^h(\omega) + \text{returns}(\varphi^h)(\omega) - \text{returns}(\psi^h)(\omega)$ all $\omega \in \Omega$

Note that (i) says that consumers finance purchases and sales of securities from date 0 consumption, while (ii) says that consumer h's consumption in state ω is the sum of his endowment and the returns on his purchases, less his payments on liabilities.[20]

An *equilibrium* for the security market $\mathcal{E}$ is a 4-tuple $(q,(x^h),(\varphi^h),(\psi^h))$, where $q \in \mathbb{R}^N$ is a vector of security prices, each x^h is a feasible consumption plan (consumer h's *equilibrium consumption plan*) and φ^h, ψ^h are non-negative portfolios such that:

(i) $\sum (x^h - e^h) \le 0$ (commodity markets clear)

(ii) $\sum (\varphi^h - \psi^h) = 0$ (securities are in zero net supply)

(iii) for each h, $(x^h,\varphi^h,\psi^h) \in B^h(q, e^h,U^h)$
(plans are budget feasible)

(iv) if $(y^h,\alpha^h,\beta^h) \in B^h(q, e^h,U^h)$, then $U^h(x^h) \ge U^h(y^h)$
(consumers optimize in their budget sets)

For $\varepsilon > 0$, an *ε-equilibrium* is a 4-tuple $(q,(x^h),(\varphi^h),(\psi^h))$ of security prices, feasible consumption plans, and portfolios, satisfying (i), (ii), (iii) and

(iv′) if $(y^h,\alpha^h,\beta^h) \in B^h(q, e^h,U^h)$, then $U^h(x^h) \ge U^h(y^h) + \varepsilon$
(consumers ε-optimize in their budget sets)

The basic fact about security markets is that equilibria exist; we defer the simple proof to Section 6.

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20. It is customary not to separate sales and purchases, but rather to consider only portfolios θ of net security trades. That would make no difference here. We maintain the separation in the interests of parallelism with the default model, where the separation plays an important role.

THEOREM 1: Every security market has an equilibrium.

Underlying the security market $\mathcal{E}$ is a complete markets (Arrow-Debreu) economy $\mathcal{E}^{CM}$ with the same consumers, in which all contingent consumption patterns are available for sale. In $\mathcal{E}^{CM}$, a *price* is a positive linear functional $\pi : C \to \mathbb{R}^+$ whose restriction to each bounded set is continuous.[22] Continuity implies that we may interpret π as a price list: There is a (unique) function $\tilde{\pi} : \Omega^* \to \mathbb{R}^+$ with the property that, for each $x \in C$, the *value* of the consumption plan x at the price π is:

$$\pi \cdot x = \tilde{\pi}(0)x(0) + \sum \tilde{\pi}(\omega)x(\omega)$$

Hence $\tilde{\pi}(\omega)$ is the value (at the price π) of the consumption plan guaranteeing 1 unit of consumption at ω and 0 elsewhere. Since $\pi \cdot x$ is finite for all consumption plans x, this guarantees that $\sum \tilde{\pi}(\omega) < \infty$.

As usual, a (*Walrasian*) *equilibrium* for $\mathcal{E}^{CM}$ is a pair $(\pi, (x^h))$ consisting of prices π for contingent claims, and feasible consumption plans x^h, satisfying:

(i) $\sum(x^h - e^h) \leq 0$

(ii) $\pi \cdot (x^h - e^h) \leq 0$ for each h

(iii) if $U^h(y^h) > U^h(x^h)$ then $\pi \cdot (y^h - e^h) > 0$.

22. Equivalently, prices are continuous with respect to the strongest vector space topology that coincides, on bounded sets, with the topology of simple convergence; this is usually called the *bounded weak star topology*. In principle, we might consider price functionals which are not continuous, but nothing would be gained by doing so, since Bewley (1972) has shown that the assumptions in force here guarantee that equilibrium prices are necessarily continuous.

3. ASYMPTOTIC INEFFICIENCY OF SECURITY MARKETS

Since the state space is infinite, no finite number of securities can form a complete set (span all the uncertainty). The failure of securities to span will usually mean that equilibria need not be efficient. In this section, we show that this gap between equilibrium and efficiency need not disappear even as the set of securities expands to a complete set. We motivate our discussion with three simple examples.[23]

We fix utility functions and endowments and an infinite sequence of securities, and examine the effect of increasing the number of available securities. We consider a two consumer world, in which consumers have identical utility functions:

$$U(x) \quad = \quad x(0) + \sum 2^{-n} [1+x(n)]^{1/2}$$

but differing endowments:

$$e^1 = (1;1,7,1,1,1,1,\ldots) \qquad e^2 = (1;7,1,1,1,1,1,\ldots)$$

(i.e., $e^1(0) = 1$, $e^1(1) = 1$, etc.) ; aggregate endowment is $\bar{e} = (2;8,8,2,2,\ldots)$. Note that the Pareto optimal allocations are all of the form $(\alpha\bar{e},(1-\alpha)\bar{e})$ for some α , $0 \le \alpha \le 1$, and that the unique Walrasian allocation is $(\bar{x}^1,\bar{x}^2) = ((1/2)\bar{e},(1/2)\bar{e})$.

To examine the effect of expanding the set of available securities, we fix an infinite sequence $\{A_n\}$ of securities; for each N we consider the security market in which the first N securities are available for trade. Since the state space Ω and the set $\{A_n\}$ of securities are each countably infinite, it is convenient to represent the array of security returns by an infinite by infinite matrix, in which the entry in the i-th row, j-th column represents the return on the j-th security in the i-th state.

EXAMPLE 1: Consider the sequence $\{A_n\}$ of *Arrow securities* ; i.e., the security A_n returns 1 unit in the n-th state, and 0 in every other state. In matrix notation, we may represent this sequence of securities as:

23. For related examples, see Green and Spear (1987, 1989) and Zame (1988).

$$
A = \begin{matrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{matrix}
$$

If the securities $A_1, \ldots A_N$ are available for trade, then the two consumers can jointly obtain precisely those allocations (y^1,y^2) for which $y^1(n) = e^1(n)$ and $y^2(n) = e^2(n)$ for $n > N$. If N is sufficiently large, consumers can therefore jointly obtain allocations arbitrarily close to Pareto optimal allocations. Indeed, it may easily be seen that equilibrium allocations of the security market converge (as $N \to \infty$) to the unique Walrasian allocation $(\overline{x}^1,\overline{x}^2) = ((1/2)\overline{e},(1/2)\overline{e})$.[24]

EXAMPLE 2: For each n, let B_n be the security that returns 1 unit in the n+1-th state, and 0 in every other state. In matrix notation, we may represent this sequence of securities as:

$$
B = \begin{matrix} 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{matrix}
$$

Since none of the securities B_n yield non-zero returns in the first state, if the allocation (y^1,y^2) is obtained by trading date 0 consumption and (any number of) these securities, then $y^1(1) = e^1(1) = 1$ and $y^2(1) = e^2(1) = 7$. The only Pareto optimal allocation which is a limit of such allocations is $(1;1,1,1, \ldots)$, which is not individually rational (it yields consumer 1 less utility than his endowment). Since security market equilibria (and hence limits of such) *are* individually rational, it follows that no sequence of

24. For details, see Zame (1988) or Green and Spear (1987, 1989).

security market equilibrium allocations can converge to a Pareto optimal allocation.

EXAMPLE 3: Let $D_n = A_n + 2B_n$, so that

$$D = \begin{matrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 2 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{matrix}$$

Consider any feasible allocation (y^1, y^2) which can be obtained by trading the first N securities, so that there is a portfolio θ (the net security trade of consumer 1), involving only the first N securities, such that $y^1 = e^1 + \text{returns}(\theta)$ and $y^2 = e^2 - \text{returns}(\theta)$. Feasibility requires that $y^1 \geq 0$ and $y^2 \geq 0$; evaluating successively in states $N+1, N, \ldots 1$ yields the following pairs of inequalities:

$$\begin{array}{llll} 1 + 2\theta(N) & \geq 0 & \qquad 1 - 2\theta(N) & \geq 0 \\ 1 + 2\theta(N-1) + \theta(N) & \geq 0 & \qquad 1 - 2\theta(N-1) - \theta(N) & \geq 0 \\ \vdots & & \qquad \vdots & \\ 1 + 2\theta(2) + \theta(3) & \geq 0 & \qquad 1 - 2\theta(2) - \theta(3) & \geq 0 \\ 7 + 2\theta(1) + \theta(2) & \geq 0 & \qquad 1 - 2\theta(1) - \theta(2) & \geq 0 \\ 1 + \theta(1) & \geq 0 & \qquad 7 - \theta(1) & \geq 0 \end{array}$$

Solving this system of inequalities yields $-1 \leq \theta(1) \leq +1$ and $-1 \leq \theta(2) \leq +1$. Since

$y^1 = e^1 + \text{returns}(\theta)$, this entails $y^1(1) \le 2$ and $4 \le y^1(2)$. If (x^1, x^2) is a Pareto optimal allocation, then $x^1(1) = x^1(2)$; it follows that $d_E(x^1, y^1) \ge \min(\mu^1(1), \mu^1(2)) > 0$, independently of the number N of securities available for trade.

