MULTILATERAL BARGAINING PROBLEMS

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ABSTRACT

In many situations, there are gains from forming coalitions but conflict over which coalitions to forms and how the gains should be distributed. The distribution of gains presents each potential coalition with a bargaining problem, but the presence of conflicts over which coalitions to form means that these bargaining problems cannot be treated in isolation. This paper presents an approach to such *multilateral bargaining problems* . A *solution* to such a multilateral bargaining problem consists of an agreement in each coalition which is consistent with the bargaining process in every coalition. We show that, under mild conditions, solutions exist and are determined by reservation prices, and characterize the range of possible solutions.

KEYWORDS:

bargaining problems

multilateral bargaining

cooperative games

characteristic function games

NTU games

bargaining solutions

Nash bargaining solution

1. INTRODUCTION

In many situations in Economics and Political Science, there are gains from forming various coalitions but conflicts about which coalitions should be formed and how the gains should be shared. Examples abound: the formation of a government by political parties in a parliamentary system, trade in an exchange economy, the formation of jurisdictions and the production of public goods in a local public goods economy, etc. In these and many other situations, we would like to predict the coalitions that are likely to form and the rewards that agents are likely to receive for their participation in these coalitions.

In such situations, the distribution of gains within each coalition is determined by bargaining within the coalition. If - as is the case in many situations of interest - each agent can actually participate in at most one coalition, participation in a particular coalition entails an opportunity cost: the foregone rewards the agent could have received from participating in another coalition. Assuming that the bargaining within each coalition takes these opportunity costs into account, the bargaining problem of each coalition is related to the bargaining problem of every other coalition. We call such an interrelated set of bargaining problems a *multilateral bargaining problem* .

The class of multilateral bargaining problems includes situations in which there is a unique coalition whose formation can yield positive gains, so that the only conflict is over the distribution of these gains. Such situations amount to simple bargaining problems, and have been extensively studied by Nash [1950] and many subsequent writers. In this paper, our interest is in situations in which there are *many* coalitions whose formation would yield positive gains. In such situations, there will typically be conflict over *which* coalitions to form, as well as over the distribution of gains. The object of this paper is to examine the effects of these interrelated conflicts.

The ingredients of a multilateral bargaining problem are: a set of agents (players) and, for each coalition of players, a description of the potential gains that can be realized by the coalition and a summary of the bargaining process within the coalition. We formalize the potential gains that can be realized by a coalition as a set of attainable utility vectors; a decision about the distribution of gains within the coalition is thus a choice of a single vector from this set of attainable utility vectors. We formally summarize the bargaining process within the coalition by a bargaining function, a mapping from opportunity costs to agreement vector for the coalition. We view the sets of attainable utility vectors and the bargaining functions as parts of the description of a multilateral bargaining problem, in much the same way that endowments and utility functions are part of the description of an exchange economy. [1]

A solution to a multilateral bargaining problem specifies an agreement utility vector for each coalition which is consistent with the bargaining within every coalition. We think of a multilateral solution as a set of consistent conjectures players hold about the "market"; i.e., a set of rational expectations about the eventual outcomes of bargaining within and across coalitions.

The agreements of a multilateral bargaining solution are feasible distributions of gains for some, but not necessarily all, coalitions. For "affluent" coalitions (those coalitions whose agreements are feasible), the agreement represents the agreed-upon distribution of gains from forming the coalition, conditional on the actual formation of the coalition. For "less affluent" coalitons (those whose agreements are infeasible), the agreement represents a vector of final demands which are mutually incompatible.

It might appear that at this level of generality, nothing could be said about the nature of solutions to multilateral bargaining problems, but this is not so. In Section 2, we show that multilateral bargaining problems always have solutions (using very weak and standard assumptions about sets of attainable utility vectors and bargaining functions). In Section 3, we show that the competition between coalitions enforces a homogeneity of payoffs across coalitions: At any solution, the agreement utilities for each player are the same in every (potential) coalition of which he is a member. As a consequence, each multilateral bargaining solution can be characterized by a "reservation price" for each player. In Section 4, we characterize the range of possible multilateral solutions by showing that the reservation price vector of every multilateral bargaining solution is an aspiration (in the sense of Bennett and Zame [1988]), and conversely, that every aspiration is the reservation price vector of a multilateral bargaining solution for some specification of (admissible) bargaining functions.

In Section 5 we present a number of examples to develop some intuition about multilateral bargaining problems and solutions. Finally, Section 6 concludes the paper by discussing related cooperative and noncooperative models.

2. ELEMENTS OF MULTILATERAL BARGAINING

In this section we give a formal description of multilateral bargaining problems and their solutions, and establish the existence of solutions.

The set of *players* (*agents*) is a finite set $N = \{1, 2, \ldots n\}$. We write C for the set of nonempty subsets of N (the set of *coalitions*). For each $S \in C$, $V(S) \subset \mathbb{R}^S_+$ is the set of attainable utility vectors for the coalition S. Throughout, we assume that $V(S)$ is a compact, strongly comprehensive[2] subset of $\mathbb{R}^S_+$ that contains the origin.

In our approach to multilateral bargaining problems, the agreements within a coalition and the opportunities available outside a coalition are interdependent. The agreement reached in the coalition S depends on the opportunities of its players in other coalitions, and these opportunities depend in turn on the opportunities of their players in their other coalitions, including the coalition S. We first discuss how agreements are reached within a coalition, given an "outside option vector" which reflects the opportunities available to the members of the coalition in other coalitions.

