FOLK THEOREMS FOR THE PROPOSAL-MAKING MODEL

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A companion paper (UCLA WP 586) introduces the proposal-making model for NTU games, (a noncooperative bargaining model for characteristic function games without sidepayments) and characterizes its stationary subgame perfect outcomes. The present paper shows that in the absence of stationarity, "anything" can be a bargaining outcome of the proposal-making model.

1. INTRODUCTION

Selten [1981] presents the following bargaining procedure (the "proposal-making" procedure) for characteristic function games. Nature selects a player to be given the initiative. A player with the initiative can pass the initiative to another player or make a proposal (consisting of a proposed coalition and a payoff vector for the coalition). Players in the proposed coalition sequentially consider the proposal; if all of them accept, the game ends with the formation of the coalition; any player who rejects the proposal takes the initiative (and can pass the initiative or make a proposal...). If the games ends with the acceptance of the proposal to form the coalition S with the payoff vector q^S then the outcome of the game is that the coalition S forms; players in S obtain their components of q^S while other players obtain nothing. Selten considers the class of sidepayment games (TU games) and formulates this bargaining procedure as a recursive game. Bennett [1990] considers the broader class of nonsidepayment games (NTU games) and formulates the bargaining procedure as an extensive form game. Both Bennett and Selten characterize the stationary subgame perfect equilibrium outcomes of this bargaining procedure.

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One might expect the restriction to stationary strategies to induce a certain uniformity across the outcomes of each subgame perfect strategy profile; this is indeed true. If, in a stationary subgame perfect equilibrium, the same coalition is proposed (and accepted) with positive probability (at different terminal nodes), then the proposed payoffs must be equal. Moreover, if the same player appears in two *different* coalitions that are accepted with positive probability then the player has the same proposed payoff. Sets of outcomes that have this uniformity property we call *price generated* since each player would appear to have a reservation payoff level, i.e., a reservation price.

The outcomes of stationary subgame perfect strategy profiles proposals are also coalitionally rational; i.e., if the proposal to form S with payoff vector q^S is such an outcome then no subcoalition of S can unilaterally improve upon its members payoffs.

In the absence of stationarity what can be said about the subgame perfect outcomes? The present paper shows that neither coalitional rationality nor uniformity of payoffs is maintained. Indeed, we show that nearly anything can happen. Such results are commonly referred to as folk theorems. Our first folk theorem asserts that any individually rational and realizable vector can be the price vector of a subgame perfect strategy profile. In particular, given a coalition S and an individually rational payoff vector q^S which is feasible for S, there is a strategy profile σ such that the proposal (S,q^S) has a positive probability of occuring, given that players follow σ. Our second folk theorem asserts that given a coalition S and an individually rational payoff vector q^S which is feasible for S, there is a strategy profile σ such that (S,q^S) is the *unique* outcome with a positive probability of occuring, given σ if and only if 0 is individually rational for every player not in S. In a concluding example we show that the uniformity of payoffs is also lost; the outcomes of subgame perfect strategy profiles need not be price-generated.

The paper is organized in the following way. Section 2 provides basic definitions and presents the bargaining procedure as an extensive form game. Section 3 presents the folk theorems and example.

2. THE MODEL

NTU Games and Unique Opportunities

A game in characteristic function form without sidepayments (an ***NTU game***) is a pair $\langle N,V\rangle$ where $N = \{1,\ldots,n\}$ is a nonempty set of ***players*** and V, the ***characteristic function***, is a function which assigns to each nonempty subset S of N (a ***coalition***) a compact subset $V(S)$ of R^S_+ which contains the origin and is ***strongly comprehensive*** (i.e., if $x^S \in V(S)$, $y^S \in R^S_+$, $y^S_i \leq x^S_i$ for each $i \in S$ and $y^S \neq x^S$, then

$y^S \in$ INT V(S), the interior of V(S) with respect to R^S_+). We do not require that any V(S) be convex or that the game be superadditive. We use **C** to denote to set of all coalitions and C_i to denote the set of coalitions in **C** that contain player i.

If x^S, y^S are in R^S we write $x^S \leqslant y^S$ if $x^S_i \leqslant y^S_i$ for each $i \in S$; we write $x^S < y^S$ if $x^S \leqslant y^S$ and $x^S \neq y^S$, and $x^S \ll y^S$ if $x^S_i < y^S_i$ for each i. For $x \in R^N$ and S a subset of N, by x^S we mean the restriction of x to S (thinking of vectors in R^N as functions from N to R).

The proposal-making model is appropriate for modeling situations with a unique opportunity, in the sense that, once a proposal is accepted the game ends; i.e., no further coalition formation can take place. The "unique opportunity" assumption imposes no restrictions on NTU games but instead limits the range of applications of the model.