We see that, as the set of available securities expands, the gap between equilibrium and efficiency may disappear (Example 1) or it may persist (Examples 2, 3). It is important to note that, although the gap persists in both Examples 2 and 3, it does so for different reasons. In Example 2, the available securities do not span all the uncertainty (even in the limit). In Example 3, the available securities span all the uncertainty (in the limit), but portfolios yielding certain return patterns in the first two states necessarily create large - and unsatisfiable - liabilities in later states, so these portfolios cannot be traded. (We might say that the "effective span" of the securities is too small.)

In general, convergence to efficiency will depend (in a rather complicated way) on the specific sequence of securities and on the specific endowments. What further moral (if any) we should draw depends to some extent on which of these Examples we view as "typical" and that in turns depends on our model of the process which gives rise to securities. Unfortunately, no convincing model seems available (and we have none to offer). Instead we take what seems to be a reasonable shortcut. We parametrize the set of endowments and the set of sequences of securities as compact metric spaces, and appeal (as is frequently done) to a topological notion of size to represent the "typical" situation. Using this framework, we show below that it is Example 3 that represents the typical situation, and Examples 1 and 2 that are atypical. Inefficiency and spanning are the rule, rather than the exception.[25]

25. The topological framework we use is certainly open to criticism. An alternative model, which is probabilistic rather than topological, could be constructed in the following way. Equip the interval $[0,1]$ with Lebesgue measure. The set of securities is the countably infinite product of intervals, and so inherits a product probability measure; the set of sequences of securities is a countably infinite product of the set of securities, and again inherits a product probability measure. The set E of endowments is a finite product of intervals $[1/8H, 1]$, and also inherits a product measure. In the probabilistic framework, we interpret "almost all" to mean "except for a set of measure 0." Then (the analog of) Theorem 2 remains valid. However, this probabilistic model is also open to criticism. For instance: Why should we assume that securities (and security payoffs in various states) are independent of each other and follow the same probability distribution?

To describe the formal framework, we fix the number H of consumers, and their utility functions $\{U^h\}$ (for reasons that will become clear in a moment, we leave endowments unspecified). Given an infinite sequence $\{A_n\}$ of securities, we consider equilibrium allocations of the security market in which the first N of these securities are available for trade; and we ask whether these equilibrium allocations (or any feasible allocations) are near to Pareto optimal allocations if N is large.[26]

To parametrize endowments, recall that we have, throughout, required that endowments be bounded above and bounded away from 0. For the present purpose it is convenient to restrict attention to endowments bounded above by 1 and below by $1/8H$; write

$$E = \{(e^1, \ldots e^H) : 1/4 \leq e^h \leq 1 \text{ for each } h\}$$

This is a compact metric space.

To parametrize sequences of securities, recall that we have required that security returns be positive and bounded; there is no loss of generality in requiring that they be bounded by 1. Write $\mathbf{S}$ for the set of such securities,

$$\mathbf{S} = \{A : 0 \leq A(\omega) \leq 1 \text{ for each } \omega\}$$

and let $\mathcal{A}$ be the set of (infinite) sequences of securities (in $\mathbf{S}$). $\mathbf{S}$ is a compact metric space; if we equip $\mathcal{A}$ with the product topology, it too becomes a compact metric space.

Recall that a set is *residual* if it is the intersection of a countable family of dense open sets. It is customary to view residual subsets of a compact metric space as large, and their complements as small; in particular, the Baire category theorem asserts that residual sets are dense. We say that a property is valid for *almost all* endowments (or sequences of securities) if the set for which it is valid contains a residual set; such a property is sometimes said to be *generic*.

Given endowments $e = (e^1, \ldots e^H) \in E$ and a sequence of securities $\mathbf{A} = \{A_n\}$, write

26. Because utility functions are continuous and the set of feasible allocations is compact, convergence of allocations to Pareto optimal allocations and convergence of utilities to Pareto optimal utilities are equivalent (up to a passage to subsequences).

PO(e) for the set of Pareto optimal allocations (from endowments e) and $F^N(e)$ for the set of allocations that can be obtained from the endowments e by trading date 0 consumption and the securities $A_1, \ldots A_N$. We say that the sequence of securities $A = \{A_n\}$ is *asymptotically inefficient (from e)* if there is a $\delta > 0$ such that $d_E(F^N(e), PO(e)) \geq \delta$ for all N . Informally: asymptotic inefficiency means that feasible allocations (and in particular, security market equilibrium allocations) are uniformly bounded away from Pareto optimal allocations, independently of the number of securities available for trade. As we have noted before, if feasible allocations are bounded away from Pareto optimal allocations, then utilities of feasible allocations are bounded away from Pareto optimal utilities.[27,28]

As the examples above show, the asymptotic behavior of security market allocations depends on the particular sequence of securities, but it also depends on endowments. In particular, equilibrium allocations will always be Pareto optimal if endowments are themselves Pareto optimal - even if no securities are available for trade. It seems natural therefore to focus on a generic set of endowments.

As we have noted in Example 2, inefficiency may arise simply because securities do not span all the uncertainty, but it is the combination of inefficiency and spanning that is of most interest. In the case of a finite state space, spanning (completeness) has an unambiguous meaning: every wealth pattern can be obtained as the returns on a (finite) portfolio. When the state space is infinite, it seems natural to require only that every wealth pattern be *approximable* by the returns on a finite portfolio. We say that the sequence $A = \{A_n\}$ of securities *spans all the uncertainty* if every wealth pattern (in C^1) is the limit (in the topology of convergence in expectation) of returns on finite portfolios of the securities $\{A_n\}$.

As the following result shows, spanning and inefficiency are the rule. Looking ahead to Section 5, we take this opportunity to record the fact that almost all sequences of securities are also linearly independent.

27. Keep in mind that we regard utility functions as fixed.
28. We caution the reader that Zame (1988) uses the term "asymptotically inefficient" with a slightly different meaning.

THEOREM 2: (a) Almost all sequences of securities are linearly independent.

(b) Almost all sequences of securities span all the uncertainty.

(c) For almost all endowments, almost all sequences of securities are asymptotically inefficient.

4. THE DEFAULT MODEL

In this section, we describe an adaptation of the security market model which allows for the possibility of endogenous default. The model we describe is a variant of a model due to Dubey, Geanakoplos and Shubik (1988), developed further by Dubey and Geanakoplos (1989). Some of our discussion is also adapted from these sources; we refer the reader to the original papers for further discussion.

The data of the model are precisely the same as the data of the security market model described in Section 2. However, we modify the definitions of the budget set and of optimizing behavior (and consequently of equilibrium) to allow for the possibility that agents do not meet their liabilities.[29] To be more precise, we allow each agent h to choose, in addition to a consumption plan and portfolios of purchases and sales, the amount $D^h(n,\omega)$ that he actually delivers on security A_n in the state ω. (Recall that sale of a security is really a promise to pay in the future. Since we require securities to have non-negative returns, we avoid the need to interpret failure to deliver on promises of negative quantities.) Of course, if agent h chooses not to sell security A_n, he will have no obligation to meet; in this case we require that his delivery $D^h(n,\omega) = 0$. In every case, we require that $D^h(n,\omega) \geq 0$. (To allow $D^h(n,\omega)$ to be negative would amount to allowing additional borrowing at date 1.) Since additional consumption is always desirable, consumer h will never choose to deliver more than the full amount of his promises, which is $\psi^h(n)A_n(\omega)$; hence we will always have $0 \leq D^h(n,\omega) \leq \psi^h(n)A_n(\omega)$.

