Vickrey-Clarke-Groves Mechanisms in Continuum Economies: Characterization and Existence[1] [2]

Louis Makowski[3] and Joseph M. Ostroy[4]

UCLA Department of Economics Working Paper No. 607
September 1990

[1]Research supported by the National Science Foundation. Thanks to Bryan Ellickson and Neil Gretsky for helpful discussions.

[2]Forthcoming in the *Journal of Mathematical Economics*.

[3]Department of Economics, UC Davis

[4]Department of Economics, UCLA

Abstract

The equivalence in the finite agent model between the families of efficient dominant strategy and Vickrey-Clarke-Groves mechanisms is extended to continuum economies. The concept of an individual's marginal product is used to link the two families of mechanisms when agents are non-atomic.

Unlike the finite agent model, feasible and efficient dominant strategy mechanims exist in the continuum, but these mechanisms do not guarantee individual rationality. For individual rationality to hold, the environment must also satisfy *full-appropriation*: each individual receives a payoff exactly equal to his/her marginal product. Full-appropriation is shown to be equivalent to the condition that each individual fully internalizes any external effects he/she creates. Environments and examples are given that exhibit or fail to exhibit full-appropriation.

1 Introduction

There are now many results about designing mechanisms that implement Pareto optimal allocations, both for private good and for public good economies. Restricting our attention to dominant strategy implementation, the results vary dramatically with the size of the economy, with impossibility results prevailing in small economies—e.g., Hurwicz (1972), Green and Laffont (1978), Laffont and Maskin (1980), Walker (1980), and Hurwicz and Walker (1983)—and possibility results prevailing when there is a (nonatomic) continuum of agents—e.g., Hammond (1979), Kleinberg (1980), Champsaur and Laroque (1981), McLennan (1982), Mas-Colell (1987), and Mas-Colell and Vives (1989). In this paper we prove no new implementation results, but we do provide a new path for proving many of the known results about implementation in the continuum. We believe this path is valuable because it gives some perspective and unity to the results on dominant strategy implementation—both for small and large economies. Our path gives a unified perspective because it involves an *extension* of the method used to prove many of the results for finite models, so one can more clearly see why impossibility in the small numbers case turns into possibility in the continuum.

Specifically, many of the known small economy results are proved assuming quasi-linear preferences. For such economies, Vickrey (1961), Groves (1970), and Clarke (1971) discovered mechanisms that were both dominant strategy incentive compatible and Pareto optimal *except* perhaps in achieving budget balance in the money commodity. Groves identified a whole family of such mechanisms, which were later shown to be exhaustive (Green and Laffont (1977), (1978), Walker (1978), and most generally, Holmstrom (1979)). Thus, under quasi-linearity, the existence of a dominant strategy and fully Pareto optimal ($DSPO$, for short) mechanism is equivalent to the existence of a balanced mechanism in the Vickrey-Clarke-Groves (VCG) family. Schematically:

$$(*) \qquad DSPO \iff \text{a balanced VCG mechanism exists.}$$

Based on this characterization, it is shown in the literature that in finite economies with private or public goods, there generally do not exist such mechanisms; for very general impossibility results see Walker (1980) and Hurwicz and Walker (1983).

We show that in the continuum, under the assumption of quasi-linear preferences, $DSPO$ mechanisms are still equivalent to balanced VCG mechanisms. But in the continuum, both for private and public goods economies, balanced VCG mechanisms *do* exist; indeed, they correspond precisely, respectively, to the Walrasian and "equal-cost-sharing" mechanisms. (The unique $DSPO$ mechanism for allocating public goods is sometimes called the "privately fair Lindahl mechanism" (Hammond, 1979) or the "fair-efficient mechanism" (Groves and Ledyard, 1987).

Extending the Vickrey-Clarke-Groves path to the continuum requires solving a puzzle which is the main mathematical contribution of this paper. At first viewing, one might conjecture that it is impossible to extend the (small economy) concept of a VCG mechanism to the continuum. Recall that any mechanism in the Groves family always rewards each individual with *the total gains from trade* minus a lump sum. But in the continuum, the size of each (infinitesimal) individual is of *a different (smaller) order of magnitude* than the total gains from trade. Is it possible to make the units commensurable?

In Makowski and Ostroy (1987a) we showed that in small economies any Vickrey-Clarke-Groves mechanism is equivalent to a marginal product (*MP*) mechanism, i.e., one that always rewards each individual with his marginal product to society minus perhaps a lump sum. Hence, (*) can be equivalently, schematically expressed as:

(**) $\qquad DSPO \Longleftrightarrow$ a balanced *MP* mechanism exists.

The equivalence between Vickrey-Clarke-Groves and marginal productivity schemes, while mathematically trivial in the finite model, turns out to be the key to extending the concept of VCG mechanisms to the continuum.

More specifically, regarding the total gains from trade as a function of the distribution of individuals in the economy, in small economies an individual's marginal product is defined as the *difference* between the gains with the individual in the economy and the gains with the individual out. Because of the "lumpiness" of persons, in finite models an individual's marginal product is a finite difference, not a derivative; but in the continuum, the lumpiness disappears and an individual's marginal product is naturally defined as the *directional derivative* of the gains function, in the "individual's" direction. Hence, even in the continuum where each individual is infinitesimal, an individual's marginal product is of an order of magnitude commensurate with his size, namely, a derivative. So, it is meaningful to reward individuals with their marginal products in both small *and large* economies. (Readers familiar with the *value* of a nonatomic game (Aumann and Shapley, 1974) will see a connection here that is described in more detail in Remark 7, below.) Once the concept of a marginal product mechanism is extended to the continuum, the extension of the Vickrey-Clarke-Groves characterization (in the form (**)) readily follows.

Extending the Vickrey-Clarke-Groves principles to the continuum allows for attractive intuitions about the similarities and differences between small and large economies, and between private and public goods economies. These intuitions are based on the "adding-up properties" of the gains-from-trade function coupled with Euler Theorem-type considerations familiar from neoclassical production theory under perfect competition (in which all factors are rewarded with their marginal products). Note it is in the continuum, in which an individual's marginal product is a derivative rather than a finite difference, that the analogies are most striking. The possibility of successful implementation—in both small and large economies—is seen as a variation on the classical question, "When can you successfully reward all factors with their marginal products?". This theme will be the subject of Sections 5 and 6 below.

Within the family of *MP* mechanisms, the central one is what we call the *full appropriation* mechanism—the *MP* mechanism without any lump sum transfers—which is balanced only if there is exact "adding-up". This centrality is reinforced in Section 5. We identify conditions under which a mechanism that is both *DSPO and* individually rational (*DSPOIR*, for short) exists and show that

$(***)$ $\qquad DSPOIR \iff$ the mechanism exhibits full appropriation.

While the main goal of this paper is an extension of Vickrey-Clarke-Groves mechanisms to the continuum, there is an accompanying secondary theme. It has a Pigovian spirit. The theme is that (a) rewarding individuals with their (social) marginal products and (b) requiring individuals to internalize any external effects they may impose on others are two sides of the same coin. This is not an unfamiliar idea: the Clarke pivot mechanism (a marginal product mechanism) for collective goods is often referred to as requiring individuals to pay for their externalities. A systematic emphasis of these "two sides" of the rule for successful *DSPO* implementation runs throughout the paper.

Hammond (1979) followed a different path for proving the possibility of finding *DSPO* mechanisms in continuum environments, both for allocating private goods and collective goods. His results are based on an alternative characterization of *DSPO* mechanisms: he shows *DSPO* mechanisms are equivalent to efficient mechanisms that are decentralizable under appropriate budget sets. While his alternative characterization result—like the VCG characterization—also applies to large and small economies, only the continuum version of the concept of decentralizability has been used to address the possibility question. The consequence is that while Hammond's method for addressing the implementation question does help in unifying our knowledge about the similarities and differences between private goods and collective goods *in the continuum*, it leads to a compartmentalization of our knowledge about the similarities and differences between small and large economies. In this regard, following the Vickrey-Clarke-Groves path is more unifying since it has been intensively travelled for proving finite impossibility results.

The sequel is organized as follows. Section 2 presents our model, a continuum-of-agents extension of a standard demand-revealing model, applicable to private or public good environments. It also discusses some exigencies of non-atomic models and introduces regularity conditions on continuum mechanisms, to meet these exigencies. Section 3 develops the general equilibrium extension of the marginal product concept that is the key to succesful extension of finite VCG mechanisms to the continuum. Section 4 then displays our continuum extension of the finite characteriation results for VCG mechanisms. Section 5 displays a general existence result for $DSPO$ mechanisms in the continuum. It also displays some special characterization and existence results for individually rational mechanisms and for mechanisms on homogeneous environments. Section 6 discusses related large economy results in the literature. The proofs of all our results are collected in the final section, Section 7.

(N.B.: The primary purpose of this paper is to extend the characterization of $DSPO$ mechanisms from finite to nonatomic models, but we are also interested in showing how these characterizations can be translated to more familiar-looking conclusions in terms of their implications for prices and quantities. A consequence of these priorities is that we shall appear to treat the important case of private goods exchange economies in an "incidental manner" because that model only appears explicitly as the subjects of Remarks of 2, 5, and 6, below. Nevertheless, the contents of these Remarks are representative of the pricing implications of VCG mechanisms in nonatomic models.)

2 The Model

2.1 Individual Characteristics

Let $v : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ be an extended real-valued function with effective domain $Y_v \equiv \{y : v(y) > -\infty\}$. An individual's characteristics are completely described by v: Y_v represents his/her *individually feasible trading capacities*, while the restriction of v onto Y_v represents his/her *tastes.*

Let V be the set of possible individual characteristics. We shall assume that for each $v \in V$,

- Y_v is non-empty, closed and convex
- v is strictly concave and twice continuously differentiable.

The differentiability of v should be understood as follows. There is a twice continuously differentiable function $\tilde{v} : \mathbf{R}^\ell \to \mathbf{R}$ that coincides with v on Y_v and is such that $\partial^2 v(y) = \partial^2 \tilde{v}(y)$ whenever $y \in Y_v$.

