

**THREE APPROACHES TO BARGAINING  
IN NTU GAMES**

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# THREE APPROACHES TO BARGAINING IN NTU GAMES<sup>1</sup>

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*This paper presents a noncooperative model of bargaining in characteristic function games and relates its outcomes to those of a cooperative model and a bargaining theory model. Despite the differences in the approach of these three models and the resulting differences in the nature of their solutions, all three models make similar predictions of bargaining outcomes.*

## 1. INTRODUCTION

In many situations in Economics, there are gains from cooperation and conflict over how these gains should be shared. Examples include trade in an exchange economy, firm formation and profit sharing in a production economy, and formation of jurisdictions and production of public goods in a local public good economy. For each of these situations we would like to predict the coalitions that are likely to form and the rewards that agents are likely to receive for their participation.

These and many other situations can be modeled as characteristic function games with sidepayments (TU games) or without sidepayments (NTU games). An NTU game specifies for each potential coalition the set of attainable utility vectors available to it. Since each coalition can select only one utility vector from this set, there is conflict over which utility vector will be chosen. In the situations we consider, each agent can participate in at most one coalition "at a time", so participation in a particular coalition entails a cost: the foregone opportunity to partici-

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pate in another coalition. Hence, players in each coalition must bargain over the distribution of gains within the coalition and one important element in the bargaining will be the players' opportunities in other coalitions.

This paper presents a noncooperative model and an overview of two other models of bargaining in NTU games, a cooperative model and a "bargaining theory" model. These models represent three very different ideas about how to model bargaining in NTU games. Despite the differences in their approach and the resulting differences in the very nature of what is considered a solution, all three models make similar predictions of the bargaining outcomes.

The *aspiration model* takes a cooperative game theory approach. The focus here is on the problem each player faces in setting an appropriate reservation price for his coalitional participation. This approach assumes that each player sets a price for his coalitional participation and that the bargaining that takes place within coalitions is bargaining over the "reasonableness" of these prices. A solution in the aspiration model is a price vector that satisfies certain bargaining criteria.

The *multilateral bargaining model* takes a bargaining theory approach. The focus here is on the problem each potential coalition faces in deciding on the distribution of payoff within the coalition. This approach assumes that each coalition, *given* a disagreement vector, bargains systematically to an agreement utility vector; and that *given* agreements in all other coalitions, each coalition systematically determines its disagreement vector. A solution in the multilateral bargaining problem is a specification of an agreement payoff vector for *each* coalition which is consistent with the bargaining in *every* coalition.

The *proposal-making model* takes a noncooperative approach. The focus here is to make explicit each step in the bargaining process. The bargaining process begins when nature randomly selects the first player to have the initiative. A player with the initiative can propose a coalition, a payoff vector, and a member of the coalition to respond to the proposal (or else pass the initiative to another player). A player responding to a proposal can either accept the proposal and pass it on for consideration by another member of the coalition, or else reject the proposal and take the initiative to make an alternative proposal and select a player to respond.... A solution in the proposal-making model is a strategy for each player which is "rational" given the strategies of others and given the rules of the bargaining procedure. The rationality notion we adopt is that of stationary subgame perfection.

The solutions of each of these models make predictions about the likely coalitions and the payoff distributions that result from each char-

acteristic function game. In the aspiration model the solution is a price vector. Given such a price vector, the coalitions that are likely to form are those that can afford their players' prices; within each coalition, payoff is distributed according to their players' prices. In the multilateral bargaining model, a solution is a consistent set of agreements, only some of which are feasible. Given such agreements, the coalitions that are likely to form are the coalitions whose agreements are feasible; within each coalition, payoff is distributed according to these agreements. In the proposal-making model a solution is a strategy profile. Given such a strategy profile, the likely coalitions and payoff distributions are those corresponding to proposals that are made and accepted with positive probability.

We show that, in all three models, underlying each solution is a price vector, i.e., a specification of a reservation price (in utility terms) for each player. In the aspiration model the price vector *is* the solution. The price vectors underlying solutions of the multilateral bargaining model and the proposal-making model are less obvious. In multilateral bargaining models, we call  $p$  the *price vector* of a multilateral bargaining solution if, in every "possible" coalition, every player's agreement payoff is his price. In the proposal-making model we call  $p$  the *price vector* of a solution if, in every proposal that has positive probability of occurring, each player is paid his price.

The common feature of these models is that, despite the differences in approach and the nature of solutions, all three agree on the range of possible price vectors and possible coalitions which are the outcomes of bargaining.

The paper is organized in the following way. Following the Introduction, Section 2 provides basic notation and definitions. Sections 3 and 4 provide overviews of the cooperative and the bargaining theory approaches: Section 3 describes the aspiration model and Section 4 the multilateral bargaining model. Section 5 presents a noncooperative model of bargaining called the proposal-making model, while Section 6 characterizes its stationary subgame perfect outcomes. A discussion of related literature in Section 7 completes the paper.

## 2. UNIQUE OPPORTUNITIES AND NTU GAMES

### *Unique Opportunities*

In this paper we focus attention on situations in which there is a unique opportunity in the sense that, once a set of players commits themselves to forming a coalition (for a stated distribution of payoff), no additional gains from any further coalition formation remain. For example:

- after elections in parliamentary systems where no party obtains a majority of the seats, various coalitions of parties can form a government, but only one government can form,
- various consortiums of companies can launch a communications satellite, but currently there is only demand for one geostationary satellite.

In the first example, the characteristic function will reflect the fact that no two disjoint coalitions can both form a (majority) government. In the second example, however, disjoint coalitions could be profitable even though only the first coalition to form will actually reap the rewards. The limitations implied by having a "unique opportunity" are restrictions *not* on the class NTU games to be considered but rather on the class of environments to which the theory applies. The unique opportunity assumption is crucial to the description of the proposal-making model, but not for the description of the aspiration or multilateral bargaining models. However, the interpretation of their solutions would be different in the absence of the unique opportunity assumption.

#### *NTU Games*

A game in characteristic function form with nontransferable utility (an *NTU game* or *nonsidepayment game*) is a pair  $\langle N, V \rangle$  where  $N = \{1, \dots, n\}$  is a nonempty set of *players* and  $V$ , the *characteristic function*, assigns to each nonempty subset  $S$  of  $N$  (a *coalition*) a convex, compact subset  $V(S)$  of  $R_+^S$  which contains the origin and is *strongly comprehensive* (i.e., if  $x \in V(S)$ ,  $y \in R_+^S$ ,  $y_i \leq x_i$  for each  $i \in S$  and  $y \neq x$ , then  $y \in \overset{\circ}{V}(S)$ , the interior of  $V(S)$  with respect to  $R_+^S$ ). Let  $C$  denote the set of coalitions. It is convenient to 0-normalize the game by requiring  $V(\{i\}) = \{0\}$  for each player  $i \in N$ . We do not require that the game be superadditive.

If  $x, y$  are in  $R^S$  we write  $x \leq y$  if  $x_i \leq y_i$  for each  $i \in S$ ; we write  $x < y$  if  $x \leq y$  and  $x \neq y$ , and  $x \ll y$  if  $x_i < y_i$  for each  $i$ . If  $T$  is a subset of  $S$ , then by  $x^T$  we mean the restriction of  $x$  to  $T$  (thinking of vectors in  $R^S$  as functions from  $S$  to  $R$ ).

We interpret a vector in  $V(S)$  as a vector of utilities which the coalition  $S$  can achieve without the cooperation of players not in  $S$ . Strong comprehensiveness allows for free disposal of utility and means that for each feasible utility vector for a coalition, any player receiving a nonzero utility can improve the utility of all other players by sacrificing some of his own utility. That is, if  $x \in V(S)$  and there is a vector  $y \in V(S)$  such that  $y > x$  then there exists a vector  $z \in V(S)$  such that  $z \gg x$ . When  $V(S)$  is strongly comprehensive the weak and strong Pareto efficient boundaries of  $V(S)$  coincide.

### 3. THE ASPIRATION MODEL

In situations where the agreement to form a coalition is a binding commitment, members of a coalition are unlikely to make such a commitment without a parallel commitment to the choice of a utility vector (from the coalition's attainable utility vectors) which each member finds "acceptable" in light of his other opportunities in other coalitions. As Shapley and Shubik (1972, p. 116) put it,

"A prudent 'economic' man playing this game would be loath to enter a partnership for a stated share of the proceeds until he had satisfied himself that more favorable terms could not be obtained elsewhere. We can imagine that each player would set a price on his participation, and that no contracts would be signed until the prices on both sides of each partnership formed are in harmony..."

The aspiration approach to bargaining assumes that each player sets a reservation utility level (his "price") for his coalitional participation, and that the bargaining that takes place is bargaining over the "reasonableness" of these prices.