Several things are important to note. First of all, the decision not to deliver on a security is *voluntary*; in particular, there is no requirement that agents meet their obligations whenever they are able. Default may occur either out of necessity or for strategic reasons. Second, by separating purchases from sales, we have allowed for the possibility that agents go long and short in (i.e., buy and sell) the same security. We have implicitly contemplated this possibility in the security market model, but when default is not possible, such an action is irrelevant. However, when default is possible, such an

29. It does not seem possible to incorporate default in the model by simply respecifying securities and states to account for what is actually delivered, as opposed to what is promised. As Dubey, Geanakoplos and Shubik (1988) puts it "... what is delivered is determined endogenously and cannot be predicted without solving for the equilibrium. Moreover, different agents will make different deliveries on the same asset even though the lenders receive the same aggregated payoffs."

action is not be irrelevant; it may benefit an agent to go long and short in the same security *if he does not intend to meet all his obligations*. In particular, there is nothing to prevent an agent from buying one share of a security, selling one share of the same security, collecting the returns from his purchase, and defaulting on his obligations.[30]

With the possibility that others will default on their obligations, a rational agent will make conjectures about this default, and act accordingly. We view purchases and sales of securities as implemented through some central bank, and assume that shortfalls on promised deliveries are spread uniformly among all creditors. In this sense default is anonymous, and each agent observes (and is affected by) only the aggregate default.[31] (Mortgage securities provide a reasonable real world approximation to such securities.) Moreover, although we consider explicitly only a model with a finite number of consumers, we implicitly take the view that there are actually a continuum of consumers, but only a finite number of *types*. Hence, each consumer correctly views the effect of his own actions on the aggregate level of default as negligible. Write $K^h(n,\omega)$ for agent h's conjecture about the aggregate fraction of promised deliveries on security n that will actually be made in state ω, so that $0 \le K^h(n,\omega) \le 1$.

If consumer h's conjectures are correct, then 1 share of security A_n will yield the actual return $K^h(n,\omega)A_n(\omega)$ in state ω, rather than the nominal return $A_n(\omega)$. Consumer h's budget set $B^h(e^h,q,K^h)$, given endowment e^h, security prices q, and conjectures K^h, will then consist of 4-tuples $(x^h, \varphi^h, \psi^h, D^h)$, where x^h is a consumption plan, φ^h is a portfolio of security purchases, ψ^h is a portfolio of security sales, and D^h is a plan of delivery on liabilities, such that:

(i) $\quad q\cdot(\varphi^h - \psi^h) = e^h(0) - x^h(0)$

(ii) $\quad x^h(\omega) = e^h(\omega) + \sum_n K^h(n,\omega)\, \varphi^h(n)A_n(\omega) - \sum_n D^h(n,\omega)$

As before, (i) says that consumers finance purchases and sales of securities from date 0 consumption; (ii) says that consumer h's consumption in state ω is the sum of his endowment and the returns on his purchases (taking into account the default against him),

30. Without this possibility, default equilibria might not exist.

31. Of course, the identity of defaulters must be known to some central authority, responsible for the imposition of penalties.

less his own deliveries on liabilities.

To this point we have not spoken of the consequences of default. In reality, creditors might be able to seize assets and be awarded judgments against future earnings, defaulters might be barred from future credit markets, etc. To simplify matters as much as possible, we assume here that the only consequences of default are penalties assessed against the defaulters, and that these penalties are assessed directly in terms of utility. Such penalties might be interpreted as extra-economic (debtor's prison, flogging indentured servitude, public humiliation, etc.), but we prefer to interpret them simply as proxies for economic penalties (seizure of assets, loss of future credit, etc.).[32] Although it would be more satisfying to explicitly model the economic consequences of default, it would also be more difficult, and would involve many institutional details. (For instance: what assets can be seized? and when?[33])

We shall also assume that default penalties are independent of the state of nature and of the security, are the same for all agents, and are proportional to the amount of default. However, these latter assumptions are made solely for notational convenience. There would be no difficulty in allowing for default penalties that depend on the state and the security, are agent-specific, and are not proportional to the amount of default. All that is really required is that default penalties be concave and become sufficiently severe, with large default, to ensure that no agent will acquire liabilities that exceed the aggregate resources of society.

Given a default penalty of λ , the utility consumer h will achieve by following the plan $(x^h,\varphi^h,\psi^h,D^h)$ is:

$$W^h(x^h,\varphi^h,\psi^h,D^h) = U^h(x) - \int \lambda \left\{ \sum \psi^h(n)A_n(\omega) - D^h(n,\omega) \right\} d\mu^h$$

That is, agent h enjoys the utility of his consumption, less his expected penalties. (Note that $\psi^h(n)A_n(\omega) - D^h(n,\omega)$ is the amount of agent h's default on the n-th security.)

32. It might be kept in mind, however, that extra-economic penalties have played an important role in the past.

33. Kehoe and Levine (1989) have constructed a model in which the penalty for default is denial of access to credit markets in succeeding periods. However, in their model, markets are complete and there is no equilibrium default. In work in progress, Geanakoplos and Zame study a model in which collateral may be seized.

A *default equilibrium* is a list q of security prices, a collection $\{K^h\}$ of conjectures, and a collection $\{(x^h, \varphi^h, \psi^h, D^h)\}$ of choices such that:

(i) $\sum (x^h - e^h) \leq 0$

(ii) $\sum (\varphi^h - \psi^h) = 0$

(iii) $(x^h, \varphi^h, \psi^h, D^h) \in B^h(e^h, q, K^h)$ for each h

(iv) if $(\tilde{x}^h, \tilde{\varphi}^h, \tilde{\psi}^h, \tilde{D}^h) \in B^h(e^h, q, K^h)$, then $W^h(x^h, \varphi^h, \psi^h, D^h) \geq W^h(\tilde{x}^h, \tilde{\varphi}^h, \tilde{\psi}^h, \tilde{D}^h)$

(v) every conjecture K^h is correct

In other words, commodity markets clear, security markets clear, agents optimize (among budget feasible plans) given security prices and their own conjectures, and agents have correct (and therefore identical) conjectures.

It remains to formalize requirement (v), that conjectures be correct at equilibrium. This is straightforward for conjectures about securities that are traded and yield positive returns in a given state ω; the requirement becomes:

(v.a) if $\sum \psi^h(n) A_n(\omega) > 0$ then $K^h(n,\omega) = \left\{\sum D^h(n,\omega)\right\} / \left\{\sum \psi^h(n) A_n(\omega)\right\}$

For securities that yield no returns, conjectures are irrelevant:

(v.b) if $A_n(\omega) = 0$ then $K^h(n,\omega)$ may be arbitrary

However, if the security A_n yields positive returns in state ω and is *not* traded at equilibrium, potential buyers and sellers receive no signals on which to base their expectations. Were we to allow for arbitrary conjectures in this case, there would always be trivial equilibria in which no assets are traded because all agents conjecture total default - even if default penalties were so high that purchasers of securities would not actually be willing to default. A similar difficulty arises in the theory of extensive form games. As Selten (1975) points out, the Nash equilibrium notion imposes no restrictions off the equilibrium path. Requiring that agents optimize at all decision nodes - even those off the equilibrium path - leads to the stronger notion of subgame perfect equilibrium. Of course the bite of subgame perfection is in the restrictions it imposes on beliefs. In our framework, insisting that consumers conjecture that others always choose optimal actions - even out of equilibrium - rules out the trivial equilibria described above.

There are several ways to formalize the requirement that consumers always conjecture optimal behavior by others. Two such formalizations are given by Dubey, Geanakoplos and Shubik (1988) and by Dubey and Geanakoplos (1989). Our formalization is weaker than either of these, so that we allow more equilibria (as the reader will see, this is in keeping with our aims).

To motivate our requirement, we show that, if the default penalty λ is sufficiently high, then - if consumers are optimizing - the probability of default will be low, and there will be no voluntary default at all. To make these assertions precise, let the aggregate endowment be $\bar{e}$. Fix a consumer k, and consider an optimal budget-feasible plan $(x^k, \varphi^k, \psi^k, D^k)$. Let Ω_k be the set of states in which consumer k defaults on some security ($\psi^k(n)A_n(\omega) - D^k(n,\omega) \neq 0$ for some n) and let $\tilde{\Omega}_k$ be the set of states in which consumer k's default is voluntary ($\psi^k(n)A_n(\omega) - D^k(n,\omega) \neq 0$ for some n and $x^k(\omega) \neq 0$). We assert that:

(*) If $\lambda > u_k'(0)$ then $\tilde{\Omega}_k = \emptyset$

(**) $\mu^k(\Omega_k) \leq (1/\lambda)[u_k'(0) + v_k'(0)]\,(\sup\{\bar{e}(\omega) : \omega \in \Omega^*\})$

Note that (*) means that consumer k will not default voluntarily if the penalty is sufficiently high; and (**) bounds the probability that consumer h will default at all.

