Money and Specialization in Production[1]

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Abstract

This paper investigates the connection between specialization and the institution of money in a search general equilibrium model with a a continuum of differentiated commodities. In a model with agents of different productivity, we provide a condition for monetary exchange to enlarge the extent of the market and to enhance specialization. It is shown how individuals sort themselves out among autarkic, pure barter and money types. Specifically, the most productive individuals will produce for the market and engage in monetary exchange. Implications for economic development and economic history are highlighted.

We construct a model which allows for gift-giving "exchanges" to appraise the welfare cost of barter. In this context, barter is viewed as a social device to overcome incentive problems in the exchange process by imposing a very extreme form of *quid pro quo*. It is shown that monetary exchange is a more efficient way of solving these problems, and therefore it is Pareto superior to barter.

Finally, we generalize our monetary model allowing agents to accumulate more than one unit of (fiat) money. We show that this generalization rationalizes models of money in the utility function. In particular, our fundamentals imply diminishing marginal utility of money.

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1 Introduction

This paper explores the relationship between the degree of specialization and the institution of money. It is conventional wisdom that in a world with trading frictions, the introduction of money enlarges markets and improves the efficiency of the exchange process. But the consequences of monetary exchange are not limited to trade. Adam Smith's often-cited claim that "specialization and division of labor are limited by the extent of the market" suggests that gains in productivity, a primary source of economic growth, are another channel through which money can be welfare-improving. Money along with other institutions such as capital markets, financial intermediation, and technologies of communication and transportation is a determinant of the extent of the market.

The link between specialization and growth was acknowledged early by the profession. Besides Adam Smith's seminal work on division of labor, Allyn Young's [1928] contribution stands out. Romer [1987] provides the first rigorous model of Young's intuition of "economic progress" driven by increasing returns based on specialization. Edwards and Starr [1987] single out indivisibilities in labor inputs as critical for the division of labor to generate increasing returns.

There is a substantial body of work, which will be discussed shortly, dealing with the connection between growth of real output and monetary and financial factors. This literature, in most cases highly aggregative, includes specialization and division of labor in their stories but not in their explicit models. An early exception is Ostroy [1970] which includes a model of money and division of labor.

The first group of papers dealing with the subject belongs to the strand of research on "Money and Growth" inspired by Tobin [1965] and associated with the names of Sidrauski [1967], Levhari and Patinkin [1968], Brock [1975] and Calvo [1978]. However, the concern of this literature differs from ours in several aspects. To begin with, growth is geared by capital accumulation in a single commodity world rather than by specialization and increasing returns. Secondly, real cash balances are introduced in either the utility or production functions. Lastly, money is viewed as a store of value, and its influence on growth hinges on portfolio effects.

A different branch of the literature, which we might call "Money, Capital and Development", is associated with the books by McKinnon [1973] and Shaw [1973]. Even though there is not a great deal of formal modelling, they include a host of insightful remarks. For instance, they point out that as monetary and financial institutions are not within the reach of potential entrepeneurs in the "traditional" or "subsistence" sector of the economy, project selection is distorted by a liquidity constraint. Under this constraint, cash balances and physical capital tend to be complements rather than substitutes as generally assumed in the standard portfolio theory underlying neoclassical growth models.

Attempts at modelling financial intermediaries, such as banks, and their role in the growth of real output are a more recent and far from finished business. For instance, Townsend [1983] and Bencivenga and Smith [1989] represent efforts in such a direction. The latter is based on Romer's [1986] model in which increasing returns are based on an externality rather than on specialization. Even though not directly

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linked to our work, we mention the work of King and Plosser [1986] tackling the problem of the selection of assets to serve as a medium of exchange, and Economides and Siow [1988] delving into the formation of markets and liquidity.

As a general matter, we believe that the incentive problems behind the exchange process are critical to understand the *raison d'etre* of money and how credit markets function. In particular, money is viewed as a social device to solve, in a more efficient way, the incentive problems posed by the enforcement of budget constraints. Loosely speaking, monetary exchange imposes *quid pro quo* in a softer manner. With different nuances, Ostroy [1973], Ostroy-Starr [1974], Lucas [1980], Green [1987], Townsend [1987] and Aiyagari-Wallace [1990] highlight incentive problems in monetary theory.

Since we model exchange processes, we emphasize the transactions role of money. Further, our framework specifies how agents meet and carry out bilateral trades. Previous work on bilateral trading includes: Ostroy [1973], Ostroy-Starr [1974], Harris [1979], Jones [1976], Townsend [1980], Diamond [1982, 1984], Oh [1988], Kiyotaki-Wright [1988] and Aiyagari-Wallace [1990].[3]

Our work belongs to the search general equilibrium paradigm proposed by Diamond [1982, 1984], which, in turn, is built on earlier contributions by Mortensen [1978] and Diamond and Maskin [1979] on matching and bargaining processes as non-cooperative games. Diamond's search models tell something about how total output and employment are determined in an economy with "trading frictions". A relevant feature of this family of models is that agents are forced to "trade" a unique commodity. Kiyotaki and Wright [1988] extend the framework so as to analyze the transaction role of money in an environment of heterogeneous individuals and a continuum of differentiated commodities.

The point of departure of this research is Kiyotaki-Wright [1988]. They concentrate on the existence, characterization and robustness of monetary equilibria. In contrast, our focus is to study how specialization responds to different trading regimes.[4]

Search general equilibrium models have two building blocks: a production sector and a trading sector. Agents search for both production opportunities and trading partners. Regarding the former sector, the choice of search policy will determine the (stochastic) production rate. This is natural in single-commodity environments. Although this feature is less compelling in a multicommodity world, Kiyotaki and Wright stick to it in their model. Kiyotaki-Wright's producers care about total output rather than the type of commodity produced. At this stage, we depart from the literature: the producer's problem will be to choose the type of commodity in which he will *specialize*. Individuals are endowed with the technological possibility of producing any type of commodity according to some schedule of comparative advantages.

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[3]Grossman-Weiss [1983] and Rotemberg [1984] link money and unemployment emphasizing the transaction role of money. Nonetheless, that role is motivated by cash-in-advance constraints, which, in some sense, tend to be less explicit about the mechanics of the trading process. Moreover, specialization plays no particular role in changes of output and employment levels.

[4]After this paper was written, the author became aware of Kiyotaki-Wright [1990] by a personnal communication from Randall Wright. They developed an specialization model in the same spirit of Section 8.

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In order to model the trading sector, we draw heavily from Kiyotaki and Wright's model. In particular, we take their specification of the matching technology and transaction costs. However, differences in preferences, production sectors and trading states lead to different implications. For instance, our model is more conclusive about the Pareto superiority of monetary exchange.

We carry out most of our investigation by modelling an economy with agents of different productivity. A bit of semantics is in order, for the venerable ideas of specialization and division of labor are not free from ambiguity. In most of the paper we have to distinguish individual specialization from the economy-wide specialization. At an individual level, it is an "all-or-nothing" phenomenon: an agent specializes if he produces with the prospect of a sale; otherwise, he produces for self-consumption. The first individual will be a "market type" and the second one an "autarkic type". At an economy-wide level, specialization is a matter of degree: the *fraction* of the population choosing to produce for the market will be our measure of the economy-wide specialization.

The aim of the paper is to pin down how people sort out themselves in some broad classes or types. In a pure barter regime, these are autarkic and (specialized) market types. In a monetary economy, besides autarkic types, we further partition market types in pure barter and money types. Pure barter types only engage in barter transactions; money types participate in both barter and monetary exchange. We call these distributions "exchange profiles". In principle, the environment can generate any "exchange profile". To a large extent, all of them are meaningfull from a historical or current viewpoint.

The basic result of the paper is the statement of a (necessary) condition for monetary exchange to enhance the economy-wide specialization. In a weak sense, the introduction of (fiat) money never decreases the specialization of the economy. But specialization increases strictly only if the exchange profile is exclusively made up of autarkic and money types. Loosely speaking, the result conveys the idea that a "high" degree of monetization is necessary for monetary exchange to further gains from specialization. In this regard, the well-known empirical fact that higher levels of monetization and economic development move together is somehow suggestive.

We can always observe barter transactions in this model even if all individuals are sorted out as money types. However, barter will be less frequent as the number of consumer types and commodities gets larger. Casual empiricism goes with this remark: the most developed economies display such a variety of commodities that there is little wonder that people do not barter.

Exchange profiles containing autarkic, pure barter and money types can be better understood in the light of the prediction of the model that the most productive types will produce for the market. Moreover, the most productive among the market types will choose to accept money in exchange for goods, and the more efficient the individual, the higher cash balances he will build up. Historical and current evidence supports this implication: monetary exchange is anchored in foreign trade. Almost by definition, participants in foreign trade are among the most specialized and efficient

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sectors of the economy.[5] This is also consistent with the development economics literature. Our implication is indeed reminiscent of the dual structure of most LDC's: a modernized enclave, often related to foreign trade and highly monetized, along with a substantial "subsistance" or "traditional" sector.[6]

The model is too simple to yield concrete policy implications. However, our work is consistent with McKinnon's [1973] idea that "financial market fragmentation" is a clue to understand the poor economic performance of the LDC's. In this context, "fragmentation" means that potential entrepeneurs in traditional sectors are left out by credit and money markets characterized by distorted prices and distorting regulations. Thus, the extent of the market participation of the less efficient sectors of the economy should be considered as a critical factor when designing financial and monetary institutions in LDC's.

The extent of the conclusions is not limited to the connection between money and specialization. From a pure monetary theory point of view, we construct a model allowing for gift-giving to appraise the welfare cost of barter as an extreme form of *quid pro quo*, where barter is viewed as a incentive-compatible constraint to overcome the problem of dishonesty in the exchange process. Monetary exchange solves the same incentive problems by imposing a softer constraint, for its *quid pro quo* is not longer based on "double coincidence of wants". For this reason, and unlike Kiyotaki-Wright [1988] and Diamond [1984], our monetary exchange is always Pareto superior to barter.

In our setup, no agent is forced to accept or hold money, which is illustrated by the pure barter types mentioned above. Typically, cash-in-advance models so popular in the literature, for instance Lucas [1980] and Diamond [1984], impose a constraint on exchange such that it is hard to assert *a priori* whether it is more or less binding than that of barter. Our model does not impose any Clower-constraint whatsoever.

As in other bilateral exchange models, we model money from fundamentals, that is, specifying preferences, technology and information and time structure from scratch. Our environment enable us to propose search general equilibrium models as a rationale for reduced form models with money in the utility and production function. In Section 7 we allow for the accumulation of more than one unit of fiat money so that we can study the accumulation and distribution of cash balances. The main property assumed in those reduced form models are either decreasing marginal utility or productivity of money. We are able to prove that our setup implies that property.[7]

The model presented in Section 8 is a brief excursion to show that money furthers specialization even though agents are equally productive.[8] Here we take up a different notion of specialization: people are more specialized when they produce a greater output of a narrower set of commodities. Since section 8 model is virtually a representative agent model, there is no difference between individual and economy-

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[5]See Braudel [1972, 1981, 1982] and Cipolla [1967].

