

The Existence of Perfectly Competitive Equilibrium à la Wicksteed^{1 2}

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Abstract

Historical documentation is provided to show that the *no-surplus* approach to perfectly competitive equilibrium is a descendant of ideas originating in the marginal productivity theory of distribution. We rely on connections with the work of J. B. Clark and especially P. H. Wicksteed. These connections are put to the test by showing how our reformulation of the marginal productivity principle can be used to give a self-contained proof of the existence of perfectly competitive equilibrium. Links are also established between the no-surplus approach to competitive equilibrium, the Shapley value, and the “productivity ethics” of Clark.

The no-surplus approach to perfectly competitive equilibrium [Ostroy (1980) and Makowski (1980)] is a descendant of ideas originating in the marginal productivity theory of distribution. In this essay, we want to emphasize a kind of “loop” — not only does our work descend from marginal productivity theory, but it also can be turned around to point to the marginal productivity principle as the centerpiece of the theory of value. A few of the early contributors to the theory had an appreciation of its centrality, but their ideas on this score were insufficiently well-developed to be understood. Reviewing the contributions of Clark and especially Wicksteed, we identify some of their more far-reaching claims which have been ignored or dismissed by commentators as the result of reading them without the aid of an appropriate theoretical map. This will provide the historical backdrop to the main theorem of this paper on the existence of perfectly competitive equilibrium. Such an equilibrium will be *defined* as an allocation yielding each individual the marginal product of his/her contribution.

The paper is organized as follows: In Section 1.1 we review the more modest position assigned to marginal productivity theory within the traditional theory of value. In preparation for what follows, we focus on claims for marginal productivity theory, by Wicksteed and Clark, that go beyond its usual interpretation and we review the critiques aimed at their extraordinary claims. After collecting the arguments on which the traditional synthesis of value theory rests, in Section 1.2 we shall informally outline another approach to marginal productivity. While this alternative approach is heuristically similar to the traditional one, it is quite different in the role that it assigns to marginal productivity. Instead of regarding marginal productivity theory as one piece of the competitive theory of value, in this approach it will emerge as “the whole thing”. Our aim is to show that when marginal productivity theory is viewed from this different perspective, the bolder claims of Wicksteed and Clark are

not overreaching. In Section 2, we put this assertion to the test by showing formally how our reformulation of the marginal productivity principle can be used to give a proof of the existence of perfectly competitive equilibrium in the spirit of Wicksteed. Our main theorem is closely related to the equivalence between the Shapley value and Walrasian equilibrium (Shapley [1953], Aumann and Shapley [1974], Aumann [1975], Champsaur [1975]) and in Section 3 we shall summarize the links between our result and the Value Equivalence Theorem. We shall also briefly discuss the connections between the Value and Clark's interpretation of marginal productivity as well as an historical irony concerning Edgeworth's views on marginal productivity theory.

Remark: Throughout our discussion of the history and of the formulation of our theorem, we shall call attention to the following distinction: the view that the *commodity* is the fundamental margin versus the view that the *individual* is the fundamental margin of marginal analysis. It is the change in this margin, from the commodity to the individual, that underlies the historical loop in our formulation of marginal productivity theory.

1 Some History of Marginal Productivity Theory

1.1 Marginal Productivity Theory As One Part of the Theory of Value

The marginalist revolution in the theory of value in the 1870's is mainly associated with a break with the classical theory in which utility, and in particular marginal utility¹, was given coeval status with cost of production as a determinant of price. Wicksteed and Clark are known for their contributions to the second generation of this revolution, in the 1890's, in which marginalist principles were applied to the demand for factors of production via the marginal productivity theory of distribution.² Both of these contributors were criticized for overreaching in their claims for marginal productivity theory. Compared to Wicksteed, whose claims were oblique and technical (see below), Clark's were regarded as something of an obvious embarrassment.

In the opening paragraph of the Preface of the *Distribution of Wealth* (1899), Clark summarized his objective:

It is the purpose of this work to show that the distribution of the income to society is controlled by a natural law, and that this law, if it worked

¹It was Wicksteed who introduced the term "marginal" utility to replace Jevons' "final degree of" utility.

²Stigler's *Production and Distribution Theories* (1941) is a masterful survey of the development of marginal productivity theory.

without friction, would give to every *agent* of production the amount of wealth which that agent creates. However wages may be adjusted by bargains freely made between individual men, the results of pay that result from such transactions tend, it is here claimed to equal that part of the product of industry which is traceable to the labor itself; and however interest may be adjusted by similarly free bargaining, it naturally tends to equal the fractional product that is separately traceable to capital. At the point in the economic system where titles to property originate, — where labor and capital come into possession of the amounts that the state afterwards treats as their own, — the social procedure is true to the principle on which the right of property rests. So far as it is not obstructed, it assigns to every one what *he* has specifically produced. (Italics added.)

Phrases such as “natural law” and “true to the principle on which the right of property rests” invited the charge that marginalist theory was an apology for capitalism.³ Neoclassical economists after Clark either ignored these remarks or distanced themselves from such interpretations. Although this is not our principle focus, we shall remark, below, that recent developments concerning perfectly competitive equilibrium and the Shapley value indicate that what has been referred to as Clark’s “naive productivity ethics” may not be so naive after all.

The bold claims made by Clark for the marginal productivity principle as the unique unifying principle underlying the theory of distribution were significantly diluted in the mature neoclassical synthesis of value theory. Marginal productivity theory was, after all, merely an interpretation of the first order conditions for profit-maximization by the individual firm. As such, we might say it was a principle underlying only one-fourth of the theory of value, namely (1) the demand for factors of production. The other three-fourths were (2) the determinants of the of the supply of factors of productions and (3) the determinants of the demands and (4) supplies of outputs. As a less than majority contributor to the theory of value, it was in no position to dictate what the income of society would be and how it would be distributed.⁴

Before moving on to the more interesting case of Wicksteed, we call attention to a change in the margin of analysis that occurs between the first and last sentence in Clark’s opening paragraph. In the first sentence, the “agent of production” is evidently a disembodied factor of production. However, it is clear from the last sentence that Clark intends his marginal analysis in terms of commodities as a means

³Stigler, *ibid.* p. 297.

⁴Blaug [1985, p.425] is particularly emphatic in wanting to limit the claims of the theory to nothing more than the underpinnings for the theory of factor demand. In contrast, Schumpeter [1954, p. 912-913], coming from the Mengerian tradition where marginal productivity is more seamlessly incorporated into value theory, gives it a more generous interpretation.

to an end, i.e., to extend his conclusions for the marginal productivity theory of factors of production to a marginal productivity theory for persons.

In *An Essay on the Coordination of the Laws of Distribution* (1894), Wicksteed recognized the importance of a logical consistency condition as an essential element of marginal productivity theory. What set the marginalist explanation of the pricing of inputs apart from the classical theory was that the rewards to all of the cooperating factors of production would be determined by one principle, marginal productivity. However, the validity of the principle depended upon the conclusion that when the rewards to all of the factors were added up, they would have to equal the total output. Otherwise, if they added up to more than the total, there would be a contradiction; and if they added up to less than the total, it would be subject to the criticisms of the classical theory where, for example, profits were the residual. To arrive at the required consistency condition, Wicksteed appears to have rediscovered Euler's Theorem on Homogeneous Functions because he points out that adding-up would require that the production function be linearly homogeneous.⁵

Unlike Clark whose analysis was in the verbal tradition, Wicksteed adopted a mathematical model that is still quite familiar: it is that of a function producing one output from many inputs. The key demonstration is that if the function is homogeneous of degree one, then payments to the factors of production according to the marginal productivity principle will just exhaust the total product.

Although the style of the *Essay* is quite distinct from that of Clark, there is one key element that the two works have in common: the production function is frequently regarded not at the level of the firm, but as an aggregate function applying to the economy as a whole. From a micro-economic theory perspective, this would seem to imply an inability to recognize or an unwillingness to face the difficult problems of aggregation. The issue of which level to use in interpreting the production function — the micro-economic level of the single firm or the macro-economic level of the economy as a whole — is, along with the distinction between the commodity margin and the individual margin, at the heart of our “rehabilitation” of Wicksteed and of the loop we shall emphasize in turning marginal productivity theory into the centerpiece of the theory of value.

It might appear that there really is no choice in the interpretation. A single aggregate function may be posited in the manner of a representative firm, but since production does take place at the level of the individual firm this is just a convenient simplification. Just as the production function is a micro-economic concept applying to the firm, so marginal productivity theory derives from the first order conditions for profit maximization by the firm: the two ideas fit together like hand and glove.

⁵Flux (1894) is credited with placing Wicksteed's contribution as an application of Euler's Theorem.

This is certainly the standard interpretation and the one that underlies the criticisms that Wicksteed received from especially Walras and Wicksell.

The critics pointed out that the requirement of constant returns to scale in the production functions of individual firms was an unnecessarily restrictive assumption for the validity of marginal productivity theory, i.e., adding-up could be obtained under much weaker conditions. The critics argued that by emphasizing the importance of constant returns, Wicksteed was ignoring the role of perfect competition. Even if firm production functions did not display constant returns but merely had U-shaped average costs, competition would drive the equilibrium allocation to the point where all producing firms would be operating at an efficient scale, at the bottom of their average costs, and therefore the zero-profits condition of competitive equilibrium would imply that the total payments to the factors of production would add up to the total value of the output.