Let $\bar{v}_i$ denote the maximum payoff player i can obtain on his own, i.e., $\bar{v}_i = \max \{ t \mid t \in V(i)\}$. We say that $p^S \in R^S$ is ***individually rational*** if for every player $i \in S$, $p_i \geqslant \bar{v}_i$.

The Rules of the Game

In the proposal-making model players bargain by making, accepting and rejecting proposals. A ***proposal*** (S,q^S) made by player i specifies a proposed coalition, S (any coalition containing i) and a proposed payoff distribution, q^S, (any individually rational payoff vector $q^S \in V(S)$).

Bargaining begins when nature randomly selects a player to have the initiative. A player with the initiative can either pass the initiative to another player or else make a proposal, (S,q^S) and call on a player in S to respond to the proposal. A player responds to a proposal by accepting or rejecting it. If he accepts the proposal, he names the next player to respond; if he rejects it he takes the initiative and makes an alternative proposal (or passes the initiative to another player). The game ends when a proposal is accepted by all the players in the proposed coalition. If the games ends with the acceptance of (S,q^S) then each player j in S obtains a payoff of q^S_j and each player not in S obtains a payoff of 0. Infinite plays result in 0 payoff for all players.

A sketch of the game tree is provided as Figure 1.

Strategies and Equilibria

In this model each player has two types of nodes: nodes at which he can pass the initiative or make a proposal (***initiator*** nodes), and nodes at which can accept or reject the current proposal (***responder*** nodes). A ***strategy*** σ_i for player i specifies for each node a probability distribution with finite support over his possible actions at that node. A ***strategy profile*** σ specifies a strategy for each player. The strategy profile σ is a ***Nash equilibrium*** if, for each player i σ_i is a best response to σ and is a ***subgame perfect equilibrium*** if σ is a Nash equilibrium in every subgame. We call σ_i a stationary strategy for player i if σ_i prescribes the same action at every node where he has the initiative and σ_i prescribes the same action at every node where he is to respond to the same proposal made by the same player with the same set of players having already accepted.

Outcomes and Prices

Each terminal node in the game corresponds to a proposal (S,q^S) which has been made by some player in S and accepted by every player in S. A strategy profile $\sigma = (\sigma_1,\ldots,\sigma_n)$ determines a set (possibly empty) of proposals $\mathcal{P}(\sigma)$ corresponding to the terminal nodes that are reached with positive probability. We call the proposals in $\mathcal{P}(\sigma)$ ***bargaining outcomes***[2] of σ.

We say the set $\mathcal{P}(\sigma)$ of bargaining outcomes is ***price generated*** by $p(\sigma) \in \mathbf{R}^N$ if for every proposal $(S,q^S) \in \mathcal{P}(\sigma)$, $q^S = p^S(\sigma)$. We refer to $p(\sigma)$ as the ***price vector*** of σ.

Aspirations

For each player $i \in N$, we say that $p \in \mathbf{R}^N$ is ***realizable for i*** if there is a coalition S with $i \in S$ such that $p^S \in V(S)$. We say that p is ***realizable*** if p is realizable for every player i.

We say that $p \in \mathbf{R}^N$ is ***maximal for player i*** if for every coalition S containing i, and every vector $q^S \in \mathbf{R}^N$ such that $q^S > p^S$ then $q^S \notin V(S)$. We say that p is ***maximal*** if it is maximal for every player $i \in N$. We say $p \in \mathbf{R}^N$ is an ***aspiration*** if it is both realizable and maximal.

For $p \in \mathbf{R}^N_+$ we use $\mathbf{C}(p)$ to denote the coalitions in $\mathbf{C}$ which can realize p, i.e., $\mathbf{C}(p) = \{ S \in \mathbf{C} \mid p^S \in V(S)\}$. We use $\mathbf{C}_i(p)$ to denote those coalitions of $\mathbf{C}(p)$ that contain player i. Notice that when p is realizable, $\mathbf{C}_i(p)$ is nonempty for each player i.

Bennett [1990] shows that, for every NTU game and every *stationary* subgame perfect strategy profile σ, the set of bargaining outcomes $\mathcal{P}(\sigma)$ is price generated and that such price vectors are aspirations. Moreover, every aspiration is the price vector of a *stationary* subgame perfect strategy profile.

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2. In Bennett [1990] $\mathcal{P}(\sigma)$ is referred to as the set of possible outcomes.