[6]See McKinnon [1973] and Shaw [1973].

[7]Along this line, we mention that in another setup Feenstra [1986] shows that money in the utility function can be derived from some transaction costs models. However, both are reduced form models and there is not derivation from a more fundamental setting.

[8]For another model in a similar vein see Kiyotaki-Wright [1990].

wide specialization. In this setup each individual likes only one commodity from a continuum. As a producer, each individual is endowed with the possibility of producing "commodity options", which are represented by arcs in a circle. The length of the arc is a choice variable but its location is stochastic. Longer arcs are more costly to produce. On the other hand, it is more likely that the trading partner's most preferred commodity is included in a longer interval. The narrower the option produced, the more specialized the agent is. It is shown that narrower commodity options are produced in a monetary economy.

The structure of the paper is as follows: Section 2 describes the environment of the main model, Sections 3, 4, 6 analyze autarky, barter and a simple monetary economy, Section 5 studies the welfare costs of barter, Section 7 introduces a generalized monetary model allowing agents to accumulate higher cash balances, Section 8 takes up another notion of specialization, Section 9 includes all the proofs and some supporting lemmae, and finally an Appendix includes several formulae and algebraic derivations.

2 The environment

The commodity space of this model is a continuum of indivisible commodities. It will be useful to think of them as points in a circle C with circumference $2(M/(M-1))$, where M is an integer strictly greater than two. In addition there is a continuum of infinitely-lived agents with a normalized mass of one. Individuals are indexed by $\gamma \in \Gamma$. Lastly, the model is framed in continuous time and the environment is assumed to be stationary.

In order to define consumer preferences, consider the circle C divided into M connected closed arcs of equal length. Arcs are denoted by A_i with $i = 1, 2, \ldots, M$. Each agent derives an instantaneous and constant utility $u > 0$ from the consumption of any commodity x belonging to only one arc A_i, and zero otherwise. Therefore all the points in A_i are perfect substitutes. We will often speak of the arc A_i as the "consumption type i". For sharper results, we will keep the number u invariant across types.

Given the consumption type i, the arc distance from the set A_i provides a natural parameterization of the points in the circle C. For each $x \in C$:

$$y = d(x, A_i) = \inf_{z_i \in A_i} |x - A_i|.$$

Notice that the circumference set at the beginning implies $0 \leq y \leq 1$. Futher, we have $y = 0$ for all the commodities in his or her consumption type ($x \in A_i$). From now on we will refer to the commodity x as commodity y. Clearly, there is not a unique parameterization since it is conditional on the consumption type. Figure 1 depicts the commodity space with the parameterization induced by the arc distance.

As for the production side, every individual γ can choose to produce any commodity in the circle according to an instantaneous continuous V-shaped effort function $e_\gamma : [0, 1] \rightarrow R_{++}$. The function e_γ is defined on the parameterized commodity space $[0,1]$ rather than on the original C. Thus, to produce one unit of commodity y, agent
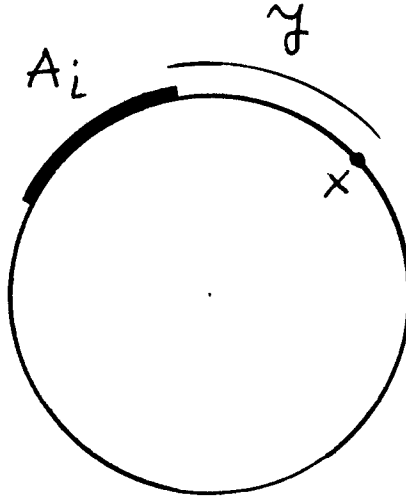
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Figure 1: Commodity space.

γ incurs a disutility cost of $e_\gamma(y)$. Continuity of e_γ on $[0,1]$ implies the existence of an effort-minimizing commodity y_γ, that is, $e_\gamma(y_\gamma) = \min_{y \in [0,1]} e_\gamma(y)$. It is assumed that y_γ is unique.

We shall analyze a sub-class of economies in which the set of effort functions $\{e_\gamma : \gamma \in \Gamma\}$ satisfy further restrictions:

1. If $y_\gamma = y_{\gamma'}$, then $e_\gamma = e_{\gamma'}$.

2. If $y_\gamma < y_{\gamma'}$, then $e_\gamma(y_\gamma) > e_{\gamma'}(y_{\gamma'})$ and $e_\gamma(0) < e_{\gamma'}(0)$.

3. $\{Y \in [0,1] : e_\gamma(Y) = \min e_\gamma(y) \text{ for some } \gamma\} = [0,1]$.

In words, the first property establishes that two individuals whose effort- minimizing commodities are at the same distance from their consumption type sets A_{i_γ} and $A_{i_{\gamma'}}$ are indistinguishable from a productive point of view. Thus we can partition the set of agents Γ in the following equivalence classes:

$$\Gamma_Y = \{\gamma : y_\gamma = Y\}.$$

The second property says that the further away is the effort-minimizing commodity from the set of more preferred ones (commodities zero), the less advantageous autarky is. In fact, the statement implies that the relative costs of producing the effort-minimizing commodity relative to the commodities in his consumption type set, $e_Y(0)/e_Y(Y)$, increases in Y.

By the third condition the set of equivalence classes is the closed interval $[0,1]$. Now it is more natural to index effort functions by equivalence class Y. In particular, $e_Y = e_\gamma$ with $\gamma \in \Gamma_Y$. In the rest of the paper we will speak of Y as the "productive type Y". Hence, the set of types Y is the interval $[0,1]$.

An agent's characteristics are summarized by the pair (A_i, Y), where the first component stands for the consumption type and the second one denotes the production type.
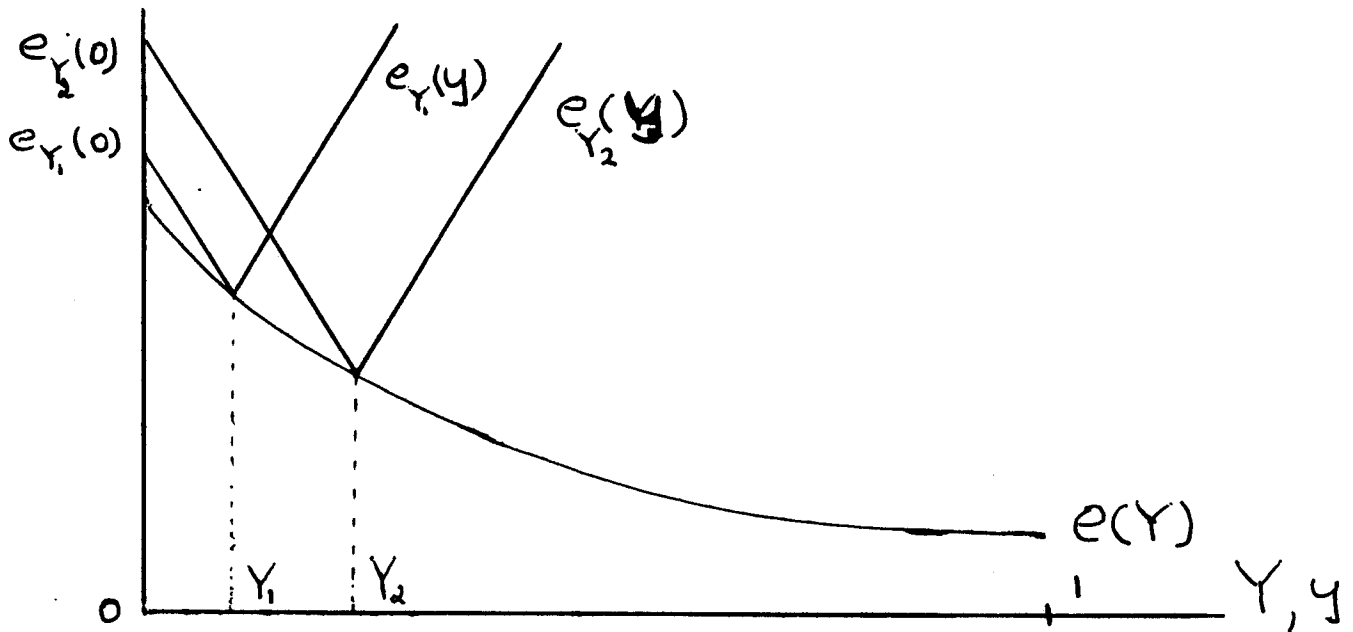
Figure 2: Economy-wide effort function e and individual effort functions e_Y.

Definition 1 *Let SC denote the space of characteristics:*

$$SC = \{(A_i, Y) : i = 1, 2, \ldots, M; Y \in [0, 1]\},$$

and $\mathcal{T}$ the mapping from individuals to characteristics:

$$\mathcal{T} : \Gamma \to SC.$$

Assumption 1 *The space of characteristics $\{(A_i, Y) : i = 1, 2, \ldots, M; Y \in [0, 1]\}$ induces a uniform distribution on the population Γ.*

An immediate implication of the assumption is that consumption types are independent of production types.

For our purposes, *Y will be the relevant type for this model.* On these grounds it will be convenient to introduce a construction, called the economy-wide effort function, which summarizes the collection of individual effort functions $\{e_Y : Y \in [0, 1]\}$.

Definition 2 *The economy-wide effort function is a mapping e from [0,1] to the strictly positive reals R_{++} such that $Y \mapsto e_Y(Y)$.*

The function e is a sort of envelope of the functions e_Y's. Figure 2 illustrates this relation.

Each individual has to make a decision as to whether to produce for self-consumption ($y = 0$) or for the market ($y > 0$). Generally speaking, that decision will be contingent on the type Y. Thus, the economy will have "autarkic" types choosing $y = 0$ and "market" types setting $y > 0$. A market type chooses a particular y on the basis of his or her effort function alone. Hence it will be an specialized type. The *fraction* of market types in the population will tell us *the extent to which the economy is specialized.* The aim of this paper is to pin down that distribution.