So, the judgement is that while Wicksteed was an important pioneer, as pioneers often do, he did not get the story entirely right. In particular, what he failed to appreciate in his partial equilibrium analysis was the importance of the general equilibrium implications of perfect competition. That he should have fallen into such an error was to be expected since he attempted to get at the laws determining distribution solely through the application of the marginal productivity principle (that one-fourth of the theory of value), without, for example, invoking the importance of marginal utility. His remarkable originality with respect to the one-fourth that he helped to develop was no match for the emerging full synthesis of value theory found in Walrasian general equilibrium.

1.2 Marginal Productivity as a Complete Theory of Value

Our work on the no-surplus theory of perfect competition suggests a more sympathetic reading of Wicksteed's *Essay*. This reading puts us in a position with respect to Wicksteed's critics that is similar to how Wicksteed might have appeared to them: we would say that his critics did not have a sufficiently comprehensive command of their theory to appreciate the significance of perfect competition!

Wicksteed's claims make more sense, and the critical observations leveled against him lose their force, if the *Essay* is interpreted as making a much bolder claim than the one for which he is generally credited. What this bolder claim is, and what evidence exists that Wicksteed was attempting to make it, is the subject of this section.

We begin with a basic change in the interpretation of the production function from the usual concrete relation between quantities of inputs and their resulting outputs to something that is, by comparison, a "metaphorical" function. Consider a function whose output is total gains from trade, or social utility, to the economy as a whole.

Thus, it is an aggregate function rather than one applying to a single micro-economic unit and, more importantly, its output is measured in some abstract utility — or, to use Wicksteed’s language, in units of “satisfaction” — rather than units of physical output. Not only does the output of this function differ from the usual production function, but its inputs are also not the same. The inputs to this “production function for social utility” are *individuals*, where the productivity of an individual is based on the tastes and endowments or production possibilities (if a firm) which characterize the individual.

To make our metaphorical function resemble the usual production function, in our formal analysis we impose two further requirements: (i) there are only a finite number of different types of individuals (similar to a finite number of different types of factors of production) and (ii) each individual is of infinitesimal scale compared to the economy as a whole (so that single individuals can be regarded as infinitesimal inputs). The quantity, or mass, of individuals of type i is denoted by r_i and the gains from trade associated with the population $r = (r_1, \dots, r_n)$ is denoted by $g(r)$. Although we have changed the *interpretation* of the function, we may still apply the concept of marginal productivity. An (infinitesimal) individual of type i is paid his/her marginal product if the quantity of “social utility” the individual receives is equal to $\partial g(r)/\partial r_i$, (assuming that this partial derivative exists). Ignoring, for the moment, what this social utility is and how the payment can be made, if this reward scheme is to be a complete and logically consistent theory of distribution, it will require that when the total output of social utility is distributed to the population according to the marginal productivity principle, the total rewards must add up, i.e.,

$$\sum \frac{\partial g(r)}{\partial r_i} r_i = g(r).$$

A distribution of the total output of social utility that satisfies the marginal product reward principle and also satisfies the above adding-up requirement will be called a marginal productivity distribution. Evidently, therefore, to guarantee that this condition will be satisfied for every r , *the gains function g will have to be homogeneous of degree one*. Fortunately, for this function such a restriction is quite reasonable. If the population were literally to double, then the gains from trade would also double. For example, assuming that there are no common property resources, then all of the population would own all of the land; so doubling the population would mean that the number of acres of land would also have to double. Indeed, for the gains function to be other than homogeneous of the first degree would require certain phenomena *not* typically admitted into the theory of value.⁶

⁶For example, common property resources are not typically admitted into the standard presentation of the theory of value. Their presence would, however, imply that a doubling of the population would not double all the relevant resources and thus may lead to decreasing returns.

The first degree homogeneity of g is perfectly consistent with the U-shaped costs discussion cited above. Doubling the size of the economy simply doubles the number of producing firms, each operating at the bottom of their U. Indeed, one can claim even more. If the aggregate value of g were to more than double when the population doubled, this would imply that economies of scale at the level of the individual firm were so large that they would be incompatible with perfect competition. The point of this observation is that while it is certainly true that with respect to variations in the level of *commodity* inputs, there is no need to assume first degree homogeneity of the production function for *the single firm*, when we are considering the aggregate function g whose inputs include the firms themselves, its homogeneity is essential to maintain perfect competition.

Granting this change to the metaphorical aggregate gains function whose inputs are households and firms, what does this have to do with the theory of value? We know that social utility is not a measurable quantity, let alone a fungible commodity which can be transferred from one person to another. So, how do these metaphorical marginal products relate to the central issues of value theory — the pricing of inputs and outputs? While a formal response to this question is the subject of Section 2, we give a brief and informal description here.

In the Walrasian (also the Marshallian) approach to value theory, there is no direct emphasis on the distribution of individual welfare. Rather, the direct focus is on the equilibrium conditions for the pricing of commodities. Once these (market-clearing) prices are known, the implications for the the distribution of the gains from trade follow from, for example, the household's indirect utility function which relates prices to the maximized value of utility. But this is not the only way to address the theory of value. It may be approached from the "opposite direction" where the direct emphasis is on the distribution of welfare. In this opposite direction, the equilibrium condition for the competitive theory of value is that each *individual* receives the marginal product he/she adds to the social gains from trade, more or less as Clark claimed. Just as the distribution of welfare follows from the commodity market equilibrium approach to the theory of value, so — when approached from the opposite direction — the theory of competitive market pricing follows from the existence of a marginal productivity distribution. In this way, the marginal productivity principle becomes the centerpiece of the theory of value.

What does this modern revision of marginal productivity theory have to do with Wicksteed? At the beginning of his *Essay*, Wicksteed makes a distinction between two interpretations of his production function $F(A, B, C, \dots)$.

When we have safeguarded this statement by all the explanations necessary to enable us to speak of communal desires and satisfactions, we may say that the total satisfaction (\mathbf{S}) of a community is a function (F)

of the commodities, services, etc. (A, B, C, \dots) which it commands; or $S = F(A, B, C, \dots)$. And the exchange value of each commodity or service, if purchasable, is determined by the effect on the total satisfaction of the community which the addition or the withdrawal of a small increment would have, *all the other variable remaining constant*. (p.8, italics in original.)

A more “concrete” interpretation of the function is:

Let the special product to be distributed (P) be regarded as a function (F) of the various factors of production (A, B, C, \dots). Then the (marginal) significance of each factor is determined *by the effect upon the product of a small increment of that factor, all the others remaining constant*. (p.8, italics in original.)

The first interpretation of F as yielding the output S is regarded as applying to the theory of exchange⁷ while the second interpretation of F as yielding the output P applies to the usual theory of distribution. Wicksteed acknowledges the measurability problems associated with satisfaction, whereas “in the case of the distribution of the product, we have something external to the claimants, *something not themselves*, which is actually sliced up and divided amongst them.” (Italics added.) It is Wicksteed’s first interpretation of his production function F that we identify as a version of our gains function g .

Further on, the question arises as to reasonableness of the constant returns hypothesis for F . Wicksteed makes a distinction here based on the two interpretations of the function.

The question we are examining, then, is this: If every one of the abilities, efforts, materials and advantages which contribute to production were severally increased in identical ratio, would the product also be increased in that ratio?

The crucial limitation is not simply the change in output, but what Wicksteed calls the “industrial position or vantage”, by which he means revenue.

... it is still unsafe to say that if all the factors were equally multiplied the product would be increased proportionally, because we have defined the product, not as a material product but as an industrial position or vantage; and, in order that this may be doubled or trebled, it is necessary not only

⁷“Indeed the law of exchange value is itself the law of distribution of the general resources of society.” p.7-8.

that all of the factors of production should be doubled or trebled, but also the *area of operations* should be capable of corresponding enlargement on the same terms that have ruled hitherto; or in other words that there should be a fresh supply of people who want, or can be made to want, the commodity or service in question, on the same terms as those who now enjoy it. And to assume this is obviously unwarrantable. (p. 34, italics in original.)

The above quote refers to the *P* interpretation of his production function *F*. By contrast, Wicksteed observes that constant returns are built into the *S* interpretation of *F*.

And at this point comes dimly into view a problem of the utmost interest and importance, which will be touched upon, on one side, under section 7, paragraph (f) of this essay, and which suggests itself under another aspect here, but is far too vast to be dealt with incidentally. For the truth is that the real or “social product” is the total satisfaction accruing from the processes of industry to the whole community, including both the customer and the manufacturer; and in this sense the body of customers and their desire for the product, themselves constitute the factors of production. If *these* factors, like the others, receive a proportional increment then obviously the conditions are exactly repeated, and the product too will receive a proportional increment. (p. 34, italics in original.)

As our final quotation from Wicksteed, we reproduce section 7, paragraph (f), referred to above.

Lastly, I would just touch upon the fascinating subject of the analogies between the curves of Production and the curves of Satisfaction. In the ordinary individual or person curve of consumption, the satisfaction is regarded as accruing to the individual, and the price paid is regarded as subtracted from that amount. The early amounts are regarded as yielding a surplus over their price because they yield a higher satisfaction than is represented by the marginal significance that regulates their price. Our investigations suggest another way of looking at the matter. Satisfaction may be regarded as a function of certain factors, one of them being a psyche or sensitive subject. If this be followed out it will be found that the “surplus” or “consumer’s rent” is neither more nor less than the differential coefficient or marginal significance of *psyche* as a factor in the production of satisfaction! Here we once again see that when the marginal distribution has been completed there is no surplus. Our ideal is for the

whole satisfaction, without deduction, to fall in distribution to psyche, so that increments of psyche would be identical with increments of a satisfaction ideally maximized in its amount per unit psyche. (p. 48-49, italics in original.)