3. FOLK THEOREMS

By tradition, the term "folk theorem" has come to mean a proposition that "anything conceivable can happen". In this context there are two possible folk theorems: (1) that "anything" can be the price vector of a subgame perfect strategy profile or (2) that "any" payoff vector for any coalition can be a subgame perfect bargaining outcome. In this section we prove versions of each. In the first folk theorem, we show that any realizable and individually rational payoff vector can be the price vector of a subgame perfect strategy profile. (Conversely: price vectors are realizable by definition; equilibrium behavior implies that they are individually rational.) It follows as a simple corollary that any individually rational payoff vector for any coalition occurs with positive probability as a bargaining outcome; this is a weak form of the second folk theorem. A stronger form of the folk theorem answers the question: Given an outcome (S,q^S), when is there a subgame perfect strategy profile such that (S,q^S) is its *only* bargaining outcome? The answer is: if and only if q^S is individually rational for players in S and 0^{N-S} is individually rational for players not in S.

We conclude the section by turning our attention to the question of whether all subgame perfect strategy profiles are price-generated. Although all *stationary* subgame perfect strategy profiles are price-generated, and the strategy profiles we construct to prove the folk theorems are price-generated, we show that some subgame perfect strategy profiles are *not* price-generated.

The basic building block of the strategies to follow are "price strategies". Let $p \in R^N_+$. We say that σ^p_i is a ***price strategy*** (with parameter p) for player i if:

1. At each of i's initiator nodes, σ^p_i assigns 0 probability to passing the initiative to another player.
2. At each of i's initiator nodes, if σ^p_i assigns a nonzero probability to making the proposal (S,q^S) then $q^S = p^S$. (We do not insist that σ^p_i assigns positive probability to all proposals (S,q^S) with $q^S = p^S$.)
3. At each of his responder nodes, player i responds to the proposal (S,q^S) by accepting whenever $q^S_j \geqslant p_j$ for every player j of S (including i) who has not yet accepted the proposal and rejecting otherwise.

We say that $\sigma^p \in \Sigma$ is a ***price strategy profile*** (with parameter p) if for every player i, σ^p_i is a price strategy with parameter p. For each $p \in R^N$, let Σ^p denote the set of all price strategy profiles corresponding to the vector p. Notice that many price strategy profiles correspond to a single parameter; they differ by the probabilities assigned to proposing the various coalitions.

The following is our first version of the folk theorem.

Theorem 1: *Every individually rational and realizable payoff vector is the price vector of a subgame perfect equilibrium strategy profile.*

Proof: Outline of the proof. Let r be individually rational and realizable. We construct a strategy profile σ^* with price vector r, (i.e., $r = p(\sigma^*)$) and prove that it is subgame perfect. The construction proceeds in two steps. In the first step, for every proposer and every proposal, we find a "punishment" aspiration with the property that any price strategy profile with that parameter "punishes" the player for making the proposal. In the second step we select a price strategy for r and for each punishment aspiration and label every nonterminal node with either r or the appropriate punishment aspiration. For each node η, the desired strategy profile σ^* is given by $\sigma^*(\eta) = \sigma^{p*}(\eta)$ where $p*$ is the label of the node η , i.e., if it is i's turn to move at η, σ^* prescribes the action specified by the price strategy for player i with parameter $p*$ at the node η. Three lemmas are then used to show that this σ^* is subgame perfect.

Step 1: For each proposer $\hat{\imath}$ and each proposal (S,q^S) we construct a punishment aspiration $\beta(S,q^S,\hat{\imath})$ such that: (1) $\beta_{\hat{\imath}}(S,q^S,\hat{\imath}) = \bar{v}_{\hat{\imath}}$; (2) there is a player $i*$ in S with $\beta_{i*}(S,q^S,\hat{\imath}) \geqslant q^S_{i*}$ (such a player would not lose by rejecting (S,q^S) and enforcing the punishment aspiration $\beta(S,q^S,\hat{\imath})$; and (3) if $S \neq [\hat{\imath}]$ and q^S is individually rational for player $\hat{\imath}$ then $i* \neq \hat{\imath}$.

For each coalition S and each player $k \in S$ we define a function $\gamma^S{}_k : \mathbf{R}^N_+ \to \mathbf{R}_+$ by:

$$\gamma^S_k(x) = 0, \text{ if } (x_1,\ldots,x_{k-1}, 0, x_{k+1},\ldots,x_n)^S \notin V(S);$$

$$\gamma^S_k(x) = \sup\{ t \mid (x_1,\ldots,x_{k-1}, t, x_{k+1},\ldots,x_n)^S \in V(S)\} \text{ otherwise.}$$

We define $\gamma_k : \mathbf{R}^N_+ \to \mathbf{R}_+$ by:

$$\gamma_k(x) = \sup\{ \gamma^S_k(x) \mid k \in S, \; S \text{ in } N\}.$$

$\gamma_k(x)$ is the largest value for t consistent with the vector $(x_1,\ldots,x_{k-1}, t, x_{k+1},\ldots,x_n)$ being realizable for player k. (Keep in mind that $0 \in V(k)$ so that $(x_1,\ldots,x_{k-1}, 0, x_{k+1},\ldots,x_n)$ is always realizable for player k.)