Each player sets his price with some knowledge or expectation of the prices that other players demand. We expect players to lower their prices whenever they must and raise their prices whenever they can. The aspiration approach assumes that a player lowers his prices when *no* coalition can afford to pay him and his partners their prices; and that a player in a coalition will raise his price if the coalition can afford to pay its players their prices and still have payoff leftover.

Formally, we regard vectors  $p \in \mathbb{R}_+^N$  as vectors of prices and  $p_i$  as player  $i$ 's price. We say that player  $i$ 's price is *realizable* at  $p$  if there exists a coalition  $S$  containing player  $i$  such that  $p^S \in V(S)$ . If every player's price is realizable at  $p$  we call  $p$  *realizable*. We say player  $i$ 's price is *maximal* if for every coalition  $S$  containing  $i$ , and every vector  $q^S$  in  $V(S)$  it is not the case that  $q^S > p^S$ . If every player's price is maximal at  $p$  we say that  $p$  is *maximal*. (A price vector is maximal if no player can raise his payoff demand in any coalition without making his price "unrealizable".) If a vector  $p$  is both realizable and maximal we call  $p$  an *aspiration*. Bennett and Zame [1988] prove that the set of aspirations is nonempty for every NTU game.

Realizability and maximality are minimal desiderata for "reasonable" reservation prices. Several authors have imposed additional desiderata in order to make more precise predictions of bargaining outcomes. See, for example, Albers [1974, 1980] for TU games or Bennett and Zame [1988] for NTU games.

## Outcomes

For each price vector  $p \in \mathbb{R}_+^N$ , define  $C(p)$  to be the set of coalitions that can afford to pay their players their prices, i.e.,

$$C(p) = \{ S \in C \mid p^S \in V(S) \}.$$

For each player  $i \in N$ , we define  $C_i(p)$  to be the set of coalitions in  $C(p)$  which contain  $i$ . For some values of  $p$ ,  $C(p)$  or  $C_i(p)$  may be empty; however if  $p$  is realizable then every  $C_i(p)$  (and therefore  $C(p)$ ) is nonempty. If the price vector  $p$  is maximal then each player  $i$  finds the coalitions in his  $C_i(p)$  equally desirable because he can obtain his price from any coalition in  $C_i(p)$  and, given the prices of other players, he can't obtain more. Hence, whenever  $p$  is an aspiration, all players can agree that all the "desirable" coalitions to form are those in  $C(p)$ .

The selection of a particular aspiration  $p$  is the prediction that one of the coalitions in  $C(p)$  is likely to form and that each player in the coalition is paid his price.

## 4. THE MULTILATERAL BARGAINING MODEL

The multilateral bargaining approach views an NTU game as a set of interrelated bargaining problems. This approach decomposes the factors that lead to the choice of a particular payoff vector in each coalition into internal considerations (such as the coalition's standards of fair division) and external considerations (such as the opportunities available to each of its members in other coalitions). This approach assumes that the effects of these factors can be summarized by two functions; the first takes players' outside opportunities and produces a utility vector referred to as the "outside option vector"; the second takes outside option vector, and produces an agreement utility vector. (That these effects are summarized by functions simply means that the same data always lead to the same resolution.) These functions are viewed as part of the description of the problem just as the initial endowments and utility functions are part of the description of an exchange economy.

Given an outside option vector, a coalition's bargaining function reflects the coalition's standards of fair division, the institutional rules for bargaining within the coalition, and the bargaining skills of its members. Let  $f^S : \mathbb{R}^S \rightarrow \mathbb{R}^S$ , denote the *bargaining function for S*.

We assume that the bargaining function for each coalition  $S \in C$  satisfies the following conditions:

1. For each  $S$  and each outside option vector  $d^S \in V(S)$ , the agreement vector for  $S$ ,  $x^S = f^S(d^S)$  satisfies:

- a. **Individual Rationality:**  $x^S \geq d^S$ , and
  - b. **Pareto Optimality:**  $x^S$  is on the Pareto optimal frontier<sup>2</sup> of  $V(S)$ .
2. **Continuity:** For each  $S$ , the bargaining function,  $f^S$ , is a continuous function of its outside option vector.
  3. **Agreeing to Disagree:** If for  $S$  the outside option vector  $d^S \notin V(S)$ , then the agreement vector for  $S$  is the outside option vector, i.e.,  $x^S = f^S(d^S) = d^S$ .

Given  $d^S$ , the pair  $(V(S), d^S)$  is a "bargaining problem" in the sense of Nash [1950]. For  $d^S \in V(S)$  the conditions given above are minimal requirements for solutions to the bargaining problem and  $f^S(d^S)$  has the usual interpretation as the agreement utility vector that results from bargaining within the coalition. In condition 3, we have extended each bargaining solution to allow for outside option vectors outside the attainable set because we are going to determine an outside option vector  $d^S$  for each coalition  $S$ , to reflect its members' opportunities in other coalitions. The outside option vector determined in this way will lie outside  $V(S)$  if, for members of  $S$ , the opportunities available in other coalitions are more attractive than those available in  $S$ . In this case, the members of  $S$  "agree to disagree", i.e., they agree to settle for their outside opportunities. We view the agreement to disagree as the result of bargaining; the players of  $S$  negotiate but their "final offers" aren't compatible given the resources  $S$ . In setting the agreement vector equal to the outside option vector we are assuming that, as a result of bargaining, each player would be willing to form the coalition for his component of the outside option vector if some other player would take the necessary loss to make the utility vector feasible for the coalition.<sup>3</sup> (For example, if the coalition  $[1,2]$  can divide 3 in any way it chooses and the outside option vector is  $(2,2)$ , agreeing to disagree here means that each player would be willing to form the coalition if he could obtain 2 (i.e., if the other player would accept a payoff of only 1.)

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2. Strong comprehensiveness implies that the weak and strong Pareto frontiers of each  $V(S)$  coincide.

3. The assumption that players will settle for their outside options when their outside option vector is infeasible is in keeping with the usual assumptions of solution concepts for the simple bargaining problem. Virtually every solution concept for the simple bargaining problem (e.g., the Nash bargaining solution) requires players to settle for their outside option utility levels whenever the outside option vector is on the Pareto boundary of  $V(S)$ .



We now turn to the question of how outside options are determined. In multilateral bargaining, if the players in one coalition fail to reach an agreement may enter into other coalitions. Thus, for each coalition  $S$  and for each player  $i \in S$ , we want to use as  $i$ 's component of the outside option vector,  $d^S$ , the utility he would receive if he broke off negotiations in  $S$  and took the initiative to form his best alternative coalition. Of course,  $i$ 's alternatives depend on the agreements that will be reached in other coalitions. We assume that the players in  $S$  make accurate (and therefore identical) conjectures about these agreements. To see what this implies, fix a (conjectured) agreement vector  $x^T$  for each coalition  $T \neq S$ ; given these conjectured agreement vectors what are the utilities of player  $i$ 's alternatives? If  $i \in T$  and the agreement vector  $x^T$  is attainable for the coalition  $T$ , then player  $i$  can certainly obtain  $x_i^T$  in the coalition  $T$ . However, if the agreement vector  $x^T$  is not attainable for  $T$ , player  $i$  cannot obtain as much as  $x_i^T$ . In view of our previous discussion about the meaning of "agreeing to disagree", the most that player  $i$  can obtain in  $T$  is the largest utility which allows all of the other members of  $T$  to obtain their agreement utilities. That is, the utility to player  $i$  of the unattainable agreement  $x^T$  in the coalition  $T$  is  $\max \{ t_i \mid x^T/t_i \in V(T) \}$ . (We use  $x^T/t_i$  to denote the vector obtained from  $x^T$  by replacing the  $i$ -th component by  $t_i$ ). If there is no value of  $t_i$  for which  $x^T/t_i \in V(T)$ , then the unattainable agreement  $x^T$  has no utility for player  $i$ ; by convention we agree to take 0 to be the maximum in this case.

Formally, given agreements  $\{x^T\}_{T \neq S}$  in all other coalitions, we define the *outside option vector*  $d^S(\{x^T\}_{T \neq S})$  for the coalition  $S$  in the following way. For each  $i \in S$  and each coalition  $T \neq S$  with  $i \in T$ , set:

$$u_i^T(x^T) = \begin{cases} x_i^T, & \text{if } x^T \in V(T) \\ \max \{ 0, t_i \mid x^T/t_i \in V(T) \}, & \text{otherwise.} \end{cases}$$

and

$$d_i^S(x) = \max \{ u_i^T(x^T) \mid i \in T \text{ and } T \neq S \}.$$

By definition,  $d^S$  is a function from collections  $\{x^T\}_{T \neq S}$  to  $R_+^S$ ; however it is convenient to view  $d^S$  as a function defined over collections of agreement vectors for *all* coalitions (although  $d^S$  won't depend on  $x^S$ ). We refer to the function  $d^S$  as the *outside option function* for the coalition  $S$ .