In virtually all respects, we find this to be an accurate summary of the conditions underlying the main theorem of this paper. Translating Wicksteed's notation, we have $\partial g(r)/\partial r_i \sim \partial F/\partial K$. The function g is derived from the basic data of the economy which includes tastes and resources. Letting v_i be the utility function of an individual of type i , there is some allocation $z = (z_i)$, where z_i is the *net trade* to individual i , that achieves the total satisfaction $g(r)$, i.e.,

$$g(r) = \sum v_i(z_i)r_i.$$

With certain qualifications, it can be shown that $\partial g(r)/\partial r_i$ exists. Therefore, since g is homogeneous,

$$g(r) = \sum \frac{\partial g(r)}{\partial r_i} r_i.$$

So,

$$(I) \quad \sum \left[\frac{\partial g(r)}{\partial r_i} - v_i(z_i) \right] r_i = 0.$$

The marginal productivity condition for equilibrium is the existence of a z such that for each i ,

$$(II) \quad \frac{\partial g(r)}{\partial r_i} = v_i(z_i).$$

When the allocation is such that each individual is receiving his/her marginal product, then each "psyche" is extracting as a reward all the surplus contributed and we can agree with Wicksteed that "when the marginal distribution has been completed there is no surplus."

The implications for commodity pricing follow from the result that for any z satisfying (I), there is a price vector p such that

$$\frac{\partial g(r)}{\partial r_i} = v_i(z_i) - pz_i.$$

Therefore, the marginal productivity condition for equilibrium (II) requires that $pz_i = 0$ and this will be shown to imply that the pair $[p, z]$ is a Walrasian equilibrium.

Remark: The marginal productivity equilibrium condition is something more than a definition of Walrasian equilibrium in disguise. For example, in models with a small

(i.e., finite) number of individuals Walrasian equilibrium rarely satisfies the marginal productivity equilibrium condition. (See Ostroy [1980] and Makowski [1980] and Anderlini [].) Even in models with a continuum of individuals, the two can differ. The requirement that ∇g exist is a substantive restriction on the definition of equilibrium which expresses a kind of *perfect substitutability among individuals* that is part of the heuristic meaning of perfect competition. When $q = \nabla g(r)$, there are well-defined rates at which *individuals* can be substituted for each other while holding total satisfaction constant. These well-defined rates of person-by-person substitutability translate to well-defined rates of commodity-by-commodity substitutability in the sense that $q = \nabla g(r)$ implies that at the associated commodity prices p each individual faces perfectly elastic demands and supplies for the goods he/she sells and buys. Recalling the price-taking assumption of Walrasian equilibrium, this perfect substitutability/perfect elasticity need not be a feature of Walrasian equilibrium, even in a nonatomic model. In Section 4, we shall use an example due to Edgeworth to illustrate the consequences of the distinction.

The quotations and discussion above are not intended to show that Wicksteed anticipated the *proof* of the main result of Section 3. What they are intended to show is while developments in economic theory did not prepare him to construct such a proof, the conclusion confirms his speculations on the implications of the marginal productivity theory of distribution as it applies to the theory of exchange!

Wicksteed has received considerable recognition for his part in the development of marginal productivity theory. Of necessity, this recognition has been limited by the commentators' own understanding of that theory and, as a result, some parts of his *Essay* have not been appreciated. It is easy to see that with a conventional view of marginal productivity theory, Wicksteed's excursions into an exchange interpretation where the product is "satisfaction" and the inputs of the function are "psyche" might appear, at best, as poetic flights of fancy to be separated from his real contribution to the development of economics. As an illustration, in commenting on Wicksteed's *Essay* Stigler writes:

As an alternative, the product may be social utility. Since marginal utility decreases, the theorem [Euler's] then holds only if consumers are included among the factors of production. This result is of no practical significance; let us pass to the third and important concept [commercial product]. (p. 331.)

What is dismissed here as having no import, we regard as pointing to an understanding of the marginal productivity principle which makes it the centerpiece of the theory of value.

2. Definitions, Theorems and Proofs

2.1 Statement of Theorems 1 and 2

2.1.1 The Model

Consider a non-transferable utility model of a pure exchange economy in net trade form. The economy can be described by a set $\mathcal{E}_{NTU} = (v_i: i = 1, \dots, n)$, where $v_i: \mathbb{R}^\ell \rightarrow \mathbb{R} \cup \{-\infty\}$ represents the preferences of i over trades $z_i \in \mathbb{R}^\ell$. Our convention is that positive (negative) elements of z_i represent purchases (sales); and the effective domain of v_i , $Z_i = \{z_i \in \mathbb{R}^\ell: v_i(z_i) > -\infty\}$, represents the set of feasible trades for i . \mathcal{E}_{NTU} may be interpreted as either an economy with n individuals, or as an economy with a non-atomic continuum of individuals consisting of n types, with a unit mass of each type. The second interpretation will be primary.

The following assumptions will be maintained throughout. Letting $\Omega = \mathbb{R}_+^\ell$, for all i :

$$(A.1) \quad Z_i = \Omega - (\omega_i) \quad \text{for some } \omega_i \in \Omega$$

$$(A.2) \quad v_i \text{ is concave, continuous on } Z_i, \text{ and for each } z_i, \text{ there exists } z'_i \text{ s.t. } v_i(z'_i) > v_i(z_i).$$

$$(A.3) \quad 0 \in \text{int}(\Sigma Z_i) \quad (\text{i.e., } \omega = \Sigma \omega_i \gg 0).$$

Interpreting ω_i as i 's resource endowment, (A.1) implies that i 's consumption set is Ω .

2.1.2 Embedding \mathcal{E}_{NTU} Into A Family of Economies

In preparation for stating and proving the main result, it will be convenient to regard \mathcal{E}_{NTU} as a particular member of a two parameter family of economies,

$$\{\mathcal{E}_{NTU}(r, w): r = (r_1, \dots, r_n) \in \mathbb{R}_+^n \text{ and } w \in \mathbb{R}^\ell\}.$$

Interpret $\mathcal{E}_{\text{NTU}}(r,w)$ as an economy with a mass r_i of each type i and with w extra resources. Within this family \mathcal{E}_{NRU} has a unit mass of each type i and has no extra resources, i.e., $\mathcal{E}_{\text{NTU}} = \mathcal{E}_{\text{NTU}}(e,0)$, where $e = (1,1,\dots,1) \in \mathbb{R}^n$.

For any economy $\mathcal{E}_{\text{NTU}}(r,w)$, an allocation $z = (z_1, \dots, z_n)$ is *attainable* if $\sum r_i z_i = w$, i.e., if the sum of trades equals the total extra resources in the economy. For any $r \in \mathbb{R}_+^n$ and $w \in \mathbb{R}^\ell$, let

$$A(r,w) = \{z \in \prod_i Z_i : \sum r_i z_i = w \text{ and } z_i = 0 \text{ if } r_i = 0\},$$

$$A = A(e,0).$$

$A(r,w)$ represents the set of attainable allocations in $\mathcal{E}_{\text{NTU}}(r,w)$. It is a standard exercise to verify that, under our maintained assumptions,

(1) $\forall r, \forall w$: $A(r,w)$ is compact, convex; and $A(r,w) \neq \emptyset$ iff $w \in \sum r_i Z_i$.

Remark: We have implicitly restricted our attention, above, to equal-treatment allocations. For our purposes this is merely a felicitous and harmless simplification. As is well-known, given our assumptions that each v_i is concave and each Z_i is convex, for any attainable allocation there exists an attainable, utility-equivalent, equal-treatment allocation.

2.1.3 The Marginal Product of an Individual

Throughout, let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$, $\lambda \neq 0$. For any such λ , define the function $\psi_\lambda: \mathbb{R}_+^n \times \mathbb{R}^\ell \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$\psi_\lambda(r,w) = \sup \{ \sum r_i \lambda_i v_i(z_i) : \sum r_i z_i = w \}.$$

(Our convention is that $0 \cdot (-\infty) = -\infty$; so the above supremum equals $-\infty$ unless $z_i \in Z_i$ for all i , even i with $\lambda_i = 0$.) The number $\psi_\lambda(r,w)$ represents the maximum potential *gains from trade* in the economy $\mathcal{E}_{\text{NTU}}(r,w)$, when social utility is formed by adding the individual utilities weighted

according to λ . Also define $f_\lambda: \mathbb{R}^l \rightarrow \mathbb{R} \cup \{-\infty\}$ and $g_\lambda: \mathbb{R}_+^n \rightarrow \mathbb{R}$ by:

$$\forall w, f_\lambda(w) = \psi_\lambda(e, w)$$

$$\forall r, g_\lambda(r) = \psi_\lambda(r, 0).$$

So, $f_\lambda(0) = g_\lambda(e)$. The function f_λ measures the gains from trade when extra *resources* are added to E_{NTU} , while g_λ measures the gains from trade when r -e extra *individuals* are added to E_{NTU} . It is a standard exercise to verify that f_λ and g_λ are well-defined (e.g., the sup in the definition of f_λ never equals $+\infty$), concave functions. More specifically, for future reference:

(2) $\forall \lambda$, f_λ is concave and satisfies for any w :

$$\bullet f_\lambda(w) = -\infty \text{ iff } A(e, w) = \emptyset,$$

$$\bullet f_\lambda(w) = \max \{ \sum \lambda_i v_i(z_i) : z \in A(e, w) \}$$

if $A(e, w) \neq \emptyset$ (and this maximum exists).