To verify (*), suppose to the contrary that $\lambda > u_k'(0)$ and that there is a state ω in which consumer k defaults voluntarily. If ε is less than consumer k's consumption and default in state ω, then consumer k can alter his plan by making a larger payment on his state ω debt, thereby decreasing both his consumption and default in the state ω by ε. Reducing default by ε reduces the utility penalty by $\varepsilon \lambda \mu^k(\omega)$; reducing consumption by ε reduces the utility of consumption by at most $\varepsilon u_k'(0)\mu^k(\omega)$. If $\lambda > u_k'(0)$, this contradicts the optimality of $(x^k, \varphi^k, \psi^k, D^k)$; this contradiction establishes (*).

To verify (**), let t be a real number with $0 < t < 1$, and consider the alternate plan $(e^k + t(x^k - e^k), t\varphi^k, t\psi^k, D^k)$. Since $e^k >> 0$, this plan is budget-feasible for t sufficiently close to 1. This alternate plan would entail a utility cost of foregone consumption not exceeding $(1-t)[u_k'(0)+v_k'(0)](\sup\{\bar{e}(\omega) : \omega \in \Omega^*\})$, while avoiding a utility penalty of $(1-t)\lambda\,\mu^k(\Omega_k)$. Since the plan $(x^k, \varphi^k, \psi^k, D^k)$ is optimal, this alternate plan cannot be an improvement; this yields (**).

Our rationality requirement is that conjectures should be consistent with the above observations. To express this requirement most simply, define $\tilde{\mu} = \inf(\mu^1, \ldots, \mu^H)$ by $\tilde{\mu}(\omega) = \inf\{\tilde{\mu}^1(\omega), \ldots \tilde{\mu}^H(\omega)\}$, and write $M_0 = \sup\{u_k'(0) + v_k'(0) : 1 \le k \le H\}$ and $\Omega^h = \{\omega : K^h(n,\omega) \neq 0$ for some $n\}$. Applying (**) for each consumer leads to the requirement:

(v.c) $\quad \tilde{\mu}(\Omega^h) \le H(M_0/\lambda)(\sup\{\bar{e}(\omega) : \omega \in \Omega^*\})$ for each h

As we have said, this is weaker than the rationality requirement imposed in Dubey, Geanakoplos and Shubik (1988) and Dubey and Geanakoplos (1989). In particular, our requirement is vacuously satisfied when $H(M_0/\lambda)(\sup\{\bar{e}(\omega) : \omega \in \Omega^*\}) > 1$ (i.e., when the default penalty is sufficiently small). Hence, we allow here for a potentially larger set of default equilibria.

For $\varepsilon > 0$, we define an *ε-equilibrium* to be a list q of security prices, a collection $\{K^h\}$ of conjectures[34], and a collection $\{(x^h,\varphi^h,\psi^h,D^h)\}$ of choices satisfying (i), (ii), (iii), (v) and

(iv′) if $(\tilde{x}^h,\tilde{\varphi}^h,\tilde{\psi}^h,\tilde{D}^h) \in B^h(e^h,q,K^h)$, then $W^h(x^h,\varphi^h,\psi^h,D^h) \ge W^h(\tilde{x}^h,\tilde{\varphi}^h,\tilde{\psi}^h,\tilde{D}^h) - \varepsilon$

This completes the description of the default model. The next step is to show that the model is consistent; i.e., that equilibria exist. Again, the proof is deferred to Section 6.

THEOREM 3: If securities are linearly independent, then for each λ, $0 \le \lambda \le \infty$, a default equilibrium exists.[35]

34. Our arguments for the rationality requirement (v.c) lose some of their force in the case of ε-equilibria. However, the ε-equilibria we construct (in Section 5) have the property that all available securities are actually traded, so the issue becomes moot.

35. The requirement that securities be linearly independent is necessary here. The problem is that,

It is instructive to consider the two extreme cases $\lambda = 0$ and $\lambda = \infty$. If $\lambda = 0$, there is no penalty for default. In such case, optimizing agents will never honor any of their commitments, and agents with correct expectations will never lend (i.e., sell securities). Thus, at equilibrium, there will be no trade in securities, and hence (given that there is a single commodity), no trade at all. If $\lambda = \infty$, optimizing agents will never default (else they would incur infinite penalties). Hence such default equilibria coincide with security market equilibria as described previously.

So, in extreme cases, we see no equilibrium default. However, for all intermediate penalties, there will generally be *some* (probability of) default at equilibrium. We have already noted that, if the default penalty is high, the probability that default will occur will be small. As we shall see, if the default penalty is high, the expected amount of default will also be small.

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since default patterns may depend on the names of the securities (and not just on their returns) redundant securities cannot necessarily be priced by arbitrage; see Dubey, Geanakoplos and Shubik (1988) for further discussion.

5. THE ROLE OF DEFAULT

Default creates inefficiencies: the direct losses of imposing the default penalties, and the indirect losses due to the reluctance of lenders to lend and the resulting inability of borrowers to borrow. In this section we show that, despite the inefficiencies it creates, default may promote overall efficiency. Example 3 and Theorem 2 of Section 3 provide the intuition necessary for understanding this apparent contradiction. When default is not possible, the requirement that consumers keep all their promises (i.e., that terminal wealth constraints be met) may severely restrict the portfolios that can actually be traded, and hence the effective span of securities. (Indeed, this will "typically" be the case.) Default gives consumers some ability to tailor security returns to their own requirements. In this way, default endogenously increases the effective span of securities, and this in turn more than compensates for the direct and indirect losses associated with default.

We make this intuition precise by showing that, if the default penalty is sufficiently large and enough securities are available for trade, then every equilibrium allocation of the security market is close to a Walrasian equilibrium allocation of the underlying complete markets economy. Moreover, the corresponding utilities are close, even when we take default penalties into account. The precise statement is a bit clumsy, since we must account for the dependence on the default penalty as well as on the number of securities.[36]

In what follows, we fix an infinite sequence (A_n) of securities, assumed to be linearly independent and span all the uncertainty. (In view of Theorem 2, almost all sequences of securities satisfy these requirements.) We also fix a set of H consumers, with utility functions U^h and endowments e^h. For $1 \le N < \infty$ and $0 < \lambda < \infty$, denote by $\mathcal{E}^{N,\lambda}$ the security market (with these consumers) in which the securities $(A_1, \ldots, A_N)$ are available for trade, and the default penalty is λ. As before, we let d_E be the metric of convergence in expectation. Recall that, given conjectures K^h, consumer h's utility for the consumption plan x^h, purchases φ^h, sales ψ^h, and delivery plans D^h is:

36. As we have noted before, our rationality requirement is weaker than that of Dubey, Geanakoplos and Shubik (1988) or Dubey and Geanakoplos (1989), so we allow for more default equilibria. In our framework therefore, that all default equilibria are close to Walrasian equilibria is a stronger assertion.

$$W^h(x^h,\varphi^h,\psi^h,D^h) = U^h(x) - \int \lambda \{\sum \psi^h(n)A_n(\omega) - D^h(n,\omega)\}\, d\mu^h$$

THEOREM 4: For every $\varepsilon > 0$ there is a $\lambda(\varepsilon)$ and a function $N(\varepsilon,\cdot) : (\lambda(\varepsilon),\infty) \to (0,\infty)$ such that:

If $\lambda > \lambda(\varepsilon)$ and $(\bar{q},(K^h,x^h,\varphi^h,\psi^h,D^h))$ is a default equilibrium of $\mathcal{E}^{N(\varepsilon,\lambda),\lambda}$, then there is a Walrasian equilibrium allocation y of $\mathcal{E}^{CM}$ such that, for each h,

$$|d_E(x^h,y^h)| < \varepsilon \quad \text{and}$$

$$|W^h(x^h,\varphi^h,\psi^h,D^h) - U^h(y^h)| < \varepsilon$$

Thus, default equilibria are close to Walrasian equilibria, provided the default penalty and number of securities are sufficiently large. If we view the underlying complete markets economy as the limit of incomplete markets economies, then this result establishes - in the presence of default - upper hemi-continuity of the equilibrium correspondences. (Section 3 shows that, in the absence of default, upper hem-continuity fails to hold.) It is natural to ask about lower hemi-continuity too: are all Walrasian equilibria approximated by default equilibria? For default ε-equilibria, the answer is yes.[37] Write

$$\lambda_0 = \max\{ u_h'(0) : h = 1,2,\ldots H\}$$

As we have noted in Section 4, if $\lambda > \lambda_0$ there will be no voluntary default.