BARGAINING FUNCTIONS

A coalition's bargaining function specifies the agreement utility vector which is the result of bargaining within the coalition, given players' potential gains within the coalition and their opportunities in other coalitions. A coalition's bargaining function may reflect the coalition's standards of fair division, the institutional rules governing bargaining in the coalition, or the relative bargaining skills of its members. [3]

For the coalition S, we summarize its players opportunities in other coalitions by a single "outside option vector" $d^S \in \mathbb{R}^S_+$. (Since one opportunity available to each player is that of remaining alone and since $V(i)$ is a subset of $\mathbb{R}_+$, each player's outside option value is non-negative.) Agreements will always be non-negative (by individual rationality), so the bargaining function for the coalition S is a function $f^S : \mathbb{R}^S_+ \rightarrow \mathbb{R}^S_+$. We assume that it enjoys the following properties:

1. For each outside option vector $d^S \in V(S)$, the agreement vector $x^S = f^S(d^S)$ satisfies:
   (a) **Individual rationality:** $x^S \geq d^S$;
   (b) **Pareto optimality:** x^S is on the Pareto frontier[4] of $V(S)$.
2. **Agreeing to disagree:** For each (infeasible) outside option vector $d^S \notin V(S)$, the agreement vector is the outside option vector; i.e., $x^S = f^S(d^S) = d^S$.
3. **Continuity:** The function f^S is a continuous function (of the outside option vector d^S).

These assumptions require some explanation. If the outside option vector d^S belongs to $V(S)$ then there are attainable utility vectors for the coalition which allocate to each member of the coalition at least the utility of his outside opportunities; i.e., there are gains from making an agreement. In this case, the first assumption requires that the agreement $f^S(d^S)$ be efficient and allocate to each member of the coalition at least as much as he could obtain by *not* participating. In this case, the agreement $f^S(d^S)$ has the usual interpretation as the division of the gains, conditional on the formation of the coalition S .

The case $d^S \in V(S)$ is the only one considered in traditional bargaining theory. We allow for the possibility that the outside option vector d^S does not belong to $V(S)$, because the outside option vector d^S represents the opportunities of members of S in *other* coalitions, and it is certainly possible that these opportunities might be more attractive than any possibilities available within S . In this case, the second assumption says that the members of S "agree to disagree;" i.e. they agree to settle for their outside opportunities.

We view this agreement to disagree as the outcome of a bargaining process; the players of S negotiate, but their "final demands" are incompatible (given the resources of the coalition). Implicit here is the assumption that each member i of S would be willing to participate in S for a payoff of d^S_i if some other player(s) would take the necessary loss. [5]

Finally, the third assumption requires that small changes in players' outside options lead to small changes in the agreement.

Note that the Nash bargaining solution (and most traditional solutions to the simple bargaining problem), when extended to allow for infeasible outside option vectors, enjoys these properties.

For notational convenience, we let Q be an index set that contains a distinct index for each occurrence of a player in one of his coalitions (i.e., for each player position).[6] We identify an element $x \in \mathbb{R}^Q$ with a set of vectors, one for each coalition: $\{x^S : S \in C\}$, where $x^S \in \mathbb{R}^S$. If $f = \{f^S : S \in C\}$ is the set of bargaining functions, it is convenient to regard f as a function from $\mathbb{R}^Q$ to $\mathbb{R}^Q$ (with $f(x) = \{f^S(x^S)\}$).

OUTSIDE OPTIONS

We turn now to the question of how "outside options" are determined. In this context, if players in one coalition fail to reach an agreement, they have opportunities in other coalitions. Thus, for each coalition S and each player $i \in S$, we want to use as i's component of the outside option vector d^S the utility he would obtain if he broke off negotiations in S and took the initiative to form his best alternative coalition. Of course, i's alternatives depend on the agreements that will be reached in other coalitions. We assume that the players in S make accurate (and therefore identical) conjectures about these agreements. To see what this implies, fix a (conjectured) agreement x^T for each coalition $T \neq S$; given these conjectured agreements, what are the utilities of player i's alternatives?

If $i \in T$ and the agreement x^T is a feasible utility vector for the coalition T (i.e., $x^T \in V(T)$), then player i can certainly obtain $x^T{}_i$ in T .

If $i \in T$ and the agreement x^T is not feasible for T (i.e., $x^T \notin V(T)$), player i cannnot obtain $x^T{}_i$. In view of our previous discussion about the meaning of "agreeing to disagre," the most that player i can obtain in T is the largest utility which allows all of the *other members* of T to obtain *their* agreement utilities. That is, the utility to player i of the unattainable agreement x^T in the coalition T is $\max\{t_i : x^T/t_i \in V(T)\}$. (We use x^T/t_i to denote the vector obtained from x^T by replacing the i-th component by t_i .) If there is no value of t_i for which $x^T/t_i \in V(T)$, then the infeasible agreement x^T has no utility for player i ; by convention, we take 0 to be the maximum in this case. [7]

Formally, given agreements $\{x^T : T \neq S\}$ in all other coalitions, we define the *outside option vector* $d^S(\{x^T : T \neq S\})$ for the coalition S in the following way. For each $i \in S$ and each coalition $T \neq S$ with $i \in T$, set:

$$u^T{}_i(x^T) = \begin{cases} x^T{}_i & \text{if } x^T \in V(T) \\ \max\{0, t_i : x^T/t_i \in V(T)\} & \text{otherwise} \end{cases}$$

and

$$d^S{}_i(\{x^T : T \neq S\}) = \max\{u^T{}_i(x^T) : i \in T \text{ and } T \neq S\} .$$

By definition, d^S is a function from collections $\{x^T : T \neq S\}$ to $\mathbb{R}^S_+$; however, it is convenient to view d^S as a function defined on collections of agreement vectors for *all* coalitions (although d^S will not depend on x^S), so that d^S is a function from $\mathbb{R}^Q_+$ to $\mathbb{R}^S_+$. We refer to the function d^S as the *outside option function* for the coalition S . We write $d = \{d^S : S \in C\}$, and view d as a function from $\mathbb{R}^Q_+$ to $\mathbb{R}^Q_+$.

MULTILATERAL SOLUTIONS

A *multilateral bargaining problem* $\langle N,V,f \rangle$ specifies a set N of players and for each coalition $S \in C$, a strongly comprehensive set $V(S)$ of attainable utilities and a bargaining function f^S .

A *solution* for a multilateral bargaining problem specifies an agreement payoff vector for each coaltion which is consistent with the description of bargaining in every coalition. Formally, $x = \{x^S : S \in C\}$ is a *solution* to the multilateral bargaining problem $\langle N,V,f \rangle$ if $x^S = f^S(d^S(x))$ for every coalition $S \in C$.