(3) $\forall \lambda$, g_λ is concave, positively homogeneous (of degree one), and satisfies for any r :

$$g_\lambda(r) = \max \{ \sum \lambda_i v_i(z_i) r_i : z \in A(r, 0) \} \text{ (which exists).}$$

Further, letting

$$\zeta(\lambda) = \arg \max \{ \sum \lambda_i v_i(z_i) : z \in A = A(e, 0) \},$$

Then:

ζ is a u.h.c, convex-valued correspondence on $\Lambda = \{ \lambda \in \mathbb{R}_+^n : \sum \lambda_i = 1 \}$.

Now we are prepared for the key definitions. The main idea of our approach to perfectly competitive equilibrium is that, under perfectly competitive conditions, the rewards that individuals receive are equated to their marginal social contributions. When there is a continuum of agents,

the latter can be calculated using the calculus, just like ordinary marginal products. More specifically, for any given λ , the "marginal product of individual i " is the marginal social contribution of an infinitesimal additional mass of type i to the economy \mathcal{E}_{NTU} , when social utility is calculated according to the weights λ . Formally, the idea is expressed in terms of the directional derivative of the gains function g_λ .

Definition: The *marginal product* of an infinitesimal individual of type i under λ is given by $g'_\lambda(e; e_i)$, where

$$g'_\lambda(e; e_i) = \lim_{t \rightarrow 0^+} \frac{g_\lambda(e + te_i) - g_\lambda(e)}{t}$$

and e_i is the i^{th} unit vector in \mathbb{R}^n .

Our main result is an existence theorem specifying conditions under which each individual receives his/her marginal product.

Definition: The pair (z°, λ°) is a *marginal productivity (MP) equilibrium* for \mathcal{E}_{NTU} if $z^\circ \in A$, $\lambda^\circ \gg 0$ and for each i ,

$$\lambda_i^\circ v_i(z_i^\circ) = g'_{\lambda^\circ}(e; e_i)$$

It will also be convenient to have the following terms based on an MP equilibrium. Say that z° is an *MP allocation* for \mathcal{E}_{NTU} if $\exists \lambda^\circ \gg 0$ such that (z°, λ°) is an MP equilibrium. Also, say that $q = (q_i)$ is an *MP distribution* for \mathcal{E}_{NTU} under λ if $q = \nabla g_\lambda(e)$, where $\nabla g_\lambda(e)$ is the gradient of g_λ at e .

2.1.4 Two Additional Assumptions

A key step in proving the existence of an MP equilibrium will involve showing that, $\forall \lambda$, g_λ is differentiable at e . For this part of the proof, an additional assumption is crucial: we will require preferences to be

smooth. For each $z_i \in Z_i$ define the cone generated by Z_i from z_i as

$$C_i(z_i) = \{d \in \mathbb{R}^\ell : \exists t > 0 \text{ s.t. } (z_i + td) \in Z_i\}.$$

$C_i(z_i)$ represents the set of feasible directions in which it is possible to go from z_i while staying in Z_i . It is well-known that C_i inherits the closure and convexity properties of Z_i . Let $v'_i(z_i; d)$ represent the directional derivative of v_i at z_i in the direction d .

Definition. v_i is smooth if $\forall z_i \in Z_i \exists \nabla v_i(z_i) \in \mathbb{R}^\ell$ s.t.

$$v'_i(z_i; d) = \nabla v_i(z_i)d \text{ for all } d \in C_i(z_i),$$

and $\nabla v_i(\cdot)$ is continuous on Z_i .

It should be obvious that v_i is smooth if there exists a continuously differentiable function \tilde{v}_i defined on an open set containing Z_i , with the property that $\tilde{v}_i(z_i) = v_i(z_i)$ for all $z_i \in Z_i$.

To prove the main result, we will assume:

(A.4) $\forall i, v_i$ is smooth.

At almost the final step in proving the existence result, we will need a resource relatedness assumption (in the sense of Arrow and Hahn, 1971).

Definition: Individual j is resource related to individual k if, $\forall z \in A, \exists w \in \Omega$ and $\exists z' \in A(e, w)$ s.t.

- (a) $\forall i v_i(z'_i) > v_i(z_i)$, with $v_k(z'_k) > v_k(z_k)$
- (b) for all commodities h ($h = 1, \dots, \ell$): $w^h > 0$ only if $\omega_j^h > 0$.

The idea is that the economy would be no worse off, and k could be made better off, with w extra resources, where only resources h are added that j could supply.

Definition: Individual j is *indirectly resource related* to individual k if there is a sequence $i_1, \dots, i_{m'}, \dots, i_m$ with $i_1 = j$, $i_m = k$, and $i_{m'}$ is resource related to $i_{m'+1}$ for $m' = 1, \dots, m-1$.

Our last assumption will be,

(A.5) Every individual is indirectly resource related to every other individual.

The maintained assumptions, (A.1-3), along with the assumption to guarantee the differentiability of g_λ at e , (A.4), and the indirect resource relatedness assumption, (A.5), will suffice to prove that an MP equilibrium exists for \mathcal{E}_{NTU} . As will be seen, there is an intimate connection between an allocation satisfying the MP definition of equilibrium and a Walrasian allocation for \mathcal{E}_{NTU} .

Definition: $[p, z]$ is a *Walrasian equilibrium* for \mathcal{E}_{NTU} , denoted by $[p, z] \in WE(\mathcal{E}_{NTU})$, if $p \in \mathbb{R}^l$, $z \in A$, and for each i :

- $pz_i = 0$
- $v_i(z_i) = \sup\{v_i(z'_i) : pz_i = pz'_i\}$.

Main Result (Theorem 1). *In addition to the maintained assumptions, assume (A.4-5). Then there exists a MP equilibrium, i.e., a $\lambda^\circ \gg 0$ and a $z^\circ \in \zeta(\lambda^\circ)$ s.t. $\forall i$:*

$$g'_{\lambda^\circ}(e; e_i) = \lambda^\circ_i v_i(z^\circ_i).$$

Moreover, for any such z° and λ° there exists $p(\lambda^\circ) = \nabla f_{\lambda^\circ}(0)$ such that $[p(\lambda^\circ), z^\circ] \in WE(\mathcal{E}_{NTU})$ where $p(\lambda^\circ)$ is the gradient of f_{λ° at 0 (which exists).

We know from Theorem 1 that marginal product allocations are Walrasian. But, if preferences are not smooth, the converse may not hold. (A classical

example is Edgeworth's master-servant example. See Section 3.2.) As our other result, we show that, assuming smoothness, the converse does hold. The result is implied by Aumann's proof of the equivalence between value allocations and Walrasian allocations in continuum NTU economies (see in particular, Aumann, 1975, Lemma 14.1). However, we prefer to give an alternative proof that is more in the spirit of this paper, with its emphasis on the geometric properties of concave functions.

Theorem 2. *In addition to the maintained assumptions, assume (A.4-5) and that preferences are weakly monotonic (i.e. $\forall i \forall z_i \in Z_i: z'_i \geq z_i$ implies $v_i(z'_i) \geq v_i(z_i)$). Then for any $[p, z] \in WE(\mathcal{E}_{NTU})$, $\exists \lambda \gg 0$ s.t. $\forall i$:*

$$g'_\lambda(e; e_i) = \lambda_i v_i(z_i).$$

2.2 Proof of Theorem 1

2.2.1 A Supplementary Construction

For any given λ , it is convenient to associate \mathcal{E}_{NTU} with a transferable utility economy with $l+1$ commodities, in which each i evaluates any trade $(z_i, m_i) \in \mathbb{R}^l \times \mathbb{R}$ according to the quasi-linear utility function $u_i^\lambda(z_i, m_i) = \lambda_i v_i(z_i) + m_i$. The associated economy can thus be described by the set $\mathcal{E}_{TU}^\lambda = (u_i^\lambda: i = 1, \dots, n)$.

The proof of Theorem 1 builds on the interesting fact that for any given λ there is a one-to-one correspondence between the subgradients of f_λ at 0 and the Walrasian prices for \mathcal{E}_{TU}^λ . Formally expressed, for any λ , let $\partial f_\lambda(0)$ represent the set of subgradients of f_λ at 0. Note that this set is non-empty since, by (2), f_λ is concave and, by (A.3), 0 is in the interior of f_λ 's effective domain.

Definition: $[p, z]$ is a Walrasian equilibrium for \mathcal{E}_{TU}^λ , denoted by $[p, z] \in WE(\mathcal{E}_{TU}^\lambda)$, if $p \in \mathbb{R}^l$, $z \in A$, and for each i :

$$\lambda_i v_i(z_i) - pz_i = \sup(\lambda_i v_i(z'_i) - pz'_i : z'_i \in \mathbb{R}^{\ell}).$$

Note that the last condition in the definition just says, $\forall i$, p is a subgradient of $\lambda_i v_i$ at z_i , i.e., $p \in \partial \lambda_i v_i(z_i)$. More conventionally expressed, it says that, $\forall i$, $(z_i, -pz_i)$ maximizes u_i^λ over all (z'_i, m'_i) s.t. $pz'_i + m'_i = 0$. But the former viewpoint is more useful here, because it is a well-known fact (e.g., see Mas-Colell, 1985, Ch. 4) that:

$$(4) \quad \forall \lambda \forall z \in \zeta(\lambda): p \in \partial f_\lambda(0) \text{ iff } p \in \partial \lambda_i v_i(z_i) \forall i.$$

Recall from (3) that $\zeta(\lambda)$ is the set of all attainable allocations that maximize the gains from trade under λ . The characterization, (4), implies that any $p \in \partial f_\lambda(0)$, when combined with any $z \in \zeta(\lambda)$, forms a $WE(C_{TU}^\lambda)$.