We will resort to the often-used island methaphor to describe the economy. In order to produce, agents must spend time and effort on the "production island" (PI). In addition, those producing for sale must search for trading partners and trade on the "trading island" (TI).
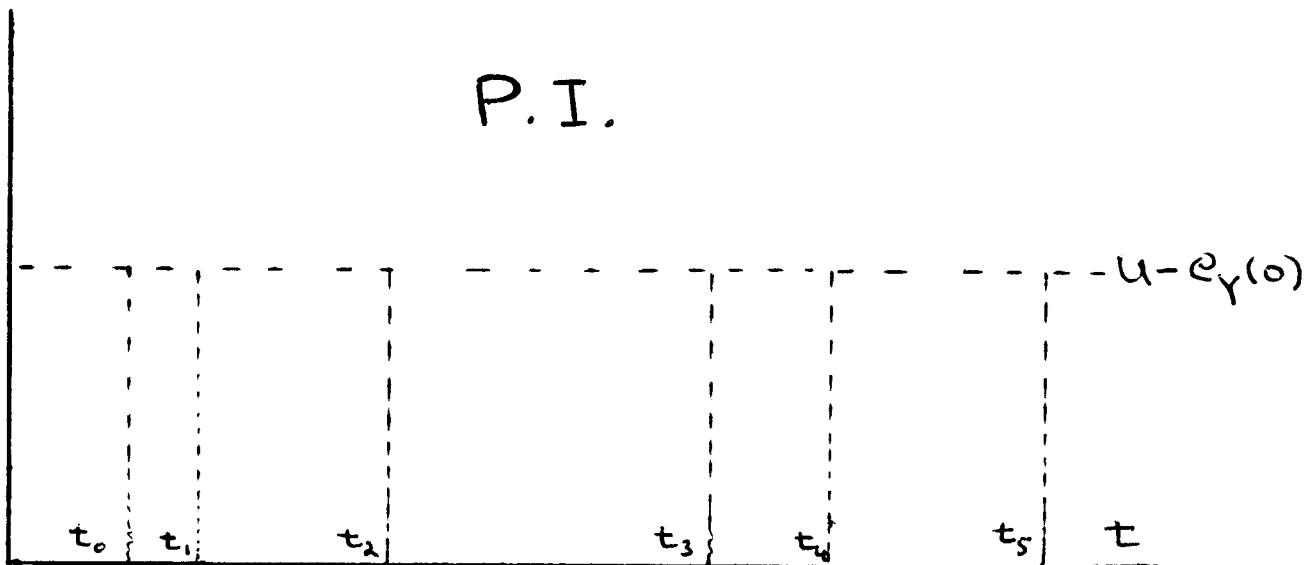
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Figure 3: Timing for an autarkic type.

The production process is extremely simple. Suppose that an agent enters the PI with no inventory of commodities at time t_0. From then he will wait a random period t so that he can apply effort $e_Y(y)$ to produce one unit of commodity y at time $t_0 + t$. After production, commodities are immediately available either for self-consumption or sale. The level of commodity inventory is either 0 or 1. The random waiting period t follows an exponential distribution with parameter α. In case of production for self-consumption, the individual will consume it right away and restart the process without leaving the PI. If the output is channeled to the market, he will go the the TI and remain there until he sells it. After that, he goes back to the PI to replenish his depleted inventory.

Therefore, an autarkic type Y will produce and consume at random times $t_0, t_1, t_2, \ldots$ and obtain an instantaneous utility of $u - e_Y(0)$. Since the waiting times follow an exponential distribution with parameter α, the random number of points t_j in fixed time interval of length h follows a Poisson process with parameter α. Figure 3 illustrate the timing of activities for an autarkic type.

To give a similar illustration for a market type we have to provide more details about trading.

Trade is carried out in a decentralized fashion through bilateral deals. Thus, when we speak of "markets", we mean a flow of bilateral transactions. When trading, agents are assumed to interact in a non-cooperative fashion in an environment in which they are paired at random. The details of the strategic interaction depend upon the trading regimes, either barter or monetary, and are spelled out below.

The costs of trade are partially costs of search. The latter is costly because it is time-consuming and individuals discount the future. Moreover, following Kiyotaki and Wright, it is assumed that traders incur a lump sum transaction cost ϵ whenever they accept a good in exchange. To illustrate, suppose that an individual exchanges goods with a sequence of n people *before* he gets one of his most preferred commodities from the $n + 1$ trader. Roughly speaking, his net utility from consumption will be $u - (n + 1)\epsilon$. Alternatively, if the individual withholds his original inventory until he

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meets the $n + 1$ trader, his payoff will be $u_\epsilon = u - \epsilon$. This strongly suggests that indirect trading cannot be an equilibrium strategy. In other words, ϵ is a transaction cost which discourages individuals from accepting commodities for indirect trading. If there were no transaction costs ($\epsilon = 0$), every commodity could be used as a means of exchange and therefore the model would not be useful for analyzing monetary exchange.

The paper sticks to the following search story: after producing one unit of commodity Y, the typical trader holds it as inventory until he meets a partner with whom he can close a transaction. Meanwhile, he engages in a time-consuming search for trading opportunities, which arrive at random. In order to make this precise, we draw from Kiyotaki-Wright's specification of Diamond's matching technology. In this setup, potential trading partners arrive according to a simple Poisson process with parameter $B(N_s)$, where N_s is the fraction of market types engaged in search for a match at a point of time, that is, the ones in the TI at time t. The function B is given by:

$$B(N_s) = \begin{cases} \beta & \text{if } N_s > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\beta > 0$.[9]

With these elements we can describe the timing for a market type. After he produces one unit of output in the PI, he carries his inventory to the TI and remains there until he depletes it as a result of a deal. This is another random waiting period following an exponential distribution with a parameter g which depends on the rate of arrival of trading opportunities β, on the number of consumption types M, and in a monetary economy, on the distribution of real cash balances. After leaving the TI, he returns to the PI. Figure 4 represents the timing for a market type.

Notice that, while autarkic types live on the PI for good, market types visit both islands infinitely often. Therefore, while only market types can be on the TI, both autarkic and market types can be on the PI.

After setting up the environment, we can define an economy:

Definition 3 *An economy $\mathcal{E}$ is a mapping from population to characteristics $\mathcal{T}$, a collection of effort functions $E = \{e_Y : Y \in [0,1]\}$, and a vector of positive numbers $(\alpha, \beta, \epsilon, u)$:*

$$\mathcal{E} = [\mathcal{T}, E, (\alpha, \beta, u, \epsilon)].$$

3   The case of autarky

The primary purpose of this section is to compute the expected discounted utility conditional on type Y in an economy without trading possibilities. In particular, it

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[9]In the search literature, the functional dependence between B and N_s is usually referred to as "search externality". The specification given virtually ignores the latter. Kiyotaki and Wright justify the assumption saying that it " ...reduces but does not eliminate the multiplicity of equilibria that can arise with such externalities." On more substantive grounds, the assumption chosen is a sort of "constant returns to people". In any case, stronger search externalities are likely to reinforce our results.
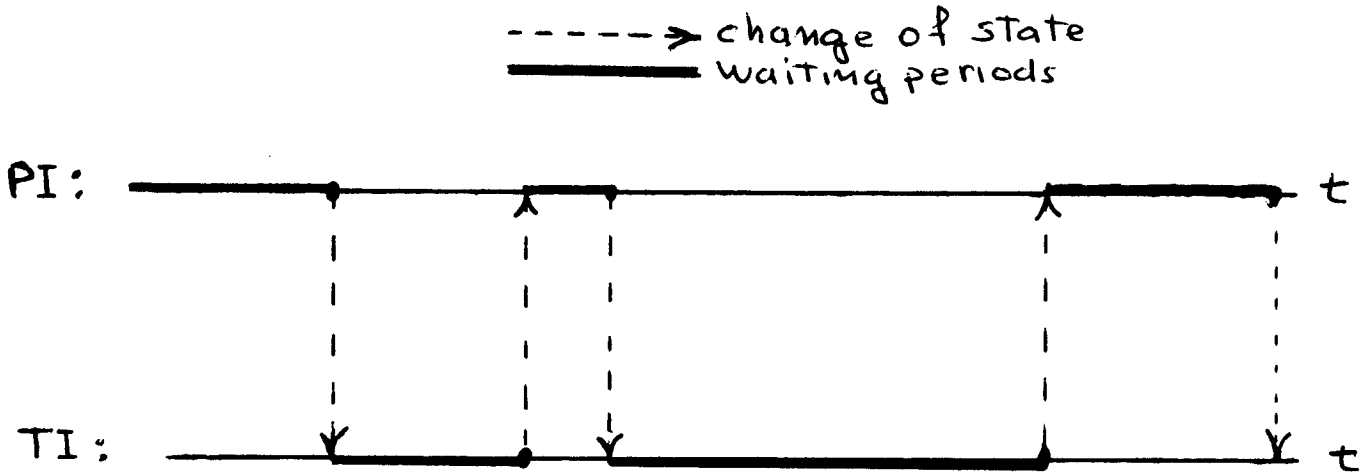
Figure 4: Timing for a market type.

is shown that the production of commodities in the agent's consumption set is utility maximizing and therefore the degree of specialization is zero. Also, this will be an illustration of some limiting arguments used to obtain relations in continuous time from relations framed in discrete time. These arguments will be taken for granted throughout the rest of the paper.

Autarky is the benchmark for the analysis of this paper. Every type can guarantee the autarkic payoff by producing and consuming his or her commodity zero.

It will be useful to state an assumption to ensure that the expected discounted utility of autarky is set at a positive level:

Assumption 2 *The instantaneous utility of producing and consuming a commodity in his or her consumption set is strictly positive. More formally, for all types Y : $u - e_Y(0) > 0$.*

Let I_0 be the indicator of the number 0 in the real line, that is, $I_0(0) = 1$, and $I_0(y) = 0$ otherwise. Further, the lifetime expected discounted utility under autarky for type Y will be denoted by $V_a(Y)$.

Keeping in the background our description of the timing under autarky, we will write a recursive equation which is an analytical version of Figure 3. Consider the time axis divided in a sequence of discrete intervals of length h. The individual has the choice of producing one unit of commodity y in a period of length h with a probability αh. The value of the option can be written as:

$$\max_y \{u I_0(y) - e_Y(y) + V_a(Y), V_a(Y)\}.$$

The first component inside the curly brackets is utility from exercising the option, which is the sum of the instantaneous utility $u I_0(y) - e_Y(y)$ and the value of repeating the process in the future $V_a(Y)$. The second component reflects the value of giving up the choice now and waiting for a chance of repeating the process some time later.

With probability $1 - \alpha h$, the option of producing a commodity is not available, and hence the individual will wait for future chances, the value of which is again $V_a(Y)$.

Since $V_a(Y)$ is the expected discounted value of future payoff, we can write:

$$V_a(Y) = e^{-rh}[\alpha h \max\{uI_0(y) - e_Y(y) + V_a(Y), V_a(Y)\} + (1 - \alpha h)V_a(Y)] + o(h)$$

where r is the discount rate and $o(h)$ is a residual term which collects the effects on utility of more than one unit consumed and produced in a period of length h, which happens with a small probability. The usual definition of the function $o(h)$ in elementary real analysis is given by the limit:

$$\lim_{h \to 0} o(h)/h = 0.$$

Rearranging terms we obtain:

$$V_a(Y)(1 - e^{-rh})/h = \alpha \max_y\{uI_0(y) - e_Y(y), 0\} + o(h)/h.$$

Taking limits when $h \to 0$, we get the continuous time version:

$$rV_a(Y) = \alpha \max_y\{uI_0(y) - e_Y(y), 0\}.$$

It is immediate that the solution of this maximization problem is $y = 0$, namely, with no trading possibilities, people choose to produce commodities in his or her consumption set. Hence, in more explicit terms we have:

$$rV_a(Y) = \alpha(u - e_Y(0)).$$

4 Trading arrangements: The barter case

Now we modify the environment to include barter trading possibilities. This opens the gates to specialization since individuals can produce for self-consumption or for the market. In either case, the timing of the production activities is determined by the previously defined exogenous Poisson rate α.

