

# DYNASTY: A SIMPLE STOCHASTIC GROWTH MODEL\*

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## ABSTRACT

This paper combines the two 'workhorses' of macroeconomics: the Ramsey model and the overlapping generations model. A new dynasty of infinite horizon families appears every period. Each dynasty is characterized by the same Kreps-Porteus utility functional that I refer to as RINCE, which is an acronym for the characteristic properties of Risk Neutrality and a Constant Elasticity of intertemporal substitution. Preferences in this class allow exact solutions to be obtained to the agent's intertemporal choice problem. The solutions to each problem are aggregated across agents to arrive at a low order stochastic difference equation that characterizes equilibria in the economy.

## 1 Introduction

This paper introduces a class of stochastic growth models that provides a bridge between the two major paradigms that are currently in use. The model consists of a countable number of dynastic families each of which solves an infinite horizon stochastic programming problem. A new generation of agents enters the model in every period which implies that the number of families grows through time. Stationary equilibria are characterized as limiting distributions of per-capita variables. The model is similar to non-stochastic economies that are described by Blanchard [1] and Weil [10]. It behaves, for most parameter values, exactly like a stochastic two period overlapping generations economy. But for a particular value of a parameter that describes the rate of creation of new dynasties the economy *is* a Ramsey representative agent economy.

The paper introduces a technique that allows one to analyse stochastic equilibria in situations for which the welfare theorems fail. Most existing work begins with the idea that equilibria can be represented as the solution to a *single* maximization problem and it characterizes equilibria as solutions to stochastic Euler equations. The alternative approach in this paper is to exploit a risk neutrality assumption in order to be able to solve each agents' stochastic programming problem explicitly. This allows one to construct equilibria in stochastic economies by building up the aggregate market clearing equations and searching for a fixed point in function space. This technique is likely to be useful in other stochastic models.

## 2 The Generational Structure of the Economy

The economy, at each date, is populated by a finite number of infinitely lived agents. These agents are referred to as families or alternatively as dynasties and it is assumed that all members of a given generation are alike. Time is discrete and each period of time, a new dynasty is created. Superscripts on a variable refer to the date of birth of a dynasty and subscripts refer to calendar time.

The world has a beginning, referred to as date 1, at which there exists a finite

number,  $n^1$ , of agents of generation 1. The growth factor:

$$\gamma = \sum_{\tau=1}^t n^\tau / \sum_{\tau=1}^{t-1} n^\tau,$$

is constant and greater than or equal to 1. When  $\gamma$  is equal to one the number of agents in the economy is constant and equal to  $n^1$ . This case is a very special one since the economy collapses to a representative agent economy. For positive growth rates the definition of  $\gamma$  allows one to compute the number of agents of generation  $\tau$  for all  $\tau \geq 1$ .

### 3 The Individual's Problem

I will introduce a number of assumptions about the stochastic structure of endowments and prices. This structure is taken as given by agents in the economy each of whom solves a stochastic dynamic programming problem. In section 8 this structure will be shown to be justified in the sense that there exists an equilibrium in which prices and endowments behave in the manner that agents believe them to.

#### 3.1 Characterizing the Distribution of Uncertainty that the Agent Takes as Given

The purpose of this section is to define two sets  $\Omega$  and  $Z$  which represent the spaces in which endowments and interest factors may evolve. I assume that agents may trade a single asset that pays a stochastic return  $R_t$ ; the space  $Z$  defines the values that can be taken by  $R$ .<sup>1</sup> The sets  $\Omega$  and  $Z$  are also used to define the sample space which is the space from which infinite sequences of endowments and interest factors are drawn.

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<sup>1</sup>The assumption of a single asset would follow from more primitive restrictions on the observability of endowments in economies with idiosyncratic risk although this explanation is not available here since the endowment is assumed to be common. I have chosen to explore the incomplete markets assumption mainly for tractability. It leads to a economy that may be characterized by a single difference equation. Complete market economies generate equilibria that are characterized by an infinite dimensional set of assets.

Agents may transfer resources between periods by buying or selling short a single security that pays a real stochastic return  $R$ .<sup>2</sup> For the purpose of solving the individual's problem I assume that the interest factor and the endowment of the agent are described by a known Markov process. The state space of this process is defined below:

**Definition 1** *Let the sets  $\Omega$  and  $Z$  be defined by:*

$$\Omega = \{\omega \in [A, B] \mid [A, B] \subset R_+\},$$

$$Z = \left\{ z \in [C, D] \mid C > 1, D < \frac{1}{\beta^\beta(1-\beta)^{(1-\beta)}} \right\}.$$

*Let  $X = Z \times \Omega^2$ ,  $S = X \times \Omega$  and let  $S$  be the Borel sets of  $S$ .*

I will refer to the vector  $s \in S$  as the *state* of the economy where the first element of  $s$  is the interest factor at date  $t$  and the subsequent three elements represent the agent's endowments at periods  $t$ ,  $t-1$  and  $t-2$  respectively. Notice that the definition of the set  $S$  implies that the interest factor will be greater than one in every period and that the endowment lies in a compact interval of the positive real line. The upper bound on the set  $Z$  is used later in the paper to demonstrate that a certain operator is a contraction. It is a stronger condition that is necessary but not a particularly restrictive one.

For some purposes it will be useful to be able to refer to the set of all possible realizations of infinite sequences of the form  $\{s_v\}_{v=t}^\infty$ . The following notation is introduced for this purpose:

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<sup>2</sup>I refer to this security as debt since, under one interpretation of the model, debt offers a fixed nominal return and its yield fluctuates randomly in response to a set of self-fulfilling price level movements. Alternatively one can think of this security as an equity contract in the gross national product. This alternative interpretation allows me to describe equilibria in which the value of government liabilities is negative. Negative debt equilibria are not supportable by a security with a fixed nominal yield although I suspect that if one were to introduce a medium of exchange function for holding fiat money (as opposed to the store of value function that is captured by the overlapping generations model) one might rescue the interpretation of debt as a fixed nominal claim even in the case of negative debt equilibria.

**Definition 2** *The sample space  $\Phi$  is the collection of all infinite sequences  $\{s_v\}_{v=t}^{\infty}$  such that  $s_v \in S$ . A single realization of  $\{s_v\}_{v=t}^{\infty}$  will be denoted by  $\phi$ .*

The next assumption uses the definition of the state space to summarize the information that is taken as given by the consumers in the economy:

**Assumption 1** *The sequence of endowments is independently distributed with a time invariant marginal density  $\mu(\omega) : \Omega \mapsto [0, 1]$ , which places positive probability weight on every element of  $\Omega$ . The interest factor evolves according to a known stochastic difference equation  $f : S \mapsto Z$ . This difference equation defines a stable<sup>3</sup> Markov transition function  $\chi(S, S), S \times S \mapsto [0, 1]$ .*

There is little to be gained from generalizing the assumption that endowments are independently distributed although the generalization is not difficult. Section 8 shows that a stochastic difference equation  $f$ , defined on a set  $Z$  with these properties, may be derived as part of a competitive equilibrium. The assumption that  $\mu$  places positive weight on every element of  $\Omega$  is used to establish convergence, of the Markov process that characterizes equilibria, to a unique invariant distribution.