For each player k let $\phi_k(x)$ be the vector obtained from x by replacing x_k by $\gamma_k(x)$; i.e., $\phi_k(x) = (x_1,\ldots,x_{k-1}, \gamma_k(x), x_{k+1},\ldots x_n)$. Thus each ϕ_k is a map from R^N_+ onto itself.

Each map ϕ_k enjoys the following properties:

(i) $\phi_k(x)$ is realizable and maximal for player k.

(ii) if x is realizable for player k, then $\phi_k(x) \geqslant x$;

(iii) if x is maximal for player k, then $\phi_k(x) \leqslant x$;

(iv) if x is maximal for player ℓ, then $\phi_k(x)$ is maximal for player ℓ (whether $\ell = k$ or $\ell \neq k$).

The first three properties are easily checked. To see that the fourth is true, write $y = \phi_k(x)$ and suppose that y is not maximal for ℓ. Then there is a coalition S containing player ℓ with $y^S \in \text{INT } V(S)$. If $k \in S$, this implies that y is not maximal for player k, which contradicts property (i). On the other hand, since applying ϕ_k to x alters at most the k-th component, if $k \notin S$ then $y^S = x^S$ which means that $x^S \in V(S)$ and this contradicts the assumption that x is maximal for player ℓ.

For each $k = 1, 2,\ldots,n$ define Φ^k: $R^N_+ \to R^N_+$ as follows:

$$\Phi^k = \phi_k \circ \phi_{k-1} \circ\ldots\phi_2 \circ \phi_1$$

Let $\bar{m} \in R_+$ be a number sufficiently large that each $V(S)$ fits in a cube with a diagonal of length $\bar{m}$. For each coalition S let m^S ($\in R^S$) denote the vector which assigns $\bar{m}$ to each player in S .

Let (S,q^S) be a proposal of player $\hat{\imath}$ and set $s = |S|$. We renumber the players (if necessary) so that $\hat{\imath} = s$, $S = \{1,2,\ldots,s\}$, and $N-S = \{s+1,\ldots,n\}$.

Define $x \in R^N$ by $x_i = q^S_i$ for $i = 1,\ldots,s-1$, $x_s = \bar{v}_{\hat{\imath}}$ and $x_i = \bar{m}$ for $i = s+1,\ldots,n$. Set $z = \Phi^s(x)$. By repeated application of properties (i) and (iv) (above) we see that z is maximal for all $i \in S$ and realizable for s.

We first prove that player $s = \hat{\imath}$ is limited to his individually rational payoff, i.e., $z_s = \bar{v}_s$. Suppose not; then, since z is maximal for s, $z_s > x_s = \bar{v}_s$. By (i), $z = \Phi^s(x)$ is realizable for s; let S'' be a coalition containing s with $z^{S''} \in V(S'')$. Clearly S'' cannot contain a player from $N-S$ (since each player in $N-S$ demands $\bar{m}$). Thus S'' is a subset of S. Since $z_s > \bar{v}_s$, $S'' \neq [s]$. Let $i \neq s$ be a player of S''. Repeated applications of (i) and (iv) show that $\Phi^{s-1}(x)$ is maximal for player i hence $((\Phi^{s-1}(x))^{S''-s}, \bar{v}_s)$ -- an element of $R^{S''}$ -- is not in the interior of $V(S'')$. Since $z_s > \bar{v}_s$, the vector

$z^{S''} = ((\Phi^{s-1})^{S''-s}, z_s)$ cannot be in $V(S'')$, a contradiction. This contradiction establishes that $z_s = \bar{v}_s$.

We now select a player to fill the rôle of $i*$. There are two cases to consider.

Case 1: $q^S_{\hat{i}} \geqslant \bar{v}_{\hat{i}}$ and $S \neq [\hat{i}]$. Then $x^S \in V(S)$ and by property (ii), there is at least one player $i \neq \hat{i}$ in S such that $z_i \geqslant x_i$. Let $i*$ be the last such player. By property (i), $\Phi^{i*}(x)$ is realizable for player $i*$. Let $S*$ be a coalition for which $\Phi^{i*}(x)$ is realizable. Since player $i*$ was the last player with $z_i \geqslant x_i$, and $z_s = \bar{v}_s$ and since each ϕ_j only changes player j's component, $z^{S*} \in V(S*)$.