A barter transaction requires that the trader make a deal with another who wants what he has and and has what he wants. This condition is the well-known "double coincidence of wants".

An agent on the PI (TI) is said to be in state "p" ("t"). Let $V_p(Y)$ and $V_t(Y)$ be the expected discounted utilities for type Y under those states. These values satisfy a system of two recursive equations, one of which reflects the choices open on the PI and the other in the TI.

Let us take up the optimal production choice. Consider the discrete time paradigm first. In an interval of length h the individual faces with probability αh the option of producing commodity 0 for self-consumption, which yields a payoff of $u - e_Y(0) + V_p(Y)$, or any other commodity $y > 0$ for sale yielding $V_t(Y) - e_Y(y)$. With probability $(1 - \alpha h)$ no option is available and he remains in state p with utility $V_p(Y)$. Putting these two pieces together we get the following Bellman equation:

$$V_p(Y) = e^{-rh}[\alpha h \max_y\{u - e_Y(0) + V_p(Y), V_t(Y) - e_Y(y)\} + (1 - \alpha h)V_p(Y)] + o(h)$$

which is an expanded version of the equation used to calculate the utility under autarky.[10]

Remark. Notice that the optimal choice for y is Y, namely, the effort-minimizing commodity. Therefore, *the individual will specialize completely should he produce for sale.* Using the economy-wide effort function, function e defined as $e(Y) = e_Y(Y)$, and taking limits when $h \to 0$, we obtain the continuous time version:

$$rV_p(Y) = \alpha \max\{u - e_Y(0), V_t(Y) - V_p(Y) - e(Y)\}.$$

The individual will produce his effort-minimizing commodity Y and go to the market if

$$u - e_Y(0) < V_t(Y) - V_p(Y) - e(Y),$$

and commodity zero and remain under autarky otherwise. This inequality says that production for sale requires that the payoff from producing commodity Y (RHS) be larger than the instantaneous autarkic utility (LHS). On these grounds we can state an important definition:

Definition 4 *The set of market types under barter is given by:*

$$MT_b = \{Y : u - e_Y(0) < V_t(Y) - V_p(Y) - e(Y)\}.$$

The set of autarkic types is the complement of MT_b in [0,1].

Notice that, if the environment makes trading difficult, i.e., the rate of arrival of trading partners β is very small, MT_b can be empty. On the other hand, MT_b can be the whole economy should trade be very easy (large β). In general we shall focus on the intermidiate outcomes in which MT_b is proper subset of the interval [0,1].

Turning our attention to trading activities, we will determine the expected discounted value of being a trader under barter $V_t(Y)$. We defined β as the exogenous rate of arrival of trading opportunities. However, meeting a partner does not mean that a trade will take place. To close a deal under barter requires that the other party accept what he has. Since people commit themselves to production choices and search before knowing the tastes of the future potential partners, they need to hold some beliefs about the other partner's willingness to trade. We define θ as the belief about the probability that his match will accept in exchange the commodity held in inventory. In game theoretic terminology, θ's are the "Bayesian" beliefs of this game of incomplete information.

We now can set up the representative trader's optimal search problem. Let $V_t(Y)$ be the expected discounted lifetime utility of a trader producing commodity Y. As above, consider the discrete time paradigm first. In a short interval h, with probability $1 - Bh\theta$, the agent does not meet a partner willing to make a deal; therefore he keeps on searching with an optimal return of $V_t(Y)$. Alternatively, he meets a trader willing

---

[10]Here we do not give the choice of inaction, i.e., the alternative of not producing when an opportunity is at hand, as in the autarky case. However, we made enough assumptions on the range of parameters so as to ensure that individuals will produce when the possibility arises. Thus, in this framework, inaction is a dominated strategy.

to make a exchange, that is, willing to accept what he has, with probability $Bh\theta$ and an expected return of

$$(1/M)\max\{V_t(Y), u_\epsilon + V_p(Y)\} + ((M-1)/M)V_t(Y),$$

where $u_\epsilon = u - \epsilon$. We interpret this return as follows: since people are paired at random, the type of commodity held in inventory by his partner can be considered a random drawing from a uniform distribution. Therefore, with a probability $1/M$ that commodity will belong to his consumption type, and with probability $(M-1)/M$ it will not. In the first case he has the choice of accepting that commodity and thereby closing the deal. Otherwise, there is only a single coincidence of wants and therefore he will keep on searching.

As a result, $V_t(Y)$ satisfies the Bellman equation:

$$V_t(Y) = e^{-rh}\{(1 - Bh\theta)V_t(Y) + Bh\theta[(1/M)\max\{V_t(Y), u_\epsilon + V_p(Y)\} + ((M-1)/M)V_t(Y)]\} + o(h).$$

In a similar fashion, we can obtain the continuous time version:

$$rV_t(Y) = B\theta(1/M)\max\{0, u_\epsilon + V_p(Y) - V_t(Y)\}.$$

Since θ was defined as the trader's belief about the probability that his trading partner will accept his inventory, and people accept commodities with a probability $1/M$, consistency of beliefs with equilibrium behavior requires $\theta = 1/M$.

We will concentrate on equilibria with a positive fraction of searchers in the TI, that is, $N_s > 0$. Hence from now on we set $B(N_s) = \beta$.[11] Next we compute the positive steady state equilibrium for N_s. The law of motion for the fraction of searchers N_s is given by

$$\dot{N}_s = \alpha(1 - N_s) - (\beta/M^2)N_s.$$

This says that the population in the TI increases by an inflow proportional to the mass of market type people in the PI, $1 - N_s$, but it decreases by an outflow proportional to its own population N_s. The constant α gives the rate at which people complete the production process and go to the TI. The parameter β/M^2 tells us the rate at which double coincidences take place, and then the flow of individuals leaving the TI to replenish their inventories. In steady state we have $\dot{N}_s = 0$ and then

$$N_s = \frac{\alpha M^2}{\alpha M^2 + \beta}$$

In Section 8 we show that, for any market type Y, the expected discounted utility from trading $V_t(Y)$ is strictly positive, provided that the fraction of searchers out of MT_b, N_s, is also strictly positive (Lemma 1). Therefore, the value of being a trader can be written as

$$rV_t(Y) = (\beta/M^2)(u_\epsilon + V_p(Y) - V_t(Y)).$$

---

[11]Notice that there is always a trivial equilibrium with $N_s = 0$. Hence, $B(0) = 0$ and the expected discounted utility of the trading state, $V_t(Y)$, is also 0 for all types Y. As a consequence, everybody will be in autarky.
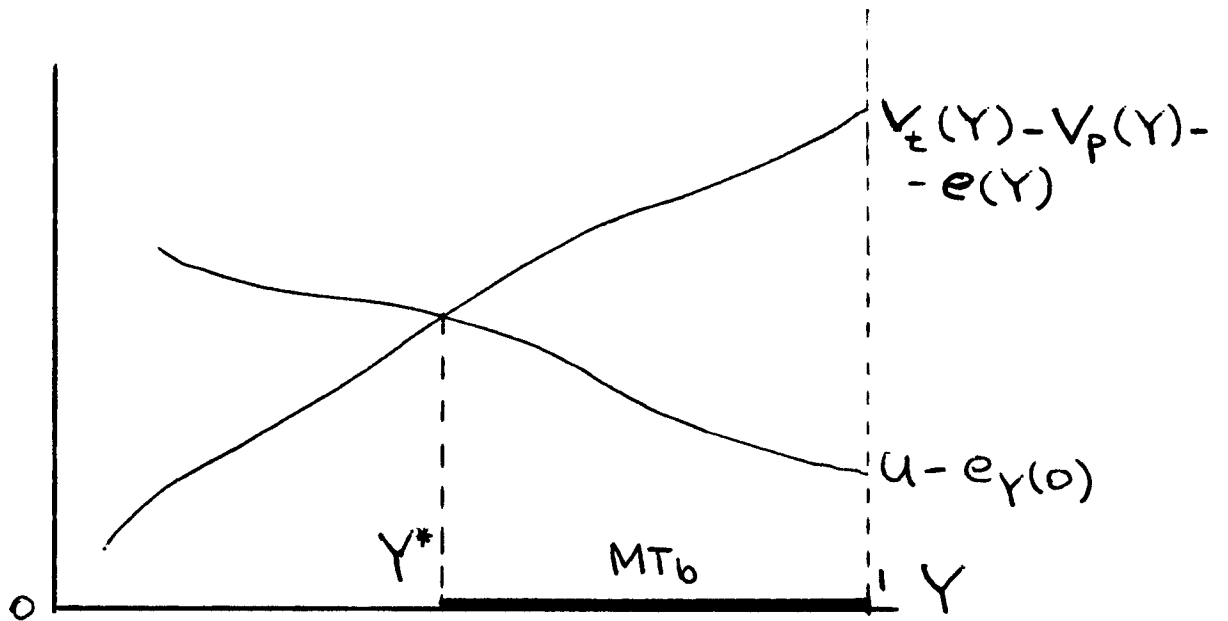
Figure 5: Economy-wide degree of specialization under barter.

After setting up the workings of the barter regime, we can determine how individuals are sorted out between market (specialized) and autarkic types. The *size* of the set MT_b tells us *the extent of the market and the degree of specialization of the economy.* The Lebesgue measure $\mu(MT_b)$ is a measure of the size of this set. Obviously,

$$0 \le \mu(MT_b) \le 1.$$

Now we can state the following proposition:

Proposition 1 *The extent of the market and the degree of specialization are given by:*

$$\mu(MT_b) = 1 - \inf(MT_b),$$

where $\inf(MT_b)$ *is the borderline between the autarkic and the market types, provided that* MT_b *is nonempty. Otherwise* $\mu(\emptyset) = 0$.

The proof is in Section 9. Figure 5 illutrates the Proposition. The autarkic payoff conditional on the type Y is the declining curve $u - e_Y(0)$, the utility of producing for sale is the increasing function of Y: $S_b(Y) = V_t(Y) - V_p(Y) - e(Y)$. Since these curves represent continuous functions, the borderline type $Y^* = \inf(MT_b)$ is at the intersection of both.[12] Market (specialized) types are to the right of Y^*, and the autarkic ones make up the remaining left side. Even though the case of incomplete economy-wide specialization, as depicted in the diagram, is more likely, the extreme instances $Y^* = 0$ or 1 are also meaningful. For example, a completely specialized economy is characterized by $Y^* = 0$, $\mu(MT_b) = 1$ and $S_b(Y) \ge u - e_Y(0)$ for all Y. If the borderline type is $Y^* = 1$, the degree of specialization is zero.[13]

---

[12]Continuity is not necessary for the analysis.