It is worth observing that the notation does not index endowment uncertainty by generation. This key assumption implies that endowment shocks are economy-wide rather than idiosyncratic. The assumption is used to show that the solution to the consumer's optimizing problem does not violate non-negativity constraints on consumption. The non-negativity constraints will not be satisfied for all possible probability distributions over sequences of endowments and interest rates but, in equilibrium, rates of return are determined endogenously by the transition function,  $\chi$ . In the case in which all agents face the same endowment sequence, one may show that the solution to the choice problem that I will characterize in theorem (1) induces an equilibrium transition function for which these constraints do not bind.<sup>4</sup>

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<sup>3</sup>A Markov transition function is *stable* if its associated Markov operator takes the space of bounded continuous functions into itself. In this case  $\chi$  is said to possess the *Feller property*, (Stokey Lucas and Prescott [9] page 20).

<sup>4</sup>It is possible to derive a version of the model in which there is idiosyncratic risk but one must

### 3.2 Placing Bounds on the Agent's Lifecycle Choices

In the section 3.3 I introduce a stochastic programming problem that is solved by each dynasty. For the solution to this problem to be well defined, one must show that the agent's wealth is bounded above. The main result that is established in section 3.2 is the existence of a such a bound.

**Definition 3** *The present value shadow price is a function  $P_t^j : Z^{j-t} \mapsto R_+$ , defined by:*

$$P_t^j = 1 \quad \text{if } j = t;$$

$$P_t^j = \frac{1}{\prod_{v=t+1}^j R(s_v)} \quad \text{if } j > t.$$

$P_t^j$  is the shadow price of a date  $j$  state dependent commodity at date  $t$ . The adjective "shadow" is attached since the market for contingent commodities is closed and agents are unable to trade at these prices.

**Definition 4** *The agent's human wealth is a function  $h : \Phi \mapsto R_+$  defined by:*

$$h(\phi) = \lim_{v \rightarrow \infty} \sum_{j=t}^v \omega_j P_t^j. \quad (1)$$

Human wealth is a random variable that takes values which are functions of the realizations of  $\phi$ . It represents the net present value of the family's endowment stream if the sequence of states is  $\phi$ . Since agents are unable to trade in a complete set of contingent claims markets, there is a different value of  $h$  for every possible value of  $\phi$ , each of which enters a separate infinite horizon budget constraint. Each of these constraints places an upper bound on the possible values of current consumption. The following lemma establishes that the support of  $h(\phi)$  is compact which implies that there exists one constraint which is most binding.

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be much more careful about the structure of assets. It seems likely that one might find equilibria, in such a model, that are supported by assets with payoffs that are linked to the realization of individual endowments but this issue is beyond the scope of the current paper.

**Lemma 1** *The value of  $h(\phi)$  lies in a compact interval  $H \subset R_+$ , for every realization of  $\phi \in \Phi$ .*

**Proof**

Since the support of  $\Omega$  is bounded above, by  $B$ , and below, by  $A \geq 0$ , each element of  $\{\omega_v\}_{v=t}^\infty$  is uniformly bounded above and below. Since the lower bound of  $Z$  is strictly greater than one, every possible realization of the sequence of interest factors is strictly greater than one. It follows immediately that every element of the weighted sum  $\sum_{j=t}^\infty \omega_j P_t^j$  exists and is bounded above and below. Let  $H$  represent the interval  $[\inf_\phi h(\phi), \sup_\phi h(\phi)]$ .  $H$  is closed since  $\Phi$  is closed. It follows that  $H$  is compact.  $\square$

In any state,  $s$ , there will exist a subset of  $\Phi$  with first element  $s$ . Call this subset  $\Phi_s$ . The set  $\Phi_s$  consists of the possible realizations of  $\phi$  conditional on the observation that the current state is  $s$ . Since the set of possible realizations of future endowments is closed and bounded it follows that the agent's human wealth will have a smallest element. This observation motivates the following definition:

**Definition 5** *The minimum value of human wealth is a function  $\hat{h} : S \mapsto R_+$ :*

$$\hat{h}(s) = \min_{\phi \in \Phi_s} h(\phi). \quad (2)$$

With one further assumption, this definition may be used to place an upper bound on the consumption of a family in any period:

**Assumption 2** *The present value of the limit of the family's assets, for every realization of  $\phi$ , is greater than or equal to zero.*

$$\lim_{T \rightarrow \infty} P_1^{T-1} a_T \geq 0.$$

Assumption 2 may be used to construct an intertemporal budget constraint in each state of nature from the sequence of state contingent period by period constraints. This set of intertemporal budget constraints places an upper bound on consumption



that will be attained if the family consumes all of its financial wealth and, in addition, it borrows enough resources to consume the present value of its endowment sequence in the worst possible realization of  $\phi_s$ .<sup>5</sup>

Let the variable  $a^\tau(s)$  represent the quantity of the single financial security that is held by a family that comes into existence at date  $\tau$ ; (this variable may be positive or negative). The upper bound on consumption in any state is given by the expression:

$$c^\tau(s) \leq R(s)a^\tau(s) + \hat{h}(s), \quad (3)$$

where the assets  $a^\tau$  evolve according to the stochastic difference equation:

$$a^\tau(s') = R(s)a^\tau(s) + \omega(s) - c^\tau(s); \quad (4)$$

$$a_1^\tau = 0, \quad \tau = 2, \dots, \infty, \quad (5)$$

$$R_1 a_1^1 = T. \quad (6)$$

Equation (5) represents the assumption that generations born in periods greater than or equal to 2 come into existence with zero wealth. I will treat the initial generation differently by endowing it with initial wealth  $T$  which may be positive or negative. This initial wealth represents a tax or transfer from the government.

In addition to the constraints that are imposed by the budget identity (4) and by the initial condition, equation (5), I require consumption to be non-negative:

$$c^\tau(s) \geq 0. \quad (7)$$

### 3.3 The Family's Choice Problem

A family of generation  $\tau$  is assumed to solve an infinite horizon stochastic programming problem that has the following value function representation, where I have

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<sup>5</sup>This definition of the constraint set implies that borrowing must be repayed with probability one in every state. It might be argued that this constraint is too strong, and that it would be more reasonable to permit bankruptcy in some states. The definition that I employ, however, is implied by the assumption that there exists a *single* security. A model in which bankruptcy is permitted in some states, would require a richer set of financial markets.

dropped the superscript  $\tau$ , to cut down on notational clutter:

$$v[R(s)a(s), s] = Tv[R(s)a(s), s]. \quad (8)$$

The function  $v : R \times S \mapsto R_+$  represents the maximum possible utility that can be attained in state  $s$  where the operator  $T$  is given by the expression:

$$Tv[R(s)a(s), s] \equiv \sup_{c(s) \in \Gamma} [c(s)]^{1-\beta} \left[ \int_S v[R(s')a(s'), s'] \chi(s, ds') \right]^\beta, \quad (9)$$

the constraint set  $\Gamma$  is defined as:

$$\Gamma = \{0 \leq c(s) \leq R(s)a(s) + \hat{h}(s)\}, \quad (10)$$

and the assets of the agent evolve according to equation (4).