Case 2: Either $q^S_{\hat{i}} < \bar{v}_{\hat{i}}$ or $(S,q^S) = ([\hat{i}],\bar{v}_i)$. For either possibility $z_s \geqslant q^S_i$. Let $i* = \hat{i}$ and let $S* = [s]$.

Renumbering the players, if necessary, we assume that $i* = 1$. We next show that $\Phi^n(z)$ is an aspiration. Repeated applications of properties (i) and (iv) proves that $z = \Phi^s(x)$ is maximal for every player in S. By construction of $\bar{m}$, z is maximal for every player in N-S. Since z is realizable and maximal for $i* = 1$, properties (ii) and (iii) imply $\phi^1(z) = z$ so $\Phi^1(z)$ is realizable for player 1. By property (iv), $\Phi^1(z)$ is maximal for player 1. By (iii), $\Phi^2(z) \leqslant \Phi^1(z) = \phi^1(z)$ so $\Phi^2(z)$ is also realizable for player 1. Continuing in this way, we see that $\Phi^k(z)$ is realizable and maximal for all players with indices k and below. Hence $\Phi^n(z)$ is realizable and maximal for all players and therefore an aspiration. Set $\beta(S,q^S,\hat{i}) = \beta = \Phi^n(z)$. By construction $\beta_{i*} \geqslant x_{i*}$ and therefore $\beta_{i*} \geqslant q^S_{i*}$. An argument similar to the one above shows that if $\hat{i} \neq i*$, $\beta_{\hat{i}} = z_{\hat{i}} = \bar{v}_{\hat{i}}$. By construction β satisfies (1) $\beta_{\hat{i}} = \bar{v}_{\hat{i}}$, (2) there exists an $i* \in S$ with $\beta_{i*} \geqslant q^S_{i*}$, and (3) if q^S is individually rational for i and $S \neq [\hat{i}]$ then $i* \neq \hat{i}$. Hence β is the desired aspiration. We will say that $\beta = \beta(S,q^S,\hat{i})$ is the ***punishment aspiration*** for the proposal (S,q^S) by player i. This completes step 1.

Step 2: Loosely speaking we can describe the strategy profile that implements the price vector r in the following way. Every player makes proposals and accepts proposals consistent with r until a player, call him i, deviates by making a proposal (S,q^S) which is not consistent with r (all other deviations from the prescribed strategy are ignored). Then all players make proposals and accept proposals consistent with the punishment aspiration $\beta(S,q^S,i)$ until a player, call him i' deviates by making a proposal $(S',q^{S'})$ which is not consistent with $\beta(S,q^S,i)$. Then all players make proposals and accept proposals consistent with the punishment aspiration $\beta(S',q^{S'},i')$. The play continues in this way with the last deviator (if any) being punished until the end (if it ends) of the game.

In order to make this precise we next label every node η with a vector $p*(\eta) \in \mathbf{R}^N$ which we refer to as the "required" payoff vector at that node. Fix a node $\bar{\eta}$ in the tree. We label $\bar{\eta}$ by labeling every node in the path from the initial node to $\bar{\eta}$. The initial node is labeled r and every subsequent node on the path is labeled r until we come to $\bar{\eta}$ or to a node η_1 where some player, call him i_1 makes a proposal, call it (S,q^S), such that $q^S \neq r^S$. The node η_1 is labeled r and the following node is labeled $\beta(S,q^S,i_1)$. Every subsequent node is labeled $\beta(S,q^S,i_1)$ until we come a node, η_2 at which a player, call her i_2, makes a proposal, call it $(S',q^{S'})$ with $q^{S'} \neq p_i^{S'}$. In this case the node η_2 is labeled $\beta(S,q^S,i_1)$ and the following node is labeled $\beta(S',q^{S'},i_2)$. Continuing in this way, we eventually reach and label the node $\bar{\eta}$. In this way we label each node η with the required payoff vector, $p*(\eta)$.

For each $p \in \mathbf{R}^N$, fix a price strategy profile $\sigma^p \in \Sigma^p$. Define σ^* by $\sigma_i^*(\eta) = \sigma_i^p(\eta)$ for $p = p*(\eta)$ for every player i and every node for player i. We assert that σ^* is a subgame perfect strategy profile with a price vector of r.