THEOREM 5: For every $\varepsilon > 0$ and every $\lambda > \lambda_0$ there is an index N' such that:

If y is a Walrasian equilibrium allocation of $\mathcal{E}^{CM}$ and $N > N'$ then there is a default ε-equilibrium $(q,(K^h, x^h, \varphi^h,\psi^h,D^h))$ of $\mathcal{E}^{N',\lambda}$ such that, for each h,

37. This is the most we should expect; exact equilibrium correspondences are usually not lower hemi-continuous.

$$|d_E(x^h, y^h)| < \varepsilon \quad \text{and}$$

$$|W^h(x^h, \varphi^h, \psi^h, D^h) - U^h(y)| < \varepsilon$$ [38]

Note that the conclusions of Theorems 4 and 5 are *not* precisely parallel. In order to assure that all default equilibrium allocations are close to Walrasian equilibrium allocations, the default penalty might need to be extremely large. However, in order to assure that each Walrasian equilibrium allocation is close to a default ε-equilibrium allocation, the default penalty will only need to be sufficiently large to discourage *voluntary* default.

A final point. In the proof of Theorem 4 we show that, if number of securities and the default penalty are sufficiently large, then the probability of default and the expected magnitude of default are both small; in particular, there is no default in most states of the world. However, we have nothing to say about the magnitude (or fraction) of default, *conditional* on the event that default actually occurs. In particular, we do not rule out the possibility that when default occurs it is total: no deliveries at all are made.

38. The ε-equilibria we construct have the property that all available securities are traded, so the rationality requirement does not enter.

6. PROOFS

We first prove that security market equilibria exist; in our one commodity world, the argument is quite simple.[39]

PROOF OF THEOREM 1: There is no loss of generality in assuming that the securities A_n have linearly independent returns (since redundant securities can be priced by arbitrage). We reduce the existence of a security market equilibrium to the existence of a Walrasian equilibrium for a shadow economy in which the commodity bundles represent date 0 consumption and portfolios of securities.

This Walrasian shadow economy is defined in the following way. The commodity space for the shadow market is $\mathbb{R} \times \mathbb{R}^N$ (where N is the number of securities). For $(t,\theta) \in \mathbb{R} \times \mathbb{R}^N$, we interpret t as date 0 consumption and θ as a portfolio (not necessarily non-negative) of securities. The consumption set for consumer h is the set X^h of pairs $(t^h,\theta^h) \in \mathbb{R} \times \mathbb{R}^N$ such that $e^h + (t^h, \text{returns}(\theta^h)) \geq 0$; it is easily seen that X^h is a closed convex subset of $\mathbb{R} \times \mathbb{R}^N$, and is bounded below (because security returns are linearly independent.) The utility function $V^h : X^h \to \mathbb{R}$ of consumer h is defined by

$$V^h(t^h,\theta^h) = U^h(e^h + (t^h, \text{returns}(\theta^h))$$

It is easily seen that V^h is continuous, quasi-concave, and strictly monotone (because security returns are non-negative). Finally, consumer h's endowment is $E^h = (e^h(0),0) \in \mathbb{R} \times \mathbb{R}^N$. (Keep in mind that securities are in zero net supply.) Our assumption that security returns are bounded and that endowments are bounded away from 0 guarantee that E^h belongs to the interior of X^h.

It follows that this Walrasian economy has an equilibrium $(q, (\bar{t}^h,\bar{\theta}^h))$. Write φ^h for the positive part of $\bar{\theta}^h$ and ψ^h for the negative part of $\bar{\theta}^h$, and set

39. See also Zame (1988) and Green and Spear (1989). Unlike the arguments given in those papers, the present argument has the virtue that it remains valid with a continuum of states (maintaining the assumption of a single commodity). With a continuum of states and several commodities, existence of equilibrium is problematical; the case of separable preferences is addressed by Hellwig (in progress) and Mas-Colell and Zame (in progress).

$x^h = e^h + (\bar{t}^h, \text{returns}(\bar{\theta}^h))$. It is easily checked that the 4-tuple $(q,(x^h),(\varphi^h),(\psi^h))$ is an equilibrium for the security market $\mathcal{E}$, as desired. ∎

We note that this construction provides an equivalence between the equilibria of the security market $\mathcal{E}$ and the equilibria of the Walrasian shadow economy. It follows that equilibria of the security market $\mathcal{E}$ are *constrained optimal* , in the sense of being Pareto undominated by any allocation attainable by trading date 0 consumption and available securities. This is an observation first made by Diamond (1967), in the context of a finite state space.

Before embarking on the proof of Theorem 2, it is convenient to collect some notation and isolate two portions of the argument as lemmas. If $e \in E$ is an endowment vector, recall that $PO(e)$ is the set of Pareto optimal allocations from e . For each state ω , set $E_\omega = \{e \in E :$ for every $(x^h) \in PO(e)$, there is an h such that $|x^h(\omega) - e^h(\omega)| > 1/3$ or $|x^h(\omega+1) - e^h(\omega+1)| > 1/3 \}$. Write $E_0 = \bigcup E_\omega$. The first lemma establishes that, for almost all endowments, every Pareto optimal allocation entails at least one large net trade.

LEMMA A: E_0 is a dense open subset of E .

PROOF: That each E_ω , and hence E_0 , is open follows directly from the definition and the fact that the Pareto correspondence is compact-valued and upper hemi-continuous. To see that E_0 is dense, fix an endowment vector $e \in E$ and a state ω ; define new endowment vectors $\tilde{e}$ and $\hat{e}$ by:

$$\tilde{e}^h(\tau) = \hat{e}^h(\tau) = e^h(\tau) \qquad \tau < \omega$$

$$\tilde{e}^1(\omega) = \tilde{e}^2(\omega+1) = 1$$

$$\hat{e}^2(\omega) = \hat{e}^1(\omega+1) = 1$$

$$\tilde{e}^h(\tau) = 1/9H \ , \ \hat{e}^h(\tau) = 1/9H \qquad \text{otherwise}$$

The endowment vectors $\tilde{e}$ and $\hat{e}$ represent different distributions of the same aggregate, so $PO(\tilde{e}) = PO(\hat{e})$. Suppose that neither $\tilde{e}$ or $\hat{e}$ belong to E_ω. Then we could find Pareto optimal allocations $(\tilde{x}^h)$, $(\hat{x}^h) \in PO(\tilde{e}) = PO(\hat{e})$ satisfying all the following inequalities:

$$\tilde{x}^1(\omega) > 2/3 \quad , \quad \tilde{x}^1(\omega+1) < 1/2$$

$$\tilde{x}^2(\omega) < 1/2 \quad , \quad \tilde{x}^2(\omega) > 2/3$$

$$\hat{x}^1(\omega) < 1/2 \quad , \quad \hat{x}^1(\omega+1) > 2/3$$

$$\hat{x}^2(\omega) > 2/3 \quad , \quad \hat{x}^2(\omega) < 1/2$$

We now use separability of preferences to compare marginal rates of intertemporal substitution at the allocations $(\tilde{x}^h)$ and $(\hat{x}^h)$: consumer 1's increases and consumer 2's decreases. Since marginal rates of substitution are equal at a Pareto optimal allocation, this means that $(\tilde{x}^h)$ and $(\hat{x}^h)$ cannot both be Pareto optimal allocations. We conclude that at least one of $\tilde{e}$, $\hat{e}$ belongs to E_ω. By choosing ω sufficiently large, we can make $\tilde{e}$ and $\hat{e}$ as close to e as we like, so it follows that E_0 is dense in E, as asserted. ∎

For $x, y \in C^1$, write $\|x\|_\infty = \sup(x(\omega))$, and $d_\infty(x,y) = \|x-y\|_\infty$. If I is a set of indices and $x \in C^1$, write $x[I]$ for the vector that agrees with x on I and is 0 elsewhere. We abbreviate $x[\{1,\ldots,n\}]$ by $x[n]$ and $x[I \cap \{1,\ldots,n\}]$ by $x[I[n]]$. If $A = \{A_n\}$ is a sequence of securities, we write $A[I] = \{A_n[I]\}$. If $A = \{A_n\}$ is a sequence of securities, write $\operatorname{span} A$ for the linear subspace of C^1 spanned by A; i.e., the set of finite linear combinations of the securities A_n, or equivalently, the set of returns on finite portfolios of elements of A. Write $\mathcal{F}$ for the set of sequences in C^1 that are 0 from some point on. If $v \in \mathcal{F}$, write $Q_I(v)$ for the set of security sequences A such that $d_\infty(v[I], \operatorname{span} A[I]) \geq (1/2)\|v[I]\|_\infty$; set $Q_I = \bigcap Q_I(v)$, the intersection extending over all $v \in \mathcal{F}$. The following lemma is closely related to a result in Zame (1988).