The proof of Theorem 1 for E_{NTU} , is based on the following analogous result for any economy E_{TU}^λ .

Theorem 3. *In addition to the maintained assumptions, assume preferences are smooth, (A.4). Then for any λ , f_λ is differentiable at 0 with gradient denoted by $p(\lambda)$. Further, for any λ , any $z \in \zeta(\lambda)$, and all i :*

$$g'_\lambda(e; e_i) = \lambda_i v_i(z_i) - p(\lambda)z_i,$$

where $p(\cdot)$ is continuous on Λ .

The proof of Theorem 3 appears in the next section. First we show that, with the aid of a standard fixed-point argument, Theorem 3 implies our main result.

2.2.2 A Fixed-Point Argument

For any $\lambda \in \Lambda$, let

$$\eta(\lambda) = (\xi - (\xi_i) \in \mathbb{R}^n:$$

$$\xi = (-p(\lambda)z_1, \dots, -p(\lambda)z_n)$$

for some $z \in \zeta(\lambda)$.

Note that $(\eta(\lambda): \lambda \in \Lambda)$ is a bounded subset of \mathbb{R}^n . Indeed, $(p(\lambda): \lambda \in \Lambda)$ is bounded since, by Theorem 3, $p(\cdot)$ is a continuous function on a bounded set; while $(\zeta(\lambda): \lambda \in \Lambda)$ is contained in $A = A(e, 0)$, which is a compact set by (1). Let B be any compact, convex subset of \mathbb{R}^n containing $(\eta(\lambda): \lambda \in \Lambda)$. We can then regard η as a correspondence from Λ to B . Further, it is a u.h.c., convex-valued correspondence since, by (3), ζ is u.h.c. and convex-valued on Λ and, by Theorem 2, $p(\cdot)$ is continuous on Λ .

Given any $\xi \in B$, let $\mu(\xi) = \{\lambda \in \Lambda: \lambda \xi \geq \lambda' \xi \text{ for all } \lambda' \in \Lambda\}$. It is well-known that μ is a u.h.c., convex-valued correspondence on B (e.g., see Debreu, 1959, 5.6(1)).

Now consider the correspondence ϕ from $\Lambda \times B$ to itself defined by

$$\phi(\lambda, \xi) = \mu(\xi) \times \eta(\lambda).$$

Since it is u.h.c. and convex-valued, it has a fixed point $(\lambda^\circ, \xi^\circ)$. Thus,

$$(i) \quad \lambda \xi^\circ \leq \lambda^\circ \xi^\circ \quad \forall \lambda \in \Lambda, \text{ where}$$

$$(ii) \quad \xi^\circ = (-p(\lambda)z_1^\circ, \dots, -p(\lambda)z_n^\circ) \text{ for some } z^\circ \in \zeta(\lambda^\circ).$$

Note that $z^\circ \in \zeta(\lambda^\circ)$ implies $z^\circ \in A$, consequently $\sum \xi_i^\circ = 0$. So, $\lambda^\circ \in \mu(\xi^\circ)$ implies $\lambda_i^\circ = 0$ whenever $\xi_i^\circ < 0$. But if $\lambda_i^\circ = 0$ then, since $p(\lambda^\circ) \in \partial \lambda_i^\circ v_i(z_i^\circ)$ by (4),

$$\xi_i^\circ = -p(\lambda^\circ)z_{0i} \geq -p(\lambda^\circ)Z_i \geq 0$$

where the last inequality follows from (A.1), specifically from the fact that $0 \in Z_i$. We conclude that

$$(iii) \quad \xi_i^\circ = 0, \quad \forall i.$$

It now follows from Theorem 3 that $\forall i$:

$$(iv) \quad g'_{\lambda^\circ}(e; e_i) = \lambda_i^\circ v_i(z_i^\circ).$$

There only remains to be shown that

$$(v) \quad \lambda_i^\circ > 0 \quad \forall i,$$

and that $[p(\lambda^\circ), z^\circ] \in WE(\mathcal{E}_{NTU})$. The former will readily imply that latter.

It is a straightforward, although slightly involved, exercise to verify that

(5) $\forall \lambda$, and all individuals j and k : if j is resource related to k and if $\lambda_j = 0$ while $\lambda_k > 0$, when

$$g'_\lambda(e; e_j) > 0.$$

(See the Appendix for a proof.) Now observe that since $\lambda^\circ \in \Lambda$, \exists a type k s.t. $\lambda_k^\circ > 0$. Further, by (A.5), any type j is indirectly resource related to k . So, \exists a sequence $i_1, \dots, i_m, \dots, i_m$ with $i_1 = j$, $i_m = k$, and i_m , resource related to $i_{m'+1}$ for $m' = 1, \dots, m-1$. Since i_{m-1} is resource related to k and $\lambda_k^\circ > 0$, if $\lambda_{i_1}^\circ = 0$ then (5) would imply that $g'_{\lambda^\circ}(e; e_{i_{m-1}}) > 0$, which would contradict (iv) for $i = i_{m-1}$. So, $\lambda_{i_{m-1}}^\circ > 0$. Repeating the above argument, since i_{m-2} is resource related to i_{m-1} and $\lambda_{i_{m-1}}^\circ > 0$, would again lead to a contradiction. So, $\lambda_{i_{m-2}}^\circ > 0$. Continuing to repeat the argument, we finally come to the conclusion that $\lambda_{i_1}^\circ = \lambda_j^\circ > 0$. So, for any individual j , $\lambda_j^\circ > 0$, as was to be shown.

There only remains to be shown that $[p(\lambda^\circ), z^\circ] \in WE(\mathcal{E}_{NTU})$, in particular that, $\forall i$, $v_i(z_i^\circ) = \sup\{v_i(z_i) : p(\lambda^\circ)z_i = 0\}$. But since by (4), $p(\lambda^\circ) \in \partial \lambda_i^\circ v_i(z_i^\circ) \quad \forall i, \quad \forall i \forall z_i :$

$$\lambda_i^\circ v_i(z_i) - \lambda_i^\circ v_i(z_i^\circ) \leq p(\lambda^\circ)z_i - p(\lambda^\circ)z_i^\circ.$$

Or, since $\lambda_i^\circ > 0$ and $p(\lambda^\circ)z_i^\circ = 0$, $v_i(z_i) \leq v_i(z_i^\circ)$ whenever $p(\lambda^\circ)z_i = 0$. So, Theorem 1 is proved.

Remark. The mapping ϕ is an adaptation of the mapping used in Debreu (1959, 5.6(1)) to prove the existence of a WE. There is also some relation

to the mapping in Arrow and Hahn (1971, Ch. 5).

2.3 Proof of Theorem 3

2.3.1 The Relation Between the Marginal Product of Individuals and the Marginal Product of Commodities

For any given λ , let $\partial g_\lambda(e)$ represent the set of subgradients of g_λ at e . This set is non-empty since, by (3), g_λ is concave, and e is in the interior of g_λ 's domain, \mathbb{R}_+^n .

The proof of Theorem 3 is based on a simple relation between the prices of commodities, $p \in \partial f_\lambda(0)$ -- reflecting the marginal social valuations of resources when social utility is formed by adding the individual utilities weighted according to λ -- and the prices of people, $q \in \partial g_\lambda(0)$ -- reflecting the marginal social valuations of individuals under the weights λ . As we shall see, the relation between the two (Theorem 4 below) can be interpreted as a Core Equivalence Theorem.

Following the terminology in Rockafellar (1970), for any given λ , let $v_{\lambda_i}^* : \mathbb{R}^\ell \rightarrow \mathbb{R}$ represent the conjugate of $\lambda_i v_i$, defined by

$$v_{\lambda_i}^*(p) = \sup(\lambda_i v_i(z'_i) - pz'_i : z'_i \in \mathbb{R}^\ell).$$

Note that $v_{\lambda_i}^*$ is well defined (i.e., for some p , $v_{\lambda_i}^*(p) < \infty$) since, by (4), $\forall p \in \partial f_\lambda(0)$, $\forall z \in \zeta(\lambda)$, $\forall i$:

$$v_{\lambda_i}^*(p) = \lambda_i v_i(z_i) - pz_i.$$

Let $v_\lambda^*(p) = (v_{\lambda_1}^*(p), \dots, v_{\lambda_n}^*(p))$. The relation between $\partial g_\lambda(e)$ and $\partial f_\lambda(0)$ is given by:

Theorem 4. $\forall \lambda, \partial g_\lambda(e) = v_\lambda^*[\partial f_\lambda(0)]$.

Once these terms are suitably interpreted, Theorem 4 is an instance of the well-known Core Equivalence Theorem for the non-atomic TU economy

consisting of n types of individuals where type i has the utility function $\lambda_i v_i$ and there is unit mass of each type. The set $\partial g_\lambda(e)$ represents the elements of the core of this economy, i.e., the set of $q = (q_1, \dots, q_n)$ such that

- $qe = g_\lambda(e)$
- $0 \leq r \leq e$ implies $qr \geq g_\lambda(r)$

This description takes advantage of the fact that in a type economy members of the same type are always equally treated in the core in terms of utility.

Since g_λ is positively homogeneous, it is well-known that $q \in \partial g_\lambda(e)$ if and only if

- $qe = g_\lambda(e)$
- $\forall r \geq 0, qr \geq g_\lambda(r)$.