Although we allow different coalitions to have different bargaining functions, we assume that all coalitions use the same outside option function. Hence, a complete description of a multilateral bargaining problem specifies a set of coalitions  $C$ , and for each coalition  $S \in C$ , an attainable utility set  $V(S)$  and a bargaining function  $f^S$ . We refer to the triple  $\langle N, V, f \rangle$  as a *multilateral bargaining problem*.

A solution for the multilateral bargaining problem is a agreement payoff vector for each coalition which is consistent with the description of bargaining in every coalition. Formally, the vector  $x = \{x^S\}_{S \in C}$  is a *consistent conjecture* (or simply a solution) for bargaining problem  $\langle N, V, f \rangle$  if for every coalition  $S \in C$ ,  $x^S = f^S(d^S(x))$ .

A consistent conjecture is a stable set of beliefs about the outcomes of the bargaining because, given the nature of bargaining in each coalition, no player can improve his payoff in any coalition by renegotiating his utility allocation.

### Prices

Let  $x$  be a consistent conjecture for the bargaining problem  $\langle N, V, f \rangle$ . We say that player  $i$ 's *price* is  $p_i$  if he obtains exactly  $p_i$  as his agreement utility from every one of his coalitions. We say that the price vector  $p$  *generates*  $x$ , if for every player  $i$  and for every coalition  $S$  containing player  $i$ , we have  $x_i^S = p_i$ .

Bennett [1985] shows that for every bargaining problem  $\langle N, V, f \rangle$ :

1. There exists a consistent conjecture for  $\langle N, V, f \rangle$ ;
2. Every consistent conjecture is price generated;
3. Every price vector that generates a consistent conjecture is an aspiration.

Moreover, for every NTU game  $\langle N, V \rangle$  and every aspiration  $p$  for  $\langle N, V \rangle$ , there exists a bargaining rule  $f$  and a vector  $x = \{x^S\}_{S \in C}$  such that  $x$  is a consistent conjecture for  $\langle N, V, f \rangle$  and  $p$  generates  $x$ .

### Outcomes

Given a multilateral solution  $x = \{x^S\}_{S \in C}$ , the "possible" coalitions to form are those with feasible agreement vectors (i.e., coalitions that *don't* agree to disagree). If the multilateral solution  $x$  has the price vector  $p$ , the coalitions that are likely to form are precisely the coalitions in  $C(p)$ .

The selection of a particular multilateral solution  $x$ , with price vector  $p$  is the prediction that one of the coalitions in  $C(p)$  will form and that within the coalition that forms, each player is paid his price.

## 5. THE PROPOSAL-MAKING MODEL

The proposal-making model takes a noncooperative approach to modeling bargaining in NTU games. The proposal-making model specifies a particular bargaining procedure as an extensive form game with perfect information.

In this model players bargain by making, accepting and rejecting proposals. A *proposal*  $(S, q^S)$  made by player  $i$  specifies a proposed coalition,  $S$  (any coalition in  $C$  containing  $i$ ) and a proposed payoff distribution,  $q^S$ , (any payoff vector  $q^S \in V(S)$ ).

The game is played according to the following rules:

0. Nature has the first move. Nature selects each player  $i \in N$  with probability  $a_0(i) > 0$ . The selected player has the initiative at the next move and follows rule 1.
1. **The Initiator Role.** The player with the initiative, call him  $i$ , can either pass the initiative or make a proposal and name a responder. If  $i$  passes the initiative to  $j$ , player  $j$  will have the initiative at the next move and follow rule 1. If player  $i$  makes the proposal  $(S, q^S)$ , he also designates a player in  $S$  to respond to the proposal. The next move is made by the designated responder who follows rule 2.
2. **The Responder Role.** The designated responder, call her  $j$ , either accepts or rejects the proposal. If  $j$  accepts the proposal  $(S, q^S)$  and every other player in  $S$  has already accepted then the game ends (see 3.), otherwise  $j$  designates any player in  $S$  who has not yet accepted as the next responder (this player then follows rule 2). If  $j$  rejects the proposal she takes the initiative and follows rule 1.
3. **Game's End.** If a proposal, call it  $(S, q^S)$ , is accepted by every player in  $S$ , the game ends. The players in  $S$  obtain their components of  $q^S$  and players not in  $S$  obtain nothing.
4. Infinite plays result in a payoff of 0 for each player.

A sketch of the game tree is provided as Figure 1.

### *Nodes and Actions*

After the initial move by nature, at each nonterminal node there is a player whose turn it is to move. Let  $\eta^0$  denote the node at which it is nature's move (the initial node). For each player  $i$  let  $X_i$  denote the set of nodes where it is player  $i$ 's turn to move. At  $\eta \in X_i$  either  $i$  has the initiative or is responding to a proposal made by another player. Let  $X_i^I$  be the set of nodes at which  $i$  has the initiative and let  $X_i^R$

be the set of nodes at which  $i$  responds to a proposal. Let  $Z$  denote the set of terminal nodes.

For each player  $i$ , and each node  $\eta \in \mathcal{X}_i$ , let  $A_i(\eta)$  denote the set of actions available to  $i$  at  $\eta$ . At  $\eta \in \mathcal{X}_i^I$  player  $i$  has the initiative and  $A_i(\eta)$  contains two type of actions: player  $i$  can designate which player  $j \in N$  will have the initiative (and the next move), or else player  $i$  can make a proposal, call it  $(S, q^S)$ , and designate a player  $j \in S$ , to respond to the proposal (this responder will have the next move). At  $\eta \in \mathcal{X}_i^R$  player  $i$  is responding to a proposal, call it  $(S, q^S)$ , and  $A_i(\eta)$  again contains two types of actions: player  $i$  can reject the proposal (this action leads to a node where player  $i$  has the initiative) or else player  $i$  can accept the proposal and designate a player  $k \in S$  who has not yet accepted to be the next responder. (If, at  $\eta$ , every other player in  $S$  has already accepted the proposal, acceptance of the proposal leads to a terminal node  $\zeta$  where the game ends with the formation of the coalition  $S$ .)

### *Strategies and Stationarity*

A behavioral strategy for player  $i$ ,  $\sigma_i$ , specifies for each node  $\eta$  in  $\mathcal{X}_i$  a probability distribution  $\sigma_i(\eta)$  with finite carrier over the actions in  $A_i(\eta)$  i.e., at each node in  $\mathcal{X}_i$ ,  $\sigma_i(\eta)$  assigns a positive probability to at most a finite number of actions in  $A_i(\eta)$ . Let  $\sigma = (\sigma_1, \dots, \sigma_n)$  denote a strategy profile, i.e., a strategy for each player, let  $\sigma_{-i} = (\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$  denote a strategy for everyone except  $i$ . Let  $\Sigma_i$  denote the set of strategies for player  $i$ , and let  $\Sigma$  denote the set of all strategy profiles.

Informally, the strategy  $\sigma_i$  is a stationary strategy for player  $i$  if it is independent of the history of the game. By this we mean that  $\sigma_i$  makes the same prescription at every node at which player  $i$  has the initiative and that  $\sigma_i$  makes the same prescription at every node at which player  $i$  is responding to the same proposal by the same proposer with the same set of players having already accepted the proposal.

To formalize this notion let  $\mathcal{X}_i^R(S, q^S, j, T)$  denote the set of nodes at which player  $i$  is to respond to the proposal  $(S, q^S)$  made by player  $j$  and already accepted by the players in  $T$ . We say  $\sigma_i$  is a **stationary strategy for  $i$**  if:

1. For  $\eta, \eta' \in \mathcal{X}_i^I$ ,  $\sigma_i(\eta) = \sigma_i(\eta')$ .
2. For  $\eta, \eta' \in \mathcal{X}_i^R(S, q^S, j, T)$ ,  $\sigma_i(\eta) = \sigma_i(\eta')$ .

The profile  $\sigma$  is a **stationary strategy profile** if for each player  $i$ ,  $\sigma_i$  is a stationary strategy for  $i$ .

## Expected Payoffs and Equilibria

Given a strategy profile  $\sigma$ , a nonterminal node  $\eta$  and a terminal node  $z$ , we let  $\Pr_\sigma(z | \eta)$  be the probability that  $z$  is reached starting from  $\eta$  given that players follow the strategy profile  $\sigma$ . Since we have required strategies to have finite carrier at each node, it is easily seen that  $\Pr_\sigma(z | \eta)$  is well defined. (Notice, that in particular, that  $\Pr_\sigma(z | \eta) = 0$  if  $z$  does not follow  $\eta$ .) For the probability that, starting from  $\eta$ , the game does not terminate<sup>4</sup>, given that players follow  $\sigma$ , we write  $\Pr_\sigma(\infty | \eta) = 1 - \sum_z \Pr_\sigma(z | \eta)$ .