The preferences that are defined by the operator  $T$  are a generalization to the infinite horizon case of RINCE utility studied in Farmer [2] and axiomatized by Kreps and Porteus [7]. These preferences are able to separate the families attitudes to risk from its attitude to the timing of consumption by dropping one of the von-Neumann Morgenstern axioms applied to a class of intertemporal lotteries. By choosing to aggregate current consumption and expected future utility with a function that is homogenous of the first degree, one introduces a kind of risk neutrality in the sense that the agent with these preferences is indifferent to the distribution of one-step-ahead endowment uncertainty. This risk neutrality property allows one to find a closed form solution for the consumption rule that can be aggregated across individuals. In the case of perfect certainty these preferences generate the same ordinal choices that would be made by a family with logarithmic utility and a discount factor equal to  $\beta$ .

## 4 The Solution to the Family's Choice Problem

### 4.1 Some Definitions

The solution to the agents' choice problem makes use of two concepts. One of these concepts is average human wealth and the other is a stochastic discount factor that

summarizes the influence of future prices on current decisions. These two concepts are defined in this section.

The first definition summarizes the way that prices influence decisions.

**Definition 6** *The function  $Q : S \mapsto R_+$  is the unique solution to the equation:*

$$Q(s) = (TQ)(s), \quad (11)$$

where;

$$(TQ)(s) = \int_S R(s') F[Q(s')] \chi(s, ds'),$$

and,

$$F[x] = (1 - \beta)^{(1-\beta)} \beta^\beta x^\beta.$$

The uniqueness of  $Q$  follows from the N-stage contraction theorem applied to the operator  $T : B(S) \mapsto B(S)$  where  $B(S)$  is the space of bounded continuous functions from  $S$  to  $R_+$ .<sup>6</sup> The function  $Q$  appears in the solution to the consumer's problem in which terms of the form  $F[Q(s')]/Q(s)$  play the role of stochastic discount factors. When there is no uncertainty  $F[Q(s')]/Q(s)$  is just the inverse of the interest factor. In the case of non-degenerate uncertainty these terms are used by the family to discount the value of its future endowments.

The second definition, of *average human wealth* plays an important role in the solution to the agent's optimization problem:

**Definition 7** *Average human wealth  $\bar{h}(s) : S \mapsto R_+$  is the unique solution to the functional equation:*

$$\bar{h}(s) = (T\bar{h})(s), \quad (12)$$

where,

$$(T\bar{h})(s) = \omega(s) + \int_S \frac{\bar{h}(s') F[Q(s')] \chi(s, ds')}{Q(s)}.$$

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<sup>6</sup>Stokey Lucas and Prescott [9] page 53. For large N the operator  $T^N$  is a contraction of modulus  $\delta = \beta^\beta (1 - \beta)^{(1-\beta)} \sup_S R(s)$ . The definition of the set  $Z$  implies that,  $\delta < 1$ .

This definition, of average human wealth, collapses to the standard definition of the discounted present value of endowments in the case when all uncertainty is degenerate. The proof of the existence and uniqueness of a function  $\bar{h}$  that satisfies equation (12) follows from observing that the operator  $T^N$ , for large enough  $N$ , satisfies Blackwells sufficient conditions for a contraction on the space of continuous bounded functions from  $S$  to  $R_+$ .<sup>7</sup>

## 4.2 Characterizing the Solution

The solution to the family's choice problem is summarized by the following theorem:

**Theorem 1** *Let the function  $a^*(s) : S \mapsto R_+$  represent the family's assets when it follows the consumption rule:*

$$c^*(s) = (1 - \beta)W(s), \quad (13)$$

where,

$$W(s) \equiv R(s)a^*(s) + \bar{h}(s), \quad (14)$$

and  $a^*(s)$  evolves according to the budget equation;

$$a^*(s') = R(s)a^*(s) + \omega(s) - c^*(s). \quad (15)$$

*If  $c^*$  remains non-negative in all states then  $c^*$  is the policy function that solves equation 8. The value function  $v^*(s)$  is given by:*

$$v^*(s) = F[Q(s)]W(s). \quad (16)$$

The proof of theorem 1 is presented in appendix 1. The condition of the theorem, that consumption remains positive, requires one to check that the proposed solution generates a sequence of values for the family's assets that does not violate the borrowing constraint in any state. This condition will not be satisfied for arbitrary Markov processes that define the evolution of endowments and interest factors. However, for the case when endowments are identical across individuals (no idiosyncratic risk) it is possible to show that there exists an equilibrium transition function  $\chi$  for which this solution is valid.

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<sup>7</sup>See Stokey Lucas and Prescott [9] theorem 3.3.

## 5 The Aggregate Economy and the Definition of Policy

There are two purposes to this section of the paper. The first is to define a procedure for aggregating quantities across dynasties and the second is to introduce the equations that define market clearing. The discussion of market clearing is used to define the policies that the government follows in equilibrium.

Although it proved convenient to drop the superscript  $\tau$  when referring to the solution to the problem of a single dynasty it is important, when discussing the aggregate economy, to keep the notion of individual and aggregate variables distinct. In the remainder of the paper, I use a superscripted variable to refer to the individual dynasty and the same variable with no superscript will refer to the average of the individual variable across all dynasties. For example, for any variable  $x^\tau(s)$ , define the aggregate per capita value,  $x(s)$ , as:

$$x(s) = \frac{\sum_\tau n^\tau x^\tau(s)}{\sum_\tau n^\tau}, \quad (17)$$

where the summation is over all existing dynasties. To illustrate the use of this notation, consider the definition of aggregate consumption. The assumption of RINCE preferences implies that the family's consumption  $c^\tau$  is a linear function of wealth. This property allows one to define aggregate consumption as:

$$c(s) = (1 - \beta)[R(s)a(s) + \bar{h}(s)]. \quad (18)$$

A similar expression defines aggregate asset accumulation:

$$\gamma a(s') = R(s)a(s) + \omega(s) - c(s). \quad (19)$$

In this case one is weighting future assets by a greater number of generations which explains the appearance of the term  $\gamma$  on the left-hand-side of the equation. The expression for aggregate average human wealth is simpler, since all agents regard the future in the same way. This assumption implies that average aggregate human wealth is equal to individual average human wealth,  $\bar{h}(s)$ , a variable which is the same for each dynasty.

To describe an equilibrium one must impose a market clearing condition. Goods market clearing requires that aggregate per-capita consumption should be equal to the aggregate per-capita endowment in each period, that is:

$$\omega(s) = c(s). \quad (20)$$

To define asset market equilibrium one must describe the behavior of the supply of assets. In the absence of an outside agency the aggregate private stock of assets would be required, in equilibrium, to equal zero. But in an economy with a government that has the authority to tax future generations it is possible for the aggregate stock of outside assets to be either positive or negative. This stock will be referred to as government debt, although I will allow government debt to be either positive or negative.