The positive homogeneity of g_λ implies that the second condition in the above description of the core implies the second condition in the characterization of the sub-differential.

To prove Theorem 4 in one direction, suppose $p \in \partial f_\lambda(0)$. We will show that $q = v_\lambda^*(p) \in \partial g_\lambda(e)$. Let $z \in \zeta(\lambda)$. Then, by (4), $[p, z] \in WE(\mathcal{E}_{TU}^\lambda)$. So q is in the core of \mathcal{E}_{TU}^λ ; which implies, as argued above, that $q \in \partial g_\lambda(e)$.

To prove the converse, suppose $q \in \partial g_\lambda(e)$ or, equivalently, q is in the core of \mathcal{E}_{TU}^λ . We need to show that $q = v_\lambda^*(p)$ for some $p \in \partial f_\lambda(0)$. Let $\Gamma_i = \{(z_i, m_i) : \lambda_i v_i(z_i) + m_i > q_i\}$, $\Gamma = \text{co}(\cup \Gamma_i)$. From this point onward, the proof follows the classic proof of core equivalence by Debreu and Scarf, 1963. One shows that there exist prices $(p, 1) \in \mathbb{R}^\ell \times \mathbb{R}$ that separate Γ from 0 and then, using (4), one shows $p \in \partial f_\lambda(0)$. The details are left to the interested reader to fill in.

2.3.2 The Differentiability of f_λ at 0 and of g_λ at e

While g_λ is not necessarily differentiable at e (i.e., $\partial g_\lambda(e)$ is not necessarily single-valued), it will be under the additional assumption that preferences are smooth, (A.4). Indeed, given (A.4), f_λ will be differentiable at 0, which immediately implies, by Theorem 4, that g_λ will be differentiable at 0.

Lemma: *In addition to the maintained assumptions, assume (A.4). Then, for any λ , f_λ is differentiable at 0.*

To prove the lemma, we must show $\partial f_\lambda(0)$ is single-valued. Suppose $p \in \partial f_\lambda(0)$ (recall $\partial f_\lambda(0) \neq \emptyset$). Then, by (4), $\forall z \in \zeta(\lambda) \forall i$:

$$p \in \partial \lambda_i v_i(z_i),$$

i.e.,

$$p(z'_i - z_i) \geq \lambda_i (v_i(z'_i) - v_i(z_i)) \quad \forall z'_i.$$

Let $z'_i = z_i + t \bar{e}_h$, when \bar{e}_h is the h^{th} unit vector in \mathbb{R}^l . Then $\forall i \forall t > 0$:

$$t p^h \geq \lambda_i (v_i(z_i + t \bar{e}_h) - v_i(z_i)).$$

Dividing by t and letting $t \rightarrow 0_+$ shows:

$$(i) \quad \forall i, p^h \geq \lambda_i v'_i(z_i; \bar{e}_h).$$

Similarly one shows:

$$(ii) \quad \forall i, -p^h \geq \lambda_i v'_i(z_i; -\bar{e}_h).$$

Now, by (A.3), $\omega \gg 0$. Hence, since $z \in A$, $\sum z_i^h - 0 > \sum (-\omega_i^h)$ for each commodity h . That is, $\forall h \exists i$ s.t. $z_i^h > -\omega_i^h$; or, alternatively expressed, $\forall h \exists i$ s.t. \bar{e}_h and $-\bar{e}_h$ are both in $C_i(z_i)$. So, $\forall h \exists i$ s.t. $v'_i(z_i; \bar{e}_h) = -v'_i(z_i; -\bar{e}_h) = \nabla v_i(z_i) \cdot \bar{e}_h$. Or, using (i)-(ii),

(iii) $\forall p \in \partial f_\lambda(0) \forall h: p^h = \lambda_i \nabla v_i(z_i) \bar{e}_h$ for some i .

Thus $\partial f_\lambda(0)$ is single-valued, so the lemma is proved.

2.3.3 Completing the Proof of Theorem 3

The lemma, when combined with Theorem 4, implies that for any λ , g_λ is differentiable at e . In particular, letting $q(\lambda) = (q_1(\lambda), \dots, q_n(\lambda))$ represent the gradient of g_λ at e , we have, $\forall i$:

$$q_i(\lambda) = v_{\lambda_i}^*(p(\lambda)),$$

where $p(\lambda)$ is the gradient of f_λ at 0. More specifically, recall from (4) that if $p \in \partial f_\lambda(0)$ then $\forall z \in \zeta(\lambda)$, $\forall i$:

$$v_{\lambda_i}^*(p) = \lambda_i v_i(z_i) - pz_i.$$

Since $g'_\lambda(e; e_i) = q_i(\lambda)$, we conclude that for any λ , any $z \in \zeta(\lambda)$, and all i :

$$g'_\lambda(e; e_i) = \lambda_i v_i(z_i) - p(\lambda)z_i.$$

At this point the proof of Theorem 3 is essentially complete. There only remains to be shown that

(6) Under the assumptions of Theorem 2, $p(\cdot)$ is continuous on Δ .

Since this is entirely routine, the proof of (6) is relegated to an appendix.

2.4 Proof of Theorem 2

As a preliminary to proving Theorem 2, we show that for every individual j , $p\omega_j > 0$. Since, by (A.5), j is resource related to some individual $k \neq j$, there exists a w and a $z \in A(e, w)$ satisfying conditions (a) and (b) in the definition of resource relatedness. Thus $p z_i \geq p z_i$ for all i , with $>$ for $i = k$. Summing shows $\sum p z_i > \sum p z_i - 0$; that is, $p w = p \sum z_i > 0$. But $w^h > 0$ only if $\omega_j^h > 0$, and $p \geq 0$ since preferences are monotonic. Thus, $p\omega_j > 0$.

For every individual i , define the function $V_i^*: \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$V_i^*(m) = \sup\{v_i(z'_i) : pz'_i \leq m\}.$$

It is well known (e.g., see Mas-Colell, 1985, Proposition 4.2.22) that, since $p\omega_i > 0$, V_i^* is a well-defined (i.e., $V_i^*(m) < \infty$ for all m), increasing, concave function. Further, pz_i is in the interior of V_i^* 's effective domain since $pz_i = 0 > \min pz_i = p(-\omega_i)$. Thus, letting $\partial V_i^*(pz_i)$ represent the set of subgradients of V_i^* at pz_i , $\partial V_i^*(pz_i) \neq \emptyset$.

For each i , let $\mu_i \in \partial V_i^*(pz_i)$. Then $\mu_i > 0$ since, by definition of a subgradient,

$$V_i^*(m) - V_i^*(pz_i) \leq \mu_i m - \mu_i pz_i, \quad \forall m.$$

If $\mu_i \leq 0$, then the non-satiation assumption in (A.2) would be violated.

For each i , let $\lambda_i = 1/\mu_i$ (i.e., the inverse of the marginal utility of income). Then

$$\lambda_i V_i^*(m) - m \leq \lambda_i v_i(z_i) - pz_i, \quad \forall m.$$

But, by definition of V_i^* ,

$$\lambda_i v_i(z'_i) - pz'_i \leq \lambda_i V_i^*(pz_i) \quad \forall z'_i.$$

Hence,

$$\lambda_i v_i(z'_i) - pz'_i \leq \lambda_i v_i(z_i) - pz_i, \quad \forall z'_i, \quad \text{i.e., } p \in \partial \lambda_i v_i(z_i)$$

for each i . Thus, $[p, z] \in WE(\mathcal{E}_{TU}^\lambda)$. Now, since $pz_i = 0$ for each i , Theorem 2 implies, $\forall i: g'_\lambda(e; e_i) = \lambda_i v_i(z_i)$, as was to be proved.

3 Concluding Remarks

To summarize, the results of Section 2 concern the existence and characterization of an MP equilibrium, i.e., an allocation $z^0 = (z_i^0)$ and weights $\lambda^0 = (\lambda_i^0)$ such that for each infinitesimal individual of type i ,

$$\lambda_i^0 v_i(z_i^0) = g'_{\lambda^0}(e; e_i).$$

Besides our maintained hypotheses, (A.1-3), sufficient conditions for the existence of such an equilibrium are the smoothness condition (A.4) and the resource-relatedness condition (A.5). Although an MP equilibrium does not say anything directly about commodity pricing, associated with any MP allocation there are prices which clear all commodity markets and realize the allocation as a Walrasian equilibrium. Conversely, any Walrasian equilibrium satisfying all of the above assumptions and a weak monotonicity condition will be a marginal productivity equilibrium.

Proofs of Theorems 1 and 2 could have been obtained by other means. They could have been demonstrated as corollaries of other results involving the characterization of the Value and its connection to Walrasian equilibria. The first step would be to cite the Value Equivalence Theorem of Aumann [1975], based on the work of Aumann and Shapley [1974]. In what has become an acceptably loose way of describing it, the formula for the Value yields for each individual the “average marginal product”, where the average is taken over all the groups in the economy which the individual might join. In the context of a nonatomic model of an exchange economy, the average group can be shown to be almost surely close to a representative sample of the entire population; e.g., in the type economy with population $r = (r_i)$, a representative sample will consist of $\hat{r} = (\hat{r}_i)$ where $\hat{r}_1/\hat{r}_2/\cdots/\hat{r}_n \sim r_1/r_2/\cdots/r_n$. Since the g function is positively homogeneous, there will be effectively no difference between the Value computed as the average marginal product to a representative sample subeconomy and what we have called an MP distribution, where the individual’s marginal product is defined with respect to that group consisting of the economy as a whole. Thus, in the context of nonatomic exchange economies, the Value coincides with an MP distribution. Aumann showed that it is precisely under these conditions, in which there is no difference between the *average* marginal product and what we have called *the* marginal product of an individual, that the Value coincides with Walrasian equilibria. (Aumann did not confine himself to the assumption of a finite number of types.)