The payoff function  $h$  specifies the payoff each player receives at each terminal node. If acceptance of the proposal  $(S, q^S)$  leads to the terminal node  $z$ ,  $h$  specifies that each player in  $S$  receives his component of  $q^S$  and each player who is left out receives 0. Formally, we define the **payoff function**  $h : Z \rightarrow \mathbb{R}^N$  by  $h_i(z) = q_i^S$  for  $i \in S$  and  $h_i(z) = 0$  for  $i \notin S$  where  $(S, q^S)$  is the proposal whose acceptance led to  $z$ .

Since  $h_i(z)$  is the payoff to player  $i$  at the terminal node  $z$  and 0 is the payoff to player  $i$  if the game does not terminate, player  $i$ 's expected payoff at  $\eta$  given the strategy profile  $\sigma$ ,  $E_i(\eta | \sigma)$  is given by  $E_i(\eta | \sigma) = \sum_z h_i(z) \Pr_\sigma(z | \eta)$ .

We say that the terminal node  $z$  has a *positive probability of occurring* given  $\sigma$  if  $\Pr_\sigma(z | \eta^0) > 0$ . We say the proposal  $(S, q^S)$  has a *positive probability of acceptance* given  $\sigma$  if acceptance of  $(S, q^S)$  leads to a terminal node  $z$  that has a positive probability of occurring given  $\sigma$ . (Recall that  $\eta^0$  is the initial node of the game.)

The strategy  $\sigma_i$  is a **best reply** to  $\sigma_{-i}$  at  $\eta$  if for every strategy  $\sigma'_i \in \Sigma_i$ ,  $E_i(\eta | \sigma) \geq E_i(\eta | \sigma/\sigma'_i)$ . The strategy profile  $\sigma$  is a **Nash equilibrium** if for every player  $i \in N$ ,  $\sigma_i$  is a best reply to  $\sigma$  at the initial node,  $\eta^0$ .

The strategy profile  $\sigma$  is a **subgame perfect equilibrium** if for every player  $i$ , and every nonterminal node  $\eta$ , the strategy  $\sigma_i$  is a best reply to  $\sigma_{-i}$  starting at  $\eta$ . (Since every nonterminal node in the game is the initial node of a subgame, this definition agrees with the usual notion of subgame perfection.)

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4. Although the sum  $\sum_z \Pr_\sigma(z | \eta)$  extends over an uncountable set of terminal nodes, at most countably many terms are non-zero; this point is made clear in the Appendix.

## 6. STATIONARY SUBGAME PERFECT EQUILIBRIUM OUTCOMES

According to the rules of the game, any player who is responding to a proposal can take the initiative by simply rejecting the current proposal. For a proposal to be "acceptable" every player in the proposed coalition must be offered a utility at least as high as his expected payoff as an initiator; we therefore refer to player's expected payoff as initiator as his "reservation price". Formally, for  $\sigma_i$  a stationary strategy for player  $i$ , and  $\eta_i \in X_i$  be a node at which player  $i$  has the initiative, we define player  $i$ 's *reservation price*  $p_i(\sigma)$  to be  $i$ 's expected payoff as an initiator,  $p_i(\sigma) = E_i(\eta_i | \sigma)$ . (Since  $\sigma$  is a stationary strategy  $p_i$  is independent of the choice of initiator node.) We refer to  $p(\sigma) = (p_1(\sigma), \dots, p_n(\sigma))$  as the *reservation price vector* of  $\sigma$ .

In the following we use  $(p/t_i)^S$  to denote the vector containing one component for each player in  $S$  whose components are the components of  $p$  except for player  $i$  whose component is  $t_i$ .

**Theorem 6.1:** *If  $p = (p_1, \dots, p_n)$  is the reservation price vector of a stationary subgame perfect strategy profile, then for each player  $i$ ,  $p_i = \max \{ u_i^S(p_{-i}) \mid S \in C_i \}$  where  $u_j^S(p_{-j})$  is given by  $u_j^S(p_{-j}) = \max \{ t_j \mid (p/t_j)^S \in V(S) \}$ .*

**Proof:** We first show that player  $i$ 's reservation price is at least the value  $p_i$  given above. Player  $i$  can ensure the cooperation of each partner  $j \in S$  by proposing a payoff  $q_j^S$  which is strictly greater than  $j$ 's expected payoff if he refuses, i.e.,  $j$ 's reservation price (otherwise backward induction would show that the strategy of one of the players in  $S$  was not subgame perfect). Since player  $i$  has a strategy, which for any  $\epsilon > 0$  guarantees player  $i$  a payoff of  $\max \{ t_i \mid (p/t_i)^S \in V(S) \} - \epsilon$ , his utility for forming the coalition  $S$ ,  $u_i^S(p_{-i})$ , must be at least  $\max \{ t_i \mid (p/t_i)^S \in V(S) \}$ . (If the set  $\{ t_i \mid (p/t_i)^S \in V(S) \}$  is empty player  $i$  will not propose the coalition  $S$  since  $i$  can obtain 0 without the cooperation of other players; in this case for simplicity we set  $u_i^S(p_{-i}) = 0$ .) Since player  $i$  can propose any coalition  $S \in C_i$  his reservation price must be at least  $u_i(p_{-i}) = \max \{ u_i^S(p_{-i}) \mid S \in C_i \}$ . Since any proposal that assigns a player  $j \in S$  less than his reservation price will be rejected,  $u_i(p_{-i})$  is also the largest payoff player  $i$  can obtain by making an acceptable proposal.

To show that  $u_i(p_{-i})$  is  $i$ 's reservation price we must show that  $i$ 's expected utility would not be higher if he either passed the initiative or made an unacceptable proposal. Consider any subgame that follows player  $i$  passing the initiative or having his proposal rejected and consider any terminal node that is reached with positive probability (given

SSP strategies in the subgame) in which player  $i$  participates in the successful coalition. Let  $j$  be the player who made the successful proposal. If  $i = j$  the previous argument shows that his payoff is no more than  $u_i(p_{-i})$ . For  $j \neq i$  to be willing to make the proposal, say  $(S, q^S)$ , his payoff must be at least his reservation price, i.e.,  $q_j^S \geq p_j$ . For the other players  $k \neq i$  to be willing to accept the proposal,  $q_k^S \geq p_k$ . Hence player  $i$ 's reservation price  $p_i = u_i(p_{-i})$ . ■

**Theorem 6.2:** *The reservation price vector of every stationary subgame perfect SSP strategy profile is an aspiration.*

Proof: We show that if  $p$  is a reservation price vector of a SSP strategy profile then  $p$  is realizable and maximal for each player  $i$ . By Theorem 6.1, we have that player  $i$ 's price  $p_i$  satisfies  $p_i = \max \{ u_i^S \mid S \in C_i \}$  where  $u_i^S = \max \{ t_i \mid (p/t_i)^S \in V(S) \}$ . If  $p_i > 0$ ,  $p$  is realizable for player  $i$  in any coalition  $S^*$  in which the maximum  $u_i^S$  is obtained; if  $p_i = 0$  then  $p$  is realizable the coalition  $\{i\}$ . The vector  $p$  is maximal for player  $i$  because for each coalition  $S \in C_i$   $t_i$  is the largest feasible residual in the coalition  $S$  and  $u_i^S$  is the largest of these values. Since  $p$  is realizable for each player  $i$ ,  $p$  is an aspiration. ■

**Theorem 6.3:** *Every aspiration  $p$  is the reservation price vector of an stationary subgame perfect strategy profile.*

Proof: For each vector  $p \in \mathbb{R}^N$  we describe a class of stationary strategy profiles such that each strategy profile in the class has a reservation price vector equal to  $p$  and show that if  $p$  is an aspiration then these strategy profiles are subgame perfect.

Let  $p \in \mathbb{R}_+^N$ . We say that  $\sigma_i^p$  is a *price strategy* (for the price vector  $p$ ) for player  $i$  if:

1. When  $i$  has the initiative he assigns 0 probability to passing the initiative and assigns positive probability to making a proposal  $(S, q^S)$  only when  $q^S = p^S$ .
2. When player  $i$  responds to the proposal  $(S, q^S)$  he accepts if  $q_j^S \geq p_j$  for every player  $j$  of  $S$  (including  $i$ ) who has not yet accepted the proposal and rejects the proposal otherwise.

We say that  $\sigma^p \in \Sigma$  is a *price strategy profile* if for every player  $i$ ,  $\sigma_i^p$  is a price strategy. For each  $p \in \mathbb{R}^N$ , let  $\Sigma^p$  denote the set of all price strategy profiles corresponding to the vector  $p$ . Notice that each strategy supports only one price vector  $p$  but many strategies support that same  $p$ ; these strategies differ by the probabilities that

players with the initiative assign to making various proposals (i.e., for player  $i$  the probabilities he assigns to making each of the proposals in the set  $\{(S, p^S) \mid S \in C_i(p)\}$ ).

It is easy to see that for any price strategy profile  $\sigma^P$ ,  $p$  is its reservation price vector of  $\sigma^P$  since only proposals made with positive probability given  $\sigma^P$  are of the form  $(S, p^S)$  and every player accepts such a proposal.