A solution is a *stable* set of conjectures about the outcomes of the multilateral bargaining because, given the nature of bargaining in each coalition, no player can improve his payoff in any coalition by renegotiating an agreement.

THEOREM 1: Every multilateral bargaining problem has a solution.

PROOF: Fix the multilateral bargaining problem $\langle N,V,f \rangle$. We first show that the outside option function d^S of each coalition S is continuous. (This depends on the fact that each $V(S)$ is strongly comprehensive.)

The function d^S is continuous if each $d^S{}_i$ is continuous. For each coalition $S' \neq S$ with $i \in S'$, define $h^{S'} : \mathbb{R}^Q{}_+ \to \mathbb{R}$ by

$$h^{S'}(x) = \max\{t_i : t_i \in V(i) \text{ or } x^{S'}/t_i \in V(S')\} \quad .$$

Notice that $d^S{}_i(x) = \max\{h^{S'}(x) : S' \neq S\}$, so we need only show that each function $h^{S'}$ is continuous.

Fix a coalition T with $i \in T$. Let $\{x^n\}$ be a sequence in $\mathbb{R}^Q{}_+$ converging to x; we show that the sequence $h^T(x^n)$ converges to x. Since h^T is a bounded function, we may assume (passing to a subsequence if necessary) that $h^T(x^n)$ converges, to w, say; we wish to show that $w = h^T(x)$. For notational convenience, renumber the players in T with player $i = 0$ and the remaining members of T as $1,2,\ldots k$. Set $v_0 = \max\{t : t \in V(i)\}$, $z^n = (x^n)^T$ and $z = x^T$. Clearly, $z^n \to z$. If $h^T(x) \neq w$, there are two cases to consider.

Case 1: $h^T(x) < w = \lim h^T(x^n)$. By definition, $h^T(y) \geq v_0 \geq 0$ for every y, so $\lim h^T(x^n) > h^T(x) \geq v_0$ and hence $h^T(x^n) > v_0$ for n sufficiently large. Hence, $(h^T(x^n), z^n) \in V(T)$ for n sufficiently large. Since $V(T)$ is closed, we conclude that $(w,z) \in V(T)$. But this implies that $h^T(x) \geq w$, a contradiction.

Case 2: $h^T(x) > w$. By definition, $h^T(x^n) \geq v_0 \geq 0$ for each n. Hence $h^T(x) > v_0$, so $(h^T(x),z) \in V(T)$. Since $h^T(x) > w > 0$, strong comprehensiveness guarantees that (w,z) is in the interior of $V(T)$. This means that there is an ε-ball around (w,z) contained in $V(T)$ for some $\varepsilon > 0$. In particular, $(w + \varepsilon/2, z^n) \in V(T)$ for n sufficiently large. This implies that $h^T(x^n) \geq w + \varepsilon/2$ for n sufficiently large, again a contradiciton.

We conclude that $h^T(x) = w$, so that h^T is a continuous function for each T. Consequently, the outside option function d^S is continuous, for each S.

Let m be a number sufficiently large that each $V(S)$ fits inside a cube with diagonal of length m. Let Y be the Q-fold Cartesian product of $[0,m]$. The functions f and d both map Y into itself. By Brouwer's fixed point theorem, the mapping $f \circ d$ has a fixed point $x = \{x^S : S \in C\}$. For each $S \in C$, $x^S = f^S(d^S(x))$, so x is a solution to the multilateral bargaining problem $\langle N,V,f \rangle$, as desired. ∎

The definition of a solution leaves open the possibility that all agreements are infeasible (i.e., that all coalitions agree to disagree). As we shall see, this is never the case: each player belongs to at least one coalition whose agreement vector is feasible. We defer the proof of this fact to Section 4.

3. RESERVATION PRICES

In this paper we are primarily interested in situations in which there are gains from forming many different coalitions, but conflicts over which coalitions to form. In such a situation, one potential source of conflict is that the members of a coalition, having agreed on the division of gains within the coalition, would not all want to form the coalition. This would be the case if forming this coalition would maximize the rewards of some, but not all, members of the coalition. In this section, we show that, becasue of the competition among coalitions, this potential source of conflict does not in fact exist: at any solution to a multilateral bargaining problem, if participation in a particular coalition maximizes the rewards of one player, it maximizes the rewards of all members of the coalition.

It is natural to state and view this result in terms of (reservation) prices. Given a multilateral bargaining problem $\langle N,V,f \rangle$ and a solution x, we say that player i has the *reservation price* p_i at x if he obtains precisely the payoff p_i in each coalition in which he can participate; i.e., $p_i = x^S{}_i$ for each coalition $S \in C$ for which $i \in C$. A vector $p \in \mathbb{R}^N$ of reservation prices *generates* the solution x if $p_i = x^S{}_i$ for every coalition S and every player $i \in S$. (We usually refer to p as the *reservation price vector* of the solution x.

THEOREM 2: Every multilateral solution is generated by a vector of reservation prices.

PROOF: Consider the multilateral bargaining problem <N,V,f> . Fix any player i . Let $v_i = \max\{t : t \in V(i)\}$ denote the maximum utility player i can obtain on his own. Let S be a coalition for which i's component of the agreement vector is maximal; i.e., $x^S_i \geq x^T_i$ for each T . We wish to show that in fact $x^S_i = x^T_i$ for each T . We fix T and distinguish three cases.

Case 1: $x^S \in V(S)$. Since x^S is a feasible utility allocation for S , i's outside option value d^T_i in T must be at least as large as x^S_i . Individual rationality implies that i's agreement value x^T_i in T must satisfy $x^T_i \geq d^T_i$. Hence $x^T_i \geq x^S_i$. Since x^S_i is maximal, we conclude that $x^S_i = x^T_i$.

Case 2: $x^S \notin V(S)$ and $x^S_i > v_i$. Let W be a coalition from which the outside option value d^S_i is obtained, and set $t^* = \max\{t_i : x^W/t_i \in V(W)\}$. Note that $r^* \leq x^W_i$. Since $x^S \notin V(S)$, members of S agree to disagree, and $d^S = x^S$. Thus, if $t^* < x^W_i$, then $t^* = d^s_i = x^S_i$ and $x^W_i > x^S_i$, contradicting maximality. Hence $t^* = x^W_i$, so that $x^W \in V(W)$ and $x^W_i = d^S_i = x^S_i$. We can now apply Case 1 (with W playing the role of S) to conclude that $x^T_i = x^W_i = x^S_i$.