Commodities are divided into two categories: those which are the arguments of $v(\cdot)$, referred to as y-commodities ($y \in \mathbf{R}^\ell$), and the money commodity, denoted by m. An individual with characteristics v will evaluate $(y, m) \in \mathbf{R}^\ell \times \mathbf{R}$ according to the quasi-linear utility function

$$U(y, m; v) = v(y) + m.$$

Thus, all individuals have the same characteristics with respect to the money commodity: tastes are linear and there is no capacity limitation on supply.

As a common feature of all $v \in V$, we assume

- there is a non-empty, compact set $B^o \subset \cap_{v \in V} Y_v$.

The set B^o is the individually feasible trading capacities that all individuals have in common. For example, it might include the zero element of $\mathbf{R}^\ell$ in a model where each individual always has the no-trade option.

It will also be assumed that

- V is a compact metric space.

In particular, we endow the space V with a metric such that if $v_n \to v$ then there exists $\tilde{v_n}$ and $\tilde{v}$ which are C^2 extensions of v_n and v, respectively, such that $\tilde{v_n} \to \tilde{v}$ in the (metrizable) C^2 compact-open topology (Mas-Colell, 1985) and $Y_{v_n} \to Y_v$ in the (metrizable) closed-convergence topology (Hildenbrand, 1974). The metric is described in more detail in Remark 8 at the beginning of Section 7.

An *economy* will be defined by a (positive, finite) Borel measure μ on V.

2.2 Allocations

An *allocation*, denoted by $(y(\cdot), m(\cdot))$, is a an element of $\mathbf{C}^\ell(V) \times \mathbf{C}(V)$, where $\mathbf{C}^\ell(V)$ is the set of continuous $\mathbf{R}^\ell$-valued functions on V denoting allocations of the y commodities and $\mathbf{C}(V)$ is the continuous real-valued functions denoting allocations of the money commodity.

Define $Y(\mu)$ as that subset of allocations in $\mathbf{C}^\ell(V)$ which are *feasible in the aggregate* for μ. The set $Y(\mu)$ will determine the nature of the economic environment. For example, in a private goods exchange economy the set of feasible trades would be described by

$$Y(\mu) = \{y(\cdot) : \forall v \; y(v) \in Y_v \text{ and } \int y(v) d\mu(v) = 0\}.$$

Alternatively, a public goods environment without any costs of production could be described by

$$Y(\mu) = \{y(\cdot) : \forall v, \; y(v) \in Y_v \text{ and } y(v) = \bar{y} \in \Upsilon\},$$

where Υ is a given set of possible, alternative public good projects. Environments involving public goods with costly production could similarly be defined. To illustrate, suppose there is just one public good, the ℓ^{th} commodity, which is produced using private goods as inputs, via the concave production funtion $\Phi : \mathbf{R}^{\ell-1} \to \mathbf{R}$. Then

$$Y(\mu) = \{y(\cdot) : \ \forall v, \ y(v) \in Y_v \text{ and } y(v) \equiv (y_1(v), \ldots, y_{\ell-1}(v), y_\ell) \in \mathbf{R}^{\ell-1} \times \mathbf{R}$$
$$\text{satisfies } y_\ell = \Phi(-\textstyle\int y_1 d\mu, \ldots, -\int y_{\ell-1} d\mu)\}.$$

In the spirit of mechanism theory, we shall suppress the differences to look for principles in common.

We shall assume that whatever its specification,

- $Y(\mu)$ is convex and only contains allocations $y(\cdot)$ which are individually feasible.

As a separate condition not always imposed, the money allocation, $m(\cdot)$, is said to be *feasible* if money transfers balance, i.e.,

$$\int m d\mu = 0.$$

To define optimality for μ we introduce the *gains function*

$$g(\mu) = \sup\{\int v(y(v)) d\mu(v) : y(\cdot) \in Y(\mu)\}.$$

It is assumed that this sup exists and is achieved by a $y(\cdot) \in Y(\mu)$.

The allocation $y(\cdot)$ is Y-*optimal* for μ if it belongs to $Y(\mu)$ and

$$\int v(y(v)) d\mu(v) = g(\mu).$$

The allocation $(y(\cdot), m(\cdot))$ is *fully optimal* at μ if $y(\cdot)$ is Y-optimal and $m(\cdot)$ is feasible.

2.3 Mechanisms and their Properties

We will be concerned with characterizing the dominant strategy properties of a *fixed* economy μ. Let N be a set of economies containing μ.

Definition: A *mechanism* is a mapping $f : N \to \mathbf{C}^{\ell}(V) \times \mathbf{C}(V)$. We shall write $f_{\mu'}(v) = (y_{\mu'}(v), m_{\mu'}(v))$ as the allocation to v at μ'. For all $\mu' \in N$, it is assumed that $y_{\mu'}(\cdot) \in Y(\mu')$, i.e., there is an aggregate feasibility restriction on the y allocation, but none on the money allocation.

Definition: A mechanism f is *Y-optimal*, or PO_Y, if for all $\mu' \in N$, $y_{\mu'}(\cdot)$ satisfies

$$g(\mu') = \int v(y_{\mu'}(v))d\mu'(v).$$

The concept of incentive compatibility for a mechanism is:

Definition: A mechanism f exhibits the *dominant strategy* (DS) *property* at μ if for all $w \in V$,

$$U(f_\mu(v); v) \geq U(f_\mu(w); v),$$

for all $v \in \operatorname{supp} \mu$.

Recalling the definition of U, this says that the utility an individual receives from the allocation mechanism by reporting his characteristics truthfully, $v(y_\mu(v))+m_\mu(v)$, is at least as large as the utility an agent of that type could obtain by reporting any other characteristics, $v(y_\mu(w)) + m_\mu(w)$. Therefore, if f satisfies DS at μ then v has no incentive to misrepresent himself as a w, even for $w \notin \operatorname{supp} \mu$.

Notice that while DS is defined *at* μ, the mechanism f is defined on a set of economies containing μ—not just at μ. This is preparatory for the next subsection in which the concept of a "regular mechanism at μ" will be defined. Basically, we will restrict ourselves to PO_Y mechanisms whose y-allocation is differentiable at μ in the appropriate directions. To make this precise we need to know the behavior of the mechanism (more specifically, its y-component) not just at μ, but in an appropriate neighborhood N of μ.

Definition: For brevity, a mechanism f will be called $DSPO_Y$ at μ if it is PO_Y and it exhibits the DS property at μ. Similarly, f will be called $DSPO$ at μ if it is $DSPO_Y$ at μ and feasible at μ, i.e., $\int m d\mu = 0$.

2.4 Regular Mechanisms

We will restrict our attention to PO_Y mechanisms that are "regular at μ". To define regularity, it will suffice to consider the behavior of the mechanism in a neighborhood of μ given by

$$N = \{\mu' = \mu + t\delta_w : t \in [0,1],\ w \in V\},$$

where δ_w is the Dirac measure of unit mass concentrated at w.

Crucial to the analysis of the mechanism, below, is the influence of one infinitesimal individual on another, which will be based on the following: The *directional derivative of $y_\mu(\cdot)$ at v in the direction w* is

$$Dy_{(\mu;w)}(v) = \lim_{t \to 0_+} \frac{y_{\mu+t\delta_w}(v) - y_\mu(v)}{t}.$$

The interpretation is, this directional derivative measures the influence on the y-allocation of an individual with characteristics v from adding an infinitesimal individual with characteristics w to the population μ.

Definition: f is a *regular* PO_Y mechanism at μ if it is PO_Y and

(R.1) The y-allocation is *smooth* at μ: for all v and w, $Dy_{(\mu;w)}(v)$ exists and is continuous in both v (holding w constant) and w (holding v constant).

(R.2) The y-allocation is *interior* at μ: for all v and w, if $v(y_{\mu+t\delta_w}(w)) = -\infty$ for t arbitrarily close to zero, then $v(y_\mu(w)) = -\infty$.

Certain implications of regularity are immediate. The smoothness of the y-allocation (R.1) necessarily implies that the y-allocation is continuous in the sense that for all w and v:

$$y_{\mu+t\delta_w}(v) \to y_\mu(v) \quad \text{as} \quad t \to 0.$$

This solves a problem. For PO_Y, the behavior of the mechanism at μ is only restricted for individuals $v \in \operatorname{supp} \mu$. But for DS, the behavior of the mechanism at μ is restricted even for $w \notin \operatorname{supp} \mu$ (recall an individual v can claim he is *any* type $w \in V$). By restricting ourselves to PO_Y mechanisms that are regular at μ, even the y-allocation for $w \notin \operatorname{supp} \mu$ is "pinned down", in particular

$$y_\mu(w) = \lim_{t \to 0} y_{\mu+t\delta_w}(w),$$

where clearly w is in the support of $\mu + t\delta_w$ for $t > 0$. (Of course, this problem would be otiose if $\operatorname{supp} \mu = V$, but from a strategic point of view it seems useful not to impose such an assumption. See Remark 1, below.)

(R.2) says that all y-allocations to individuals that the mechanism calls for and that are not in v's effective domain are away from the boundary of v's effective domain. This rules out the possibility of $v(y_\mu(w))$ being discontinuous (relative to the mechanism) for allocations approaching the boundary of v's effective domain. It is a strong assumption, but it is only a sufficient condition to prove our results for general DS mechanisms; it is not required to prove any of our results for (less general) demand-revealing mechanisms, where the trading capacities of all individual types are identical.

REMARK 1 (Fair Allocations): An allocation $f_\cdot(v)$ is said to be *fair* at μ if it is Pareto-optimal and for all $v, w \in \operatorname{supp} \mu$

$$U(f_\mu(v); v) \geq U(f_\mu(w); v).$$

(See Schmeidler and Vind (1972), Varian (1976), Kleinberg (1980), Champsaur and Laroque (1981), McLennan (1982), Mas-Colell (1987) and others.) Interpreting this condition in the language of misrepresentation, it says that an allocation is fair if no individual in supp μ would prefer to represent himself as any other individual in supp μ. By contrast an allocation is DS at μ if no individual in supp μ would prefer to represent himself as any individual in V. Evidently a DS allocation is fair, but the converse need not hold. However, the two definitions can lead to quite similar conclusions provided supp μ is a connected set. (See Section 6.) Characterization of the DS property requires a similar connectedness assumption, but on V rather than supp μ (see Holmstrom, 1979, and below); and the results of this paper could, with straightforward modifications, be applied to show that the marginal productivity/externality principles underlie fair allocations.