The set of equilibria will not be independent of the class of policies that is pursued by the government. In this paper I will study policies in the following class:

$$\gamma b(s') = R(s)b(s), \quad (21)$$

$$R_1 b_1 = T, \quad (22)$$

where the initial condition, equation (22), represents a tax or transfer to the first generation depending on whether it is positive or negative. If  $R_1 b_1$  is negative the government lends to the private sector, if it is positive the public lends to the government. These policies are a special case of balanced budget polices. They are a good place to begin a study of endowment economies since they may be shown to induce equilibria that are similar to the class of monetary equilibria that are known to exist in two period overlapping generations models.

In equilibrium, the value of government debt equals the net value of private assets:

$$b(s) = a(s); \quad (23)$$

It follows that private assets will evolve in the same manner as government debt:

$$\gamma a(s') = R(s)a(s). \quad (24)$$

## 6 Competitive Equilibria: A Definition

A *competitive equilibrium* is a transition function  $\chi(S, S)$  and a policy function  $c^*(s)$  with the properties:

1. The function  $c^*$  solves each family's optimizing problem (8) when agents believe that the state will evolve according to the Markov transition function  $\chi$ .
2. The actual evolution of interest factors and endowments is governed by the transition function  $\chi$  when interest rates adjust each period to ensure zero excess demand in the goods and asset markets.

## 7 Difference Equations that Characterize Equilibria

In this section I study the properties of a pair of stochastic difference equations that describe the behavior of aggregate per-capita assets and of interest factors. To begin with I assert that these equations characterize equilibria and in section 8 I state a theorem to this effect.

**Definition 8** *The asset evolution equation is a function  $g : Y \mapsto Y$  given by the expression:*

$$a_t = g(a_{t-1}; \omega_{t-1}, \omega_{t-2}) \equiv \frac{a_{t-1}\omega_{t-1}}{\gamma[\beta\omega_{t-2} - (\gamma - 1)(1 - \beta)a_{t-1}]} \quad (25)$$

*The interest factor evolution equation is a function  $f : Z \mapsto Z$  given by the expression:*

$$R_t = f(R_{t-1}; \omega_t, \omega_{t-1}, \omega_{t-2}) \equiv \frac{\omega_t}{\omega_{t-1}[\beta + \frac{1}{\gamma}] - \beta(R_{t-1}/\gamma)\omega_{t-2}} \quad (26)$$

The sets  $Y$  and  $Z$  on which the functions  $g$  and  $f$  are defined are chosen by picking upper and lower bounds for  $a$  and  $R$  that map intervals  $[0, E] \equiv Y$  and  $[C, D] \equiv Z$  into themselves for all possible realizations of  $\omega$ . For some values of  $\beta$  and  $\gamma$ , this construction leads to a set  $Y$  that is contained in  $R_+$  and for other values it leads to a set that is contained in  $R_-$ . This means that some economies will

support competitive equilibria in which the public lends to the government, and other economies will support equilibria in which the government lends to the public. The following classification defines the critical parameter values that divide the set of economies accordingly. The terminology is borrowed from David Gale [5] who suggests a similar distinction for the two period overlapping generations model under certainty:

**Definition 9** *The economy is Samuelson if:*

$$\gamma\beta \geq \frac{B}{A},$$

*and classical if:*

$$\gamma\beta \leq \frac{A}{B}.$$

Given this classification scheme the sets  $Y$  and  $Z$  are defined below:<sup>8</sup>

**Definition 10**

*Let the set  $Y$  be given by:  $Y = \{y \in [0, E] \mid E \in R\}$ , where,  $E = \frac{\gamma\beta B - A}{\gamma(\gamma-1)(1-\beta)} \geq 0$ , if the economy is Samuelson, and  $E = \frac{\gamma\beta A - B}{\gamma(\gamma-1)(1-\beta)} \leq 0$ , if the economy is classical.*

*Let the set  $Z$  be given by:  $Z = \{z \in [C, D] \mid [C, D] \in R_+\}$ , where  $C$  is the smaller root of the quadratic:  $CB \left(\beta + \frac{1}{\gamma}\right) - A \left(\frac{\beta}{\gamma}\right) C^2 - A = 0$ , and  $D$  is the larger root of:  $DA \left(\beta + \frac{1}{\gamma}\right) - B \left(\frac{\beta}{\gamma}\right) D^2 - B = 0$ .*

To prove the existence of an equilibrium one must impose bounds on the uncertainty in the economy that ensure the interest factor will remain within limits for which the solution to the consumer's problem is well defined. These bounds were introduced in the first part of the paper as exogenous restrictions on the set  $Z$ . Now that we have derived the range of  $Z$  as functions of the parameters of preferences and technology, these restrictions must be restated as an assumption about the set  $\Omega$  that allows one to define a competitive equilibrium for any given choice of  $\beta$  and  $\gamma$ .

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<sup>8</sup>To motivate the definition of  $Z$ , consider the properties of the difference equation  $f$ : This equation is graphed in figure 1. For any given value of  $R_{t-1}$ , the upper bound of the set  $f(R_{t-1}, \omega_{t+1}, \omega_t, \omega_{t-1})$  for  $\{\omega_{t+1}, \omega_t, \omega_{t-1}\} \in \Omega^3$  is given by the expression  $R_t = f(R_{t-1}; B, A, B)$ . Similarly, the lower bound is given by:  $R_t = f(R_{t-1}; A, B, A)$ . The bounds of the set  $Z$  are defined by the equations  $D = f(D; B, A, B)$  and  $C = f(C; A, B, A)$ .



**Assumption 3** *The values of  $\beta, \gamma, A$  and  $B$  satisfy the joint restrictions:*

$$C > 1, \quad D < \frac{1}{\beta^\beta(1-\beta)^{(1-\beta)}}.$$

When  $A = B$ ,  $C$  equals the smaller, and  $D$  equals the larger, of  $\gamma$  and  $1/\beta$ . It follows that for almost all values of  $\beta$  and  $\gamma$  one can always make the uncertainty in the economy small enough to meet these conditions. The exception, is the critical economy in which  $\gamma = (1/\beta)$ . In this case the only values of  $A$  and  $B$  that satisfy assumption 3 are values for which  $A = B$ , that is, the uncertainty is degenerate. In this critical economy, the set  $Y$  collapses to the singleton set  $\{0\}$ . The sets  $Z$  and  $Y$ , for Samuelson and classical economies, are depicted in figures 1, 2, 3 and 4. The behavior of the stochastic difference equations  $f$  and  $g$  that are defined on the sets  $Z$  and  $Y$  are best described in terms of the Markov transition functions that these equations induce on state spaces  $S$  and  $Q$ . The following lemma defines the spaces  $S$  and  $Q$  and establishes that the difference equations  $f$  and  $g$  can be used to define transition functions with the required properties.