The remaining step leading to the results in Theorems 1 and 2 would be to cite an existence theorem for Walrasian equilibria. In the finite type context, appeal could be made to Debreu [1959] or Arrow and Hahn [1971].

However, such a method of proof would have been contrary to our goal of putting the marginal productivity equilibrium condition on an independent footing, i.e., one

that is independent of the Value and of Walrasian equilibrium. Thus, instead of relying on the existence of Walrasian equilibrium plus some additional smoothness qualifications to demonstrate the existence of an MP allocation, we reversed the line of causation to show how the existence of an allocation satisfying the marginal productivity equilibrium condition implies Walrasian equilibrium. Similarly, we wanted to clearly demonstrate the stand-alone properties of an MP distribution apart from the Value.

Having shown that conditions for its existence can be independently obtained, we now want to comment on the relation between an MP equilibrium on the one hand and the Value and Walrasian equilibrium on the other.

3.1 The Shapley-Clark Connection

We interpret Theorem 1 as giving a certain logical respectability to Clark's zealous interpretation of marginal productivity theory. There are two interrelated parts to his zeal. The first is that marginal productivity is the central feature of the perfectly competitive theory of value. Although marginal utility is certainly acknowledged, in Clark's hands it is marginal productivity which stands out as the most important implication of the marginal revolution. Contrast this with the more conventional view that the two conceptual margins are of equal importance, or if one is more important, it is marginal utility. After all, marginal utility alone suffices to provide a theory of value in exchange economies whereas marginal productivity alone only provides a theory of the demand for factors of production. Compared to Wicksteed, however, we do not see any evidence that Clark thought of applying the marginal productivity concept to exchange and we do not credit him with this position.

The second, and more well-known, component of his zeal has to do with the "ethics" of marginal productivity and here we think that the concept of the Value lends some support to his claims. Consider Clark's position that giving each individual his/her marginal product is "the social procedure [that] is true to the principle on which the right of property rests." As it stands, it can only be true by definition because Clark did not say what the general principles are that would make it true. This is where the Value enters.

The Value is an attempt to define axiomatically how to share the surplus, or gains from trade. It can be interpreted as an "arbitration scheme"⁸ which makes each player's share depend in a kind of unbiased way on the contribution the player can make to others. The scheme is, however, conditioned on the rules of the game so that it might be described as a "conditionally neutral arbitration scheme". Taking the counterpart of the rules of the game as the initial assignment of property, Clark

⁸See Luce and Raiffa, *Games and Decisions*, p. 250 - 252.

recognizes the conditional nature of the marginal productivity reward principle.⁹

Not only did Clark *not* say what the general principles were that define a social procedure true to the principles of private property, but he also did *not* say what those principles or their consequences could possibly be in the absence of perfect competition, i.e., in environments which failed to exhibit adding-up (in terms of individuals). Shapley did both — he provided axioms that applied to conditions that were not necessarily, indeed not typically, perfectly competitive. Thus, the Value is an axiomatic characterization of an arbitration scheme applicable to all environments, not only perfectly competitive ones.

The remarkable result, the Value Equivalence Theorem, is that in perfectly competitive environments the social arbitration procedure defined by the Value coincides with an MP distribution. Thus, Clark's assertions that the division of the surplus under conditions of perfect competition exhibits a kind of "conditionally rational" division of the gains from trade is supported by its coincidence with the Value.

Remark: The existence of an MP equilibrium implies that each individual is fully appropriating the gains he/she contributes. Thus, when a Value allocation coincides with an MP allocation, the two principles of distribution evidently coincide. Is the Value, therefore, an extension of perfectly competitive theory to non-competitive environments? In our view, the answer is "no". Elsewhere (Makowski and Ostroy [1987,1988]) we have emphasized the remarkable incentive properties of full appropriation for achieving efficiency, properties that would not be shared by Value allocations — or any feasible allocation — in non-competitive environments. Without full appropriation the first-best efficiency, which is a part of the Value, is not implementable.

3.2 The Edgeworth-Wicksteed Connection

Theorems 1 and 2 describe conditions under which an MP equilibrium coincides with a Walrasian equilibrium. In this section we focus on reasons why the two equilibrium conditions differ, more specifically why a Walrasian equilibrium might not satisfy — even in a nonatomic model — the marginal productivity conditions for equilibrium. We shall point out these differences using an example due to Edgeworth.

Besides homogeneity of the first degree which is a built-in feature of g , the existence of an MP equilibrium requires that the gradient of g exist at e , i.e., that $\partial g(e)/\partial e_i$ exist for each i . Because g is an aggregate function constructed from the characteristics of the individuals in the population, there is no need to require that differentiability hold at the level of the individual since smoothing can occur through

⁹See the next to the last sentence of the opening paragraph of the Preface of his *The Distribution of Wealth* quoted above.

aggregation. However, in a model with only a finite number of types, the possibilities for such smoothing are severely limited and the only practical way to achieve differentiability in the aggregate is to assume it at the level of the individual.¹⁰

Existence problems for ∇g are related to traditional concerns about the legitimacy of marginal analysis. One of the criticisms of marginal productivity was that the marginal product need not exist. The two recognized sources of non-existence were (1) resources may be indivisible so that infinitesimal margins were irrelevant to the firm's calculations and, even if resources were divisible, (2) production functions need not be differentiable. The scepticism which greeted efforts to apply marginal productivity to the entrepreneur's return was a favorite instance of the first problem, while fixed coefficient production functions were pointed out as an instance of the second.¹¹

The above two issues are of unequal importance when dealing with the differentiability of g . The problems posed by resource indivisibility effectively vanish in an economy with a large number of participants because the individual, who is him/herself an indivisible unit, is an infinitesimal unit at the level of the economy as a whole. When evaluating the marginal product of an individual, we are considering the (infinitesimal) difference between having and not having that individual with his *entire package* of resources whether or not those resources are divisible. Thus, the marginal product of an individual supplying entrepreneurial services is not any more problematic than the marginal product of an individual supplying other kinds of labor services.¹² The second difficulty does not disappear. In those cases where smoothing is not achieved by aggregation, the failure of differentiability at the micro level can have macro consequences for the existence of an MP equilibrium.

To illustrate this point we shall consider Edgeworth's master-servant example. In his *Mathematical Psychics*, the author was concerned to show how a more fundamental approach involving bargaining could be used to demonstrate perfectly competitive equilibrium. For Edgeworth, the crux of the matter was to show that the outcome of bargaining would be *determinate*. To the extent that the outcome of bargaining was indeterminate, the market was subject to imperfections. For example, "...*contract is*

¹⁰This is the main defect of the finite type model for exhibiting the existence of an MP equilibrium. However, even in a model permitting a continuum of types, if we did not impose smoothness conditions on the characteristics of all individuals, we could only obtain a generic existence theorem on MP equilibria.

¹¹For example, Pareto commented: "Thus, in a chocolate factory, you may increase as much as you wish the labor, the area occupied by the factory, the machines and yet if you do not increase the quantity of cacao, you will not appreciably increase the chocolate." Quoted in Stigler, *ibid*, p. 365.

¹²Several years ago, Frank Hahn brought to our attention Chapman's "Remuneration of Employers" (1906). It is remarkable for its analysis of a model with a continuum of workers and employers in which the marginal product of an employer is obtained as the derivative of an aggregate production function.

more or less indeterminate according as the field is less or more affected with the first imperfection, limitation of numbers.” (p. 42, italics in original.) Edgeworth noted, however, that large numbers did not necessarily guarantee determinacy.

The *second* indeterminacy may be operative in many cases of contract for personal service. Suppose a market, consisting of an equal number of masters and servants, offering respectively wages and service; subject to the condition that no man can serve two masters, no master can employ more than one man; or suppose equilibrium already established between such parties to be disturbed by a sudden influx of wealth into the hands of the masters. Then there is no *determinate*, and very generally *unique*, arrangement towards which the system tends under the operation of, may we say, a law of Nature, and which would be predictable if we knew beforehand the real requirements of each, or of the the average, dealer; but there are an indefinite number of arrangements *à priori* possible, towards one of which the system is urged *not* by the concurrence of innumerable (as it were) neuter atoms eliminating chance, but (abstraction being made of custom) by what has been called the Art of Bargaining — higgling dodges and designing obstinacy, and other incalculable and often disreputable accidents. (p. 46, italics in original.)

The phenomenon that Edgeworth attributes to indivisibility can be reproduced under the assumption of divisible resources but non-differentiable utility functions. Let the masters have two units of the first commodity and the servants have two units of the second commodity. A master’s utility function for net trades is $v_1(z_1) = \min(2 + z_{11}, z_{12})$ and a servant’s is $v_2(z_2) = \min(z_{21}, 2 + z_{22})$. Whatever the number of masters and servants, an optimal $z = (z_1, z_2)$ will be such that $z_1 = (-1, 1) = -z_2$ whenever $z_i \neq 0$, i.e., each (employed) servant will serve only one master and each (employing) master will hire only one servant. It is readily verified that for this example $g(r) = \min(r_1, r_2)$.¹³ Therefore, with unit mass of both masters and servants, $\partial g(e) = \{(\alpha, 1 - \alpha) : \alpha \in [0, 1]\}$. As Edgeworth noted but did not emphasize, the difficulty only occurs when the numbers on each side of the market are the same ($r_1 = r_2$). In more contemporary language it can be shown that when $r_1 = r_2$, the “higgling dodges and designing obstinacy” present opportunities for *manipulation* of a Walrasian equilibrium by groups of individuals of arbitrarily small size. However, when $r_1 \neq r_2$, then even though the market continues to exhibit features of indivisibility, the outcome of bargaining is perfectly determinate in the sense that $\nabla g(r) = (1, 0)$ or $(0, 1)$ whenever $r_1 < r_2$ or $r_2 < r_1$, respectively, and the outcome is *non-manipulable*.