Despite the important similarities between the fair and DS definitions of misrepresentation in nonatomic models, for our purposes the differences are significant. For example, since connectedness of supp μ is crucial for the fair definition of misrepresentation to narrow down the class of possible allocations, there is only a very loose connection between fair allocations and the DS property in finite agent models: unless supp μ is a singleton it is necessarily disconnected in such models. But finiteness of the actual types does not preclude connectedness of V, the set of potential types, and this is what permits a single characterization of DS mechanisms applicable to finite and nonatomic models.

REMARK 2 (Regular Private Goods Economies and Regular PO_Y Mechanisms): We outline an argument that in a private goods exchange economy with quasi-linear preferences, the "regularity" of the economy in the sense of Debreu (1970) will ensure that a y-optimal mechanism is regular.

For $p \in \mathbf{R}^\ell$, let $e_p : V \to \mathbf{R}^\ell$ be a mapping such that $e_p(v)$ is a vector of y-commodities that maximizes $v(y) + m$ subject to the constraint $p \cdot y + m = \alpha$ (the price of m is unity). It is assumed that p is chosen so that $e_p(v)$ is non-empty for all v. Because v is strictly concave $e_p(v)$ is unique, and because utility is quasi-linear $e_p(v)$ is independent of α. Note that when $\alpha = 0$, then $m = -p \cdot y$ and

$$v(e_p(v)) - p \cdot e_p(v) = \sup\{v(y) - p \cdot y\} = v^*(p),$$

where v^* is the *conjugate function* of v. Therefore $(e_p(v), -p \cdot e_p(v))$ are the utility maximizing demands for the y-commodities and money when the individual faces the Walrasian budget constraint $p \cdot y + m = 0$.

Define $E_p(\mu) = \int e_p d\mu$. Suppose $y_\mu(\cdot)$ is Y-optimal for μ and utility functions are monotone. Then it may be shown that there are *efficiency prices* $p(\mu) \in \mathbf{R}^\ell_+$ such that

$$y_\mu(v) = e_{p(\mu)}(v).$$

Feasibility of net trades, $\int y_\mu d\mu = 0$, therefore implies

$$E_{p(\mu)}(\mu) = \int e_{p(\mu)} d\mu = 0.$$

Thus, $p(\mu)$ is an "equilibrium" price vector for the ℓ y-commodities.

Definition: The *economy* μ is said to be *regular* (Debreu (1970)) if $\partial_p E_{p(\mu)}(\mu)$ is non-singular.

In this case we can apply the Implicit Function Theorem to obtain

$$Dp(\mu; w) = -\left(\partial_p E_{p(\mu)}(\mu)\right)^{-1} DE_{p(\mu;w)}(\mu),$$

where $Dp(\mu; w)$ and $DE_{p(\mu;w)}(\mu)$ are the directional derivatives in the direction w of the equilibrium price mapping $p(\mu)$ and the excess demand function $E_{p(\mu)}(\mu)$, respectively.

A simple calculation shows that

$$DE_{p(\mu;w)}(\mu) = e_{p(\mu)}(w).$$

Therefore, the formula for $Dy_{(\mu;w)}(v)$ is

$$Dy_{(\mu;w)}(v) = \partial_p e_{p(\mu)}(v) Dp(\mu; w) = -\partial_p e_{p(\mu)}(v) \left(\partial_p E_{p(\mu)}(\mu)\right)^{-1} e_{p(\mu)}(w).$$

To show that the mechanism is regular, we make the following assumption:

$$e_{p(\mu)}(v) \in \text{int } Y_v \quad \text{for each } v \in V.$$

This interiority assumption evidently guarantees (R.2). To establish (R.1), first note that the existence of $Dy_{(\mu;w)}(v)$ is obtained from the formula above. Its continuity with respect to w depends upon the continuity of $e_p(\cdot)$: this follows from the fact that if w and w' are close, then so are ∂w and $\partial w'$. Its continuity with respect to v depends upon the continuity of $\partial_p e_p(\cdot)$: this follows from the fact that if v and v' are close and $e_p(v) \in \text{int } Y_v$, $e_p(v') \in \text{int } Y_{v'}$, then $\partial^2 v(e_p(v))$ is close to $\partial^2 v'(e_p(v'))$ and each of these Hessian matrices of second partial derivatives is non-singular because the functions are strictly concave. Application of the Implicit Function Theorem yields the desired conclusion.

In Remark 5, below, we shall demonstrate that a $DSPO$ mechanism is actually a Walrasian allocation.

3 The Marginal Products of Individuals and Their External Effects

The key to our characterization of $DSPO_Y$ mechanisms is the concept of an individual's marginal product. This is no less true in the finite numbers model than in the continuum (see Makowski and Ostroy, 1987a), but in the continuum the infinitesimal scale of each agent is ideally suited for the application of the calculus.

Definition: The *marginal product of w to the population μ* is

$$MP_\mu(w) \equiv Dg_\mu(w) \equiv \lim_{t \to 0_+} \frac{g(\mu + t\delta_w) - g(\mu)}{t}.$$

There is an intimate connnection between an individual's marginal product and the externalities he/she imposes on others.

Definition: The *(sum of the) external effects imposed by w on the population μ* is

$$\xi_\mu(w) \equiv \int \left\{ \partial v(y_\mu(v)) \cdot Dy_{(\mu;w)}(v) \right\} d\mu(v).$$

Substituting the definitions of $g(\mu + t\delta_w)$ and $g(\mu)$ into the definition of w's marginal product, and making the obvious substitutions (to be justified below),

$$\begin{aligned} MP_\mu(w) &= \lim_{t \to 0_+} \frac{\int v(y_{\mu+t\delta_w}(v))d\mu(v) + tw(y_{\mu+t\delta_w}(w)) - \int v(y_\mu(v))d\mu(v)}{t} \\ &= \int \left\{ \partial v(y_\mu(v)) \cdot Dy_{(\mu;w)}(v) \right\} d\mu(v) + w(y_\mu(w)) \\ &= \xi_\mu(w) + w(y_\mu(w)). \end{aligned}$$

Thus, the rate at which the total gains function g changes as an infinitesimal individual with characteristics w is added to μ, $MP_\mu(w)$, consists of two parts: (a) the sum of the "external effects" that the very presence of w creates for all the other agents in μ, $\xi_\mu(w)$, plus (b) the utility that w enjoys in the y-optimal allocation, $w(y_\mu(w))$.

To elaborate on the externality component of w's marginal product, notice that the external effect on any one agent of type $v \in \operatorname{supp} \mu$ caused by the introduction of w is the infinitesimal change in v's utility from his y-allocation, $\partial v(y_\mu(v))$, evaluated according to the directional derivative of $y_\mu(v)$ in the direction w, i.e., $Dy_{(\mu;w)}(v)$. The magnitude of this effect will be insignificant compared to the total utility of agent v, but the cumulative sum of these external effects of the addition of w on the entire population μ, $\xi_\mu(w)$, can be of the same order of magnitude as an individual's total utility. (N.B.: Even if $\xi_\mu(w) \neq 0$, w's "externalities" may still be internalized by w through a money transfer from w to the rest of the population. See the definition of *full internalization*, below.)

The following result summarizes the implications of a regular mechanism for the marginal product of an individual.

Lemma 1 *Let f be a regular PO_Y mechanism at μ. Then for any $w \in V$,*

$$MP_\mu(w) = \xi_\mu(w) + w(y_\mu(w)).$$

Moreover, $\xi_\mu(\cdot)$ and $MP_\mu(\cdot)$ are continuous on V.

3.1 The *MP* of an Individual Who Misrepresents His Type

We shall show that any mechanism f is $DSPO_Y$ if and only if it always rewards all types with their marginal products, plus perhaps a lump sum. Since any type $v \in \operatorname{supp} \mu$ may claim he is really some other type $w \in V$, as a final preliminary we need to define not only v's marginal product when he is truthful, $MP_\mu(v)$, but also his marginal product to society when he announces some other type w, denoted by $MP_\mu(w; v)$.

Just as $MP_\mu(v)$ is defined by taking limits, so define

$$MP_\mu(w; v) = \lim_{t \to 0_+} \frac{g(\mu + t\delta_w; \mu + t\delta_v) - g(\mu)}{t},$$

where

$$g(\mu + t\delta_v; \mu + t\delta_v) = \int z(y_{\mu+t\delta_w}(z))d\mu(z) + tv(y_{\mu+t\delta_w}(w))$$

is the total gains in the economy μ when t agents of type v are added to the population but announce characteristics w. Notice that for some v and w, v may be called upon to deliver a y-optimal allocation that is infeasible, i.e., $v(y_\mu(w)) = -\infty$. Certainly it is not in v's interest to make such an announcement; in terms of the above formula it leads to $MP_\mu(w; v) = -\infty$.

The implications of a regular mechanism for the marginal product of an individual who misrepresents his type are given by

Lemma 2 *Let f be a regular PO_Y mechanism at μ. Then for any $w, v \in V$,*

$$\begin{aligned} MP_\mu(w; v) &= MP_\mu(w) - w(y_\mu(w)) + v(y_\mu(w)) \\ &= \xi_\mu(w) + v(y_\mu(w)). \end{aligned}$$

Thus, the marginal product of an individual who misrepresents his type consists of the external effect of the misrepresented type, as if it were the actual one, plus the utility the actual type receives from the allocation obtained under misrepresentation. Notice that if $w = v$, the formula arrives at the required conclusion that $MP_\mu(w; v) = MP_\mu(v)$.

4 Characterization of $DSPO_Y$ Mechanisms

In the following two subsections, we shall show that the marginal productivity reward principle is necessary and sufficient for a regular PO_Y mechanism to provide incentives for truth-telling.

Because the economy μ will be fixed throughout much of the following, we shall suppress the functional dependence on μ, except for emphasis. For example, at μ the mechanism is a function on V and, from Lemma 1, we now write $MP(w) = \xi(w) + w(y(w))$.