**Lemma 2** *Define the spaces  $S = Z \times \Omega^3$  and  $Q = Y \times \Omega^2$  and let  $S$  and  $Q$  be the Borel sets of  $S$  and  $Q$ . The difference equations:*

$$R_t = f(R_{t-1}; \omega_t, \omega_{t-1}, \omega_{t-2}),$$

and,

$$a_t = g(a_{t-1}; \omega_{t-1}, \omega_{t-2}),$$

where  $f$  and  $g$  are given by expressions (26) and (25) define stable Markov transition functions  $\chi : S \times S \mapsto [0, 1]$  and  $\psi : Q \times Q \mapsto [0, 1]$ .

The proof of this lemma is a direct application of theorem 8.9 in Stokey Lucas and Prescott [9] page 234. Given the definitions of  $\chi$  and  $\psi$  one can begin to ask questions about the behavior of the random variables  $R_t$  and  $a_t$  as  $t \rightarrow \infty$ . For stochastic models of this kind the appropriate way of characterizing the limiting behavior of the system is in terms of invariant probability measures. The equations

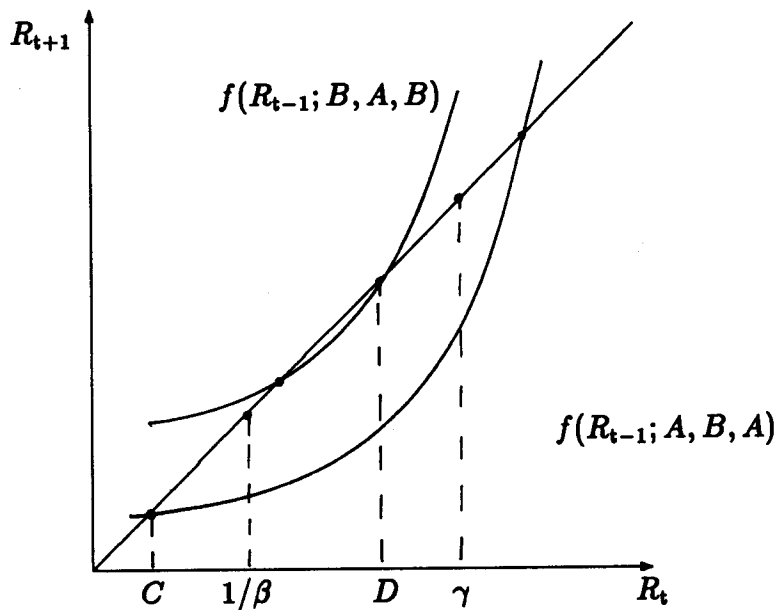


Figure 1: The Set  $Z$  for a Samuelson Economy

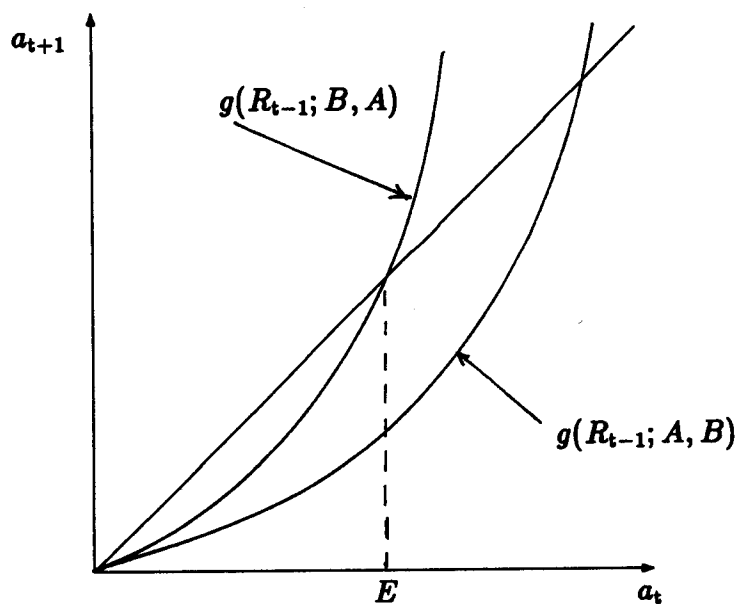


Figure 2: The Set  $Y$  for a Samuelson Economy

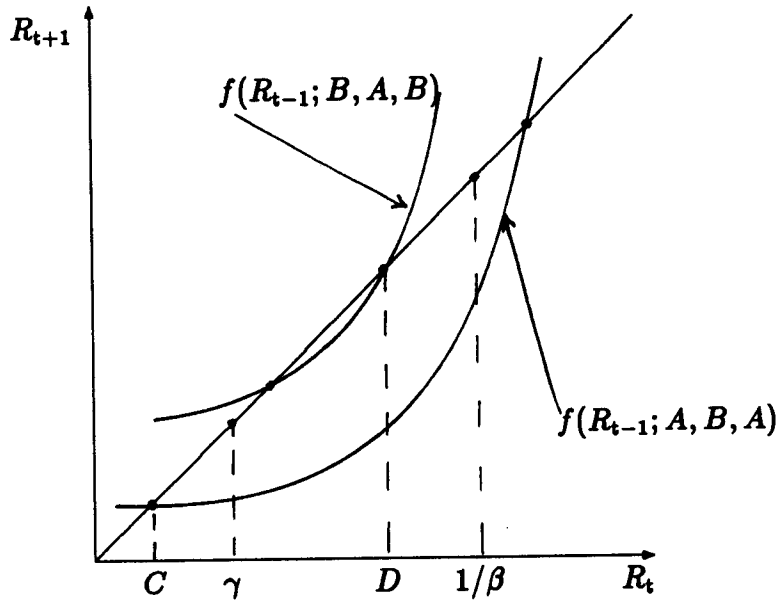


Figure 3: The Set Z for a Classical Economy

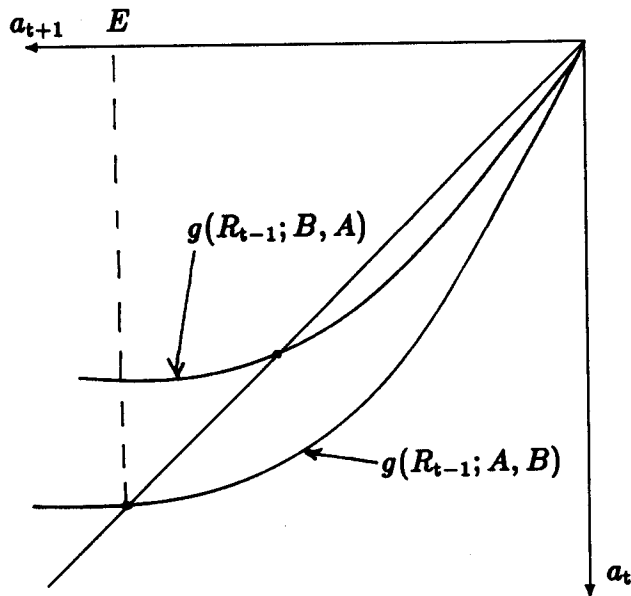


Figure 4: The Set Y for a Classical Economy

$f$  and  $g$  are non-linear but regular enough to establish the existence of, and convergence to, a pair of invariant measures  $\lambda$  and  $\theta$ . The following theorem, proved in the appendix, states this result:

**Theorem 2** *Let  $\Lambda$  and  $\Theta$  represent the spaces of probability measures on  $S$ , and  $\mathcal{Q}$  metrized with the total variation norm. Let  $\{\lambda_t \in \Lambda\}_{t=1}^{\infty}$  and  $\{\theta_t \in \Theta\}_{t=1}^{\infty}$  be sequences of probability measures. Define  $\lambda_t$  and  $\theta_t$  by recursive application of the adjoint operators  $T_\lambda^*$  and  $T_\theta^*$ :*

$$\lambda_j = T_\lambda^{*j} = \int_S \chi(S, S) T_\lambda^{*(j-1)} \lambda_{j-1}, \quad j = 2, \dots, \infty,$$

$$\lambda_1 \in \Lambda.$$

$$\theta_j = T_\theta^{*j} = \int_{\mathcal{Q}} \psi(Q, \mathcal{Q}) T_\theta^{*(j-1)} \theta_{j-1}, \quad j = 2, \dots, \infty,$$

$$\theta_1 \in \Theta.$$

*Then  $\{\lambda_t\}$  and  $\{\theta_t\}$  converge uniformly to unique invariant measures  $\lambda(S)$  and  $\theta(\mathcal{Q})$ .*

At any date  $t$ , the measure  $\lambda_t(A)$  gives the probability that the interest factor  $R_t$ , will be in the set  $A_Z$ , where  $A_Z$  is the projection of  $A$  onto  $Z$ . Similarly,  $\theta_t(B)$  is the probability that aggregate per-capita assets will be found in  $B_Y$ , the projection of  $B$  onto  $Y$ . The statement that these sequences of measures converge to invariant distributions implies that the unconditional probability of observing the state of the system in any particular set converges to a constant over time, where a complete description of the state includes current and lagged values of endowments as well as the values of the endogenous variables  $R$  and  $a$ .

## 8 Competitive Equilibria: Existence and Characterization

In the previous section we established the properties of a pair of non-linear difference equations,  $f$  and  $g$ . We are now ready to establish that these difference equations characterize a class of equilibria in the dynastic economy.

**Theorem 3** *Let the Markov transition functions  $\chi : S \times S \mapsto [0, 1]$  and  $\psi : Q \times Q \mapsto [0, 1]$  be as defined in lemma 2. For all policies in the class:*

$$\gamma b(s') = R(s)b(s)$$

*with values of the initial transfer  $T$  in the interval  $[0, E]$  the transition function  $\chi$ , and the policy function:*

$$c^{r^*}(s) = (1 - \beta)[R(s)a^{r^*}(s) + \bar{h}(s)]$$

*constitute a competitive equilibrium. The evolution of aggregate assets in equilibrium is described by the transition function  $\psi$ .*

Theorem 3 establishes the existence of an equilibrium for this economy and theorem 2 implies that interest factors and per-capita assets converge to an invariant distribution. It is also possible to establish certain characteristics of these invariant distributions.

**Theorem 4** *If the economy is Samuelson then the invariant measure  $\theta$  places all probability mass at zero. The support of the invariant measure  $\lambda$  is the closed interval  $[\frac{A}{\beta B}, \frac{B}{\beta A}]$ . If the economy is classical then the support of  $\theta$  is a closed interval  $[a, b] \subset Y$ , where  $E < a < b < 0$ , and the support of  $\lambda$  is a closed interval  $[c, d] \subset Z$ , where  $1 < c < \gamma < d < 1/\beta$ .*

Theorem 4 states that in a Samuelson economy the value of per-capita outside assets converges to zero with probability one as  $t \rightarrow \infty$ . In a classical economy the invariant measure  $\theta$  that describes the unconditional probability of observing any particular value of per-capita outside assets has a support which is an interval of the negative real line.

These results are fairly easy to understand by inspecting figures 2 and 4. The proof is omitted. Notice from figure 2 that, in a Samuelson economy,  $a_t < a_{t+1}$  for all  $a_t \in Y$  and from the geometry of the figure it is clear that  $a_t$  converges to zero for all realizations of the sequence of endowments. In the classical economy, for all

$a_t \in Y$  assets converge to an interval that is bounded by the intersection of the functions  $g(a; A, B)$  and  $g(a; B, A)$  with the 45 degree line.

The theorem implies that, in a classical economy, the support of the invariant measure  $\lambda$  contains  $\gamma$  and, in the Samuelson economy, it contains  $(1/\beta)$ . This result is analogous to known results about two period overlapping generations economies for which, in a classical economy, the interest factor converges to the growth factor and, in a Samuelson economy, it converges to the inverse of the discount factor.

## 9 Dynasties and Representative Agents

The first order difference equations that characterize equilibria in the dynastic economy are reminiscent of the two period overlapping generations economy. Since the behavior of this model is very different from that of representative agent economies it is instructive to examine the behavior of equations (26) and (25) as the parameter  $\gamma$  converges to one from above. If one holds constant the values of  $A$ ,  $B$  and  $\beta$  and lowers  $\gamma$  the set of economies will pass from Samuelson to classical. Continuing to lower  $\gamma$  towards one, the value of  $E$  in figure 4 becomes more and more negative. For all values  $\gamma > 1$  there exists a dynastic economy for which the policy of rolling over the value of (negative) government debt will support an equilibrium in which the value of per-capita assets is finite in all states. But for  $\gamma = 1$  the proof of the existence of an equilibrium breaks down as one can no longer find a finite set  $Y$  to which the stochastic difference equation  $g$  converges. When  $\gamma = 1$  equation (25) becomes linear. The construction of equation (26) involves division by  $\gamma - 1$ , (see the proof of theorem 3 in the appendix) and when  $\gamma = 1$  this equation reduces to:

$$R_t = \frac{\omega_t}{\beta\omega_{t-2}}. \quad (27)$$

For the case of  $\gamma = 1$  the economy is a representative agent economy. In this case the policy of rolling over a fixed debt is infeasible since it implies that the net present value of private borrowing is unbounded. Ricardian equivalence, in the zero growth economy, implies that the policy of an initial lump-sum tax or transfer must eventually be repaid in present value terms.

## A Proofs

### A.1 Proof of Theorem 1

This appendix provides a sketch of the proof that the closed form solution to the value function described in the text is valid.

#### A.1.1 Part 1

Given the assumption that consumption remains non-negative, the proposed solution for  $v^*$  is:

$$v^*(s') = F(Q(s'))W(s'). \quad (28)$$

Taking expectations of  $v^*(s')$  one period in advance, using the asset accumulation rule one obtains:

$$E[v^*(s')] = E[F(Q(s'))R(s')][R(s)a(s) + \omega(s) - c(s)] + E[\hat{h}(s')F(Q(s'))]; \quad (29)$$

which simplifies, using the definitions of  $Q$  and  $\bar{h}$  to

$$E[v(s')] = Q(s)[W(s) - c(s)]. \quad (30)$$

By substituting (30) into the definition of the value function and maximizing over  $c$  one obtains the first order condition:

$$(1 - \beta)[c^*(s)]^{-\beta}[Q(s)(W(s) - c^*(s))]^{\beta} - \beta Q(s)[c^*(s)]^{(1-\beta)}[Q(s)(W(s) - c^*(s))]^{(\beta-1)} = 0; \quad (31)$$

which may be rearranged to give the functional form (13). By substituting the solution for  $c^*(s)$  at a maximum (equation(13)), back into the definition of the value function one obtains the expression:

$$v^*(s) = F(Q(s))W(s). \quad (32)$$

This establishes that  $v^*$  is a fixed point of equation 8.  $\square$  To complete the proof one must establish that  $v_T = F(Q_T)W_T$  is optimal as  $T \rightarrow \infty$ . The following lemma helps to establish this result.