Now fix an aspiration  $p$ , and fix any  $\sigma^P \in \Sigma^P$ . To show that  $\sigma^P$  is subgame perfect we show that for every player  $i$ ,  $\sigma_i^P$  is a best response to  $\sigma_{-i}^P$ , starting at any decision node for  $i$ . This requires three lemmas (below). Lemma 6.4 shows that if  $i$  makes an "acceptable" proposal at any of his initiator nodes, then his proposed payoff is not greater than  $p_i$ . Lemma 6.5 shows that  $\sigma_i^P$  is a best response to  $\sigma_{-i}^P$  at any of his initiator nodes. Lemma 6.6 completes the proof by showing that  $\sigma_i^P$  is a best response to  $\sigma_{-i}^P$  at any of his responder nodes.

**Lemma 6.4:** *If player  $i$ , makes a proposal  $(S, q^S)$  that has a positive probability of being accepted given  $\sigma_{-i}^P$ , then  $q_i^S \leq p_i$ .*

**Proof of Lemma 6.4:** Suppose player  $i$  makes the proposal  $(S, q^S)$ . (The vector  $q^S$  need not be compatible with  $\sigma_{-i}^P$ .) Since  $(S, q^S)$  is a proposal,  $q^S \in V(S)$ . Suppose first that  $S = [i]$ . By maximality of aspirations  $q_i^S \leq p_i$ . Suppose instead  $S \neq [i]$ . If  $(S, q^S)$  has a positive probability of acceptance by the first responder ( $\neq i$ ) (who by assumption is following a price strategy for  $p$ ) then  $q^{S-i} \geq p^{S-i}$ . By maximality it must be the case that  $q_i^S \leq p_i$ . ■

**Lemma 6.5:** *For every player  $i$ ,  $\sigma_i^P$  is a best response to  $\sigma_{-i}^P$ , at any initiator node  $\eta_i \in \mathcal{X}_i^I$ .*

**Proof of Lemma 6.5:** For any  $\sigma' \in \Sigma_i$  and let  $\sigma' = \sigma^P / \sigma'$ . Consider any terminal node  $\zeta$  that is reached from  $\eta_i$  with positive probability given  $\sigma'$ . Let  $(S, q^S)$  be the proposal made by player  $k$  at node  $\eta$  whose acceptance leads to  $\zeta$ . If  $k = i$ , then Lemma 6.4 implies that  $h_i(\zeta) = q_i^S \leq p$ . If  $k \neq i$ , then (since  $k$  is following  $\sigma^P$ ),  $q^S = p^S$ . In this case  $h_i(\zeta) = p_i$  if  $i \in S$  and  $h_i(\zeta) = 0 \leq p_i$  if  $i \notin S$ . Since the choice of  $\zeta$  was arbitrary, we conclude that  $E_i(\eta_i \mid \sigma') \leq p_i = E_i(\eta_i \mid \sigma^P)$ . Hence  $\sigma_i^P$  is a best response to  $\sigma_{-i}^P$  at every  $\eta_i \in \mathcal{X}_i^I$ . ■

**Lemma 6.6:** *For every player  $i$ ,  $\sigma_i^P$  is a best response to  $\sigma_{-i}^P$  at any responder node  $\eta_i \in \mathcal{X}_i^R$ .*



Proof of Lemma 6.6: We first show that if  $i$  accepts the proposal  $(S, q^S)$  which is rejected with probability 1 by some future respondent (call her  $j$ ), then  $i$ 's expected payoff (given  $\sigma_j^D(\cdot)$ ) is at most  $p_i$ . To see why this is true notice that player  $j$ , having rejected the proposal will follow  $\sigma_j^D$ , and make a proposal  $(S'', q^{S''})$  with  $q^{S''} = p^{S''}$  and this proposal will be accepted for certain if  $i \notin S''$ . This would result in a payoff to  $i$  of  $0 \leq p_i$ . If  $i \in S''$ , and  $i$  accepts, so will everyone else, so  $i$ 's payoff is  $p_i$  for certain. If  $i$  rejects  $(S'', q^{S''})$ , Lemma 6.5 implies that his expected payoff is at most  $p_i$ .

To complete the proof, let  $\eta_i \in \mathcal{X}_i^R$  and suppose at  $\eta_i$  player  $i$  is responding to a proposal  $(S, q^S)$  and the players in  $S'$  (a subset of  $S$ ) have not yet accepted it. We consider three cases depending on whether or not  $q^{S'}$  is at least as large as  $p^{S'}$ .

**Case 1:**  $q^{S'} \geq p^{S'}$ . In this case  $E_i(\eta_i | \sigma^D) = q_i^S$ . If player  $i$  accepts the proposal she obtains  $q_i^S$ , if she rejects the proposal, her expected payoff is (by Lemma 6.5) at most  $p_i \leq q_i^S$ . Hence  $\sigma_i^D$  is a best response to  $\sigma_j^D(\cdot)$  at  $\eta_i$ .

**Case 2:**  $q_k^{S'} < p_k$  for some  $k \in S' - i$ . In this case  $E_i(\eta_i | \sigma^D) = p_i$ . If player  $i$  rejects the proposal, she takes the initiative and Lemma 6.5 guarantees her expected payoff to be at most  $p_i$ ; if she accepts the proposal, player  $k$  will certainly reject the proposal and, as we earlier argued,  $i$ 's expected payoff is at most  $p_i = E_i(\eta_i | \sigma^D)$ , so  $\sigma_i^D$  is a best response.

**Case 3:**  $q^{S' - i} \geq p^{S' - i}$  and  $q_i^{S'} < p_i$ . In this case  $E_i(\eta_i | \sigma^D) = p_i^S$ . If player  $i$  accepts the proposal she obtains  $q_i^S < p_i$ ; if she rejects the proposal, her expected payoff is (by Lemma 6.5) at most  $p_i \leq q_i^S$ . Hence  $\sigma_i^D$  is a best response to  $\sigma_j^D(\cdot)$  at  $\eta_i$ .

Since in each case  $\sigma_i^D$  is a best reply to  $\sigma_j^D(\cdot)$ ,  $\sigma_i^D$  is a best reply at every responder node -- this completes the proof of Lemma 6.6. Lemmas 6.5 and 6.6, together prove that  $\sigma_i^D$  is a best response at every node for player  $i$ , --this completes the proof of Lemma 6.6, and with it the proof of Theorem 6.3. ■

### Prices

Associated with each strategy profile  $\sigma$  is a set of proposals, call it  $P(\sigma)$ , corresponding to the terminal nodes that have positive probability of being reached given  $\sigma$ . The proposals in  $P(\sigma)$  are the proposals that have a positive probability of being made and accepted given  $\sigma$ . We call the proposals in  $P(\sigma)$  the *acceptable proposals* given  $\sigma$ .

We say that the vector  $p$  *generates* the set of acceptable proposals  $P(\sigma)$  if in every acceptable proposal every player is paid according to  $p$ .

Formally, the vector  $p \in \mathbb{R}^N$  generates  $P(\sigma)$  if for every  $(S, q^S) \in P(\sigma)$ ,  $q^S = p^S$ .

**Theorem 6.7:** *The reservation price vector of each stationary subgame perfect strategy profile generates its set of acceptable proposals.*

Proof: To see that the reservation price vector  $p = p(\sigma)$  of each SSP  $\sigma$  generates  $P(\sigma)$  we first note that each player's expected payoff is at least as high when he has the initiative as when any other player has the initiative (otherwise he would pass the initiative to this other player); i.e., for  $\eta \in \mathcal{X}_i^I$ ,  $\eta_j \in \mathcal{X}_j^I$ ,  $E_i(\eta_i | \sigma) \geq E_i(\eta_j | \sigma)$ .

Let  $\zeta$  be any terminal node that occurs with positive probability given  $\sigma$ . (There is a terminal node with positive probability since a strategy profile that leads only to infinite plays cannot be subgame perfect.) Let  $i$  be the proposer and let  $(S, q^S)$  be the proposal whose acceptance led to  $\zeta$  and let  $\eta$  be the node at which  $i$  makes this proposal. Consider the path from  $\eta$  to  $\zeta$ . Since  $(S, q^S)$  is accepted with positive probability  $q_j^S \geq p_j$  for every  $j \neq i$  in  $S$ . We claim that  $q_i^S \geq p_i$ , also. To see this, note that, since  $i$  made the proposal  $(S, q^S)$  (with positive probability) at  $\eta$ , his expected payoff given this action must be at least  $p_i$ . However, his expected payoff given this action is a weighted average of his payoff given that his proposal is accepted (which is  $q_i^S$ ) and his expected payoff given that his proposal is rejected. If the proposal were rejected by responder  $j$ , player  $i$  would expect to obtain  $E_i(\eta_j | \sigma)$  for  $\eta_j \in \mathcal{X}_j^I$  which, as we have noted above is at most  $p_j$ . So  $i$ 's expected payoff is a weighted average of  $q_i^S$  and a number no larger than  $p_j$ . Hence for  $i$ 's expectation to be  $p_i$ ,  $q_i^S \geq p_i$ . Hence  $q^S \geq p^S$ . Since  $p$  is maximal,  $q^S \in V(S)$ , and  $V(S)$  is strongly comprehensive,  $p^S = q^S$ . Hence  $p$  is the price vector of  $\sigma$ . ■

We have proved that the set of all reservation price vectors of stationary subgame perfect equilibria is the set of aspirations; it follows that the set of all price vectors that generate SSP possible outcomes is also the set of aspirations.