[13]If the set of MT_b is empty, then $\inf(MT_b)$ does not exist. In this case we follow the convention that the borderline type is 1.

5 The welfare cost of barter: A gift-giving arrangement

This section brings out the issue of incentives in the exchange process. More specifically, we set up arrangement for the distribution of commodities which assumes away incentive problems.

It is clear that barter is an extreme way of imposing *quid pro quo*. In a world of *honest* people, enforcement of budget constraints in trading through exchange of commodities of equal value would not be necessary. In this section we model an environment in which people are willing to make bilateral as well as unilateral transfers of commodities. In other words, we allow for barter, gift-taking and gift-giving. This will be our benchmark to appraise the cost of barter, which is basically the cost of *quid pro quo* as an incentive-compatible constraint.

This model parallels the barter one as much as possible. In particular, individuals have the choice of producing either for self-consumption or for exchange and gift-giving. As before, the decision will be contingent on the type Y. Those producing for self-consumption will be called autarkic types. It would be a terminological abuse to call market types those choosing to produce for exchange and gift-giving; so we call them *outward* types.

Autarkic types will remain on the PI for good, and outward types will switch between the PI (state P) and the TI (state T). Let $V_P(Y)$ and $V_T(Y)$ be the expected discounted life-time utilities corresponding to the mentioned states. Whenever an agent depletes his inventory by exchange or gift-giving, he must replenish by production. The Bellman equation of the production state is the same as before:

$$rV_P(Y) = \alpha \max\{u - e_Y(0), V_T(Y) - V_P(Y) - e(Y)\}.$$

Agents sort out themselves by solving this equation. As before, autarkic types produce commodity zero and outward types produce the effort-minimizing one. As a result, we have the following definition:

Definition 5 *The set of outward types is*

$$OT = \{Y : u - e_Y(0) < V_T(Y) - V_P(Y) - e(Y)\},$$

and the degree of specialization is given by the Lebesgue measure of OT, $\mu(OT)$.

The meaning of "trader" is trickier here. A trader, in this context, holds an inventory of his effort-minimizing commodity that he surrenders to his partner should the latter so desire. Since we still keep the lump-sum disutility cost ϵ, traders request commodities in their consumption types. When requesting, traders can be gift-takers should the commodity they possess not be in the partner's consumption type.

The equation for the state T does not involve any maximization. It is only a recursive relation to compute $V_T(Y)$. However, $V_T(Y)$ can be interpreted as either the result of the choice of honest people, or the outcome of a command economy in which the planner has perfect and complete information. To spell out the details

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more fully we give the equation in discrete time:

$$
\begin{aligned}
V_T(Y) \;=\; & e^{-rh}\{(1 - \beta h((1 + 2(M-1))/M^2))V_T(Y) + \\
& \beta h(M-1)M^{-2}V_P(Y) + \beta h M^{-2}(u_\epsilon + V_P(Y)) + \\
& \beta h(M-1)M^{-2}(u_\epsilon + V_T(Y))\} + o(h).
\end{aligned}
$$

The first term between curly brackets reflects the event of either no match or no coincidence of wants, the second one, gift-giving, the third one, barter and the last one, gift-taking. The continuous time version is given by

$$
rV_T(Y) = (\beta/M)(u_\epsilon + V_P(Y) - V_T(Y)).
$$

In a similar fashion we can determine the steady state proportion of searchers:

$$
N_S = \frac{\alpha M}{\alpha M + \beta}.
$$

Notice that this fraction is smaller than that of barter. This makes perfect sense, for people spend, on average, less time "trading".

The pair of propositions which follow are the main conclusions of this section:

Proposition 2 *The gift-giving arrangement is ex ante Pareto superior to barter, and the welfare costs of barter can be measured by:*

$$
1 - \frac{\beta + \alpha M}{\beta + \alpha M^2}.
$$

Notice that the larger the number of consumption types M, the higher the welfare losses of barter.

Proposition 3 *The gift-giving arrangement is statewise Pareto superior to barter:*

$$
V_T(Y) > V_t(Y) \; and \; V_P(Y) > V_p(Y) \; for \; all \; Y \in OT.
$$

Proposition 4 *The degree of specialization under the gift-giving arrangement is higher than under barter. Furthermore, if the set of outward types is non-empty, the degree of specialization under gift-giving can be written as*

$$
\mu(OT) = 1 - \inf(OT).
$$

The proof, as usual, is in Section 9. We illustrate the proposition in Figure 6. The autarky option has the same payoff in both regimes, $u - e_Y(0)$. The return of producing for exchange and gift-giving,

$$
V_T(Y) - V_P(Y) - e(Y),
$$

is increasing in Y and is higher than the return of producing for sale under barter,

$$
V_t(Y) - V_p(Y) - e(Y).
$$

The borderline type under barter, Y_b, is to the right of that of the gift-giving regime, Y_g. From the diagram it is clear that the set of outward types is larger than the set of market types under barter.
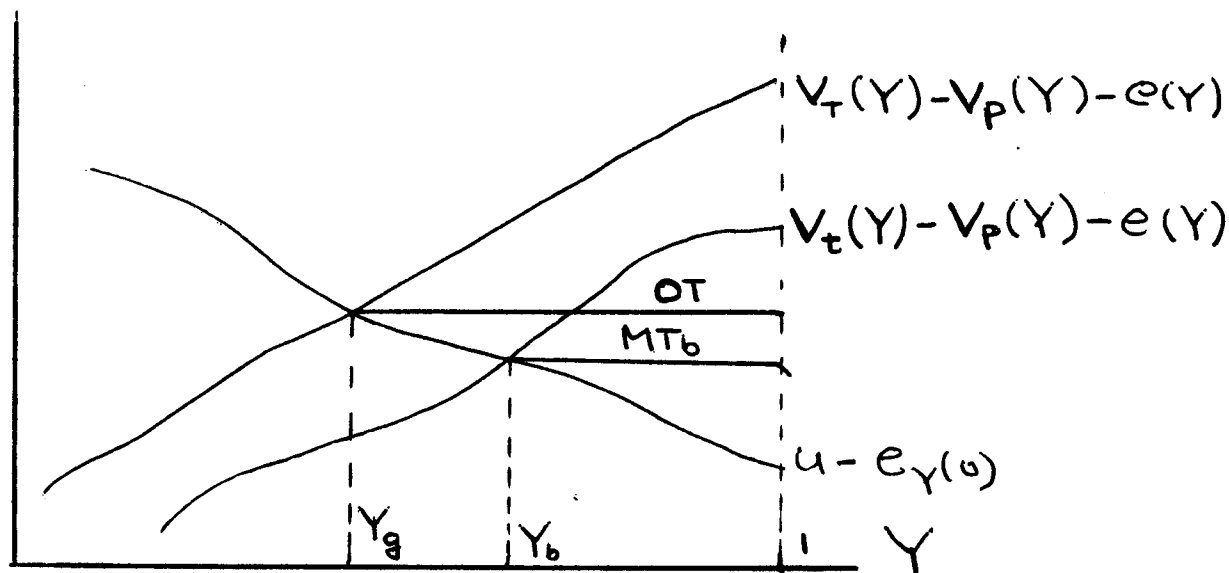
Figure 6: Specialization under barter and gift-giving.

6 Trading arrangements: The fiat money case

The monetary economy of this section is a step forward to a softer *quid pro quo* than that imposed by barter, for single coincidence of wants are enough to close transactions.

Under this trading regime, individuals have the possibility of holding one unit of a "worthless" object which will play the role of medium of exchange. In this context, "worthless" means that it has neither consumption nor production value. Thus this section deals with "fiat" money. To put money and commodities on the same footing, it is assumed that money bears the same transaction cost ϵ as any commodity. In other words, the model does not give an "unfair" advantage to money over commodities.

In principle, any trader can hold either an inventory of one commodity or the combination of a commodity and fiat money. We call these two states "c" and "cm" respectively. Sometimes we will speak of these states as types c and cm. People of these types are found on the TI.

A monetary transaction can only take place when agents in different states meet. In this section, we do not allow for accumulation of more than one unit of fiat money. As a consequence, there cannot be monetary transactions when two people of type cm meet. Individuals in the same state can only barter, while two agents in different states can barter or engage in monetary exchange. We have to expand the notation a bit:

- w_c: Proportion of people on the TI who are *currently* holding only an inventory of commodities, that is, in state c.

- θ_M: Type cm's belief about type c's probability of accepting money in exchange for a commodity.[14]

---

[14]It is important to keep clear the distinction between the fraction of individuals currently holding money on the TI, $1 - w_c$, and the proportion of market types willing to accept money in exchange for their commodities, which, in equilibrium, will be equal to the belief θ_M. Of course, an agent

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In a monetary equilibrium, money is accepted with non-zero probability, that is $\theta_M > 0$. Under barter, $\theta_M = 0$.

A critical assumption of the model is that each individual is forced to replenish his or her inventory of the commodity produced whenever it is given up in exchange for money or for any commodity in the agent's consumption type.[15] As a result, *every agent will always carry an inventory of commodities to the TI*. One way of thinking about this is the fact that, in some sense, we always hold our endowment of labor.[16] Thus, when two individuals meet, there will *always* be room for barter regardless of their trading state. When two agents in different states meet, monetary exchange opens the gates to additional transactions (purchases and sales). If we dropped the assumption, a trader in state cm could end up as a pure money holder should he sell his commodity. And between two pure money holders neither barter nor any other transaction is possible.[17] Hence, money expands the types of possible deals without undermining barter transactions.

In contrast with a recent strand of monetary theory literature, e.g. Lucas [1980] and Diamond [1984], we do not impose Clower-contraints to make money valuable in the exchange process. Also, the way in which Kiyotaki and Wright [1988] introduce money undercuts some barter possibilities, for their model allow for pure money holders.

Some would question the possibility of having monetary exchange along with barter. However, historical evidence suggests that the presence of barter and monetary exchange in the same economies is far from uncommon. [18]

In addition to the trading states c and cm, there will be two production states: "pc" and "pm". As before, individuals engage in production activities on the PI. If, after a transaction, the agent holds neither a commodity nor money, then he will switch to state pc, but, if he holds money, he will switch to pm. In Figure 7 we represent a flow diagram among states for an agent who engages in both barter and monetary transactions. Obviously, a pure barter type will only switch between states pc and c.