LEMMA B: If $I \subset \{1,2,\ldots\}$ is an infinite set, then Q_I is a residual subset of S.

PROOF: We show first that $Q_I[v]$ is residual for each v. To this end, let $v \in \mathcal{F}$; if $v[I] = 0$ there is nothing to prove, so assume $v[I] \neq 0$. Fix integers m, r such that $v(\omega) = 0$ for $\omega > r$, and let k be the first integer such that $I \cap \{1,2,\ldots k\}$ has r elements. Write $\rho = (1/2)(1-2^{-m})$, and let $Q_I(v,r,m)$ be the set of security sequences A such that

(i) $\qquad A_1[I[k]],\ldots A_r[I[k]]$ are linearly independent

(ii) $\qquad d_\infty(v[I], \operatorname{span}(A_1[I],\ldots A_r[I]) > \rho \| v[I] \|_\infty$.

It is evident that $Q_I[v] = \bigcap Q_I(v,r,m)$, so to show that $Q_I[v]$ is residual, it suffices to show that each $Q_I(v,r,m)$ is a dense open set.

Note first that (i) is equivalent to the non-vanishing of a $k \times k$ determinant, and so remains valid if we replace A by any perturbation that is small in the first k states and for the first k terms of the sequence. Hence the set of sequences satisfying (1) is dense and open. Note next that if (2) is satisfied there is a state s such that

(3) $\qquad d_\infty(v[I[s]], \operatorname{span}(A_1[I[s]],\ldots A_r[I[s]]) > \rho \| v[I] \|_\infty$

Since the vectors $v[I[s]]$, $A_n[I[s]]$ all lie in a finite dimensional space, a simple continuity argument shows that (3) remains valid if we replace A by any perturbation that is sufficiently small in the first s states and for the first r terms of the sequence. Hence the set of security sequences satisfying condition (2) is open.

To see that the set of security sequences satisfying condition (2) is dense, fix a sequence A; we must find arbitrarily small perturbations of A satisfying (ii). Since the set of sequences satisfying (i) is dense, there is no loss of generality in assuming that A already satisfies (i). We may also assume without loss that there is a state $s > r$ such that $A_i(t) = 0$ for $t \geq s$. Consider the returns operator $\mathbb{R}^r \to C^1$ defined by $\theta \to \sum \theta_n A_n[I]$; since $A_1[I],\ldots A_r[I]$ is a linearly independent set, this transformation is an isomorphism of $\mathbb{R}^r$ with a finite dimensional subspace of C^1. Let Θ be the set of portfolios $\theta \in \mathbb{R}^r$ such that

$$d_\infty(\sum \theta_n A_n[I], v[I]) \le \rho \| v[I] \|_\infty$$

Since the returns operator is an isomorphism, Θ is a compact set.

Fix $\theta \in \Theta$, and choose *any* state $t > s$; define a perturbation A' of A by $A_n'(\omega) = A_n(\omega)$ for $\omega \neq t$ and $A_n'(t) = 1$. The returns of each security A_i are non-negative and bounded by 1, and $\rho < 1/2$, so $|\sum \theta_i| > \rho \| v \|_\infty$. Since $v(\omega) = 0$ for $\omega > r$, it follows that

$$(*) \qquad d_\infty(\sum \theta_n A_n'[I], v[I]) > \rho \| v[I] \|_\infty$$

Continuity implies that (*) remains valid if we replace θ by any portfolio θ' in some neighborhood W_θ of θ; moreover, the neighborhood W_θ may be chosen independently of the choice of the state t. Since Θ is compact, we can cover it with a finite number of these neighborhoods, and make perturbations in different states to achieve a security sequence A'' such that

$$(**) \qquad d_\infty(\sum \theta_n A_n''[I], v[I]) > \rho \| v[I] \|_\infty$$

for all $\theta \in \Theta$. Since we have made perturbations only in states where v and each A_i vanish, we conclude that (**) holds for all $\theta \in \mathbb{R}^r$. since we can make these perturbations in states with s arbitrarily large, these perturbations are arbitrarily small. Hence $Q_I(v,r,m)$ is a dense set; as we have noted, this implies that $Q_i(v)$ is residual.

To see that Q_I is residual, observe that, if $v, v' \in \mathcal{F}$ then

$$d_\infty(v', \operatorname{span} A) > d_\infty(v, \operatorname{span} A) - d_\infty(v, v')$$

Hence, if $A \in Q_I(v)$ for each $v \in \mathcal{F}$ having only rational entries, then in fact $A \in Q_I(v)$ for every $v \in \mathcal{F}$. Since the subset of $\mathcal{F}$ consisting of vectors with only rational entries is countable, it follows that Q_I may in fact be written as the intersection of countably many residual sets, and is hence residual. ■

With these technical results in hand, we turn to the proof of Theorem 2.

PROOF OF THEOREM 2: (a) See the second paragraph of the proof of Lemma B.

(b) For each j let $\delta_j \in C^1$ be the sequence whose j-th entry is 1 and all of whose other entries are 0. For each j, k write $\mathcal{A}^{j,k} = \{A : d_E(\delta_j, \text{span} A) < 1/k\}$. Evidently, every sequence in $\cap \mathcal{A}^{j,k}$ spans all the uncertainty, so it suffices to show that each $\mathcal{A}^{j,k}$ is a dense open set. To this end, observe first that every vector in $\text{span} A$ is a linear combination of a *finite* number of securities. If $d_E(\delta_j, \sum \alpha_i A_i) < 1/k$, then $d_E(\delta_j, \sum \alpha_i \overline{A}_i) < 1/k$ provided that $d_E(A_i, \overline{A}_i)$ is small enough (for each i). Hence $\mathcal{A}^{j,k}$ is an open set. To see that it is dense, fix any sequence of securities A. For each i, let A^i be the sequence which is identical to A except for the i-th security, with $A^i_i = \delta_j$. Since $A^i \in \mathcal{A}^{j,k}$ and $A^i \to A$ in the product topology, we conclude that $\mathcal{A}^{j,k}$ is dense, as desired.

(c) For each ω, let $E^\omega = \{e \in E : \text{for some } \tau > \omega,\ e^h(\tau) < 1/9H \text{ for each } h\}$; set $E^0 = \cap E^\omega = \{e \in E : \text{for infinitely many } \tau,\ e^h(\tau) < 1/9H \text{ for each } h\}$. It is easily seen (by arguments such as those above) that each E^ω is a dense open set, so E^0 is a residual set. Hence $E' = E_0 \cap E^0$ is also a residual set.

Fix $e \in E'$; say that $e \in E_\omega$. Set $I = \{\omega, \omega+1\} \cup \{\tau : e^h(\tau) < 1/9H \text{ for each } h\}$; this is an infinite set of states. According to Lemma B, the set Q_I of security sequences is residual, so it suffices to prove that every sequence $A \in Q_I$ is asymptotically inefficient for e. Since $e \in E_\omega$, every Pareto optimal allocation requires a net trade of at least $1/3$, either in state ω or in state $\omega+1$. Hence every allocation which is close to a Pareto optimal allocation requires a net trade of at least $1/4$, either in state ω or in state $\omega+1$. We claim that no such allocation (x^h) cannot be obtained by trading a finite number of the securities A_n.