### Outcomes

For each SSP strategy profile  $\sigma$ , let  $C(\sigma)$  be the set of coalitions that have a positive probability of being formed given  $\sigma$ , i.e.,

$$C(\sigma) = \{ S \mid (S, q^S) \in P(\sigma) \text{ for } q^S \in \mathbb{R}^S \}.$$

One might expect that if  $p$  generates  $P(\sigma)$  then  $C(\sigma) = C(p)$ . However this is not the case. While  $C(\sigma)$  is always a subset of  $C(p)$ , when there are "many" coalitions in  $C(p)$  players can discriminate against a particular coalition by never proposing it. If players do not discriminate (i.e., each player as initiator assigns positive probability to proposing

every coalition in  $C_j(p)$ , then  $C(\sigma) = C(p)$  and the three models agree on the entire set of predicted coalitions.

## 7. RELATED LITERATURE

1. Selten [1981] presented the proposal-making model for TU games (modeled as a recursive rather than an extensive form game) and proved the analog of the Section 6 results. After showing that equilibrium points are aspirations (Selten calls them semistable demand vectors), he uses equilibrium selection arguments to isolate a particular set of aspirations, which he refers to as the set of stable demand vectors. Selten's equilibrium selection arguments could be carried over directly to the NTU case; the analog of Selten's stable demand vectors are the bargaining aspirations of Bennett and Zame [1988].

2. One way to avoid the multiplicity of equilibria in noncooperative bargaining models is to introduce friction into the bargaining procedure. For instance, following Rubinstein [1981], one might assume that drawing up a proposal takes time and that time matters (i.e., players discount the utility of future agreements). This is the approach taken by Binmore [1985] and by Chatterjee et al [1987].

Binmore [1985] presents a noncooperative model and a bargaining theory model for a limited class of 3-player NTU games (games in which the coalition on the whole earns nothing). Binmore discusses several alternative noncooperative models for this class of games including one he calls the "telephone" bargaining model. Apart from details, the telephone bargaining model is the proposal-making model with discounting. Binmore "tests" this model on a particular 3-player TU example and finds its (unique) outcome "unintuitive". Binmore then dismisses the model and goes on to the "market demand" model -- a model unrelated to the model presented here. (Binmore [1985] also presents a bargaining theory model which is quite similar in spirit to the multilateral bargaining model presented here.)

Chatterjee et al [1987] presents a noncooperative model for the class of TU games, which, apart from details, is again the proposal-making model with discounting. Although differences in "detail" can result in substantive differences in the equilibria of the model, we believe their results to be suggestive of the type of results that would be obtained (for the class of TU games) if this friction were introduced into the proposal-making model. In the model of Chatterjee et al, the bargaining outcomes for each TU game are generated by a unique price vector; and this price vector is an aspiration. Unfortunately the unique aspiration chosen is often not the most intuitively plausible price vector for the game.

Although the nonuniqueness of the equilibrium of the proposal-making model is unsatisfying, the results of Chatterjee et al and Binmore suggest

that time pressure is not the "right" friction in this context. Other frictions and other ways of selecting among price vectors in the proposal-making model should be investigated.

3. Bennett [1990] shows that, in the absence of stationarity "anything" can happen in the proposal-making model-- i.e., any individually rational payoff distribution for any coalition is a possible outcome of subgame perfect equilibrium strategies.

4. Various solution concepts on the space of aspirations have been proposed, independently and in various guises, by a number of authors. The *aspiration core* for TU games was first identified by Cross [1967]; it was later proposed independently by Albers [1974], by Turbay [1977], and by Bennett and Wooders [1979]. Bennett [1980] recognized it as the extension of the core solution concept to the space of aspirations. The *aspiration bargaining set* for TU games was first proposed by Albers [1974], and independently by Bennett [1980, 1983], who recognized it as the extension of the bargaining set to the space of aspirations. The extension of the aspiration bargaining set to the class of NTU games is in Bennett and Zame [1988]. Other aspiration solutions (kernel and nucleolus variants, in particular) are described in Albers [1980] and Bennett [1980].

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APPENDIX

For each node  $\eta$ , let  $T(\eta)$  be the set of nodes that follow  $\eta$ . Write  $T_1(\eta)$  for the subset of  $T(\eta)$  consisting of immediate successors of  $\eta$ ,  $T_2(\eta)$  for the immediate successors of elements of  $T_1(\eta)$ , etc. Note that  $T(\eta)$  is the union over all  $T_i(\eta)$ .

Fix a strategy profile  $\sigma$ ; recall that we require  $\sigma$  to have a finite support at each node. For  $\eta' \in T_1(\eta)$ , write  $\Pr_\sigma(\eta' | \eta)$  for the conditional probability that  $\eta'$  is reached from  $\eta$  given that players follow the strategy profile  $\sigma$ . This defines a probability measure on  $T_1(\eta)$  with finite support. Hence, we may define for  $\eta'' \in T_2(\eta)$  the conditional probability  $\Pr_\sigma(\eta'' | \eta)$  that  $\eta''$  is reached from  $\eta$  given that players follow the strategy profile  $\sigma$ , obtaining a probability measure on  $T_2(\eta)$  with finite support. Continuing in this way, we obtain, for each  $n$ , a probability measure on  $T_n(\eta)$  with finite support.

Let  $Z$  be the set of terminal nodes. Write  $Z(\eta)$  for the set of terminal nodes in  $T(\eta)$  and  $Z_n(\eta)$  for the set of terminal nodes in  $T_n(\eta)$ . Note that

$$\sum_{\xi \in Z_1(\eta)} \Pr_\sigma(\xi | \eta)$$

is the probability that the game terminates in one step, beginning from  $\eta$ , that

$$\sum_{\xi \in Z_2(\eta)} \Pr_\sigma(\xi | \eta)$$

is the probability that the game terminates in two steps, beginning from  $\eta$ , etc. (Note that each of these sums is in fact finite, since each of the probability measures we have constructed has finite support.)

For  $\xi$  a terminal node not following  $\eta$ , set  $\Pr_\sigma(\xi | \eta) = 0$ . The probability that the game terminates, starting from  $\eta$ , is

$$\begin{aligned} \sum \Pr_\sigma(\xi | \eta) &= \sum_{\xi \in Z(\eta)} \Pr_\sigma(\xi | \eta) \\ &= \sum \sum \Pr_\sigma(\xi | \eta) \end{aligned}$$

$$n \quad \xi \in Z_n(\eta)$$

The last double sum is countable, since each of the measures  $\Pr_\sigma(\cdot \mid \eta)$  has finite support. Thus  $\Pr_\sigma(\infty \mid \eta) = 1 - \sum \Pr_\sigma(\xi \mid \eta)$ , the probability that the game does not terminate (i.e., that the strategy  $\sigma$  leads to an infinite play), beginning from  $\eta$ , is clearly well-defined.