The optimal values of the expected discounted utilities $V_{pc}(Y)$, $V_{pm}(Y)$, $V_c(Y)$ and $V_{cm}(Y)$ satisfy the following system of Bellman equations in continuous time:

$$
\begin{aligned}
rV_{pc}(Y) &= \alpha \max\{u - e_Y(0), V_c(Y) - V_{pc}(Y) - e(Y)\} \\
rV_{pm}(Y) &= \alpha \max\{u - e_Y(0), V_{cm}(Y) - V_{pm}(Y) - e(Y)\} \\
rV_c(Y) &= \beta M^{-2} \max\{0, u_\epsilon + V_{pc}(Y) - V_c(Y)\} + \\
&\quad \beta(1 - w_c)(M - 1)M^{-2} \max\{0, V_{pm}(Y) - V_c(Y) - \epsilon\}
\end{aligned}
$$

currently holding money belongs to a type Y willing to accept it. However, there are agents willing to accept money who are not currently holding money on the TI. The latter individuals are currently TI holding only his or her commodity. This possibility arises when an individual in state cm bought a commodity but was not able to sell his so far.

[15] In a pure barter regime this the natural assumption to make, for commodities are the only purchasing power and we allow for an storage capacity of one unit.

[16] I thank Prof. Oh for this remark.

[17] After this discussion is clear that the critical assumption is built in the very definition of the trading states c and cm.

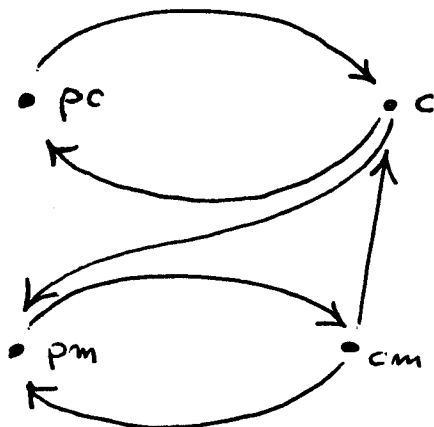[18] See F. Braudel [1972, 1981, 1982] and C.Cipolla [1967].

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Figure 7: State flow diagram.

$$rV_{cm}(Y) = \beta M^{-2} \max\{0, u_\epsilon + V_{pm}(Y) - V_{cm}(Y)\} +$$
$$\theta_M \beta w_c (M-1) M^{-2} \max\{0, u_\epsilon + V_c(Y) - V_{cm}(Y)\}.$$

The equations in discrete time are written out in Appendix B. Notice that the average flow utilities of the trading states, $rV_c(Y)$ and $rV_{cm}(Y)$, are the sum of the payoff from bartering and and the payoff from a monetary transaction (sales for state c and purchases for cm).

Definition 6 *The set of market types for a monetary economy*[19]*:*

$$MT_m = \{Y : \alpha(u - e_Y(0)) < rV_{pc}(Y)\}.$$

Type Y will choose to accept money in exchange for his good if

$$V_{pm}(Y) - V_c(Y) - \epsilon > 0.$$

Definition 7 *The set of market types willing to accept money in exchange for commodities is as follows:*

$$MYT = \{Y \in MT_m : V_{pm}(Y) - V_c(Y) - \epsilon > 0\}.$$

We will speak loosely of these types as "money types". From the definition it is clear that $MYT \subset MT_m$ and therefore *there may be market types which will exclusively engage in barter*. The latter will be referred as "pure barter types".

An interesting property of this model is that the individual gains from monetary exchange are increasing in the productivity of the agent. More formally,

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[19]This definition leaves out the set

$$\{Y : \alpha(u - e_Y(0)) = rV_{pc}(Y) < rV_{pm}(Y)\}.$$

In other words, these types go to the market provided that, at the outset, they were endowed with money. Nevertheless, they will be transient traders, namely, until they "hit" the state pc, which is an absorbing barrier. Since we consider steady states only, we can disregard them.

Proposition 5 *The utility gains from accepting money in exchange for goods,*

$$V_{pm}(Y) - V_c(Y) - \epsilon,$$

is increasing in Y.

As a result, money types are the most productive types in the economy. The reason for this is that people engaged in monetary exchange will give up and replenish their inventory more often than the pure barter types, for the money types close transactions under double as well as single coincidence of wants. This means more frequent trips to the PI, *which are less costly for the relatively more efficient types.*

For the belief θ_M to be consistent with equilibrium we need

$$\theta_M = \mu(MYT)/\mu(MT).$$

In a monetary equilibrium we have $\mu(MYT) > 0$.

In this economy everybody can guarantee the autarkic payoff. The theorem which follows makes a stronger statement: every active trader can guarantee the barter payoff.

Proposition 6 *Monetary exchange is ex ante Pareto superior to barter. If* $Y \in MT_m$, *then* Y's *average utility is bounded below by the barter payoff:*

$$\frac{\alpha\beta(u_\epsilon - e(Y))}{\beta + \alpha M^2}.$$

Proposition 7 *Monetary exchange is statewise Pareto superior to barter:*

$$V_i(Y) \leq \min\{V_c(Y), V_{cm}(Y)\} \text{ and } V_p(Y) \leq \min\{V_{pc}(Y), V_{pm}(Y)\}.$$

We are now prepared to make our first statement, in a weak sense, of the main conclusion of this model: the degree of specialization and extent of the market never decreases when introducing money. More formally,

Proposition 8 *The set of market types under the monetary economy is at least as large as that under barter, namely,* $MT_b \subseteq MT_m$.

On intuitive grounds, an individual already in the market in a barter economy looses nothing if he participates in an economy with money traders.

A necessary condition for an strict increase in the degree of specialization is that every market type is willing to engage in monetary exchange. More formally:

Proposition 9 *If* $MT_b \subset MT_m$ *and* $MT_b \neq MT_m$, *then* $MYT = MT_m$.

The barter equilibria disscused in Section 4 partition the population between autarkic and market types. When we introduce money, two exchange profiles are possible. The first one has every market willing to engage in monetary transactions, namely, the set of market types coincides with the set of money types. So the necessary condition holds. The second one partitions the set of market types between money and pure barter types. Thus, the last proposition does not hold.

Figure 8, which is analogous to Figure 6, depicts the case in which specialization increases and therefore the necessary condition is met. The set of specialized types
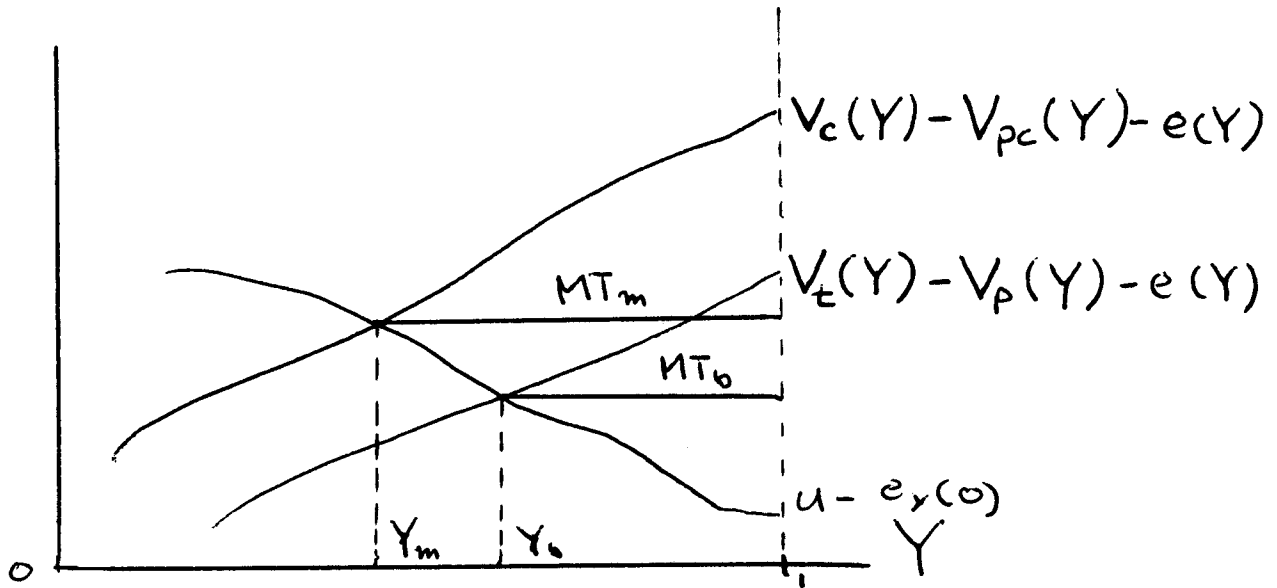
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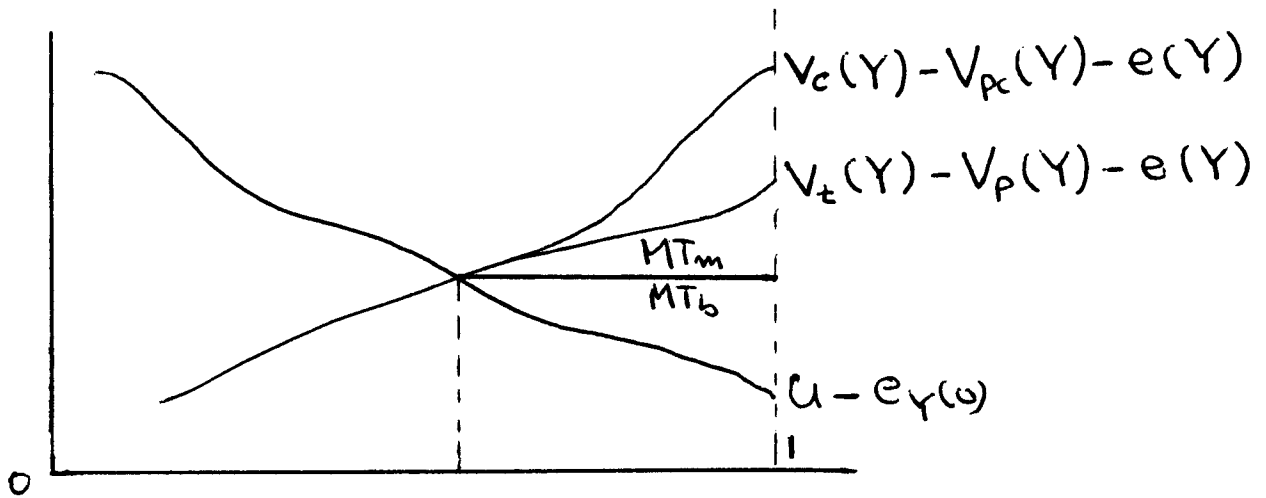Figure 8: A) Specialization under barter and monetary exchange.

Figure 9: B) Specialization under barter and monetary exchange.

under barter (monetary exchange) are the ones to the right of Y_b (Y_m). Hence we must have $Y_b > Y_m$.

An important insight of this model is that an increase in specialization requires that the benefits of monetary exchange *reach* the relatively most inefficient types. Otherwise, money is just an intramarginal advantage for the most productive and specialization is unchanged. This is the case in which the necessary condition is not satisfied and it is illustrated in Figure 9.