¹³Among game theorists, a model such as this would be called a “glove market”. See Aumann and Shapley [1974, p. 201-204].

There is an historical irony in this analysis of Edgeworth's example. Determinacy holds (fails) precisely when the formula

$$\nabla g(r) \cdot r = g(r)$$

can (cannot) be applied. One might therefore imagine that Edgeworth would have been sympathetic to marginal productivity theory in general and to Wicksteed's *Essay* in particular. In fact, he was critical of both. It is interesting to speculate why.

Wicksteed was able to make the bridge between the marginal productivity theory of distribution and the theory of exchange. Before introducing his metaphorical function, the output of which was satisfaction \mathbf{S} , he says: "The modern investigations into the theory of value have already given us the lead we require. Indeed the law of exchange value itself is the law of distribution of the general resources of society" (p. 7-8). Edgeworth, however, was not convinced that such a bridge could be made, or if there were a bridge, he seems to believe that it would carry traffic in only one direction. He was quite willing to say that "distribution is a species of exchange"¹⁴, but Edgeworth was reluctant to accept Wicksteed's position that, in effect, exchange could be regarded as a species of distribution. Indeed, he refers disapprovingly to the sentence of Wicksteed quoted in this paragraph.

Edgeworth, whose remarkable treatment of competitive equilibrium lay in his ability to go beyond price-taking in terms of *commodities* to its bargaining foundations in terms of *individuals*,¹⁵ would seem to be the one 19th century economist most capable of appreciating and reorienting marginal analysis away from commodities and towards individuals. But his comments on marginal productivity were surprisingly conventional. He insisted on treating marginal productivity theory on a par with marginal utility theory which meant that increments from 'last doses' of commodities were literally applied.¹⁶

In this paper the historical emphasis has been on some of the bolder claims of Clark and Wicksteed and more specifically on ones which we have found to be logically justifiable. The key to this justification is a reorientation of marginal analysis away from the commodity and towards the individual as the fundamental margin of analysis. Considering developments in mathematics and game theory since the marginalist revolution, it is easy to see why such a reorientation would have been beyond the technical capabilities of its founders. One can only guess how they would have responded to the claim that their revolution could have been stated, and also given more unity, by revising the margin.

¹⁴*Collected Papers*, vol. I, p.13

¹⁵"Here it is attempted to proceed without postulating the phenomenon of uniformity of price [he references Walras] by the longer route of *contract-curve*." (p. 40, italics in original.)

¹⁶*Collected Papers*, vol. I, pgs. 28 and 52.

APPENDIX I: PROOF OF (5)

We must show that $g'_\lambda(e; e_j) > 0$. Let $z \in \zeta(\lambda)$; and let w, z' satisfy the conditions in the definition of j being resource related to k .

As a preliminary step, we show $\exists w''$ and $\exists z'' \in A(e, w'')$ s.t.

(i) $\forall i, v_i(z''_i) \geq v_i(z_i)$, with $v_k(z''_k) > v_k(z_k)$,

where $0 \leq w'' \leq \omega_j$. Indeed, pick $\bar{t} \in (0, 1)$ s.t. $w'' = \bar{t}w \leq \omega_j$. (Such a \bar{t} exists since $w^h > 0$ only if $\omega_j^h > 0$.) Let $z''_i = (1-\bar{t})z_i + \bar{t}z'_i$. The convexity of Z_i implies each $z''_i \in Z_i$. Further, $z'' \in A(e, \bar{t}w)$ since

$$\Sigma z''_i = (1-\bar{t}) \Sigma z_i + \bar{t} \Sigma z'_i = \bar{t}w.$$

That z'' satisfies (i) follows from the concavity of preferences and the fact that z' satisfies condition (a) of the definition of resource relatedness.

We next show that for any $t \in (0, 1) \exists z(t) \in A(e + te_j, 0)$ s.t.

(ii) $\forall i \neq j, v_i(z_i(t)) \geq v_i(z_i)$, with $v_k(z_k(t)) > v_k(z_k)$.

Indeed, let $z_j(t) = \frac{[(1-t)z_j + tz''_j] + t[-w'']}{1+t}$ and, $\forall i \neq j$, let $z_i(t) =$

$(1-t)z_i + tz''_i$. Obviously, $\forall i \neq j, z_i(t) \in Z_i$. To see that $z_j(t) \in Z_j$ observe that $(1-t)z_j + tz''_j \in Z_j$ since Z_j is convex; and $-w'' \in Z_j$ since, by construction, $-w'' = -\bar{t}w \geq -\omega_j$. Thus, $z_j(t) \in Z_j$ since it is just a convex combination of two points in Z_j . Further, $z(t) \in A(e + te_j, 0)$ since

$$\begin{aligned} \sum_{i \neq j} z_i(t) + (1+t) z_j(t) &= (1-t) \Sigma z_i + t \Sigma z''_i + t(-w'') \\ &= tw'' - tw'' = 0. \end{aligned}$$

That $z(t)$ satisfies (ii) follows from the concaving of preferences and the fact that z'' satisfies (i). Indeed, by concavity, $\forall t \in (0, 1), \forall i \neq j$:

$$v_i(z_i(t)) \geq (1-t) v_i(z_i) + t v_i(z_i'')$$

or

$$\frac{v_i(z_i(t)) - v_i(z_i)}{t} \geq v_i(z_i'') - v_i(z_i).$$

Since $z(t) \in A(e + t e_j, 0)$, we have, $\forall t \in (0,1)$:

$$\begin{aligned} \frac{g_\lambda(e + t e_j) - g_\lambda(e)}{t} &\geq \frac{\sum \lambda_i v_i(z_i(t)) - \sum \lambda_i v_i(z_i)}{t} \\ &\geq \sum \lambda_i (v_i(z_i'') - v_i(z_i)) > 0. \end{aligned}$$

Letting $t \rightarrow 0^+$ shows $g'_\lambda(e; e_j) > 0$, which was to be proved. ■

APPENDIX II: PROOF OF (6)

Preliminary to showing that $p(\cdot)$ is continuous on Λ , we first verify that $\{p(\lambda): \lambda \in \Lambda\}$ is bounded. Observe that since, by assumption, $\nabla v_i(\cdot)$ is continuous on Z_i and since, by (1), A is compact, $\forall i$:

$$\{\lambda_i \nabla v_i(z_i): \lambda_i \in [0,1], z_i \in Z_i, \text{ and } z_i \text{ is a part of some } z \in A\}$$

is bounded. Thus since by (iii) of Section 3.2, $\forall \lambda \in \Lambda$ and $\forall h$, $p^h(\lambda)$ is in one such set, $\{p(\lambda): \lambda \in \Lambda\}$ is bounded.

Given the boundedness of $\{p(\lambda): \lambda \in \Lambda\}$, to verify that $p(\cdot)$ is continuous on Λ it suffices to show " $\lambda_m \in \Lambda$, $\lambda_m \rightarrow \lambda$, $p_m = p(\lambda_m) \rightarrow p$ " implies " $p = p(\lambda)$." Since $p_m \in \partial f_{\lambda_m}(0)$,

$$f_{\lambda_m}(w) - f_{\lambda_m}(0) \leq p_m w \quad \forall w \in \mathbb{R}^l.$$

We want to show

$$f_\lambda(w) - f_\lambda(0) \leq p w \quad \forall w \in \mathbb{R}^l.$$

If $w \notin \Sigma Z_i$ then $f_\lambda(w) = -\infty$ while $f_\lambda(0) > -\infty$, therefore the conclusion holds. If $w \in \Sigma Z_i$ then, by (2):

$$f_{\lambda_m}(w) = \max(\sum \lambda_{m,i} v_i(z_i): z \in A(e, w))$$

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We next show that for any $t \in (0, 1) \exists z(t) \in A(e + te_j, 0)$ s.t.

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or

$$\frac{v_i(z_i(t)) - v_i(z_i)}{t} \geq v_i(z_i'') - v_i(z_i).$$

Since $z(t) \in A(e + te_j, 0)$, we have, $\forall t \in (0,1)$:

$$\begin{aligned} \frac{g_\lambda(e + te_j) - g_\lambda(e)}{t} &\geq \frac{\sum \lambda_i v_i(z_i(t)) - \sum \lambda_i v_i(z_i)}{t} \\ &\geq \sum \lambda_i (v_i(z_i'') - v_i(z_i)) > 0. \end{aligned}$$

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$$f_{\lambda}^m(w) = \max\{\sum_{m,i} \lambda_i v_i(z_i) : z \in A(e,w)\}$$

and

$$f_{\lambda}(w) = \max\{\sum_i \lambda_i v_i(z_i) : z \in A(e,w)\}$$

and $f_{\lambda}^m(0) \rightarrow f_{\lambda}(0)$. (Formally, this follows from Debreu, 1959, 2.8(4).)

So, the conclusion again holds. This completes the proof of (6). ■

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