Case 3: $x^S \notin V(S)$ and $x^S = v_i$. By definition, $d^T_i \geq v_i$. Since agreements are individually rational, $x^T_i \geq d^T_i$. Since x^S_i is maximal and $x^S_i = v_i$, we have $v_i \geq x^T_i \geq d^T_i \geq v_i$. Hence $x^T_i = v_i = x^S_i$.

In every case, player i receives the same agreement value in each coalition of which he is a member. Let this common value be p_i . Clearly, the vector $p = (p_1, \ldots p_n)$ of reservation prices defined in this way generates the solution x . ∎

Since, at a solution, a player's agreement utility is the same in each coalition in which he can participate, each player is indifferent between any two coalitions for which the agreement vector is feasible; these are the coalitions in which he can "obtain his price." This means that coalitions fall into two categories: coalitions in which players agree to disagree, and coalitions in which players can obtain their prices. Players uniformly agree that coalitions in the first category are undesirable and that coalitions in the second category are "equally desirable." In this way, a multilateral solution determines the coalitions that are unlikely and likely to form in a multilateral bargaining problem.

4. CHARACTERIZATION OF MULTILATERAL SOLUTIONS

Underlying the multilateral bargaining problem $\langle N,V,f\rangle$ is a cooperative game in characteristic function form without sidepayments, $\langle N,V\rangle$ with the same players, coalitions and sets of attainable utility vectors. We might view a cooperative game as the result of omitting the bargaining functions from the description of the multilateral bargaining problem; alternatively, we might view a multilateral bargaining problem as the result of adding the specification of bargaining functions to the description of a cooperative game. It is important to keep in mind that a single cooperative game underlies many different multilateral bargaining problems.

In this Section, we describe the possible range of multilateral solutions in terms of the underlying cooperative game. We show that, for every multilateral solution to the problem $\langle N,V,f\rangle$, the corresponding vector of reservation prices is an aspiration (definition below) for the underlying game $\langle N,V\rangle$. (As a consequence, we fulfill a promise made at the end of Section 2 by showing that the agreements of a multilateral solution are not all infeasible.) Conversely, every aspiration for the underlying cooperative game is the vector of reservation prices for a multilateral solution corresponding to some specification of admissible bargaining functions.

For a multilateral bargaining problem $\langle N,V,f\rangle$ (and for the underlying cooperative game $\langle N,V\rangle$), we say that a vector $p \in \mathbb{R}^N_+$

(of reservation prices) is *realizable* if for each player $i \in N$ there is a coalition S with $i \in S$ and $p^S \in V(S)$ (where p^S is the restiction of p to S). We say that p is *coalitionally efficient* if, for each coalition S , p^S is not in the interior of $V(S)$ (with respect to $\mathbb{R}^S_+$). The vector p is an *aspiration* if it is both realizable and coalitionally efficient (see Bennett and Zame [1988]). (Note that the definitions of realizability and coalitional efficiency do not depend on the bargaining functions; realizability and coalitional efficiency are entirely properties of p relative to the underlying cooperative game.)

THEOREM 3: The vector of reservation prices of any multilateral solution is an aspiration.

This result (together with the existence of reservation prices) implies that the agreement vectors of a multilateral solution cannot all be infeasible. Indeed, let $x \in \mathbb{R}^Q$ be a multilateral solution and let $p \in \mathbb{R}^N$ be the corresponding vector of reservation prices; by definition, $x^S = p^S$ for each coalition $S \subset N$. Since p is realizable, for each player $i \in N$, we can find a coalition S with $i \in S$ such that $x^S = p^S \in V(S)$; i.e., the agreement vector x^S is feasible for the coalition S . Moreover, since p is coalitionally efficient, there does not exist any coalition T for which $x^T = p^T$ lies in the interior of $V(T)$; so none of the agreements are Pareto inefficient.

PROOF OF THEOREM 3: Let x be a solution for the multilateral bargaining problem $\langle N,V \rangle$, and let p be the corresponding vector of reservation prices. We first show that p is realizable. If not, there is a player i such that $p^S \notin V(S)$ for every coalition S containing i . Since $p^S = x^S$, this means that players in S agree to diasagree. Hence $d^S = x^S$ for every coalition S containing i . Now, fix a coalition T containing i , and let W be any coalition from which the outside option value $d^T{}_i$ is obtained. Then $x^W/d^T{}_i \in V(W)$. On the other hand, $d^T{}_i = p^T{}_i = p_i = p^W{}_i = x^W{}_i$, so this tells us that $x^W \in V(W)$, a contradiction. We conclude that p is realizable.

To see that p is coalitionally efficient, note that the assumptions on bargaining functions (Pareto optimality when the outside option vector is feasible, agreeing to disagree when it is not) guarantee that no agreement vector x^S lies in the interior (relative to $\mathbb{R}^S{}_+$) of the feasible set $V(S)$. Since $x^S = p^S$, we conclude that p^S cannot lie in the interior of the feasible set $V(S)$; i.e., p is coalitionally efficient. Since p is both realizable and coalitionally efficient, it is an aspiration. ■

Theorem 3 tells us that the vector of reservation prices corresponding to a solution of a multilateral bargaining problem $\langle N,V,f \rangle$ is an aspiration for the underlying cooperative game $\langle N,V \rangle$. The converse (Theorem 4 below) tells us that all aspirations of the cooperative game $\langle N,V \rangle$ arise in this way. More precisely, for every aspiration p of the cooperative game $\langle N,V \rangle$, there is a set f of (admissible) bargaining functions such that p is the vector of reservation prices corresponding to a solution of the multilateral bargaining problem <N,V,f> .

THEOREM 4: Every aspiration is the vector of reservation prices of a multilateral solution for some specification of bargaining functions.