Notice that the set of money types MYT is a proper subset of the set of market types MT_m. Thus, this model can have equilibria displaying autarkic, pure barter and money types all at the same time, which is consistent with the work of economic historians and evidence from development economics.

Remark. Our result about specialization is closely related to the Pareto superiority of monetary exchange. Kiyotaki and Wright's monetary trading model is not always more efficient than barter and they show an example to illustrate the point.

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They do not allow agents to carry both commodity and money, and therefore barter is to some extent restricted. In particular, if two money holders meet, which happens with a positive probability, no transaction can take place. Kiyotaki and Wright implicitly introduce a weak Clower-type constraint preventing some trades and causing eventual inefficiencies.

7 A generalized monetary model

Here we discuss briefly an extension of the monetary model which allows for accumulation of more than one unit of fiat money. To be concrete, suppose that people can accumulate up to $k > 1$ units, and the set of traders who find it rational to do so has a strictly positive measure.

Some reasons to undertake this extension are as follows:

1. To study how specialization responds to cash balances accumulation.

2. To analyze the process of cash balances accumulation and its distribution over types.

3. To construct and derive the properties of the mapping from the level of cash balances to utilities.

4. To test the consistency of constructions like money in the utility function.

There are $k + 1$ production states pj and a similar number of trading states cj, where j stands for the number of units of money kept in inventory and runs from 0 to k. The value function for type Y is given by the matrix $(V_{pj}(Y), V_{cj}(Y))_{j=0}^{j=k}$. The set of market types for this economy is defined as before:

$$MT_k = \{Y : \alpha(u - e_Y(0)) < rV_{p0}(Y)\}.$$

Similarly, we define the set of traders who accumulate up to $j \geq 1$ units of money:

$$MYT_j = \{Y \in MT_k : V_{pj}(Y) - V_{c(j-1)}(Y) - \epsilon > 0\}.$$

It is intuitive that $MYT_j \subseteq MYT_{j-1}$.

The Bellman equations for the production states in continuous time are as follows:

$$rV_{pj}(Y) = \alpha \max\{u - e_Y(0), V_{cj}(Y) - V_{pj}(Y) - e(Y)\}.$$

As for trading states:

$$
\begin{aligned}
rV_{c0}(Y) &= \beta M^{-2} \max\{0, u_\epsilon + V_{p0}(Y) - V_{c0}(Y)\} + \\
&\quad \beta(1 - w_0)(M - 1)M^{-2} \max\{0, V_{p1}(Y) - V_{c0}(Y) - \epsilon\}, \\
rV_{cj}(Y) &= \beta M^{-2} \max\{0, u_\epsilon + V_{pj}(Y) - V_{cj}(Y)\} + \\
&\quad \beta(1 - w_0)(M - 1)M^{-2} \max\{0, V_{p(j+1)}(Y) - V_{cj}(Y) - \epsilon\} +
\end{aligned}
$$

$$\beta(\sum_{i=0}^{i=k-1} w_i\theta_{i+1})(M-1)M^{-2}\max\{0, u_\epsilon + V_{c(j-1)}(Y) - V_{cj}(Y)\}, \qquad \cdot$$

$$rV_{ck}(Y) = \beta M^{-2}\max\{0, u_\epsilon + V_{pk}(Y) - V_{ck}(Y)\} +$$

$$\beta(\sum_{i=0}^{i=k-1} w_i\theta_{i+1})(M-1)M^{-2}\max\{0, u_\epsilon + V_{c(k-1)}(Y) - V_{ck}(Y)\},$$

where $j = 1, 2, \ldots, k-1$. The fraction of traders holding i units of money is denoted by "w_i". The greek letters "θ_{i+1}" stand for the trader's belief about the probability that a holder of i units of money accepts another unit in exchange. For beliefs to be consistent in equilibrium we require:

$$\theta_{i+1} = \mu(MYT_{i+1})/\mu(MT_k).$$

Monetary exchange in this generalized model is also Pareto superior to barter:

Proposition 10 *If $Y \in MT_k$, then Y's average utility is bounded below by the barter payoff:*

$$\frac{\alpha\beta(u_\epsilon - e(Y))}{\beta + \alpha M^2}.$$

Definition 8 *A monetary economy is of order k if no one is allowed to hold more than k units of money and $\mu(MYT_k) > 0$. As a convention, a barter economy is of order $k = 0$.*

Now we can restate our main result in this more general setup:

Proposition 11 *The set of market types is non-decreasing in the order of the economy k, namely, $MT_{k'} \subseteq MT_k$, for $k' < k$.*

The necessary condition for an economy of order k to be more specialized than one of order k', $(k > k')$, is that every market type is willing to engage in monetary exchange. A formal statement of this is:

Proposition 12 *If $MT_{k'} \subset MT_k$, $k' < k$ and $MT_{k'} \neq MT_k$, then $MYT_k = MT_k$.*

As an interesting by-product of this model, we can show that, for a given type Y, the marginal utility of holding money, $\Delta V_{cj}(Y) = V_{c(j+1)}(Y) - V_{cj}(Y)$, is non-increasing in the level of inventory j. This property was usually assumed in the models with money in the utility function rather than derived from fundamentals. More formally:

Proposition 13 *Let type $Y \in (MYT_i - MYT_{i+1})$ and $k \geq 1$. For $j' < j$: $\Delta V_{cj'}(Y) \geq \Delta V_{cj}(Y)$.*

An implication is that the gains from accepting money in exchange for goods, $V_{pj}(Y) - V_{c(j-1)}(Y) - \epsilon$, is non-increasing in j:

Proposition 14 *If $Y \in (MYT_i - MYT_{i+1})$ and $j' < j \leq i$, then*

$$V_{pj}(Y) - V_{c(j-1)}(Y) \leq V_{pj'}(Y) - V_{c(j'-1)}(Y).$$

8 A Variation: A model with homogeneous types.

In this section we explore a different idea of specialization. Here agents, *ex ante* identical in production, choose to produce a "commodity option". The narrower the option chosen, the greater the individual productivity and therefore the greater the degree of specialization. In similar fashion, it is shown that people tend to produce narrower options under monetary exchange.

8.1 The environment: preferences and technology

The commodity space is a circle of circumference 1. Agents' preferences are simple: each individual has only one commodity, which varies across agents, that yields a strictly positive utility u.

Agents can be in two different states: producer or trader. As a producer, each individual is endowed with the possibility of producing intervals of the commodity circle. The length of the interval is a choice variable but the location is stochastic. We will assume that, unless the agent chooses to produce the whole circle, the interval never includes the producer's most preferred commodity. The basic input in the production process is time, denoted by T. It is assumed that longer intervals take more time to produce. In particular, the input of time T is an increasing convex function φ of the length of the interval s. The function φ does not vary across agents and belongs to the class C^2. Since optimal choices will be characterized by standard FOC, we will rule out corner solutions by imposing the following boundary conditions on the first derivative of φ:

$$\lim_{s \to 0} \varphi'(s) < +\infty,$$

$$\lim_{s \to 1} \varphi'(s) = +\infty.$$

We will see that the first condition implies that agents will optimally choose nondegenerate intervals ($s > 0$), and that individuals will not consume what they produce. Thus, we do not take up the issue of production for the market or for selfconsumption. Rather, the issue is how diversified the individual supplier will decide to be.

We call these intervals "commodity options". When two potential trading partners meet, each one can deliver any commodity in his interval. Therefore, each party will order his most preferred one. The possibilities of closing a a deal will depend on the particular trading arrangement. Production according to customer specifications is a good source of example of this "options". Restaurant menus, choice of sizes and colors in retailing and accesories in car sales are other everyday life instances of this theoretical construction. It will be assumed that these options allow for only one consumer choice after which another one has to be produced.

8.2 The Producer's Choice: the barter case

The matching technology is the same as before. We will derive an expression for the expected discounted utility of a trader holding an option of size s, which will be denoted by $V_t(s)$. The function V_t is increasing in s, the reason being that a

larger s enhances the likelihood that the trading partner will find his most preferred commodity. Notice that, unlike Model I, this likelihood is under the control of the producer.

As before, we first set up the discrete time version of the Bellman equation. In a period of length h, the function $V_t(s)$ satifies the following relation:

$$V_t(s) = e^{-rh}\{(1 - \beta hs)V_t(s) + \beta hs[s'(u + V_p) + (1 - s')V_t(s)]\} + o(h)$$

where s' stands for the size of the trading partner's commodity option and V_p denotes the maximum discounted utility when the agent produces. Since all producers are *ex ante* identical, it is reasonable to look for a symmetric Nash equilibrium, that is, $s = s'$. Thus, in equilibrium, the probability of a "double coincidence of wants" is s^2. Therefore, the agent will consume and become a producer with probability βhs^2 and payoff of $(u + V_p)$. Otherwise, he will keep on searching with a probability $(1 - \beta hs) + \beta hs(1 - s)$ and a return of $V_t(s)$.

The continuous time version of the Bellman equation is given by

$$rV_t(s) = \beta ss'(u + V_p - V_t(s)).$$

Rearranging terms we obtain:

$$V_t(s) = \frac{\beta ss'}{r + \beta ss'}(u + V_p).$$

It is easy to check that V_t is an incresing concave function of s.

The utility maximizing choice of s solves the following problem:

$$\max_{s.t.\ T=\varphi(s)} \{e^{-rT}V_t(s)\}.$$

Let s^* be optimal choice given s'. Then it satifies the following FOC:

$$r\varphi'(s) = \frac{V_t'(s)}{V_t(s)} = \frac{r}{s(r + \beta ss')}.$$

After simplification the FOC becomes:

$$\varphi'(s) = \frac{1}{s(r + \beta ss')}.$$

Let s_b be the symmetric Nash equilibrium. Hence, $s_b = s^* = s'$.

Clearly, $0 < s_b < 1$. Moreover, s_b gets smaller as β gets larger. In words, individuals specialize more as the rate of arrival trading partners, the "extent of the market", increases. Finally, the more impatient the producer is, the more he will specialize. The reason being that production and search are time-consuming activities.
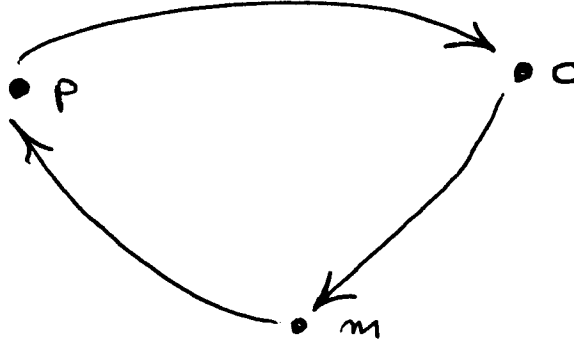
Figure 10: States in a monetary arrangement.