4.1 The Marginal Product/Internalization Principle as a Sufficient Condition

The payoff in a regular PO_Y mechanism can always be written as

$$U(f(w); v) \equiv v(y(w)) + m(w) = MP(w; v) - H(w),$$

where $H \in \mathbf{C}(V)$ is simply the residual establishing the equality.

The following result says that the marginal productivity reward principle has a built-in dominant strategy property.

Lemma 3 *For every type v,*

$$\max_w MP(w; v) = MP(v; v) \equiv MP(v).$$

Say that H is a *lump sum* function if there is a constant h such that for all $w \in V$,

$$h = H(w).$$

This might be better termed an anonymous lump sum function in contrast with the lump sum function described for finite agent models (e.g., see Groves and Loeb, 1975). In the latter, the lump sum is invariant to the individual's characteristics but may vary with the individual's "name". Of course, the distribution approach taken here builds in anonymity.

The DS property of the MP reward principle with lump-sums follows immediately from Lemma 3.

Theorem 1 *Let f be a regular PO_Y mechanism at μ. If for all w and v,*

$$U(f(w); v) = MP(w; v) - h,$$

then f is DS at μ.

Rearranging the terms in the total payoff

$$U(f(w); v) = w(y(w)) + m(w) = MP(w; v) - h;$$

and using Lemma 2 this yields,

$$\begin{aligned} m(w) &= MP(w; v) - v(y(w)) - h \\ &= \xi(w) + v(y(w)) - v(y(w)) - h \\ &= \xi(w) - h. \end{aligned}$$

Hence, the *MP* reward principle may be equivalently described as giving an individual of type v who announces w a y-allocation based on his announced type to satisfy y-optimality, and then guaranteeing that the money allocation, $m(w)$, will equal (ignoring h) the external effects associated with the type he announces, $\xi(w)$; i.e., external effects are internalized.

Let us restate Theorem 1 in terms of external effects.

Theorem 1′ *Let f satisfy the hypothesis of Theorem 1. If*

$$m(w) = \xi(w) - h,$$

then f is DS at μ.

Theorem 1′ says that when external effects are internalized, it pays to tell the truth. Note, however, that this internalization is from the individual point of view but not necessarily from the point of view of the economy as a whole. For that we would also need the budget balancing condition $\int m d\mu = 0$.

4.2 The Marginal Product/Internalization Principle as a Necessary Condition

There remains the converse, that to achieve the DS property a regular PO_Y mechanism must be specified as in Theorem 1 (or Theorem 1′). Based on the preparations given above and those to follow, we shall show that Holmstrom's (1979) demonstration of necessity for the finite agent model can be "lifted" to the nonatomic case.

For the sufficient conditions on $DSPO_Y$ to become necessary it is well-known that V must exhibit a certain amount of variety. A simple method of insuring enough variety is to assume that: ***V is a convex set***, i.e., for $v, w \in V$, $\{v_\alpha : v_\alpha = \alpha v + (1-\alpha)w$ and $\alpha \in [0,1]\} \subset V$.

The role of convexity will be to ensure that for any $v, w \in V$, the environment will contain the parameterization $v_\alpha = \alpha v + (1-\alpha)w$, $\alpha \in [0,1]$, connecting v and w. Recalling that Y_v (resp.Y_w) equals the effective domain of v (resp.w), note that

$Y_{v_\alpha} = Y_v \cap Y_w$ provided $0 < \alpha < 1$. However, $Y_{v_0} \equiv Y_v$ and $Y_{v_1} \equiv Y_w$ may differ from $Y_v \cap Y_w$, and therefore this parameterization need not be "smooth". Before dealing with this problem, we consider a simpler one.

It is common in mechanism theory to assume $Y_w = Y_v$ for all w and v. First we shall prove a converse to Theorem 1 for this special case, where only tastes may be misrepresented.

Definition: Call a mechanism f *demand-revealing* (DR) at μ if it is DS at μ and $Y_w = Y_v$ for all w and v. Analogous to the definitions of $DSPO_Y$ and $DSPO$, call a mechanism $DRPO_Y$ at μ (respectively, $DRPO$ at μ) if it is both DR at μ and PO_Y (respectively, $DRPO_Y$ at μ with $\int m d\mu = 0$).

A DR mechanism is a special case of a DS mechanism in which, as it were, information about feasible net trades of individuals is always common knowledge.

Theorem 2 *Let V be a convex set and let f be a regular PO_Y mechanism at μ. If f is DR at μ then for all w and v,*

$$U(f(w); v) = MP(w; v) - h.$$

Equivalently, if f is DR at μ then for all w,

$$m(w) = \xi(w) - h.$$

In some settings, such as models of exchange economies, we must deal with the fact that individual characteristics include, besides variations in tastes, variations in what is individually feasible. The following assumption, by providing for sufficient variation in what is individually feasible in V, allows for a more complete converse to Theorem 1.

Definition: V *is feasibly connected* at μ if for all $v, w \in V$ there exists $z \in V$ such that

(1) z could have delivered $y(v)$ or $y(w)$: $y(v), y(w) \in Y_z$,

(2) v and w could have delivered $y(z)$: $y(z) \in Y_v \cap Y_w$.

In the above, $y(\cdot) \equiv y_\mu(\cdot)$ is the y-optimal allocation at μ.

Feasible connectedness is a relatively weak assumption. To illustrate, consider a pure exchange economy in which for all v and all $y > 0$, $v(y) > -\infty$, i.e., any v's trading possibilities include the positive orthant (pure purchases). Suppose for any v, w there is a type z with a greater endowment of goods than either v or w (hence $Y_z \supset (Y_v \cup Y_w)$); but because z likes y-commodities so much, in any PO_Y allocation for μ, $y(z) \geq 0$. Then (1) and (2) would be satisfied.

Corollary 1 *Let f be a regular PO_Y mechanism at μ, and let V be convex and feasibly connected. If f is DS at μ then for all w and v,*

$$U(f(w); v) = MP(w; v) - h.$$

Equivalently, if f is DS at μ then for all w,

$$m(w) = \xi(w) - h.$$

REMARK 3: A weaker assumption would suffice. It is enough to postulate that for all $v \in \operatorname{supp} \mu$ and for all w, there exists a finite sequence $(z_0, z_1, \ldots, z_n)$ with $z_0 = v$ and $z_n = w$ such that for all $i = 1, \ldots, n-1$, z_i could have delivered $y(z_{i-1})$ or $y(z_{i+1})$ and z_{i-1} and z_{i+1} could have delivered $y(z_i)$. In Holmstrom's terminology this, along with convexity, would make V into a piecewise smoothly connected domain.

5 Existence Theorems for DSPO and DSPOIR Mechanisms

5.1 A Possibility Theorem for DSPO Mechanisms

Recall that a $DSPO$ mechanism at μ is a $DSPO_Y$ mechanism at μ in which the sum of money transfers, $\int m$, is zero. If that sum were positive, the allocation of the money commodity would not be feasible for the participants in the economy and the balance would have to be made up by some outside authority; or if it were negative, the sum would represent the departure from full utility maximization and Pareto-optimality. While the results for $DRPO_Y$ mechanisms in nonatomic models completely parallel the finite agent mechanism results (the literature concentrates on DR, rather than the more general DS mechanisms), the situation for $DRPO$ is quite the opposite. Instead of the impossibility results for $DRPO$ cited above for finite agent models, there is always possibility—even for $DSPO$ mechanisms.

From the hypothesis that the y-allocation is optimal, $\int v(y(v)) = g$. Therefore by Lemma 1

$$\int \xi = \left(\int MP\right) - g,$$

i.e., the sum of the external effects is the difference between the sum of the marginal products and the total gains from trade for the economy.

From Theorems 1 and 2 and the Corollary we know that a $DSPO_Y$ mechanism requires that each individual's money payment must equal the external effects associated with the individual minus a lump sum,

$$m(w) = \xi(w) - h.$$

Suppose the lump sum term to each agent, h, just equalled the *per capita external effects*; i.e.,

$$h = \overline{\xi} \equiv \int \xi / \overline{\mu},$$

where $\overline{\mu} \equiv \int d\mu$ is the size of the economy μ. Then

$$m(w) = \xi(w) - \overline{\xi}.$$

Summing the money payments, we evidently have,

$$\int m d\mu = \int \left(\xi - \overline{\xi}\right) d\mu = 0.$$

This leads immediately to the following conclusion.

Theorem 3 *Let f be a regular PO_Y mechanism at μ, and let V be convex and feasibly connected. Then f is $DSPO$ at μ if and only if for each w and v,*

$$U(f(w); v) = MP(w; v) - \overline{\xi},$$

or equivalently, if and only if for each w,

$$m(w) = \xi(w) - \overline{\xi}.$$

Thus, assuming the mechanism is PO_Y, the unique method to obtain $DSPO$ is: set the money payment for any announcement w, whether or not $w \in \operatorname{supp} \mu$, equal to the external effect that announcement would create for others, $\xi(w)$, minus the average external effect in the population, $\overline{\xi}$.

With a finite number of individuals this method of strategically internalizing external effects fails because each individual announcement typically changes the average so that it cannot act as a lump sum. This observation agrees with—but does not, of course, demonstrate—the conclusion that $DSPO$ mechanisms typically do not exist in finite individual models. However, as the number of individuals increases, each individual external effect will influence the average less and less, and with a continuum of individuals the influence will be nil. (This conclusion requires certain smoothness assumptions as well as large numbers.)

REMARK 4: Obviously, the conclusions of Theorem 3 also hold for $DRPO$ mechanisms, without the assumption that V is feasibly connected.

In the following Remark and the next two subsections, we shall see that there are important classes of models in which average external effects are zero. In such a case, Theorem 3 can obviously be sharpened to:

Corollary 2 *Under the hypotheses of Theorem 3, if* $\bar{\xi} = 0$, *then* f *is DSPO at* μ *if and only if for each* w *and* v,

$$U(f(w); v) = MP(w; v),$$

or equivalently, if and only if for each w,

$$m(w) = \xi(w).$$

REMARK 5 (Regular Private Goods Exchange Economies and Walrasian Equilibrium): The implication of Corollary 2 is that *if* private goods exchange economies exhibit $\bar{\xi} = 0$, then f is $DSPO$ at μ if and only if

$$\begin{aligned} U(f(v); v) &= v(y(v)) + m(v) \\ &= MP(v) \\ &= v(y(v)) + \xi(v). \end{aligned}$$

In the process of exhibiting $\bar{\xi} = 0$, the above condition will be shown to imply that the mechanism will be $DSPO$ at μ only if it is Walrasian at μ.