**Lemma 3** *The present value of the limiting value of assets in the optimal plan is equal to zero:*

$$\lim_{T \rightarrow \infty} P_1^T a_T^* = 0, \quad \text{for all } \phi \in \Phi.$$

**Proof**

Let  $z_t = \omega_t - \bar{h}_t$ . The variable  $z$  lies in a compact set since  $\Omega$  and  $H$  are compact, it follows that  $\max z$  exists and is finite. If the agent follows the optimal consumption plan, the value of assets  $a_{T+1}^*$  is given by the expression:

$$a_{t+1}^* = R_t a_t^* + z_t.$$

Iterating this expression from period 1 to period  $T$  and multiplying through by  $P_1^T$  yields the expression:

$$P_1^T a_{T+1}^* = \beta^T R_1 a_1^* + \sum_{s=1}^T \beta^{T-s} P_1^{s+1} z_s.$$

Now let  $\delta = \max\{\beta, 1/C\} < 1$ . Then it follows that

$$\beta^T R_1 a_1^* + \sum_{s=1}^T \beta^{T-s} P_1^{s+1} z_s \leq \delta^T R_1 a_1^* + \delta^T \sum_{s=1}^T \max z.$$

Since the limit of each term of this expression converges to zero as  $T \rightarrow \infty$  it follows that  $\lim_{T \rightarrow \infty} P_1^T a_T^* = 0$ .  $\square$

### A.1.2 The Value of Deviating From the Optimal Plan

To establish that the solution to the value function equation is also the solution to the sequence problem in which one forms sequences of contingent plans that maximize utility, I will obtain an expression for the value of deviating from the proposed optimal plan for the first  $T - 1$  periods. Letting  $T \rightarrow \infty$ , I will show that, the limit of the value of any such sequence of deviations exists and is bounded above by zero.

Let  $\{c^*\}$  be the proposed optimal sequence of contingent plans and let  $\{c^T\}$  be some other sequence that coincides with  $\{c^*\}$  for all dates  $t \geq T$ . Let  $u_t^T$  be the



value of  $\{c^T\}$  at date  $t$  and let  $v_t$  be the value of  $\{c^*\}$ , also at date  $t$ . From the definitions of  $v$  and  $u$  it follows that:

$$v_{T-1} = [c_{T-1}^*]^{(1-\beta)} [E_{T-1} v_T]^\beta, \quad (33)$$

$$u_{T-1}^T = [c_{T-1}]^{(1-\beta)} [E_{T-1} v_T]^\beta. \quad (34)$$

Expanding  $u_{T-1}^T$  around  $c_{T-1}^*$ , exploiting the concavity of  $u$  it follows that :

$$v_{T-1} - u_{T-1}^T \geq \frac{(c_{T-1}^* - c_{T-1})(1-\beta)[Q_{T-1} W_{T-1} \beta]^\beta}{[(1-\beta)W_{T-1}]^\beta}, \quad (35)$$

which simplifies to:

$$v_{T-1} - u_{T-1}^T \geq (c_{T-1}^* - c_{T-1}) F(Q_{T-1}). \quad (36)$$

By substituting out from the asset accumulation rule (4) one may replace the consumption terms in this expression by the value of assets;

$$v_{T-1} - u_{T-1}^T \geq [R_{T-1}(a_{T-1}^* - a_{T-1}) - (a_T^* - a_T)] F(Q_{T-1}). \quad (37)$$

Taking expected values at date T-2 it follows that:

$$E_{T-2}(v_{T-1} - u_{T-1}^T) \geq Q_{T-2}(a_{T-1}^* - a_{T-1}) - E_{T-2}(a_T^* - a_T) F(Q_{T-1}). \quad (38)$$

One proceeds by expanding  $u_{T-2}^T$  around  $(c_{T-2}^*, v_{T-1})$  to obtain the expression:

$$v_{T-2} - u_{T-2}^T \geq F(Q_{T-2})(c_{T-2}^* - c_{T-2}) + \frac{F(Q_{T-2})}{Q_{T-2}} E_{T-2}[v_{T-1} - u_{T-1}^T]. \quad (39)$$

Using (4) to eliminate  $(c_{T-2}^* - c_{T-2})$  and (38) to eliminate  $E_{T-2}(v_{T-1} - u_{T-1}^T)$  it follows that:

$$E_{T-3}(v_{T-2} - u_{T-2}^T) \geq Q_{T-3}(a_{T-2}^* - a_{T-2}) - E_{T-3} \frac{F(Q_{T-2})}{Q_{T-2}} E_{T-2} F(Q_{T-1})(a_T^* - a_T). \quad (40)$$

By proceeding backwards recursively from period  $T$  to period 1, recognizing that  $a_1^* = a_1 = 0$  one arrives at the following expression for the value of deviating from the proposed optimal path for the first  $T-1$  periods:

$$v_1 - u_1^T \geq F(Q_1) \prod_{s=2}^{T-1} E_{s-1} \frac{F(Q_s)}{Q_{s-1}} (a_T - a_T^*). \quad (41)$$

Expanding the bracket on the right-hand-side of (41) yields two terms, the first of which is given by the expression:

$$F(Q_1) \prod_{s=2}^{T-1} E_{s-1} \frac{F(Q_s)}{Q_{s-1}} a_T. \quad (42)$$

But notice from the definition of  $Q_t$ , equation (6), removing the expectations operators from (42) one as the identity:

$$R_1 F(Q_1) \prod_{s=2}^{T-1} \frac{F(Q_s)}{Q_{s-1}} a_T \equiv P_1^{T-1} a_T. \quad (43)$$

Assumption 2 implies that:

$$\lim_{T \rightarrow \infty} P_1^{T-1} a_T \geq 0, \quad (44)$$

which bounds this expression in every state. It follows from averaging across states that expression (42) must be greater than or equal to zero.

A similar argument may be applied to the second term that arises from expanding the right-hand-side of (41):

$$- \lim_{T \rightarrow \infty} F(Q_1) \prod_{s=2}^{T-1} E_{s-1} \frac{F(Q_s)}{Q_{s-1}} a_T^*. \quad (45)$$

Notice that:

$$R_1 F(Q_1) \prod_{s=2}^{T-1} \frac{F(Q_s)}{Q_{s-1}} a_T^* \equiv P_1^{T-1} a_T^*, \quad (46)$$

hence, if one can show that:

$$\lim_{T \rightarrow \infty} P_1^{T-1} a_T^* = 0, \quad (47)$$

it will have been demonstrated that the expression (45), which averages these terms across states, must also equal zero. But this result is established in lemma 3. Hence the value of any feasible deviation from the optimal plan is bounded above by zero

□

## A.2 Proof of Theorem 2

Given the following lemma (equivalent to Condition M in [9]) the proof follows directly from Stokey Lucas and Prescott theorem 11.12 page 350.