To see that σ^* implements r notice that each player's first initiator node is labeled r, so each player is to play according to σ^r and therefore make proposals of the form (S,r^S). Given these actions, the succeeding responder nodes are all labeled r so each responder is to play according to σ^r and therefore accepts the proposal (S,r^S). Continuing in this way, it is clear that all nodes along the equilibrium path are labeled r so every proposal made with positive probability is form (S,r^S). Hence r is the price vector of σ^r.

To show that σ^* is subgame perfect, we show that for each player i, and each decision node of player i, σ_i^* is a best response to $\sigma^*_{)i(}$. The proof proceeds in three Lemmas. The first asserts that if i makes an acceptable proposal then his payoff is no more than his component of the node-label. The second asserts that σ_i^* is a best response to $\sigma^*_{)i(}$ at i's initiator nodes and the third asserts that σ_i^* is a best response at i's responder nodes.

Lemma 1: ***If i makes a proposal (S,q^S) that has a positive probability of being accepted given $\sigma^*_{)i(}$, then $q_i^S \leq p_i^*(\eta)$.***

Proof of Lemma 1: Suppose first that $S = [i]$. Since every label is individually rational (all aspirations are individually rational and r is individually rational) $q_i^S \leq p_i^*(\eta)$ as desired. (Recall that all proposed payoff vectors must be feasible for their coalitions.) Suppose next that player i makes the proposal (S,q^S) at the (initiator) node η with the label $p*(\eta)$ and designates player $j \neq i$ as the responder (either as part of the proposal or else after responding himself) and that this action leads to the node η_j, a responder node for player j. If $q^S = p*(\eta)^S$

then clearly $q_i \leq p_i*(\eta)$. If $q^S \neq p*(\eta)^S$, then $p*(\eta_j) = \hat{p}(S,q^S,i)$ is a punishment aspiration. Since player j follows $\sigma^*{}_j$, j accepts only if $q^{S-i} \geq p*(\eta_j)^{S-i}$. Since $p*(\eta_j)$ is an aspiration and therefore maximal, if j accepts the proposal with positive probability, it must be that $q_i^S \leq p_i*(\eta_j)$. Since $p*(\eta_j)$ is a punishment aspiration for player i, $p_i*(\eta_j) = \bar{v}_i$. Since every label, and in particular $p*(\eta_j)$, is individually rational, $q_i^S \leq p_i*(\eta_j)$ so $q_i^S \leq p_i*(\eta)$, as desired.■

Lemma 2: ***The strategy σ_i^* is a best response to $\sigma^*)i($ at every initiator node for player i.***

Proof of Lemma 2: Consider any $\sigma' \in \Sigma_i$ and consider any terminal node ζ that is reached from η_i with positive probability given σ^*/σ'. Let (S,q^S) be the proposal made by player k at node η whose acceptance leads to ζ. As we have shown in the previous lemma, if $k = i$ then $q_i^S \leq p*(\eta)$. If $k \neq i$, he follows σ^* so $q^S = p*(\eta)$. Hence $h_i(\zeta) = p_i*(\eta)$ if $i \in S$ and $h_i(\zeta) = 0$ otherwise.

However since every player $j \neq i$ is following σ^* if the node η has positive probability of being reached from η_i given σ^*/σ', then the label of η is either $p*(\eta_i)$ or $\hat{p}(S',q^{S'},i)$ for some $S' \in C_i$, and some $q^{S'} \in V(S')$. In either case $p_i*(\eta) \leq p*(\eta_i)$. Hence $E_i(\eta_i|\sigma^*/\sigma') \leq p_i*(\eta_i) = E_i(\eta_i|\sigma^*)$, so σ_i^* is a best response to $\sigma^*)i($ at the initiator node η_i. ■

Lemma 3: ***The strategy σ_i^* is a best response to $\sigma^*)i($ at every responder node for player i.***

Proof of Lemma 3: At the node η, i is to respond to a proposal (S,q^S); let S' be the set of players in S (including player i) who have not yet accepted the proposal. In order to see that $\sigma^*{}_i$ is a best response to $\sigma^*)i($ at η, it is convenient to consider three cases.

Case 1: $q_j^S \geq p_j^*(\eta)$ for every $j \in S'$. In this case, σ^* calls for every player in S' to accept (S,q^S) so $E_i(\eta \mid \sigma^*) = q_i^S \geq p*_i(\eta)$. On the other hand, if player i rejects (S,q^S), then he becomes the initiator at a node η' that is an immediate successor to η. By Lemma 2, σ_i^* is a best response to $\sigma^*)i($ at the node η', so player i's expected payoff at η' is at most $p^*(\eta')$. However, given our labeling, $p*(\eta') = p*(\eta)$, so i's expected payoff at η' is at most $p_i^*(\eta)$. We conclude that if player i deviates from σ_i^* and rejects (S,q^S), then his expected payoff is at most $p_i^*(\eta)$. Therefore $E_i(\eta \mid \sigma_i^*) \leq p_i^*(\eta) \leq q_i^S = E_i(\eta \mid \sigma^*)$; i.e., σ_i^* is a best response to $\sigma^*)i($ at η.