Definition: A *Walrasian equilibrium* for μ is a triple $(p(\mu), y(\cdot), m(\cdot))$ satisfying

- $m(v) = -p(\mu) \cdot y(v)$ for all $v \in \text{supp } \mu$,
- $v(y(v)) - m(v) = \sup\{v(y) - p \cdot y\}$ for all $v \in \text{supp } \mu$,
- $\int y d\mu = 0$.

Recalling the definition of $\xi_\mu(w)$ applied to this model,

$$\begin{aligned} \xi_\mu(v) &= \int \left\{ \partial v(y_\mu(v)) \cdot Dy_{(\mu;w)}(v) \right\} d\mu(v) \\ &= p(\mu) \cdot \int Dy_{(\mu;w)}(v) d\mu(v) \\ &= p(\mu) \cdot (-y_\mu(w)), \end{aligned}$$

where $p(\mu)$ is the efficiency prices of Remark 2. The second line follows from the well-known equality of efficiency prices and the gradients of individual utility functions. The third line follows from the definition of $Y(\mu') = \{y(\cdot) : \int y d\mu' = 0\}$ which implies that for $\mu' = \mu + t\delta_w$

$$\frac{t y_{\mu+t\delta_w}(w)}{t} + \frac{\int (y_{\mu+t\delta_w}(v) - y_\mu(v))\, d\mu(v)}{t} = 0.$$

Taking the limits of both sides as $t \to 0_+$ and assuming the mechanism is regular (as verified in Remark 2) yields

$$y_\mu(w) + \int Dy_{(\mu;w)}(v) d\mu(v) = 0.$$

To summarize and conclude, for private goods exchange economies $\xi(v) = p \cdot (-y(v))$ and $\int y = 0$. Therefore $\int \xi = \bar{\xi} = 0$. Thus, by Corollary 2, f will be $DSPO$ at μ if and only if $m(v) = \xi(v)$, i.e., the money payment must equal the external effect. But the value of the external effect is the evaluation of $y(v)$ at the efficiency prices $p(\mu)$, so $m(v) = -p \cdot y(v)$. Therefore, in a regular private goods exchange economy, *for $DSPO$ the full allocation of y commodities and money must be a Walrasian equilibrium.*

5.2 A Characterization of $DSPOIR$ Mechanisms

There is an interesting qualification to Theorem 3, one that highlights the role of the money commodity in quasi-linear preferences as a built-in medium for making lump sum transfers. The qualification involves "individual rationality".

No matter what the value of $\int \xi$, a regular mechanism can reward each agent with his/her marginal product—thus ensuring $DSPO_Y$—and then, by requiring each agent to make a lump sum payment in the money commodity of $\bar{\xi}$, the mechanism can ensure the PO property. It is the ability to break down the construction of a $DSPO$ mechanism into the separate problems of (1) $DSPO_Y$ and then (2) PO, which we shall call the "separation phenomenon", that permits Theorem 3 to apply to a wide range of economic environments.

In this section, we show that even within the class of models with quasi-linear preferences, there is a way to "undermine" the separation phenomenon through the introduction of a voluntary participation, or individual rationality, restriction. It is as if the degree of freedom on making lump sum transfers provided by quasi-linearity is removed once this added restriction is imposed. The argument will require further definitions and assumptions.

Recall that B^o is a set of elements common to all Y_v.

Definition: The element $y^o \in B^o$ is a *status quo allocation* at μ if for all v, $v(y^o) = 0$, and $(y(v) = y^o)_{v \in V} \in Y(\mu)$.

Thus, utility functions are scaled so that the status quo has a value of zero. In addition, the status quo is feasible in the aggregate. For environments in which allocations can be described by net trades (with or without public goods), y^o would be the null trade; and for environments in which $Y_v = \Upsilon$, a fixed class of public good projects, y^o would represent the status quo project.

With a status quo allocation, we can formulate the following condition.

Definition: The mechanism f satisfies *individual rationality* (IR) at μ if for all $v \in V$,

$$U(f_\mu(v); v) \geq v(y^o) + 0 = 0.$$

For brevity, a mechanism will be called *DSPOIR* at μ if it is both *DSPO* at μ and *IR* at μ.

There is no "rationality" behind this inequality unless the mechanism gives each individual the choice of whether or not to depart from the status quo. Where the status quo is the null trade, the *IR* condition can be interpreted as a modification of the *DS* property: it gives each individual the right to receive the null trade, not only in y-commodities but also in money.

In addition to the status quo allocation, we impose more structure on the model by assuming that the following holds for μ.

(E.1) (Non-decreasing returns) $\int MP - g \geq 0$.

(E.2) (Characteristics are benign) for all w, $MP(w) \geq 0$.

(E.3) (Existence of "dummies") there exists v^o such that $MP(v^o) = 0$.

Were we to formulate more explicitly a particular model of an economy with costly private or public goods, assumptions (E.1) and (E.2) could be derived as conclusions. Here we simply assert that these conditions do not go beyond conventional restrictions. (See Makowski and Ostroy (1987a) for a demonstration and Section 6, below, for an illustration of a model in which $\int MP - g < 0$.

(E.3) postulates the existence of individuals having no effect on the gains from trade. For example, in a private goods exchange economy if p were the efficiency price vector corresponding to the y-optimal allocation in the population μ, then v^o could be taken to be those preferences for which the hyperplane $\{y \in \mathbf{R}^\ell : p \cdot y = 0\}$ is tangent to the indifference curve of v^o passing through the origin of $\mathbf{R}^\ell$; with public goods, v^o would be the tastes of someone entirely indifferent to public goods and who, furthermore, has no resources that contribute toward their production.

With the above assumptions, the following result is a simple corollary of Theorem 3.

Theorem 4 *Assume f is a regular PO_Y mechanism at μ. Also assume (E.1-3) and the existence of a status quo allocation. Then, f will be DSPOIR at μ only if the lump-sum payment h satisfies $h = \overline{\xi} = 0$. Conversely, if $\overline{\xi} = 0$ then there exists a DSPOIR mechanism at μ.*

5.3 A Characterization of *DSPO* Mechanisms in Homogeneous Environments: Full Appropriation/Internalization

We shall conclude this investigation into dominant strategy mechanisms in nonatomic economies by pointing out the connections between the condition $\overline{\xi} = 0$ and the century-old problem of "adding-up" in the marginal productivity theory of distribution.

Say that there is *adding-up* at μ if $\int MP = g$. Use this aggregate condition to prescribe individual payoffs.

Definition: The mechanism f exhibits *full appropriation* at μ if individuals receive exactly their marginal products: for each v, $U(f(v);v) = MP(v)$.

There is full appropriation in the sense that each individual fully appropriates in utility the benefits that his presence confers on the rest of the economy; or, alternatively put, others neither gain nor lose from the presence of any individual. Note that, assuming f is fully optimal at μ, full appropriation at μ is possible if and only if there is adding-up at μ.

From the fact that $\int \xi = \int MP - g$, adding-up is evidently equivalent to the condition that the *per capita* external effect $\bar{\xi}$ is zero. Given the relation between marginal products and external effects, $MP(v) = v(y(v)) + \xi(v)$, there is an alternative description of full appropriation in terms of internalizing external effects.

Definition: The mechanism f exhibits *full internalization* at μ if individuals' money payments equal their external effects: for each v, $m(v) = \xi(v)$.

It is important to recognize that adding-up (or, $\bar{\xi} = 0$) is a property of the economy μ, more precisely a property associated with Y-optimal allocations at μ and $\mu + t\delta_v$, and is *not* a property of the mechanism. However, adding-up has significant implications for efficient, dominant strategy mechanisms. The Corollary to Theorem 3 shows that if there is adding-up, any mechanism that is $DSPO$ at μ will exhibit full appropriation (or, full internalization). Further, Theorem 4 gives conditions under which adding-up at μ is necessary and sufficient for the existence of a $DSPOIR$ mechanism at μ. Therefore, we can go a long way toward recognizing the kinds of economic environments for which not only $DSPO$, but even $DSPOIR$, is possible by identifying those environments exhibiting adding-up. Traditional marginal productivity theory suggests that adding-up will require constant returns in the function g, and this is indeed the case.

To prepare the argument, we expand the domain of g from N to the smallest positive cone containing it, i.e., $\{\mu' : \mu' = t(\mu + s\delta_w),\ t > 0,\ s \in [0,1],\ w \in V\}$. For our formal analysis it will suffice to focus on the behavior of the gains from trade on the subset $C \equiv \{\mu' = t\mu : t > 0\}$.

Definition: The environment is called *homogeneous* if g is (positively) *homogeneous* on C, i.e., if $g(t\mu) = tg(\mu)$ for all $t > 0$.

In contrast to homogeneity (constant returns), the other polar cases are for all $t > 0$: $g(t\mu) > tg(\mu)$ (increasing returns) and $g(t\mu) < tg(\mu)$ (decreasing returns). We shall refer to these returns-to-scale features of an economic environment in Section 6. The remainder of this section is devoted to a continuum extension of Euler's Theorem for positively homogeneous functions. The theorem is well-known to be false unless g is differentiable. In the present context it suffices to say that g is *differentiable* at μ if,

$$(D) \qquad \int MP_{\mu}(v)d\mu(v) = Dg_{\mu}(\mu) = -Dg_{\mu}(-\mu),$$

where $Dg_{\mu}(\cdot) = \lim_{h \to 0^+} h^{-1}\{g(\mu + h(\cdot)) - g(\mu)\}$.

There is an abuse of notation in (D). On the LHS we have written $MP_{\mu}(\cdot)$ as a function on V rather than on the set of positive measures on V, $\mathbf{M}_+(V)$, because we preferred to write the directional derivative of g at μ in the direction v as $MP_{\mu}(v)$ rather than as $MP_{\mu}(\delta_v)$. In the more consistent notation, (D) is $\int MP_{\mu}(\delta_v)d\mu(v) = Dg_{\mu}(\mu) = -Dg_{\mu}(-\mu)$. The differentiability condition (D) says: (1) at μ an infinitesimal increase in the direction μ has the same effect on g as the sum of the infinitesimal increases in the component directions δ_v; and, (2) there is two-sided directional derivative at μ with respect to C.