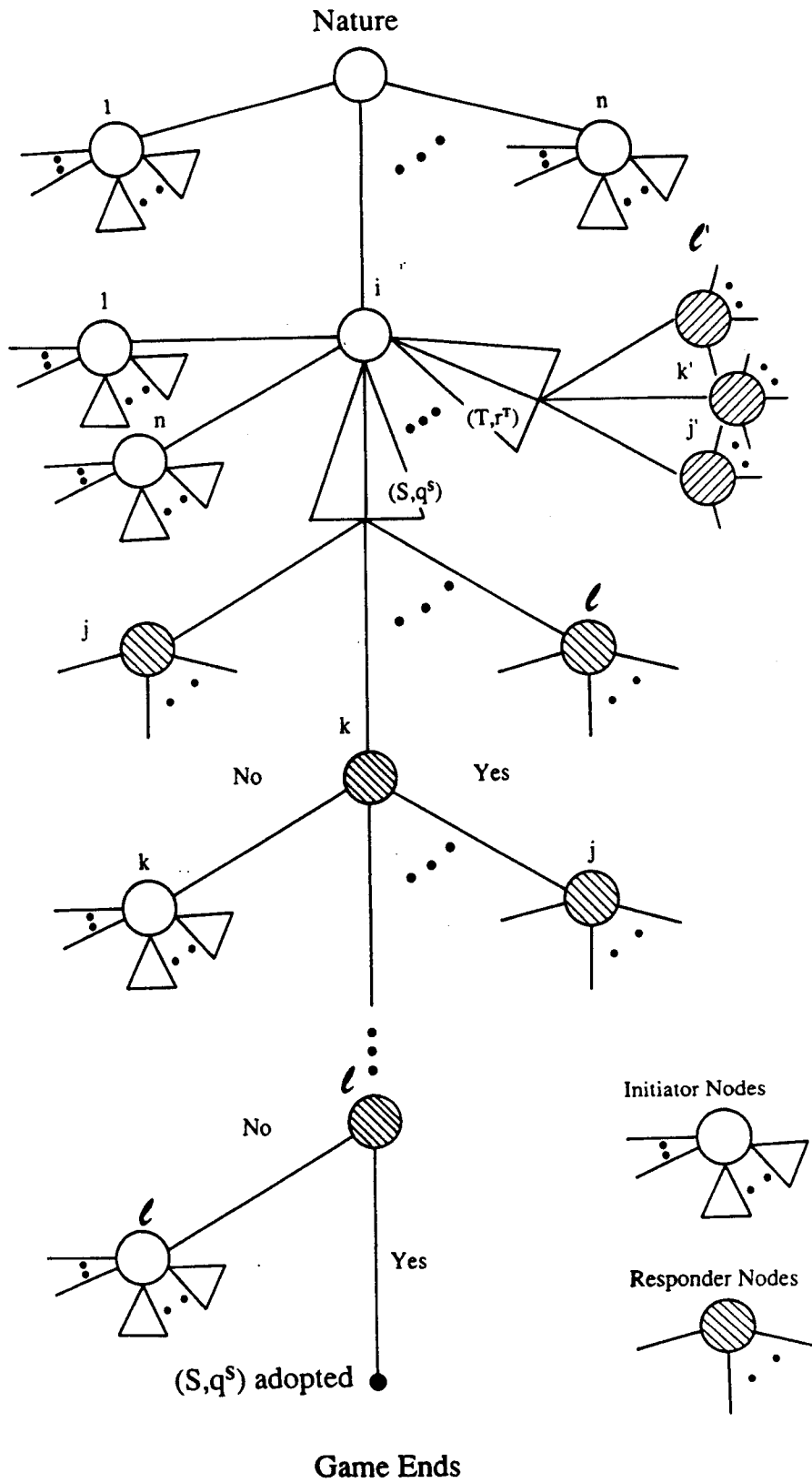


FIGURE 1: The Game Tree of the Proposal-Making Model



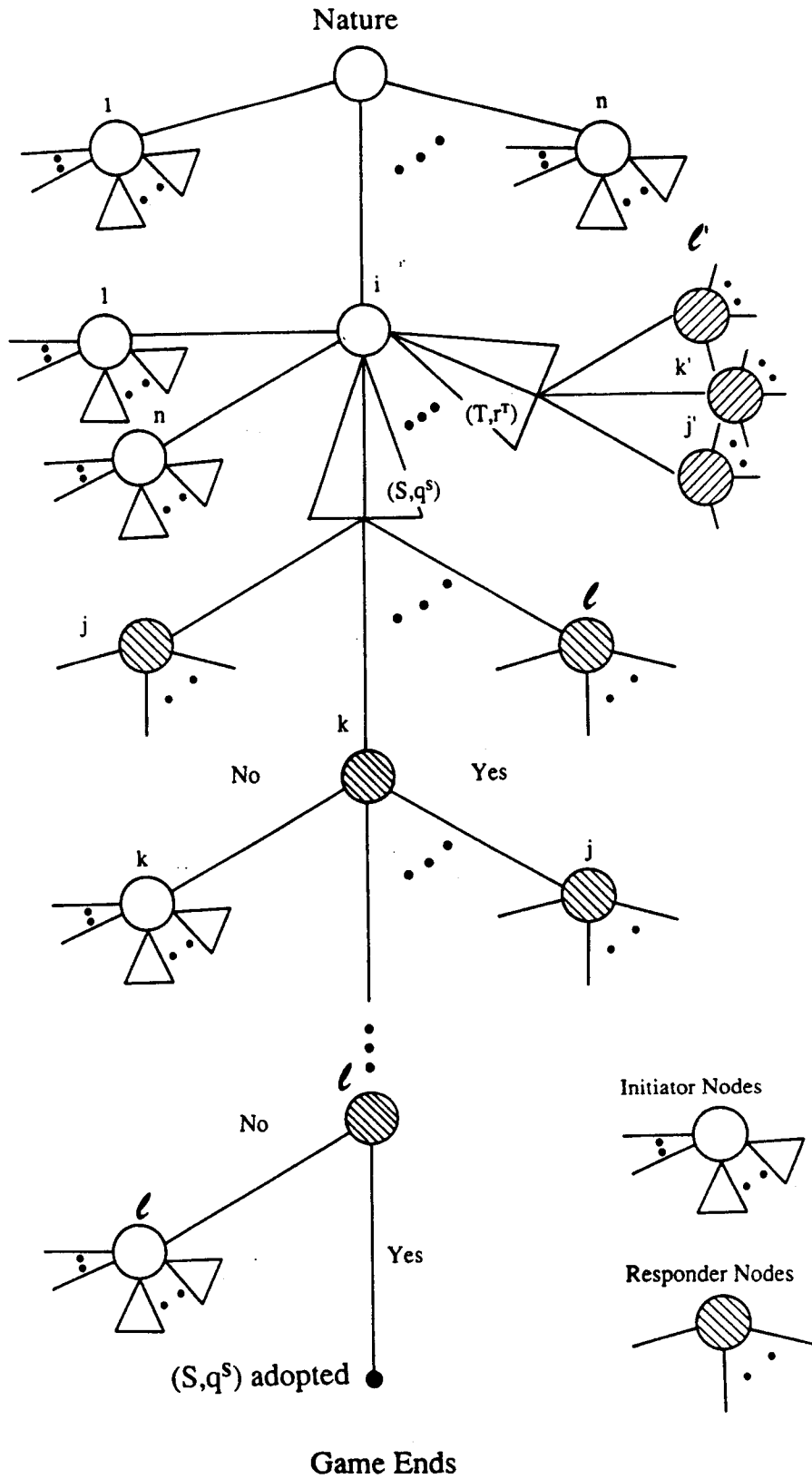


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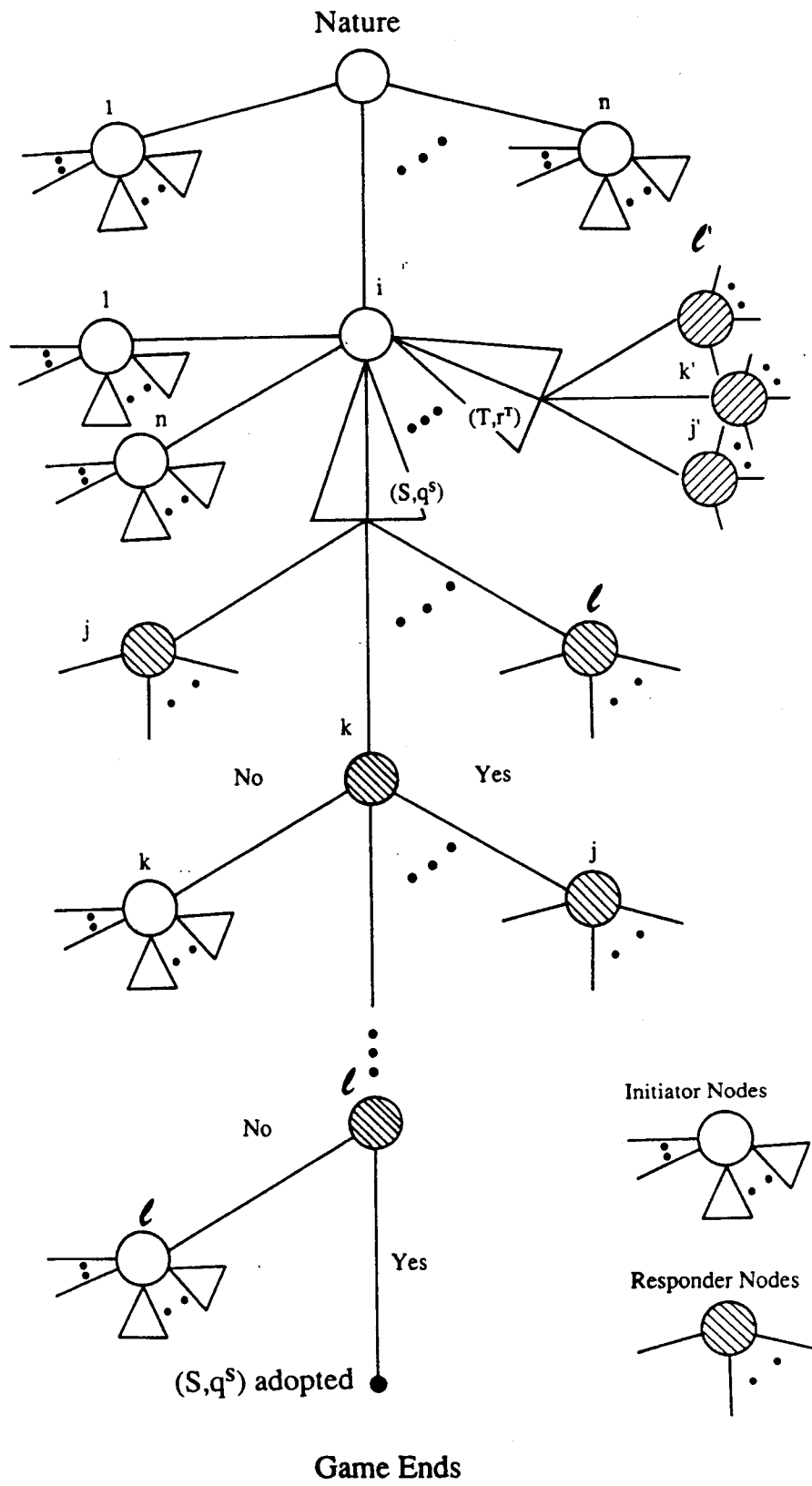