Case 2: $q_i^S < p^*{}_i(\eta)$ and $q_j^S \geq p_j^*(\eta)$ for every $j \in S'-[i]$. In this case, σ_i^* calls for player i to reject (S,q^S) and make some (counter) proposal consistent with $p*(\eta)$. Since this counter proposal will certainly be accepted (given that all other players follow σ^*), $E_i(\eta \mid \sigma^*) = p_i^*(\eta)$. On the other hand, if player i deviates from σ^*_i and accepts (S,q^S), so will all players in $S'-[i]$, so player i's expected payoff will be q_i^S. Since $q_i^S < p_i^*(\eta) = E_i(\eta \mid \sigma^*)$, it is again the case that σ_i^* is a best response to $\sigma^*_{)i(}$ at η.

Case 3: $q_j^S < p_j^*$ for some $j \in S'-[i]$. In this case, σ_i^* calls for player i to reject (S,q^S) so $E_i(\eta \mid \sigma^*) = p_i^*(\eta)$. If player i deviates from σ^* be accepting the proposal (S,q^S), it will certainly be rejected by some player $k \in S'-[i]$. (Indeed it will be rejected by whomever i designates as the next respondent.) Player k will then become initiator and, playing according to σ^* will make a proposal of the form $(T, p*(\eta)^T)$. If $i \notin T$, then this proposal will certainly be accepted by all players in T, so i's payoff will be $0 \leq p_i^*(\eta)$. If $i \in T$, then this proposal will be accepted by all players in $T-[i]$, so it follows from cases 1 and 2 that i's expected payoff will be at most $p_i^*(\eta) = E_i(\eta \mid \sigma^*)$. Once again, σ_i^* is a best response to $\sigma^*_{)i(}$ at η. This completes the proof of Lemma 3 and with it the proof of Theorem 1.■

We now turn to our second version of the folk theorem.

Theorem 2: ***Let (S,r^S) be a proposal. There exist a subgame perfect strategy profile σ^* such that (S,r^S) is the unique bargaining outcome for σ , if and only if $r = (r^S,0^{N-S})$ is individually rational for all players.***

Proof: To show that there is a subgame perfect strategy profile for which (S,r^S) is the unique bargaining outcome, we alter players' price strategies and then follow the arguments in the proof of Theorem 1.

In Step 2 of the proof of Theorem 1, we associated to each $p \in R^N$, an arbitrary price strategy profile σ^p. We alter the assignment in the following ways: (1) For $p = r$ and for each player i in S, we select the particular price strategy for player i that assigns probability 1 to proposing (S,r^S) whenever i has the initiative. (2) For every parameter p and every player i *not* in S we alter player i's strategy so that whenever player i has the initiative, he passes the initiative to one of the players in S.[3] With these alterations, we may continue as in Theorem 1 to define the composite strategy σ^*. Notice first that (S,r^S) is the unique bargaining outcome of $\sigma*$ -- since (S,r^S) is the

3. More formally, player $i \notin S$ assigns 0 probability to making any proposal and positive probability to passing the initiative to player j only if $j \in S$.

only proposal made with positive probability along the equilibrium path. To show that σ^* is subgame perfect, follow the proofs of the Lemmas keeping the following points in mind. The altered strategies for players $i \notin S$ are "as good" as the unaltered strategies, since either way (given the strategies of other players) player i's expected payoff is his individually rational level 0. No player (in or not in S) can take advantage of these altered strategies in order to improve his payoff because any proposal made which would give the proposer a payoff higher than his componenet of the current node label will be punished.

To complete the proof, we show why $(r^S, 0^{N-S})$ must be individually rational. If (S, r^S) is the unique bargaining outcome of a strategy profile σ, then each player $i \in S$ obtains exactly r_i^S when nature selects him as initiator and each player $i \in N-S$ obtains 0 when nature selects him as initiator. Since player i can always obtain at least $\bar{v}_i$ when his is the initiator (by proposing $([i], \bar{v}_i)$) and nature selects each player to the initiator with positive probability, in order for σ_i to be a best response to $\sigma_{)i(}$ we must have that $r_i^S \geqslant \bar{v}_i$ for $i \in S$ and $0 \geqslant \bar{v}_i$ for $i \notin S$.∎

The first folk theorem asserts that every individually rational and realizable vector is the price vector of a subgame perfect strategy profile. It is easy to construct subgame perfect strategy profiles that aren't price-generated. For example, for each player i, choose a price vector $s(i) \in \mathbf{R}^N$ that is individually rational and realizable. Consider the strategy profile which is identical to the one constructed in Theorem 1 except that the price vector to be supported (i.e., the vector "r") depends on the player initially chosen by nature, i.e., in the subgame in which the i-th player is chosen, all players follow the strategy constructed in Theorem 1 for $r = s(i)$. Since no player controls the choice of nature, this composite strategy profile is also subgame perfect. Since different subgames have different price vectors, no single price vector generates its bargaining outcomes.