Theorem 5 (Euler's Theorem) *Assume g is differentiable on C, i.e., differentiable at $t\mu$ for all $t > 0$. Then g is homogeneous on C if and only if g exhibits adding-up on C.*

Theorem 5 says that with differentiability, homogeneity of the environment implies adding-up and conversely, if there is always adding-up, the environment must be homogeneous. Therefore, adding-up will typically occur *only* when the environment is homogeneous.

We have not traced the returns to scale property of g back to the underlying conditions on the y-allocation because in the final analysis it is the results of the y-allocation on utility that matters. Nevertheless, the function g is derived from PO_Y allocations and we shall comment briefly on the implications for g of some relevant properties of these allocations. The homogeneity of g will derive from a condition that for all $t > 0$, $y_{t\mu}(v) = y_{\mu}(v)$, i.e., constant returns to g result if scale changes in the population cause no changes in optimal *per capita* allocations. This, in turn, will depend on the property of aggregate feasibility that for all $t > 0$, $Y(t\mu) = Y(\mu)$.

The differentiability condition (D) must ultimately be derived from a condition on the Y-optimal allocation, namely that for each v,

$$(D_Y) \qquad \int Dy_{(\mu;\delta_w)}(\delta_v)d\mu(w) = Dy_{(\mu;\mu)}(\delta_v),$$

where $Dy_{(\mu;\cdot)}(\cdot)$ has been translated from a function on $V \times V$ to $\mathbf{M}_+(V) \times \mathbf{M}_+(V)$. Condition (D_Y) says that the sum of all the separate effects on v, $Dy_{(\mu;w)}(v)$, over all individuals w in the population, is equal to the effect on v of a simultaneous infinitesimal change in the scale of the population, $Dy_{(\mu;\mu)}(v)$.

REMARK 6 (Regular Private Goods Exchange Economies and Homogeneity): It was shown in Remark 5 that a regular private goods exchange economy satisfies adding-up, i.e., $\int \xi = 0$. It should therefore come as no surprise that this environment is homogeneous. This conclusion can be demonstrated indirectly via Theorem 5 after first verifying (D_Y). For a direct demonstration, recall the aggregate feasibility condition on trades in a private goods exchange economy μ is $Y(\mu) = \{y(\cdot) : \forall v,\ y(v) \in Y_v \text{ and } \int y d\mu = 0\}$, and therefore $Y(t\mu) = Y(\mu)$, $t > 0$. From this it readily follows that such an environment is homogeneous, i.e., $g(t\mu) = tg(\mu)$, for all $t > 0$.

REMARK 7 (The Value): Readers familiar with the *value* of a nonatomic game (Aumann and Shapley, 1974) will recognize important similarities between the formulas for the value and for $DSPO$ mechanisms. This is a good point at which to make some comparisons.

Let $I = [0,1]$ be the players in a nonatomic game and $e : I \to \mathbf{M}_+(V)$ be a function describing each player's characteristics with the restriction that $e(i) = \delta_v$ so that each player is endowed with a pure characteristic.

Denote $\mu = \int e d\lambda$, where λ is the Lebesgue measure, as the total of all players' characteristics in the game; and let $\mu_S = \int_S e d\lambda$ be the characteristics of the players in $S \subset I$. (The integral $\mu_S = \int_S e d\lambda$ is taken in the Gelfand sense.)

Ignoring how the construction is obtained let $g(\mu_S)$ be the worth of coalition S. (This is an infinite-dimensional version of Aumann and Shapley's finite-dimensional vector measure game.) The value assigned to an individual of type v in a game g where the total of all players' characteristics is μ is a utility $\phi_\mu(v)$ given by the "diagonal formula,"

$$\phi_\mu(v) = \int_0^1 Dg_{t\mu}(v)dt = \int_0^1 MP_{t\mu}(v)dt.$$

The formula for the utility in a $DSPO$ mechanism is

$$\Phi_\mu(v) = Dg_\mu(v) - h_\mu = MP_\mu(v) - h_\mu,$$

where $h_\mu = \overline{\xi}_\mu$.

If the formulas do not coincide, i.e., $\phi_\mu(v) \neq \Phi_\mu(v)$, then the value allocation as a prescription for a mechanism cannot be DS because $\Phi_\mu(v)$ is *the* method of achieving $DSPO$. Alternatively put, if the two formulas differ then $(\phi_\mu(v) - \Phi_\mu(v))$ is not a lump sum.

The one environment on which the two payoffs agree is the homogeneous one.

With homogeneity, $Dg_{t\mu}(v) = Dg_{\mu}(v)$ whenever $t > 0$, from which it readily follows that $\phi = \Phi$.

Homogeneity is well-known to be important for the Value Equivalence Theorems. For example, Aumann and Shapley (1974) demonstrate that a class homogeneous games is derived from nonatomic exchange economies and for these games/economies they show that the core, the value and Walrasian equilibrium coincide. Regarding the value as a mechanism yielding utilities given by the formula $\phi_{\mu}(v)$, we are led to the following conclusion based on Theorems 3 and 5: *Assuming (D), the value is a DSPO mechanism if and only if the environment is homogeneous.*

6 The Work of Others

To conclude our analysis, we comment briefly on some of the connections between the results of this paper and the work of others mentioned in the Introduction. We wish to show that the general equilibrium extension of the marginal product concept and the characterization results derived using it provide a framework in which many, apparently unrelated, results from mechanism theory can be synthesized. Our focus is on results for models with large numbers or a continuum of individuals, but it should be re-emphasized that our characterization results depend upon constructs applicable to finite individual models. In particular, the formula for a $DSPO_Y$ mechanism—give each individual his marginal product minus a lump sum—is also the necessary and sufficient condition for a $DSPO_Y$ mechanism in finite models.

Two further preliminary remarks are in order. First, we shall not distinguish between results quoted below that apply to models with quasi-linear utility and those that apply to more general models without quasi-linearity. Second, in keeping with the mechanism approach and the emphasis of this paper in which explicit reliance on price-guided allocations is minimized (except for Remarks 2 and 5), we shall not elaborate upon the pricing interpretations of the results stated below. Demonstrations that the findings of this paper for quasi-linear utility models can be extended to models without quasi-linearity as well as pricing interpretations of DS mechanisms are the subject of Makowski-Ostroy (1987b).

We divide the literature on DS mechanisms with large numbers of individuals according to returns-to-scale properties of the models and then remark on the link with finite individual models.

6.1 Constant Returns (Homogeneous Environments)

6.1.1 Private Goods

Private goods economies have a built-in homogeneity: doubling the number of each type clearly doubles the total gains, i.e., $g(2\mu) = 2g(\mu)$. Therefore, a $DSPO$ mechanism must reward each individual with an allocation the utility of which is exactly equal to his marginal product. This agrees with the findings of Roberts and Postlewaite (1976) that the Walrasian mechanism has the DS property and that it is the *only* one to have this property (Hammond, 1979). McLennan (1982), and Mas-Colell (1987) give versions of this result under the hypothesis that the net trades in a given economy must be "fair." (See Remark 1, above).

6.1.2 Costless Public Goods

In a model with a fixed set of *costless public goods* projects among which only one will be selected, the environment is also homogeneous: if y_μ is the project chosen to maximize total utility when the population is μ, then $y_{2\mu} = y_\mu$ will be chosen when the population is 2μ, so $g(2\mu) = 2g(\mu)$. (This model does not capture the distinguishing property of pure public goods. See 6.3, below.)

Since the environment is homogeneous, we have $\overline{\xi} = 0$. But the costless public goods model can be shown to have the stronger property that *each* $\xi_\mu(v) = 0$. Thus, in a $DSPO$ mechanism, $m_\mu(v) = 0$. Asymptotic versions of this result are demonstrated by Tideman and Tullock (1976), Green and Laffont (1979), Rob (1982), and Mitsui (1983); they show that the *per capita* surplus run by the Clarke "pivot" mechanism (our marginal product mechanism with zero lump sum) converges to zero.

Warning: This does *not* imply one can design a $DSPOIR$ mechanism for costless public goods because, contrary to Theorem 4, in this environment characteristics may not be benign (recall (E.2)). In particular, any type w who does not like the efficient public good project for population μ will contribute a *negative* marginal product to μ.

6.2 Decreasing Returns

Consider a model of private goods without private property where individuals "own" their tastes but total resources *are fixed* and under the control of the mechanism. Because an individual's characteristics include only his tastes (representable by a concave utility function) and *not* resources, when the population doubles total utility less than doubles. The decreasing returns property $g(2\mu) < 2g(\mu)$ is equivalent to $\int \xi = \int MP - g < 0$. Therefore, $DSPO$ requires that the utility of each individual's allocation equal his marginal product plus a lump sum subsidy equal to $\bar{\xi}$ to make up for the difference between $\int MP$ and the total gains, g.

Varian (1976), Kleinberg (1980), McLennan (1982) and Champsaur and Laroque (1981,1982) used this model to study fair allocations. Their findings established that the only fair allocations are Walrasian equilibria arising from an initial allocation in which each individual has an equal-valued share of total resources. We note that such an allocation is the only way to realize the formula for a $DSPO$ mechanism in this model of private goods with decreasing returns.

6.3 Increasing Returns

Consider a nonatomic model with public goods produced using private goods as inputs, e.g., Meunch (1972). Two identical populations μ, each producing the same optimal quantities of public goods with the same resources— so producing total utility $2g(\mu)$—could, simply by combining to form one economy, halve the *per capita* resources contributed, maintain the same total quantity of public goods and therefore produce total gains for the population 2μ such that $g(2\mu) > 2g(\mu)$.

In this situation, $\int \xi = \int MP - g > 0$. Here a $DSPO$ mechanism gives each individual his MP and then imposes a uniform lump sum tax of $\bar{\xi}$. Hammond (1979) has given a price characterization of a $DSPO$ mechanism with public goods. It can be shown that his "privately fair Lindahl mechanism" is equivalent to the above marginal product mechanism minus lump sum.