PROOF: Let p be an aspiration for the game $\langle N,V \rangle$, and set $x = \{p^S : S \in C\}$. We construct a set $f = \{f^S : S \in C\}$ of bargaining functions, and show that x is a solution for the multilateral bargaining problem $\langle N,V,f \rangle$. (It is clear that p is the vector of reservation prices corresponding to x .)

Fix a coalition S , and consider the vector $d^S(x)$; since x is fixed, $d^S(x)$ is a fixed vector in $\mathbb{R}^S_+$. We wish to define a bargaining function f^S . For $z \notin V(S)$, define $f^S(z) = z$. For $z \in V(S)$, we distinguish two cases:

Case 1: $d^S(x)$ is in the interior (relative to $\mathbb{R}^S_+$) of $V(S)$. We consider the ray $\{z + \lambda(p^S - d^S(x)) : \lambda \geq 0\}$. Since $d^S_i \leq x^T_i = p_i$ for each i , the vector $(p^S - d^S(x))$ is strictly positive (i.e., all of its coordinates are strictly positive). Hence this ray is strictly increasing in λ . Comprehensiveness of $V(S)$ implies that this ray meets the boundary of $V(S)$ at a unique point, which we define to be $f^S(z)$. It is easily checked that the function $f^S : \mathbb{R}^S_+ \to \mathbb{R}^S_+$ defined in this way satisfies our requirements for a bargaining function. (Pareto optimality of $f^S(z)$ when $z \in V(S)$ follows from the strong comprehensiveness of $V(S)$.)

Case 2: $d^S(x)$ is not in the interior (relative to $\mathbb{R}^S_+$) of $V(S)$. Consider the ray $\{z + \lambda p^S : \lambda \geq 0\}$. As before, this ray meets to boundary of $V(S)$ in a unique point, which we define to be $f^S(z)$. Again, it is easily checked that the function $f^S : \mathbb{R}^S_+ \to \mathbb{R}^S_+$ defined in this way satisfies our requirements for a bargaining function.

By construction, the functions f^S satisfy our requirements for bargaining functions, so taking $f = \{f^S : S \in C\}$ yields a multilateral bargaining problem $\langle N,V,f \rangle$. It is easily checked that x is a fixed point of the mapping $f \circ d$, and so is a solution, as required. ∎

It is easily checked that any vector in the core of a cooperative game is an aspiration; hence we have the following special case of Theorem 4.

COROLLARY 5: Every vector in the core of a cooperative game is the vector of reservation prices corresponding to a multilateral solution for some specification of bargaining functions.

5. EXAMPLES

In this Section, we present a few examples to give some insight into the nature of multilateral solutions. The data of a multilateral bargaining problem includes the specification of a bargaining function for each coalition. The most natural choices of bargaining functions are the solutions to the simple bargaining problem, such as the Nash bargaining solution (Nash [1950]), extended to allow for outside option vectors which are infeasible.

For instance, given the coalition S and the feasible set[8] $V(S)$, we define the *(extended) Nash bargaining function* $N^S : \mathbb{R}^S_+ \to \mathbb{R}^S_+$ in the following way: For $d^S \in V(S)$, $N^S(d^S)$ is the vector $x^S \in V(S)$ which maximizes the Nash product $\prod_{i \in S}(y^S_i - d^S_i)$, over all $y^S \in V(S)$; for $d^S \notin V(S)$, $N^S(d^S) = d^S$. Anticipating the discussion below, we define the *constrained Nash bargaining function* [9] $CN^S : \mathbb{R}^S_+ \to \mathbb{R}^S_+$ in the following way: For $d^S \in V(S)$, $CN^S(d^S)$ is the vector $x^S \in V(S)$ which maximizes the Nash product $\prod_{i \in S} y^S_i$ over all $y^S \in V(S)$ with $y^S \geq d^S$; for $d^S \notin V(S)$, $CN^S(d^S) = d^S$. The egalitarian E^S and Kalai-Smorodinsky K^S bargaining functions are defined by similarly extending the egalitarian and Kalai-Smorodinsky bargaining solutions (see Roth [1979] or Kalai [1985]).

If all coalitions use the Nash bargaining function, we frequently refer to the associated multilateral bargaining problem as a *multilateral Nash bargaining problem*, and to a solution as a *multilateral Nash solution*. Analogously, a *multilateral egalitarian (Kalai-Smorodinsky, constrained Nash) solution* is a solution to a multilateral bargaining problem in which each coalition's bargaining function is the extended egalitarian (Kalai-Smorodinsky, constrained Nash) bargaining function.

EXAMPLE 1: (See Figure 1, p. 24.) We consider a simple 4-player multilateral Nash bargaining problem $\langle N,V,f\rangle$ with sidepayments: the player set is $N = \{1,2,3,4\}$, the feasible sets $V(S)$ for coalitions are:

$$V[1,2] = \{(y_1,y_2): y_1 \geq 0 , y_2 \geq 0 , y_1 + y_2 \leq 40\}$$

$$V[1,4] = \{(y_1,y_4): y_1 \geq 0 , y_4 \geq 0 , y_1 + y_4 \leq 34\}$$

$$V[3,4] = \{(y_3,y_4): y_3 \geq 0 , y_4 \geq 0 , y_3 + y_4 \leq 20\}$$

$$V[3,2] = \{(y_3,y_2): y_3 \geq 0 , y_2 \geq 0 , y_3 + y_2 \leq 34\}$$

$$V(S) = \{0\} \quad \text{for all other coalitions } S \subset N ,$$

and the bargaining functions f^S for coalitions are the Nash bargaining functions defined above. (Note that the Nash, egalitarian, and Kalai-Smorodinsky solutions coincide in the sidepayment case). It is not hard to see that there is a unique multilateral Nash solution for this problem; the vector of reservation prices is $p = (22, 22, 12, 12)$, and the agreement and disagreement vectors for the four relevant coalitions are:

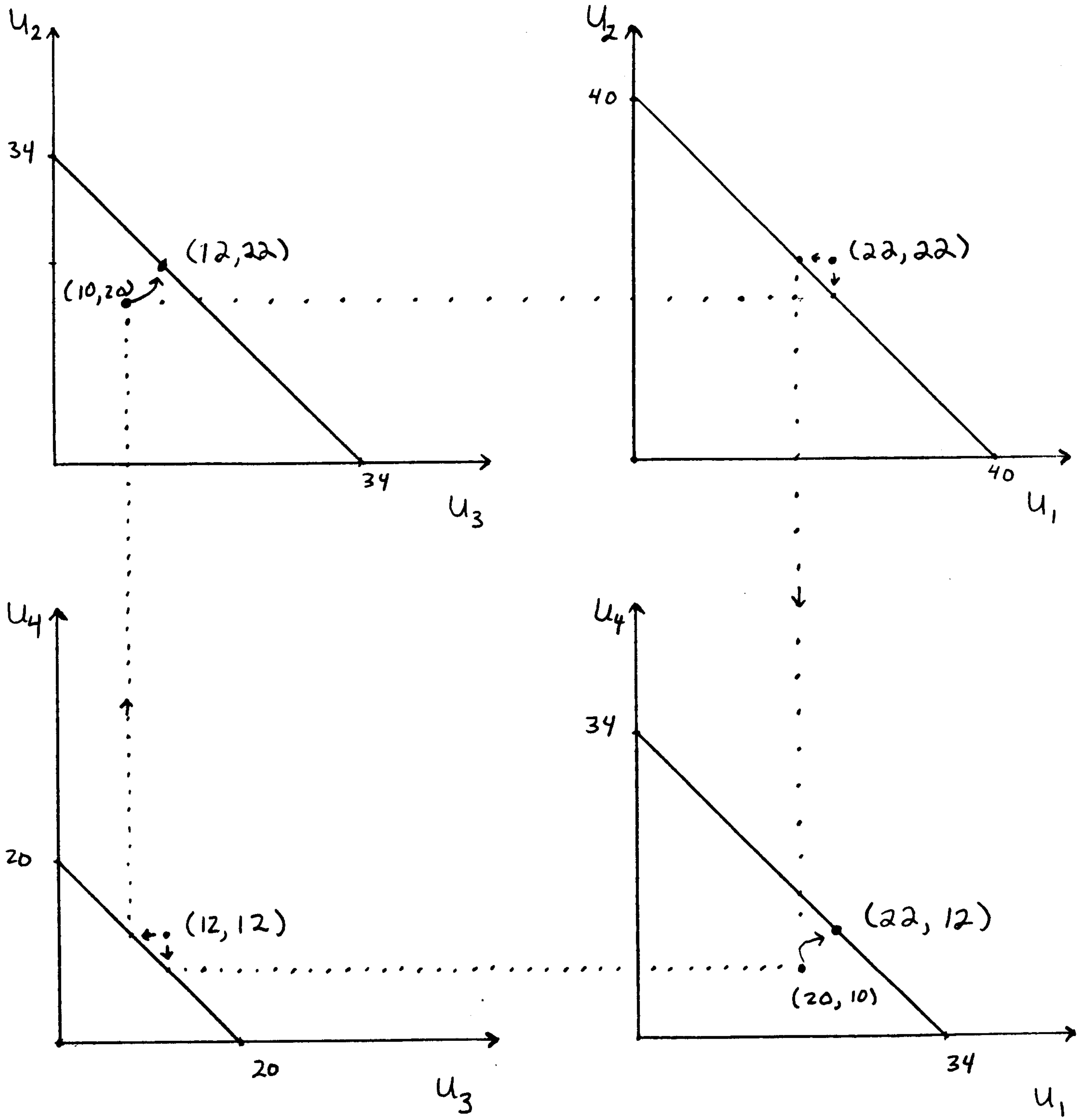
$$x^{[1,2]} = (22,22) \qquad d^{[1,2]} = (22,22)$$

$$x^{[1,4]} = (22,12) \qquad d^{[1,4]} = (18,8)$$

$$x^{[3,4]} = (12,12) \qquad d^{[3,4]} = (12,12)$$

$$x^{[3,2]} = (12,22) \qquad d^{[3,2]} = (8,18)$$

Note that, at the (unique) solution, the agreements are feasible for the coalitions [2,3] and [1,4] ; all other coalitions agree to disagree.



u2
34
(12,22)
(10,20)
34
u3
u2
40
(22,22)
40
u1
u4
20
(12, 12)
20
u3
u4
34
(22, 12)
(20, 10)
34
u1

EXAMPLE 2 (3-player/3-cake problems): Perhaps the simplest interesting class of multilateral bargaining problems arise in three player situations in which the coalition of the whole earns nothing, individuals earn nothing, and so the only profitable coalitions are the three 2-player coalitions. The coalition [i,j] has the "cake" V[i,j] to divide, but only one of these three cakes can actually be divided. (This class of problems was discussed by Binmore [1985]; see also Section 6.) In such problems, there are three possibilities for the underlying game $\langle N,V \rangle$:

Case 1: the core of $\langle N,V \rangle$ is empty. In this case, there is a unique vector $q \in \mathbb{R}^3$ such that q^S is on the Pareto efficient boundary of V(S) for S = [1,2] , [2,3] , [1,3] ; i.e., there is a unique vector q such that all three coalitions can afford to pay their members exactly their components of q . For *every* choice of bargaining functions, this vector q is the vector of reservation prices of a multilateral solution. For the Nash, or egalitarian, or Kalai-Smorodinsky bargaining functions, or indeed for any bargaining functions that satisfy strict individual rationality[10], this vector q is the vector of reservation prices of the *unique* multilateral solution. For bargaining functions (such as constrained Nash) which do not satisfy strict individual rationality, there will generally be other multilateral constrained Nash solutions. (We give an illustration below.)

Case 2: the core of $\langle N,V \rangle$ contains a unique point $q \in \mathbb{R}^3$. This core point is supported by a partition of coalitions, say {[1,2], [3]} . Hence $q_3 = 0$ and $(q_1, q_2) \in V[1,2]$. It follows from strong comprehensiveness of V[1,2] and the fact that q is the unique point in the core that $(q_2, q_3) \in V[2,3]$ and $(q_1, q_3) \in V[1,3]$. As in Case 1 above, it follows that, for every choice of bargaining functions, q generates a multilateral solution. For any bargaining functions that satisfy strict individual rationality, q generates the *unique* multilateral solution. For bargaining functions which do not satisfy strict individual rationality (such as the constrained Nash bargaining functions), there will generally be other multilateral solutions.