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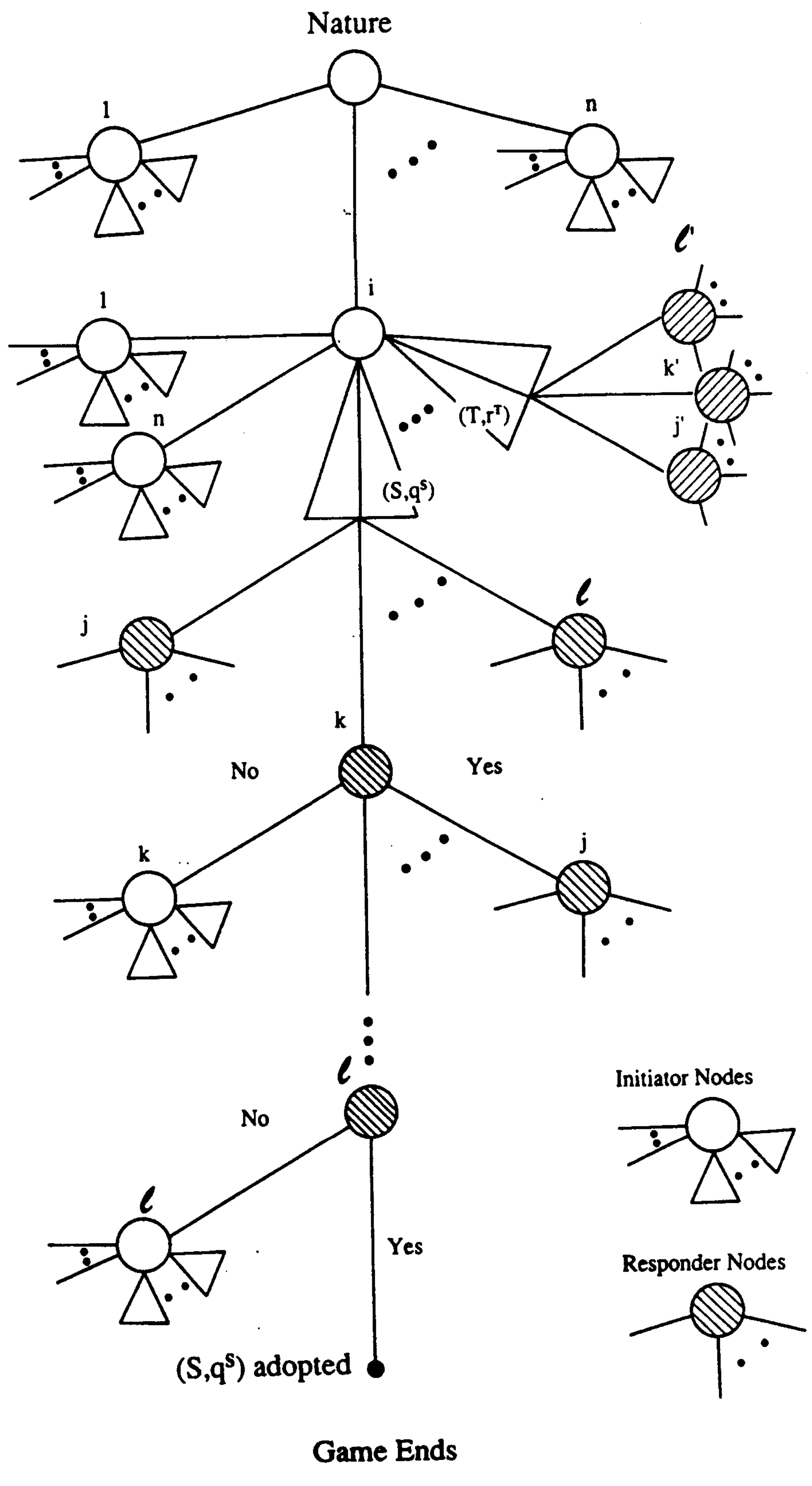

FIGURE 1: The Game Tree of the Proposal-Making Model