Does a $DSPOIR$ mechanism exist for models with costly public goods? (Clearly, they do exist for private goods, while IR may not be applicable in case 6.2.) Assuming the cost of the public good does not vary with the size of the population, a model of costly public goods satisfies the hypotheses of Theorem 4. Therefore, a $DSPOIR$ mechanism exists if and only if $\int \xi = 0$. However, if $\int \xi = 0$ holds, the gains function must be homogeneous, which contradicts the increasing returns feature of public goods models, $\int \xi = \int MP - g > 0$. So, we can conclude that a $DSPOIR$ mechanism cannot exist when there are (costly) public goods.

There is an interesting variation in Roberts (1976). Thus far in our discussion we have assumed that if one doubles the population then the cost of producing any given amount of the public good will remain constant. By constrast, to prove his asymptotic impossibility result, Roberts assumes that if an economy grows, the cost of producing any given amount of the public good will also increase. (The role of this assumption is to avoid asymptotically approaching a *costless* public good model.) The cost may even increase in direct proportion to the population's size; so, the gains from trade function may even be homogeneous. But, to be consistent with the central distinguishing property of pure public goods, Roberts also assumes that for any *given* size economy, the cost of producing any given amount of the public good will *not* vary with the number of people who actually consume it. The consequence is that individuals' characteristics may not be benign (recall (E.2)). Indeed, if w does not value the public good, $MP(w) < 0$ since his presence *only* adds to the cost of production. It is easy to show that without benign characteristics, for $DSPOIR$ one needs to subsidize individuuals (so that $MP(w) + h \geq 0$ for all w). Consequently, to finance the subsidy, one needs $\int MP - g < 0$. But this violates (E.1), which holds in any model with costly public goods; hence the Roberts impossibllity result for $DSPOIR$ mechanisms even in the limiting case of a homogeneous g.

6.4 Finite Numbers: Indivisibilities

We have used the MP theme to provide an interpretive survey of the various results for DS mechanisms in continuum economies. But what is the connection with finite individual models, to which the literature on DS mechanisms is primarily devoted? In making this connection, we will implicitly be shifting the focal point from models with small numbers to models with large numbers. This change in perspective is suggested by the parallels with traditional marginal productivity theory in which continuous variation in the factors of production is regarded as the norm and discrete variations as a special case.

It suffices to confine attention to constant returns models — Sections 6.1.1 and 6.1.2, above. While constant returns environments are the ideal setting for $DSPO$ mechanisms in nonatomic models, how to explain that $DSPO$ mechanisms do not generally exist when the number of individuals is finite?

Consider the parallels with the marginal productivity theory of distribution. In a continuum model, we have shown that in constant returns environments, the necessary and sufficient condition for $DSPO$ is to pay individuals exactly their marginal products. The feasibility of such a condition is not automatically guaranteed by constant returns; it also requires the differentiability condition (D). (Recall that there is a similar requirement in the traditional version of Euler's Theorem for Homogeneous Functions.) Going behind the g function to the economic environment from which it is derived, it can be demonstrated that while (D) need not always obtain, it will hold generically for the kinds of environments to which we have referred. Thus, in these constant returns environments, when each individual is infinitesimal it is typically possible to pay each one his/her MP and therefore to demonstrate that a $DSPO$ mechanism is feasible.

Now, while continuing to assume a constant returns environment such as would come from a private goods exchange economy or a costless public goods model, make the following modification: assume each individual is an *atom*. The fact that individuals are no longer infinitesimal is similar to the hypothesis in the theory of production that factors are *indivisible*; and, in that case, even if there is constant returns it will typically be impossible to pay each factor its marginal product. A similar interpretation appears to lie behind the non-existence results for $DSPO$ mechanisms in finite individual models. There is firm support for this interpretation in the case of costless public goods. Laffont and Maskin (1979) have shown that among all of the $DSPO_Y$ mechanisms, there is none that dominates the pivot mechanism in minimizing the absolute value of the sum of monetary transfers. (Recall that for $DSPO$, the sum must be zero.) Since the pivot mechanism rewards individuals with their MP 's, we can trace the non-existence of $DSPO$ mechanisms to the failure to obtain adding-up, which in turn can be traced to the fact that the "factors of production" in the gains function g, i.e., the individuals, are indivisible.

7 PROOFS

Before giving the proofs, as promised in Section 2, we provide a metric on V.

REMARK 8 (A Metric for V): Recall that an individual's characteristics are represented by an extended real-valued concave function $v : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ with $Y_v = \{y : v(y) > -\infty\}$. Denote by $\tilde{V}$ a set of real-valued concave C^2 functions $\tilde{v} : \mathbf{R}^\ell \to \mathbf{R}$ that is compact in the C^2 compact-open topology. It is assumed that any $v \in V$ can be represented by the pair $(\tilde{V}(v), Y_v)$, where $\tilde{V}(v) = \{\tilde{v} \in \tilde{V} : \tilde{v}(y) = v(y)$, for all $y \in Y_v\}$ is the set of possible C^2 extensions of v in $\tilde{V}$. It can be verified that $\tilde{V}(v)$ is closed. Therefore, we can define V as a compact metric space by taking the product metric on the set of closed subsets of $\tilde{V}$ endowed with the Hausdorff metric and the closed subsets of $\mathbf{R}^\ell$ endowed with the metrizable closed convergence topology.

Lemma 1 Holding μ and w fixed, let

$$\begin{aligned} K(t) &\equiv g(\mu + t\delta_w) \\ &= \int v(y_{\mu+t\delta_w}(v))d\mu(v) + tw(y_{\mu+t\delta_w}(w)) \\ &\equiv \int k(t,v)d\mu(v) + tk(t,w). \end{aligned}$$

Using (R.1), for all v, $k(t,v)$ has a right derivative at 0 given by

$$k'_+(0,v) = \partial v(y_\mu(v)) \cdot Dy_{(\mu;w)}(v).$$

(It is easy to verify that the chain rule applies even though we are taking a one-sided derivative.) Further, $k'_+(0,\cdot)$ is continuous on V since, by assumption, $y_\mu(\cdot)$ is continuous and, by (R.1), $Dy_{(\mu;w)}(\cdot)$ is continuous. Thus, we can apply the Lebesgue Convergence Theorem to obtain:

$$
\begin{aligned}
MP_\mu(w) &\equiv \lim_{t\to 0_+} \frac{K(t)-K(0)}{t} \equiv K'_+(0) \\
&= \int k'_+(0,v)d\mu(v) + k(0,w) \\
&= \int \partial v(y_\mu(v)) \cdot Dy_{(\mu;w)}(v)d\mu(v) + w(y_\mu(w)) \\
&\equiv \xi_\mu(w) + w(y_\mu(w)).
\end{aligned}
$$

Since by (R.1), for each v, $Dy_{(\mu;\cdot)}(v)$ is continuous, $\xi_\mu(\cdot)$ is continuous. The continuity of $MP_\mu(\cdot)$ now follows immediately from the continuity of $\xi_\mu(\cdot)$ and of $y_\mu(\cdot)$. $\|$

Lemma 2 Proceeding along lines similar to Lemma 1, let

$$
\begin{aligned}
H(t) &\equiv g(\mu + t\delta_w; \mu + t\delta_v) \\
&= \int z(y_{\mu+t\delta_w}(z))d\mu(z) + tv(y_{\mu+t\delta_w}(w)) \\
&\equiv \int k(t,z)d\mu(z) + th(t,v),
\end{aligned}
$$

where $k(t,\cdot)$ was defined in the proof of Lemma 1. Then, as in the proof of Lemma 1, we have:

$$
\begin{aligned}
MP_\mu(w;v) &\equiv \lim_{t\to 0_+} \frac{H(t)-H(0)}{t} \equiv H'_+(0) \\
&= \int k'_+(0,z)d\mu(z) + \lim_{t\to 0_+} v(y_{\mu+t\delta_w}(w)) \\
&= \xi_\mu(w) + v(y_\mu(w)),
\end{aligned}
$$

where the last equality follows from (R.2) and the fact that $y_{\mu+t\delta_w}(w) \to y_\mu(w)$ (which, recall, follows from (R.1)). $\|$

Lemma 3 It is evident from the definition of g that for all $t > 0$ and v,

$$
g(\mu + t\delta_w; \mu + t\delta_v) \leq g(\mu + t\delta_v; \mu + t\delta_v) \equiv g(\mu + t\delta_v),
$$

i.e., the total gains from trade cannot possibly be increased through misrepresentation. Therefore,

$$
\begin{aligned}
MP(w;v) &\equiv \lim_{t\to 0_+} \frac{g(\mu + t\delta_w; \mu + t\delta_v) - g(\mu)}{t} \\
&\leq \lim_{t\to 0_+} \frac{g(\mu + t\delta_v; \mu + t\delta_v) - g(\mu)}{t} \\
&= \lim_{t\to 0_+} \frac{g(\mu + t\delta_v) - g(\mu)}{t} \\
&\equiv MP(v). \quad \|
\end{aligned}
$$

Theorem 1 If $U(f(w);v) = MP(w;v) - h$ then by Lemma 3, $U(f(v);v) - U(f(w);v) = MP(v;v) - MP(w;v) \geq 0$. $\quad \|$

Theorem 2 Fix v and w. By convexity, for any $\alpha, \beta \in [0,1]$, V contains $v_\alpha = \alpha v + (1-\alpha)w$ and $v_\beta = \beta v + (1-\beta)w$.

Recall, without loss of generality, for any $\alpha, \beta \in [0,1]$ we can write

$$\begin{aligned} U(f(v_\beta); v_\alpha) &= MP(v_\beta; v_\alpha) - H(v_\beta) \\ &\equiv \psi(\beta, \alpha) - k(\beta), \end{aligned}$$

taking advantage of the fact that v and w are fixed.

From the hypothesis that f is a DR mechanism, we have that for all $\alpha, \beta \in [0,1]$:

$$(a) \qquad \alpha \in \operatorname{argmax}_\beta \ \psi(\beta, \alpha) - k(\beta).$$

From Lemma 3, for all $\alpha, \beta \in [0,1]$:

$$(b) \qquad \alpha \in \operatorname{argmax}_\beta \ \psi(\beta, \alpha).$$

Since by Lemma 2, $MP(v_\beta; v_\alpha) = v_\alpha(y(v_\beta)) + \xi(v_\beta)$,

$$\psi(\beta, \alpha) = v_\alpha(y(v_\beta)) + \xi(v_\beta),$$

where $v_\alpha(y(v_\beta)) = \alpha v(y(v_\beta)) + (1-\alpha)w(y(v_\beta))$.