Case 3: the core of $\langle N,V \rangle$ contains more than one point. It is not hard to see that all points in the core are supported by the same partition of coalitions, say $\{[1,2],[3]\}$. For $i = 1, 2$ let $\tilde{r}_i = \max\{r_i : (r_i, 0) \in V[i,3]\}$. The fact that the core of $\langle N,V \rangle$ contains more than one point implies that the point $(\tilde{r}_1, \tilde{r}_2)$ lies in the interior of $V[1,2]$. Let $r = N^{[1,2]}(\tilde{r}_1, \tilde{r}_2)$ be the Nash solution in the coalition $[1,2]$, given the outside option vector $(\tilde{r}_1, \tilde{r}_2)$. Then $(r, 0) \in \mathbb{R}^3$ is the price vector of the unique multilateral Nash solution. Similarly, $(E^{[1,2]}(\tilde{r}_1, \tilde{r}_2), 0)$ is the price vector of the unique multilateral egalitarian solution, and $(K^{[1,2]}(\tilde{r}_1, \tilde{r}_2), 0)$ is the price vector of the unique multilateral Kalai-Smorodinsky solution. Indeed, whenever the bargaining functions f^S satisfy strict individual rationality, $(f^{[1,2]}(\tilde{r}_1, \tilde{r}_2), 0)$ is the price vector of the unique multilateral solution. On the other hand, although $(CN^{[1,2]}(\tilde{r}_1, \tilde{r}_2), 0)$ is the price vector of a multilateral constrained Nash solution, there will (as we show below) generally be many other multilateral constrained Nash solutions.

It may be useful to apply the above analysis in a particular problem. For M a parameter in the range $0 \leq M \leq 20$, consider the

3-player/3-cake (transferable utility) problem with feasible sets:

$$V[1,2] = \{(y_1,y_2): y_1 \geq 0, y_2 \geq 0, y_1 + y_2 \leq 40\}$$

$$V[2,3] = \{(y_2,y_3): y_2 \geq 0, y_3 \geq 0, y_2 + y_3 \leq 30\}$$

$$V[1,3] = \{(y_1,y_3): y_1 \geq 0, y_3 \geq 0, y_1 + y_3 \leq M\}$$

$$V(S) = \{0\} \quad \text{for all other coalitions } S.$$

For $10 < M \leq 20$, the core is empty and the unique multilateral Nash solution is generated by the price vector $(5 + M/2, 35 - M/2, -5 + M/2)$. For $M = 10$, the core consists of the single point $(10,30,0)$, which generates the unique multilateral Nash solution. For $0 \leq M < 10$, the core contains many points, but $(10,30,0)$ continues to generate the unique multilateral Nash solution. If we consider the constrained Nash bargaining functions however, the situation is quite different. For $10 < M \leq 20$, the price vectors of multilateral constrained Nash solutions comprise the interval:

$$\{(10 + \lambda, 30 - \lambda, \lambda) : -5 + M/2 \leq \lambda \leq 10\}.$$

For $M = 10$ and for $0 \leq M < 10$, the price vectors of multilateral constrained Nash solutions comprise the interval:

$$\{(10 + \lambda, 30 - \lambda, \lambda) : 0 \leq \lambda \leq 10\}.$$

(Note that, in all cases, the unique multilateral Nash solution is an endpoint of the interval of multilateral constrained Nash solutions.)

The reader should not be misled. In the 3-player/3-cake case, strict individual rationality of bargaining functions is enough to guarantee uniqueness of the multilateral solution. For more general multilateral bargaining problems, strict individual rationality of bargaining functions substantially restricts the range of possible multilateral solutions, but is not generally enough to guarantee uniqueness.

6. RELATED NONCOOPERATIVE AND COOPERATIVE MODELS

The Proposal-Making Model

The Nash program is to provide noncooperative justification for cooperative solutions, in the sense of obtaining cooperative solutions as equilibria of appropriate noncooperative models. Bennett [1988d] takes a step in this direction by providing an extensive form model for the play of certain multilateral bargaining problems (those in which there are no remaining gains from coalition formation once the first coalition has formed), whose outcomes are consistent with the solutions described here.

Bennett's "proposal-making model" is inspired by Selten's [1981] model of the play of cooperative games with sidepayments. The game begins when nature chooses a player to have the initiative. A player with the initiative can pass the initiative or make a proposal (consisting of a coalition and a feasible payoff distribution for that coalition) and name a respondent. A respondent can accept the proposal and name the next respondent, or reject the proposal and assume the initiative. The game ends when a proposal is accepted by every member of the coalition; players in the coaltion receive their proposed payoffs, other players receive nothing.

Bennett [1988d] shows that the outcomes of each stationary subgame perfect equilibrium of this noncooperative game are generated by vectors of reservation prices (in the same sense as in Section 4), that these reservation prices are aspirations, and that every aspiration is the "price vector" of a stationary subgame perfect equilibrium.

The predicted set of coalitions of a stationary subgame perfect equilibrium may be a proper subset of those in the corresponding multilateral solution; when there are many "affluent" coalitions, a stationary subgame perfect equilibrium need not assign all of them positive probability. However, if the stationary subgame perfect equilibrium strategies assign positive probability to *all* best replies, then the predicted sets of coalitions will coincide.

Binmore [1985]

Binmore [1985] presents a multilateral bargaining model for the 3-player/3-cake division problem discussed in Section 5, based fundamental ideas are quite similar to those presented here. Binmore's multilateral solution[11] specifies an agreement *for each player* in each coalition (corresponding to which player has the "initiative"). For coalitions whose outside option vectors are feasible, these agreements coincide - and coincide with the agreements of the corresponding multilateral constrained Nash solution. However, coalitions whose outside option vectors are infeasible do not agree to disagree. They agree to different agreement vectors, depending on who has the initiative; these vectors are the projections of the outside option vector onto the coalition's attainable utility frontier. These "projected" utility levels appear in the model presented here, not as agreements but in the calculation of outside option vectors (as the utilities players can obtain from infeasible agreements). As we saw in Section 5, there can be many multilateral constrained Nash solutions; to each of them corresponds a multilateral solution in Binmore's model. Binmore applies additional criteria (which eliminate this multiplicity) to select his unique cooperative solution for the 3-player/3-cake problem.