To see this, write $z^h = x^h - e^h$ for the net trade of consumer h. For the sake of definiteness, assume that consumer 1's net trade in state ω is large: $|z^1(\omega)| = |x^1(\omega) - e^1(\omega)| \geq 1/4$. If (x^h) can be obtained by trading n securities, there is a profile of portfolios $\theta^h \in \mathbb{R}^n$ such that $z^h = \text{returns}(\theta^h) \in \text{span} A$ for each h. Set $v = z^1[\omega]$. Since $A \in Q_I$

$$d_\infty(v[I], \operatorname{span} A[I]) \geq (1/2)\|v[I]\|_\infty \geq |z^1(\omega)| \geq 1/4$$

and hence $d_\infty(v[I], z^1[I]) \geq (1/8)$. Since $v(\omega) = z^1(\omega)$, there is a state $\tau \in I$, $\tau \neq \omega$, such that $|z^1(\tau)| \geq 1/8$. On the other hand, if $\tau \in I$ then $e^h(\tau) \leq 1/9H$; since $x^h(\tau) \geq 0$ for each h and $\sum z^h = 0$, this entails $|z^1(\tau)| \leq 1/9$. This contradiction shows that (x^h) cannot be implemented by trading a finite number of the securities A_n, and completes the proof. ∎

PROOF OF THEOREM 3: We have already noted that, if $\lambda = 0$, autarky is an equilibrium, and if $\lambda = \infty$ the default model reduces to the usual security market model, so it suffices to treat the case $0 < \lambda < \infty$. We construct a default equilibrium for the security market $\mathcal{E}$ as the limit of equilibria in security markets with a finite number of states.

For each index r, define a security market $\mathcal{E}[r]$ by truncating all the data of $\mathcal{E}$ to the first r states (i.e., endowments $e^h[r]$ defined as above by $e^h[r](\omega) = e^h(\omega)$ if $\omega \leq r$, $e^h[r](\omega) = 0$ if $\omega > r$, etc.). It follows from work of Dubey, Geanakoplos and Shubik (1988) and Dubey and Geanakoplos (1989) that $\mathcal{E}[r]$ has a default equilibrium $(q[r],((K^h[r],x^h[r],\varphi^h[r],\psi^h[r],D^h[r])))$. We claim that some subsequence of these equilibria converge to an equilibrium for $\mathcal{E}$.

To demonstrate this, we need to show first that the components of equilibria lie in compact sets. Note that boundedness of the set of feasible date 0 consumptions implies that marginal utilities for date 0 consumption (evaluated at feasible consumption bundles) are bounded away from 0. By assumption, marginal utilities for date 1 consumption are bounded above. It follows that the security prices $q[r]$ are bounded (independently of r), and hence lie in a compact subset of $\mathbb{R}^N$. To see that portfolios lie in a compact set, note first that, since the collection of securities is finite, there is an index r_0 with the property that each of the securities yields a strictly positive return in at least one state $\omega \leq r_0$, and that the truncations $A_1[r_0], \ldots, A_N[r_0]$ are linearly independent. If the portfolios of sales $\psi^h[r]$ were not bounded (independently of r), there would be at least one state $\omega \leq r_0$ in which liabilities were unbounded. Since aggregate consumption is finite in each state, there would be at least one state $\omega \leq r_0$

in which default would be unbounded. Since default penalties become unbounded with unbounded defaults, such actions would be incompatible with individual rationality, and hence with equilibrium. It follows that portfolios of sales $\psi^h[r]$ are bounded; since securities are in 0 net supply, portfolios of purchases $\varphi^h[r]$ are also bounded. Hence, portfolios of purchases and sales lie in a compact subset of $\mathbb{R}^N$. It follows that promises, and hence deliveries $D^h[r]$ lie in a compact subset of C; we have already noted that consumption plans lie in a compact subset of C. Of course conjectures also lie in a compact set. Passing to a subsequence if necessary, we see that equilibria of $\mathcal{E}[r]$ converge to some tuple $(q,((K^h,x^h,\varphi^h,\psi^h,D^h)))$, which we assert is an equilibrium of $\mathcal{E}$.

With the exception of individual optimization, verification of the equilibrium conditions is straightforward, and left to the reader. To verify individual optimization, suppose that $\tilde{p}^h = (\tilde{x}^h,\tilde{\varphi}^h,\tilde{\psi}^h,\tilde{D}^h)$ is an alternative plan for consumer h, which is feasible and superior to the equilibrium plan $p^h = (x^h,\varphi^h,\psi^h,D^h)$ (given endowment e^h, conjectures K^h, and security prices q); we find an index s and a plan $\bar{p}^h$ in the economy $\mathcal{E}[s]$ which is superior to consumer h's equilibrium plan $p^h[s]$. To this end, let $\alpha, \beta, \delta > 0$ be parameters (to be chosen later) and let s be an integer (also to be chosen later). Define the plan $\bar{p}^h = (\bar{x}^h,\bar{\varphi}^h,\bar{\psi}^h,\bar{D}^h)$ in the following way:

$$
\begin{aligned}
\bar{x}^h(\omega) &= \max(\tilde{x}^h(\omega) - \alpha, 0) && \text{for } 0 \le \omega \le s \\
&= 0 && \text{for } \omega > s \\
\bar{\varphi} &= (1-\beta)\tilde{\varphi}^h && \\
\bar{\psi} &= \tilde{\psi} && \\
\bar{D}^h(n,\omega) &= \max(\tilde{D}^h(n,\omega) - \delta, 0) && \text{for } 1 \le \omega \le s \\
&= 0 && \text{for } \omega > s
\end{aligned}
$$

If we choose s sufficiently large and α, δ sufficiently small, the plan $\bar{p}^h$ achieves almost as much utility as $\tilde{p}^h$ and in particular achieves more utility than p^h. (Note that the utility achieved by a plan depends on consumption, on sales, and on deliveries, but not on purchases or on conjectures.) Since conjectures $K^h[s]$ converge to conjectures $\tilde{K}^h$, if we choose β sufficiently small and s sufficiently large, the plan $\bar{p}^h$ will be feasible (i.e., meet the non-negativity constraints) in the economy $\mathcal{E}[s]$. Since prices $q[s]$ converge to $\tilde{q}$, if we choose s sufficiently large, the plan $\bar{p}^h$ will be budget feasible in $\mathcal{E}[s]$. Finally, if we choose s sufficiently large, the plans $p^h[s]$ and p^h achieve almost the same utility, so the plan $\bar{p}^h$ will achieve more utility than $p^h[s]$. Since this contradicts the equilibrium conditions for the economy $\mathcal{E}[s]$, we conclude that $\tilde{p}^h$ cannot be superior to p^h, so that $(q,((K^h,x^h,\varphi^h,\psi^h,D^h)))$ is an equilibrium for $\mathcal{E}$, as desired. ∎

PROOF OF THEOREM 4: For each λ, N consider a default equilibrium $\eta(N,\lambda)$ of $\mathcal{E}^{N,\lambda}$; let $x(N,\lambda)$ be the vector of consumption plans. Passing to subsequences if necessary, we assume that $x(N,\lambda) \to x(\lambda)$ (as $N \to \infty$), and that $x(\lambda) \to x$ (as $\lambda \to \infty$). The desired result follows if we can show that (for all choices just made) x is a Walrasian equilibrium of $\mathcal{E}^{CM}$ and $W^h(\eta^h(N,\lambda)) \to U^h(x^h)$ for each h.

We show first that x is in the core of the economy $\mathcal{E}^{CM}$; i.e., that no group of consumers can improve on x using only their own resources. To see this, suppose not. Then there is a set of consumers (whom we may suppose to be $\{1,\dots,K\}$) and a vector y of consumption plans that is feasible for the group $\{1,\dots,K\}$ such that $U^h(y^h) > U^h(x)$ for each $h \le K$. Let $\alpha, \beta, \gamma, \delta > 0$ be real parameters and let r be an integer (all to be chosen later). Define $\bar{y}$ by $\bar{y}^h(\omega) = \max\{x^h(\omega) - \alpha, \alpha\}$ for $\omega \le r$ and $\bar{y}^h(\omega) = 0$ for $\omega > r$; define $\bar{e}^h$ by $\bar{e}^h(\omega) = e^h(\omega)$ for $\omega \le r$ and $\bar{e}^h(\omega) = 0$ for $\omega > r$. If α is sufficiently small and r is sufficiently large, then $\bar{y}$ is feasible for the group $\{1,\dots,K\}$ and $U^h(\bar{y}^h) > U^h(x)$ for each $h \le K$. Let $\hat{y}$ be the restriction of y to Ω.