Differentiating ψ with respect to α,

$$\frac{\partial \psi(\beta, \alpha)}{\partial \alpha} = v(y(v_\beta)) - w(y(v_\beta)).$$

Let $Q = \{y(v_\beta) : \beta \in [0,1]\}$. Since $y(\cdot)$ is continuous on V, a compact set, Q is bounded. Therefore, $\sup\{v(y) - w(y) : y \in Q\}$ is bounded, i.e,

$$(c) \qquad \sup_{\beta,\alpha} \left[\frac{\partial \psi(\beta, \alpha)}{\partial \alpha}\right] < \infty.$$

Having established (a)–(c), now apply the following basic result proved in Holmstrom (1979):

Lemma Let $\psi : [0,1] \times [0,1] \to \mathbf{R}$ and $k : [0,1] \to \mathbf{R}$ satisfy (a), (b) and (c), then k is constant on [0,1].

Therefore, there is a constant h such that $h = H(v) = H(w)$, as was to be demonstrated. $\quad \|$

Corollary 1 Note that what is required to prove Theorem 2 is that for all $\beta \in [0,1]$, $y(v_\beta) \in Y_v \cap Y_w$, hence ψ is real-valued. The feasible connectedness assumption says that if this does not hold there is a z such that for all $\beta \in [0,1]$, (1) $y(v_\beta) \in Y_v \cap Y_z$ when $v_\beta = \beta v + (1-\beta)z$ and (2) $y(z_\beta) \in Y_z \cap Y_w$, where $z_\beta = \beta z + (1-\beta)w$. Apply the conclusions of Theorem 2 to (1) to obtain $H(v) = H(z)$ and to (2) to obtain $H(z) = H(w)$, leading to the same final conclusion as Theorem 2 that $h = H(v) = H(w)$. ||

Theorem 3 Given Lemma 2, the two statements in the theorem are obviously equivalent. Hence we need only prove one. By Theorem 1′ if for all w, $m(w) = \xi(w) - h$, then f is $DSPO_Y$ at μ. If in addition $h = \int \xi/\overline{\mu}$, then $\int m = 0$; hence, f is $DSPO$ at μ.

Conversely, if f is $DSPO$ at μ then by Theorem 2 and its Corollary, for all w, $m(w) = \xi(w) - h$, and $\int m = 0$. Thus $\int \xi - h\overline{\mu} = 0$, or $h = \int \xi/\overline{\mu}$. ||

Theorem 4 To demonstrate that $DSPOIR$ implies $\int \xi = 0$, suppose the contrary. Then, since $\int \xi = \int MP - g$, (E.1) implies $\int \xi > 0$. Now recall (E.3). By Theorem 3, $DSPO$ implies

$$\begin{aligned} U(f(v^o), v^o)) &= MP(v^o) - \overline{\xi} \\ &= -\overline{\xi} < 0, \end{aligned}$$

contradicting IR.

To prove the converse, for each $v \in V$, let $m'(v) \equiv \xi(v)$. Let f' be the mechanism that is identical to f except for the population μ, $m(\cdot)$ is replaced by $m'(\cdot)$. By construction, $\int m' = \int \xi = 0$. Hence, by Corollary 2, f' is $DSPO$ at μ. Further, since $U(f'(v); v) = \xi(v) + v(y(v)) = MP(v)$, and since by (E.2), $MP(v) \geq 0$, f' satisfies IR at μ. Hence we have constructed a mechanism that is $DSPOIR$ at μ. ||

Theorem 5 This is a straightforward extension of the finite-dimensional version of Euler's Theorem for positively homogeneous functions.

If $g(t\mu) = tg(\mu)$, $t > 0$, then

$$\begin{aligned} Dg_\mu(\mu) &\equiv \lim_{t \to 0^+} \frac{g(\mu + t\mu) - g(\mu)}{t} = \lim \frac{(1+t)g(\mu) - g(\mu)}{t} \\ &= g(\mu). \end{aligned}$$

By (D), the L.H.S. equals $\int MP$, so there is adding-up at μ.

Conversely, if there is adding-up on C, then $\int MP_{t\mu}(\delta_v) d(t\mu(v)) = g(t\mu)$; and by differentiability at $t\mu$, $\int MP_{t\mu}(\delta_v) d(t\mu(v)) = Dg_{t\mu}(t\mu) = -Dg_{t\mu}(-t\mu)$. So, we have

$$L(t) \equiv g(t\mu) = Dg_{t\mu}(t\mu) = tDg_{t\mu}(\mu).$$

Therefore,

$$L'_+(t) = Dg_{t\mu}(\mu) = t^{-1}L(t),$$

where the first equality follows from the definition of $L'_+(t)$ as

$$\lim_{h\to 0_+} \frac{(g((t+h)\mu) - g(t\mu)}{h}$$

, and the second equality from the positive homogeneity of the directional derivative. Since $L'_-(t) = \lim_{h\to 0_-} h^{-1}\{g((t+h)\mu) - g(\mu)\} = Dg_{t\mu}(-\mu)$, the hypothesis on differentiability of C implies that $L'_+(t) = -L'_-(t)$, i.e., L is differentiable at t.

The equation $L(t) = tL'(t)$ is well-known to have the solution $L(t) = ct$. Putting $t = 1$, $c = L(1) = g(\mu)$. Therefore, $g(t\mu) == L(t) = tc = tg(\mu)$. $\|$

References

R. J. Aumann and L. Shapley, *Values of Non-Atomic Games*, Princeton University Press, Princeton, 1974.

P. Champsaur and G. Laroque, Fair allocations in large economies, *Journal of Economic Theory* **25** (1981), 269–282.

P. Champsaur and G. Laroque, A note on incentives in large economies, *Review of Economic Studies* **49** (1982), 627–635.

E. H. Clarke, Multipart Pricing of Public Goods, *Public Choice* **8** (1971), 19–33.

J. Green and J. J. Laffont, Characterization of satisfactory mechanisms for the revelation of preferences for public goods, *Econometrica* **45** (1977), 427–438.

J. Green and J. J. Laffont, *Incentives in Public Decision Making*, North Holland, Amsterdam, 1978.

J. Green and J. J. Laffont, On coalition incentive compatibility, *Review of Economic Studies* **46** (1979), 243–254.

T. Groves, *The Allocation of Resources under Uncertainty*, Ph.D. dissertation, University of California, Berkeley, 1970.

T. Groves and J. Ledyard, Incentive compatibility since 1972, in *Information, Incentives, and Economic Mechanisms*, Groves, Radner and Reiter, eds., University of Minnesota Press, Minneapolis, 1987.

T. Groves and M. Loeb, Incentives and public inputs, *Journal of Public Economics* **4** (1975), 211–226.

P. Hammond, Straightforward individual incentive compatibility in large economies, *Review of Economic Studies* **46** (1979), 263–282.

W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton 1974.

L. Hurwicz On informationally decentralized systems, in *Decision and Organization*, C. McGuire and R. Radner, eds., North-Holland, Amsterdam, 1972.

L. Hurwicz and M. Walker, On the Generic Non-Optimality of Dominant-Strategy Allocation Mechanisms, with an Application to Pure Exchange Economies, manuscript, 1983.

N. KLEINBERG, Fair allocations and equal incomes, *Journal of Economic Theory* **23** (1980), 189–200.

J. J. LAFFONT AND E. MASKIN, A differential approach to dominant strategy mechanisms, *Econometrica* **48** (1980), 1507–1520.

A. MCLENNAN, *Three Essays on Economic Theory*, Ph.D. dissertation, Princeton University, 1982.

L. MAKOWSKI AND J. OSTROY, Vickrey-Clarke-Groves mechanisms and perfect competition, *Journal of Economic Theory* **42** (1987a), 244–261.

L. MAKOWSKI AND J. OSTROY Efficient Dominant Strategy Mechanisms in Large NTU Economies: Private Goods versus Collective Goods, manuscript, 1987b.

A. MAS-COLELL, On the second welfare theorem for annonymous net trades in exchange economies with many agents, in *Information, Incentives, and Economic Mechanisms*, Groves, Radner and Reiter, eds., University of Minnesota Press, Minneapolis, 1986.

A. MAS-COLELL, *The Theory of General Equilibrium: A differentiable approach*, Cambridge University Press, Cambridge, 1985.

A. MAS-COLELL AND X. VIVES, Implementation in Economies with a Continuum of Agents, manuscript, 1989.

T. MEUNCH, The core and the Lindahl equilibrium of an economy with a public good: and example, *Journal of Economic Theory* **4** (1972), 241–255.

T. MITSUI, Asymptotic efficiency of the pivot mechanism with a general project space, *Journal of Economic Theory* **31** (1983), 318–331.

R. ROB, Asymptotic efficiency of the demand-revealing mechanism, *Journal of Economic Theory* **28** (1982), 207–220.

J. ROBERTS, The incentives for the correct revelation of preferences and the number of consumers, *Journal of Public Economics* **6** (1976), 359–374.

J. ROBERTS AND A. POSTLEWAITE, The incentives for price-taking behavior in large economies, *Econometrica* **44** (1976), 115–128.

D. SCHMEIDLER AND K. VIND, Fair net trades, *Econometrica* **40** (1972), 637–642.

T. TIDEMAN AND G. TULLOCK, A new and superior principle for collective choice, *Journal of Political Economy* **84** (1976), 1145–1159.

H. VARIAN, Two problems in the theory of fairness, *Journal of Public Economics* **5** (1976), 249–260.

W. VICKREY, Counterspeculation, auctions, and competitive sealed tenders, *J. Finance* **16** (1961), 8–37.

M. WALKER, A note on the characterization of mechanisms for the revelation of preferences, *Econometrica* **46** (1978), 147–152.

M. WALKER, On the nonexistence of a dominant strategy mechanism for making optimal public decisions, *Econometrica* **48** (1980), 1521–1540.