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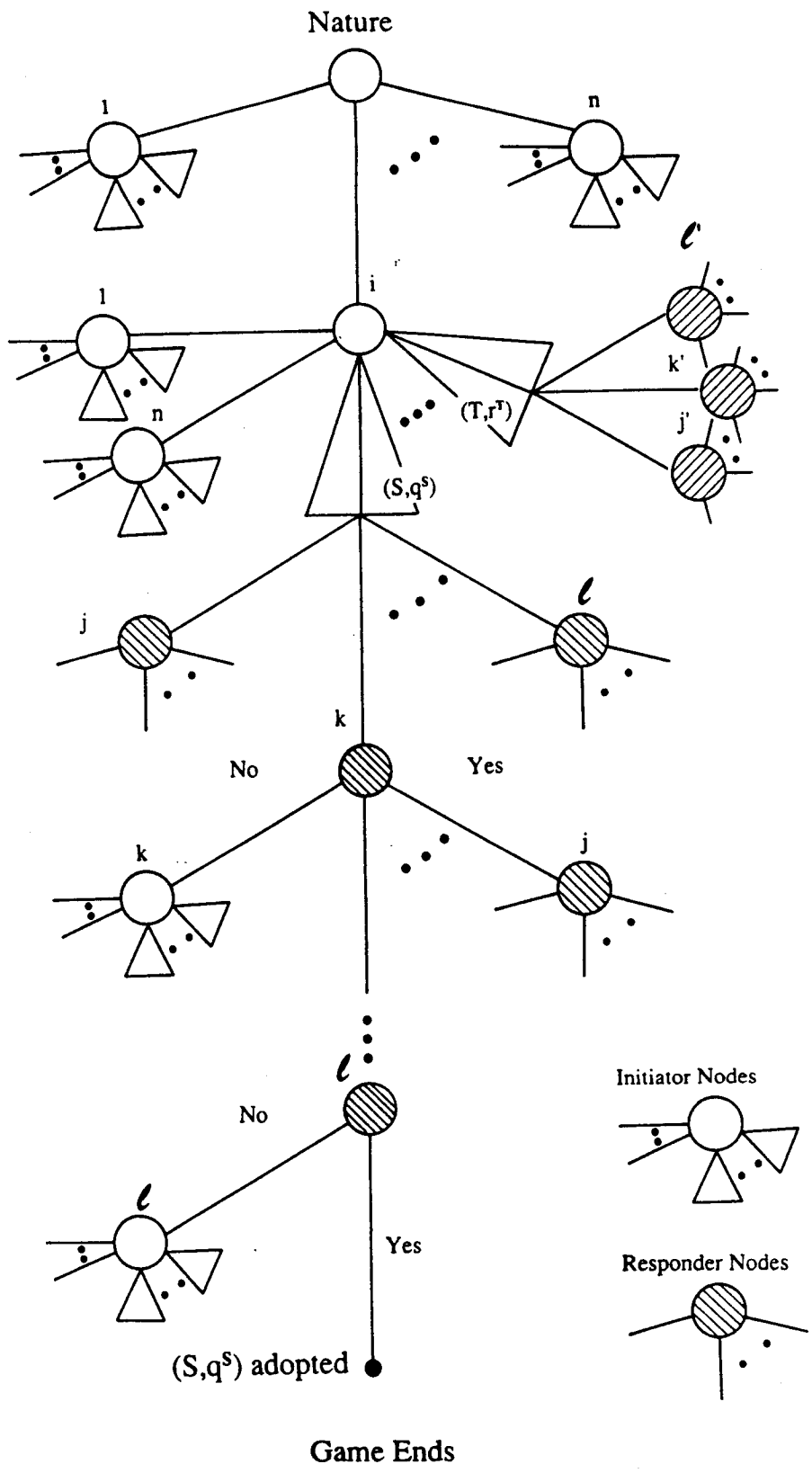


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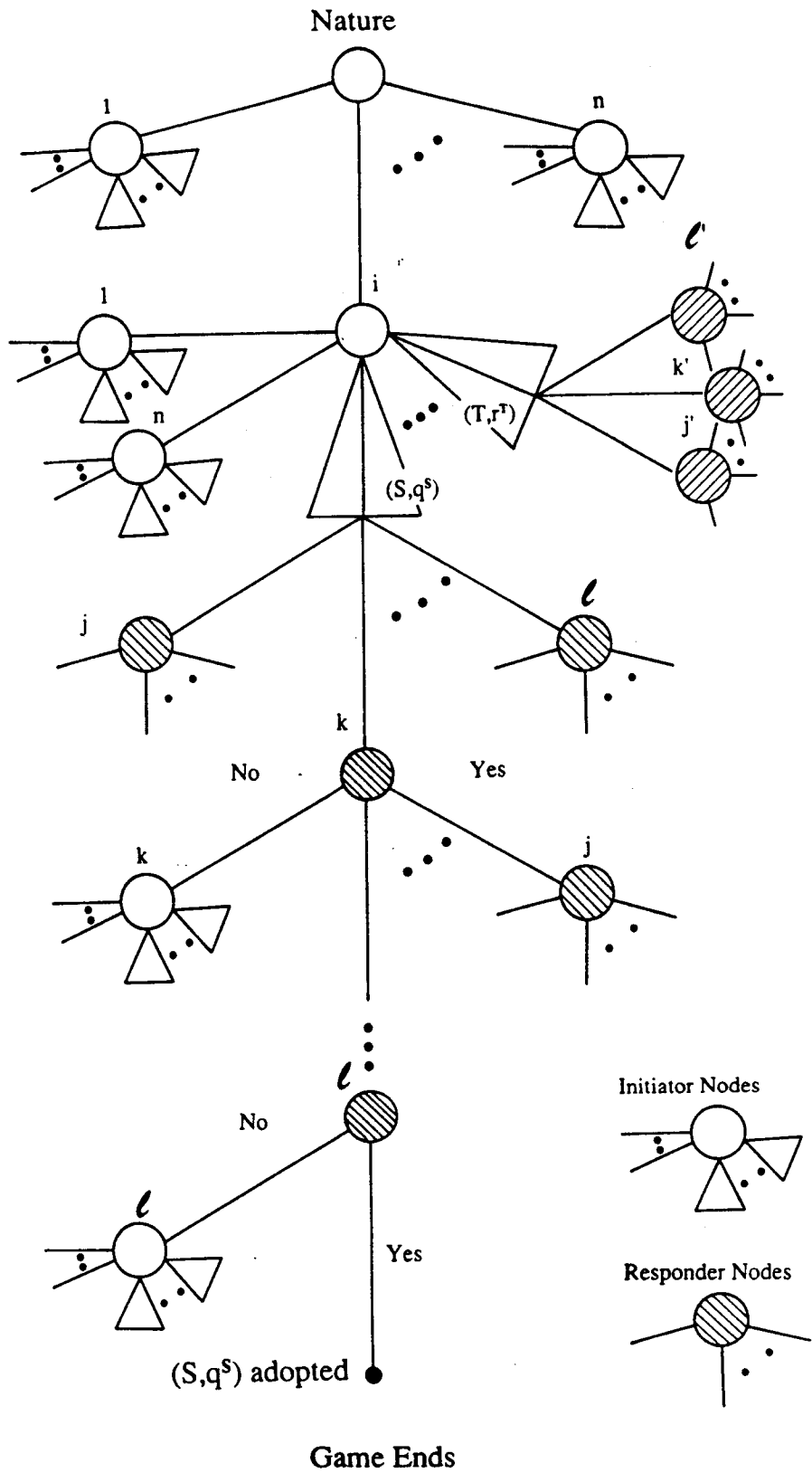


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