Binmore [1985] also presents an extensive form model (the "market model") to support his cooperative solution. it is not difficult to show that the predicted outcomes of the subgame perfect equilibria of Binmore's model are price generated, and that the generating price vectors are price vectors of multilateral constrained Nash solutions.

The Monotonic Solution

Kalai and Samet's [1985] *monotonic solution* for cooperative games can be interpreted as an alternative model of multilateral bargaining in which there is less competition among coalitions. It can be described in the following way: [12] Each coalition uses the egalitarian bargaining function, modified so that agreements are always feasible. (When the outside option vector is not feasible, the agreement is the largest feasible utility vector which gives the players equal losses from their outside option.) Each coalition's *dividend* is the vector difference between its agreement vector and its outside option vector. (The empty set's dividend is 0 .) In each coalition, a player's outside option utility is the sum of his dividends from his coalitions which are *proper subsets* of the coalition. The *monotonic multilateral solution* (which is unique) is an agreement for each coalition which is consistent with this determination of ageements, dividends, and outside options; the *monotonic solution* is the agreement utility for the coalition of the whole.

The monotonic multilateral bargaining model differs from the one presented here in two essential ways: First, the monotonic model insists that all agreements be feasible, while the model presented here allows coalitions to agree to disagree; Second, the monotonic model computes outside options for each coalition based solely on players' opportunities in *subcoalitions*, while the model presented here computes outside options based on players' opportunities in *all* other coalitions.

The differences in the way outside options are arrived at suggests that the monotonic model assumes far less competition among coalitions than is assumed in the model presented here. To see the difference this makes, we consider the following situation: the set of players is $N = \{1,2,3\}$, and the feasible sets $V(S)$ for coalitions are:

$$V[1,2] = \{(y_1, y_2) : y_1 \geq 0 , y_2 \geq 0 , y_1 + y_2 \leq 18\}$$

$$V[2,3] = \{(y_2, y_3) : y_2 \geq 0 , y_3 \geq 0 , y_2 + y_3 \leq 12\}$$

$$V[1,3] = \{(y_1, y_3) : y_1 \geq 0 , y_3 \geq 0 , y_1 + y_3 \leq 12\}$$

$$V[1,2,3] = \{(y_1, y_2, y_3) : y_1 \geq 0 , y_2 \geq 0 , y_3 \geq 0 , y_1 + y_2 + y_3 \leq 21\}$$

$$V(i) = \{0\} \quad \text{for } i = 1, 2, 3$$

The agreements y^S of the monotonic multilateral solution are:

$$y^i = 0 \text{ for each } i$$
$$y^{[1,3]} = (6,6)$$
$$y^{[2,3]} = (6,6)$$
$$y^{[1,2]} = (9,9)$$
$$y^{[1,2,3]} = (8,8,5) \quad .$$

By contrast, the agreements of the unique multilateral egalitarian solution (see Section 5) are:

$$x^i = 0 \text{ for each } i$$
$$x^{[1,3]} = (9,3)$$
$$x^{[2,3]} = (9,3)$$
$$x^{[1,2]} = (9,9)$$
$$x^{[1,2,3]} = (9,9,3) \quad .$$

If player beliefs are based on the *monotonic* multilateral solution, players 1 and 2 would prefer to form the coalition [1,2] , and player 3 would prefer to form either [1,2] or [2,3] , rather than the coalition [1,2,3] . As we have noted, this source of conflict among players does not exist for multilateral bargaining solutions (and in particular for the multilateral egalitarian solution). For this reason, the monotonic solution can best be regarded as (and indeed was proposed as) an arbitration scheme (for the coalition of the whole) for situations in which players are not free to form alternative coalitions. By contrast, the multilateral egalitarian solution should not be regarded as an arbitration scheme (for the coalition of the whole), since the agreement vector for the coalition of the whole will not generally be feasible.

FOOTNOTES

1. Our purpose here is to study how the *interrelationships* among the coalitions' bargaining problems affect the bargaining outcomes, not the bargaining within individual coalitions. Determining which bargaining function is appropriate is the domain of traditional bargaining theory (see for example Roth [1979] or Kalai [1985]) and of more recent noncooperative models of the bargaining process (see Sutton [1986]).

2. We say that $V(S)$ is *strongly comprehensive* if, whenever $x \in V(S)$, $y \neq x$ is in $\mathbb{R}^S_+$ and $y \leq x$, then y is in the interior of $V(S)$ (relative to $\mathbb{R}^S_+$).

3. The assumption that the bargaining process within each coalition can be summarized by a function is merely the assumption that there is no indeterminacy: from the same data, the bargaining process always leads to the same result.

4. Recall that the weak and strong Pareto boundaries of $V(S)$ coincide when $V(S)$ is strongly comprehensive.

5. For example if the coalition [1,2] can divide 3 in any way it chooses, and has an outside option vector of (2,2), agreeing to disagree here means that each player would be willing to form the coalition [1,2] for a payoff of 2 if the other player would accept a payoff of 1.

6. Since each player position is distinct, $|Q| = n 2^{n-1}$.

7. This is harmless because $0 \in V(i)$ for each i .

8. The Nash bargaining solution is defined only for convex sets of attainable utility vectors. When using the Nash (or constrained Nash) bargaining function for a coalition S , we will consider only situations in which $V(S)$ is convex.

9. For discussion of whether the Nash or constrained Nash bargaining solution is appropriate, see Binmore et al [1986].

10. The bargaining function f^S is *strictly individually rational* if $f^S(d^S) >> d^S$ whenever d^S is in the interior of $V(S)$.

11. Binmore's multilateral solution is $\{\underline{\sigma}^{C,P}\}$; see Binmore [1985], pp. 276-277.

12. We consider only the symmetric monotonic solution; similar comments hold for nonsymmetric monotonic solutions.

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