Now let λ^* be any default penalty so large that, if $\lambda \ge \lambda^*$, and N is arbitrary, then in the security market $\mathcal{E}^{N,\lambda}$ there is no default in states $\omega \le r$. For $1 \le h \le H-1$, we may use the fact that the securities $\{A_n\}$ span all the uncertainty to choose a finite portfolio θ^h such that:

$$|e^h(\omega) + \text{returns}(\theta^h)(\omega) - y^h(\omega)| < \alpha/K \text{ for } 1 \le \omega \le r$$

$$\| \bar{e}^h + \text{returns}(\theta^h) - \hat{y}^h \|_E < \beta/K$$

Set

$$\theta^K = - \sum_{h=1}^{K-1} \theta^h$$

Our construction guarantees that $\sum \varphi^h = \sum \psi^h$, and that

$$| e^K(\omega) + \text{returns}(\theta^K) - y^K(\omega) | < \alpha \text{ for } 1 \le \omega \le r$$

$$\| \bar{e}^K + \text{returns}(\theta^K) - \hat{y}^K \|_E < \beta$$

We now define plans $(z^h, \varphi^h, \psi^h, D^h)$ (for the security markets $\mathcal{E}^{N,\lambda}$ as follows. Consumption plans z^h are given by $z^h(0) = y^h(0)$, $z^h(\omega) = e^h(\omega) + \text{returns}(\theta^h)(\omega)$ for $1 \le \omega \le r$, and $z^h(\omega) = 0$ for $\omega > r$. Portfolios φ^h , ψ^h of purchases and sales to defined to be the positive and negative parts of θ^h , respectively. Plans of delivery D^h are defined by $D^h(n,\omega) = \psi^h(n)A_n(\omega)$ for $\omega \le r$, $D^h(n,\omega) = 0$ for $\omega > 0$. Our choice of λ guarantees that there is no conjectured default in states $\omega \le r$; it follows that these plans are feasible (i.e., satisfy the non-negativity constraints). If α is small enough then $U^h(z^h) > U^h(x^h)$ for $h \le K$. If β is small enough, these plans all incur very small default penalties; in particular, if β is small enough, $W^h(z^h, \varphi^h, \psi^h, D^h) > U^h(x^h)$ for $h \le K$. On the other hand, continuity of utility functions implies that $U^h(x^h(N,\lambda)) \to U^h(x^h(\lambda))$ as $N \to \infty$ and $U^h(x^h(\lambda)) \to U^h(x^h)$ as $\lambda \to \infty$. It follows that, if λ and N are sufficiently large then $W^h(z^h, \varphi^h, \psi^h, D^h) > U^h(x^h(N,\lambda))$. Since $U^h(x^h(N,\lambda))$- $\ge W^h(\eta^h(N,\lambda))$, we conclude that $W^h(z^h, \varphi^h, \psi^h, D^h) > W^h(\eta^h(N,\lambda))$, provided that N and λ are sufficiently large. This contradicts individual optimization at equilibrium, so we conclude that x is in the core of $\mathcal{E}^{CM}$, as asserted.

The same argument shows that (the replication of) x belongs to the core of every replication of the economy $\mathcal{E}^{CM}$. By a result of Aliprantis, Brown and Burkinshaw (1987), which is the infinite dimensional version of the Debreu and Scarf (1963) core convergence theorem, it follows that x is a Walrasian equilibrium allocation of $\mathcal{E}^{CM}$.

It remains to show that $W^h(\eta^h(N,\lambda)) \to U^h(x^h)$ for each consumer h . If not we could (passing to subsequences if necessary) find a a consumer (say consumer 1) and a $\zeta > 0$ such that $W^1(\eta^1(N,\lambda)) \le U^1(x^1) - (\zeta/2)$ for N, λ sufficiently large. We may then find a sufficiently large index r and a feasible profile of consumption plans (y^h) such that:

$$y^h(\omega) = 0 \qquad \text{for } \omega > r \text{, all } h$$

$$U^h(y^h) > U^h(x^h) \ge W^h(\eta^h(N,\lambda)) \qquad \text{for } h \ne 1 \text{, all } N, \lambda$$

$$U^1(y^1) > W^1(\eta^1(N,\lambda)) \qquad \text{for all } N, \lambda$$

Let $\alpha, \beta > 0$ be parameters. Using the same ideas as above, we may construct portfolios φ^h, ψ^h such that:

$$| e^h(\omega) + \text{returns}(\varphi^h)(\omega) - \text{returns}(\psi^h)(\omega) - y^h(\omega) | < \alpha \quad \text{for } 1 \le \omega \le r$$

$$\| \overline{e}^h + \text{returns}(\varphi^h) - \text{returns}(\psi^h) - \hat{y}^h \|_E < \beta$$

and $\sum(\varphi^h - \psi^h) = 0$. If λ is sufficiently large then there will be no default in states $\omega \le r$. If β is sufficiently small the penalty for default in all other states will also be small. Hence, if we choose α sufficiently small we may construct, just as above, a collection of plans that are superior to the equilibrium plans in $\mathcal{E}^{N,\lambda}$ (provided N is sufficiently large) and have the property that at least one of them is budget feasible. This is a contradiction, so we conclude that $W^h(\eta^h(N,\lambda)) \to U^h(x^h)$, as desired. This completes the proof. ∎

PROOF OF THEOREM 5: Fix $\varepsilon > 0$, the default penalty $\lambda > \lambda_0$, and a Walrasian equilibrium $(\pi, (x^h))$; there is no loss in assuming that the price of date 0 consumption is 1 . Let $\alpha, \beta, \gamma > 0$ be parameters, and let r be an integer (all to be chosen later). Define consumption plans $\overline{x}^h$ by:

$$\overline{x}^h(\omega) = \max\{x^h(\omega) - 2\alpha, 0\} \qquad 0 \le \omega \le r$$

$$= 0 \qquad \omega > r$$

As in the proof of Theorem 4, we may choose a collection $\{\varphi^h, \psi^h\}$ of finite portfolios of the securities $\{A_n\}$ such that:

$$0 < e^h(\omega) + \text{returns}(\varphi^h)(\omega) - \text{returns}(\psi^h)(\omega) < \bar{x}^h(\omega) + \alpha$$
$$(\text{for } \omega \le r)$$

$$\| \text{returns}(\varphi^h) - \text{returns}(\psi^h) - \bar{x}^h \| < \beta$$

$$\sum (\varphi^h - \psi^h) = 0$$

If we choose r sufficiently large and α sufficiently small then $d_E(x^h, \bar{x}^h) < \varepsilon/2$ and $|U^h(x^h) - U^h(\bar{x}^h)| < \varepsilon/4$.

The portfolios $\{\varphi^h, \psi^h\}$ involve only finitely many securities, say $A_1, \ldots, A_{N_0}$. For $N \ge N_0$, let $\mathcal{E}^{N,\lambda}$ be the security market in which the securities $A_1, \ldots, A_N$ are available for trade and the default penalty is λ ; we construct an ε-equilibrium $(q, \{(K^h, x^h, \varphi^h, \psi^h, D^h)\})$ for $\mathcal{E}^{N,\lambda}$. Define security prices q^N by $q^N(n) = q \cdot A_n$. Let consumption plans $\bar{x}^h$ be as constructed above. For securities $A_1, \ldots, A_{N_1}$, conjectures are that there is no default in states $\omega \le r$ and complete default in states $\omega > r$; for securities A_n with $n > N_0$, conjectures are that default is total in all states. For consumers $1, 2, \ldots, H-1$, security portfolios φ^h , ψ^h are as above; we require that consumer H purchase (in addition to the portfolio φ^H above) *and* sell (in addition to the portfolio ψ^H above) a small quantity γ of each security A_n with $n > N_0$. Finally, we arrange deliveries consistent with no default on securities $A_1, \ldots, A_{N_1}$ in states $\omega \le r$, total default on securities $A_1, \ldots, A_{N_1}$ in states $\omega > r$, and total default in all states on securities A_n with $n > N_0$. These plans are feasible. If β is sufficiently small, then default penalties for each consumer do not exceed $\varepsilon/2$. Using this fact, and keeping in mind the conjectures and that $\lambda > \lambda_0$, it is easily checked that $(q, \{(K^h, \bar{x}^h, \varphi^h, \psi^h, D^h)\})$ is an ε-equilibrium). Moreover, the fact that default penalties for each consumer do not exceed $\varepsilon/2$ implies that

$$|W^h(\bar{x}^h, \varphi^h, \psi^h, D^h) - U^h(x^h)| < \varepsilon/2$$

What we have accomplished is not quite what was called for, since we have chosen the index N_0 in a way that depends on the particular Walrasian equilibrium $(\pi, (x^h))$, while the statement of Theorem 5 calls on us to choose N_0 in a way that depends only on λ and ε, but this is easily arranged. The ε-equilibrium we have constructed is close to the Walrasian allocation (x^h) and hence close to every Walrasian equilibrium allocation $(\bar{x}^h)$ close to (x^h). Since the set of Walrasian equilibrium allocations is compact, uniformity follows by an obvious compactness argument. ∎

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