**Lemma 4** *Let  $\chi^N(s, U)$  be the probability of reaching the set  $U \in S$  from  $s \in S$  in  $N$  steps. Let  $S - U$  be the complement of  $U$  in  $S$ . There exists an  $\epsilon > 0$  and an integer  $N \geq 1$  such that for any  $U \in S$ ,*

$$\text{either } \chi^N(s, U) \geq \epsilon, \quad \text{all } s \in S, \quad \text{or } \chi^N(s, S - U) \geq \epsilon, \quad \text{all } s \in S.$$

Lemma 4 says that every set in  $S$  is visited with probability greater than or equal to  $\epsilon$  in  $N(\epsilon)$  steps from every point in  $S$ : or its complement is visited with probability  $\epsilon$  in  $N(\epsilon)$  steps from every point in  $S$ . The proof of lemma 4 relies on the geometry of figure 1.

## A.3 Proof of Lemma 4

First, note that any constant sequence  $\{\omega_t = \omega\}_{t=2}^N$  occurs with positive probability  $\epsilon(N)$ . Since, for large enough  $N$ , any constant sequence will bring the interest factor within a distance  $\delta(N)$  of  $1/\beta$  it follows that from any initial  $R_1$  in the set  $[C, D]$  there is positive probability of coming within  $\delta$  of the point  $1/\beta$  in  $N(\delta, R_1)$  steps. Now define the set  $(e, h)$  to be the open interval:

$$(e, h) = ( f(1/\beta; A, B, B), f(1/\beta, B, A, A) ).$$

The set  $(e, h)$  defines points that can be arrived at with positive probability from any initial  $s_1 \in S$  by constant sequences of very low endowments followed by a large endowment, or by sequences of very large endowments followed by a small endowment.

Since, by assumption 1,  $\mu$  puts positive weight on all values of  $\omega \in [A, B]$  it follows that any point  $q$  in the interval  $(e, 1/\beta]$  can be reached in finite steps from any  $R_1$  in  $\{R_1 \leq 1/\beta \leq R_1 \leq D\}$  by sequences of the form  $\{B, B, \dots, \omega_N(q)\}$  and

from any point in  $\{R_1 \mid C \leq R_1 \leq 1/\beta\}$  by sequences of the form  $\{A, A, \dots, \omega_N(q)\}$ . A similar argument holds for points  $r$  in the interval  $[1/\beta, h)$ . It follows that for any set  $U \subset (e, h)$  one can find a finite  $N$  and a positive  $\epsilon$  such that the probability of entering  $U$  from any  $R_1 \in [C, D]$  is greater than or equal to  $\epsilon$ . If  $U \subset \{S - (e, h)\}$  then there is positive probability of entering  $S - U$  in  $N$  steps from any  $R_1$ .  $\square$

### Proof of Theorem 3

This appendix proves that the difference equations (25) and (26) characterize competitive equilibria.

#### Part 1: The Asset Evolution Equation

Agents with the preferences described by equation (8) must form an expectation of the value of average human wealth in order to compute their current consumption. The equilibria that I focus on in this paper are constructed by assuming that human wealth is described by the stochastic difference equation:

$$\bar{h}_{t+1} = R_{t+1}[\bar{h}_t - \omega_t], \quad (48)$$

with initial condition:

$$\bar{h}_1 = \frac{\omega_1}{1 - \beta} - T. \quad (49)$$

For any transition function  $\chi$  describing the evolution of  $R_t$ , equation (48) defines a function  $\bar{h}$  that satisfies the functional equation given in definition (7). Providing  $\bar{h}$  remains bounded it must represent the unique solution to this equation.

Given that  $\bar{h}$  evolves in this way, the market clearing condition in subsequent periods will impose a relationship that must hold between the value of financial assets and the value of human wealth:  $c_t = \omega_t = (1 - \beta)[R_t a_t + \bar{h}_t]$  or, rearranging this expression:

$$\bar{h}_t = \frac{\omega_t}{1 - \beta} - R_t a_t. \quad (50)$$

Substituting  $\bar{h}$  into equation (48) and using the budget identity  $\gamma a_{t+1} = R_t a_t$  to eliminate  $R_{t+1}$  one arrives at the difference equation:

$$a_{t+1} = \frac{a_t \omega_t}{\gamma[\beta \omega_{t-1} - (\gamma - 1)(1 - \beta)a_t]},$$

which is the expression given in equation (25). One still needs to check that  $\bar{h}$  remains bounded. Since  $\bar{h}$  is the function of  $R_t a_t$ , given by equation (50), and since  $R_t a_t = \gamma a_{t+1}$ , from the evolution of aggregate assets, one need only show that aggregate assets remain bounded. But since  $g : Y \mapsto Y$  it follows that for  $R_0 a_0 \in Y$  that  $a_t$  remains bounded for all  $t$ .  $\square$

### Part 2: The Interest Factor Evolution Equation

Multiply both sides of equation (25) by  $\gamma$ , divide both sides by  $a_{t-1}$  and use the definition of the evolution of aggregate assets:  $\gamma a_t = R_{t-1} a_{t-1}$  to obtain the expression:

$$R_{t-1} = \frac{\omega_{t-1}}{[\beta \omega_{t-2} - (\gamma - 1)(1 - \beta)a_{t-1}]}.$$

Rearranging this equation and substituting it into equation 25 one arrives at difference equation in  $R_t$ :

$$R_t = \frac{\omega_t}{\omega_{t-1}[\beta + \frac{1}{\gamma}] - \beta(R_{t-1}/\gamma)\omega_{t-2}}.$$

which is the expression given in equation (26).  $\square$

### Part 3: The Behavior of Individual Consumption

The conditions of theorem 1 require one to check that, in any proposed equilibrium, the consumption of every dynasty remains positive. The consumption of dynasty  $\tau$  is given by the expression:

$$c_t^\tau = (1 - \beta)W_t^\tau,$$

where,

$$W_t^\tau = R_t a_t^\tau + \bar{h}_t.$$

But when human wealth is given by equation (48), it follows that the family's wealth,  $W_t^r$  evolves according to the expression:

$$W_t^r = R_t(W_{t-1}^r - c_{t-1}^r),$$

and when  $c_t^r$  is chosen optimally:

$$W_t^r = \beta R_t W_{t-1}^r.$$

Restating this equation in terms of consumption:

$$c_t^r = \beta R_t c_{t-1}^r.$$

Since the family's initial consumption is a fraction of average human wealth in its first period of life, since average human wealth is positive in all states, and since interest factors remain positive in every state, it follows individual consumption remains positive in every realization of  $\phi$  for all dynasties.  $\square$

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