8.3 The Producer's Choice: the monetary case

Fiat money is now introduced in the model. Traders can be commodity holders or money holders but not both. The first state is denoted by the letter "c" and the second one by "m". In contrast with Model I, we do not allow agents to hold both at the same time. As before, we will speak of types c and m. A type c individual can barter with another agent of the same type or sell for money to type m. On the other hand, type m can only buy from type c. People become producers when they hold neither money nor commodities. In Figure 10 represents the relation among the states.

As in Model I, w_c stands for the fraction of type c out the population engaged in trading; θ_M denotes type m beliefs about the probability that type c accepts money in exchange for goods.

The expected discounted utilities $V_c(s)$ and V_m satisfy the following Bellman equations in discrete time:

$$
\begin{aligned}
V_c(s) &= e^{-rh}\{(1 - \beta hs)V_c(s) + \\
&\quad \beta hs[w_c[s'(u + V_p) + (1 - s')V_c(s)] + \\
&\quad (1 - w_c)\max\{V_m, V_c(s)\}]\} + o(h) \\
V_m &= e^{-rh}\{(1 - \beta h\theta_M m_c)V_m + \\
&\quad \beta h\theta w_c(s'(u + V_p) + (1 - s')V_m)\} + o(h).
\end{aligned}
$$

Since we are looking at monetary equilibria, then $\theta_M = 1$ and $V_m > V_c(s)$. After getting the continuous version the previous equations and some elementary manipulations, we obtain:

$$
V_c(s) = \frac{\beta s[w_c s'(u + V_p) + (1 - w_c)V_m]}{r + \beta s[w_c s' + (1 - w_c)]},
$$

$$
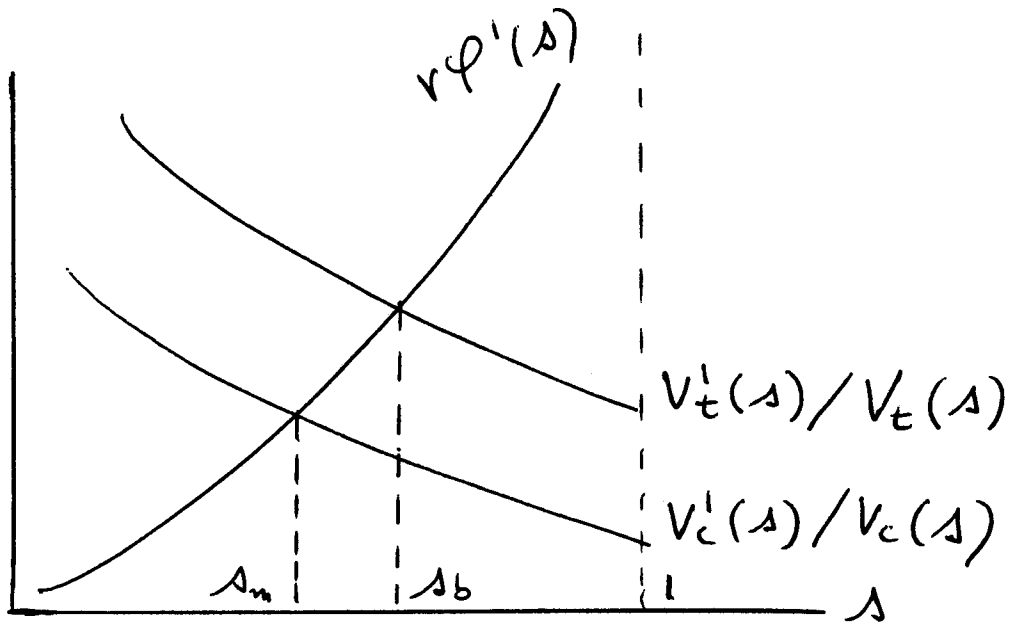V_m = \frac{\beta w_c s'(u + V_p)}{r + \beta w_c s'}.
$$

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Figure 11: Specialization under barter and under monetery exchange.

Let s_m be the symmetric Nash equilibrium. As shown before, it solves the problem

$$V_p = \max_{s.t.\ T=\varphi(s)} \{e^{-rT} V_c(s)\}.$$

Also, it satisfies the FOC:

$$r\varphi'(s) = \frac{V_c'(s)}{V_c(s)} = \frac{r}{s(r + \beta s(w_c s' + (1 - w_c)))}.$$

As long as the proportion of type m individuals is positive $((1 - m_c) > 0)$, then $w_c s' + (1 - w_c) > s'$ and therefore

$$\frac{V_c'(s)}{V_c(s)} < \frac{V_t'(s)}{V_t(s)}.$$

Consequently, agents specialize more under this monetary arrangement than under barter since $s_m < s_b$. Figure 11 illustrates the degree of specialization in both regimes.

9 Lemmae and proofs

9.1 Trading arrangements: The barter case

The following lemma says that market traders have an strictly positive expected discounted utility from trading, provided that the fraction of serchers, N_s, is also positive:

Lemma 1 *If $N_s > 0$, then $u_\epsilon + V_p(Y) - V_t(Y) > 0$ for all $Y \in MT_b$.*

Proof. Otherwise, $V_t(Y) = 0$ and hence $Y \notin MT_b$. QED.

Proposition 1 *For a non-empty market type set MT_b the extent of the market and the degree of specialization under barter for are given by:*

$$\mu(MT_b) = 1 - \inf(MT_b),$$

where $\inf(MT_b)$ is the borderline between autarkic and market types.

Proof. In Appendix A it is shown that the return of producing for the market, $S_b(Y) = V_t(Y) - V_p(Y) - e(Y)$, is increasing in the type Y. Further, we know that the effort of producing commodity zero, $e_Y(0)$, is increasing in Y. Since $e_Y(0) + S_b(Y)$ is a monotonic function defined on measurable set, $[0,1]$, it is therefore a measurable function. Then MT_b is measurable. Since MT_b is bounded below by 0, $\inf(MT_b)$ exists. QED.

9.2   The welfare cost of barter: A gift-giving arrangement

Proposition 2 *The gift-giving arrangement is ex ante Pareto superior to barter, and the welfare costs of barter can be measured by*

$$1 - \frac{\beta + \alpha M}{\beta + \alpha M^2}.$$

Proof. A direct comparison between the average flow utility under barter

$$r(N_s V_t(Y) + (1 - N_s)V_p(Y)) = \frac{\alpha \beta}{\beta + \alpha M^2}(u_\epsilon - e(Y)),$$

and under gift-giving

$$r(N_S V_T(Y) + (1 - N_S)V_P(Y)) = \frac{\alpha \beta}{\beta + \alpha M}(u_\epsilon - e(Y))$$

establishes the result. Substracting from one the ratio of the former to the latter yields the desired measure of the welfare cost of barter. QED.

Proposition 3 *The gift-giving arrangement is statewise Pareto superior to barter:*

$$V_T(Y) > V_t(Y) \text{ and } V_P(Y) > V_p(Y) \text{ for all } Y \in OT.$$

Proof. The result comes out of the relation:

$$\begin{aligned}
\beta(u_\epsilon - \alpha' e(Y)) &= (rM^2 + \beta(1 - \alpha'))V_t(Y) \\
&= (rM + \beta(1 - \alpha'))V_T(Y),
\end{aligned}$$

where $\alpha' = \alpha/(\alpha + r)$. QED.

Proposition 4 *The degree of specialization under the gift-giving arrangement is higher than under barter.*

Proof. Recall that a market type Y under barter is characterized by the inequality:

$$u - e_Y(0) < V_t(Y) - V_p(Y) - e(Y).$$

In a similar fashion, we have "market" types Y in the gift-giving arrangement:

$$u - e_Y(0) < V_T(Y) - V_P(Y) - e(Y).$$

The claim follows if we show that

$$V_T(Y) - V_P(Y) > V_t(Y) - V_p(Y).$$

By direct computation we have

$$V_t(Y) - V_p(Y) = \frac{\beta u_\epsilon + \alpha M^2 e(Y)}{\beta + (r + \alpha)M^2},$$

$$V_T(Y) - V_P(Y) = \frac{\beta u_\epsilon + \alpha M e(Y)}{\beta + (r + \alpha)M}.$$

It can be seen that the latter LHS is greater than the former one, for $u_\epsilon > e(Y) > 0$ and $M^2 > M$ imply a reweight towards the largest number. QED.

9.3 Trading arrangements: The fiat money case

The proposition which follows establishes that a producer holding money is never worse off than one without it:

Lemma 2 *For all types Y: $V_{pc}(Y) \leq V_{pm}(Y)$.*

Proof. Suppose that there exists a type Y' such that

$$V_{pc}(Y') > V_{pm}(Y').$$

Recalling our assumption that all types derive a strictly positive utility from autarky, we have $u - e_Y(0) > 0$. Hence $V_{pm}(Y') > 0$, for everybody can guarantee the autarky level. Further, $V_{pc}(Y') > u - e_{Y'}(0)$. Otherwise, Y' would choose autarky and therefore money would not matter.

Bellman equations for the production states imply:

$$\alpha(V_{cm}(Y') - V_{pm}(Y') - e(Y')) < \alpha(V_c(Y') - V_{pc}(Y') - e(Y')).$$

It follows that $V_{pc}(Y') - V_c(Y') < V_{pm}(Y') - V_{cm}(Y')$, $V_{cm}(Y') < V_c(Y')$, $V_{pc}(Y') < V_c(Y')$ and $V_{pm}(Y') - V_c(Y') - \epsilon < 0$.

Turning to the trading states:

$$rV_c(Y') = \beta M^{-2}(u_\epsilon + V_{pc}(Y') - V_c(Y')),$$

$$rV_{cm}(Y') \geq \beta M^{-2}(u_\epsilon + V_{pm}(Y') - V_{cm}(Y')).$$

Clearly, $V_{pm}(Y') - V_{cm}(Y') > V_{pc}(Y') - V_c(Y')$ implies $V_{cm}(Y') > V_c(Y')$, which contradicts the inequality obtained before. QED.

The following corollary says that a searcher holding money is never worse off than one who does not:

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Corollary 1 *For all types Y: $V_c(Y) \le V_{cm}(Y)$ and*

$$V_{pc}(Y) - V_c(Y) \ge V_{pm}(Y) - V_{cm}(Y).$$

The following is the formal statement that the reason to accept money in exchange is the expectation that it will be spent later on:

Lemma 3 *In a monetary equilibrium: $Y \in MYT$ implies $u_\epsilon + V_c(Y) - V_{cm}(Y) > 0$.*

Proof. By Corollary 1, $V_{cm}(Y) \ge V_c(Y)$ and $V_{pc}(Y) - V_c(Y) \ge V_{pm}(Y) - V_{cm}(Y)$. Then

$$\max\{0, u_\epsilon + V_{pc}(Y) - V_c(Y)\} \ge \max\{0, u_\epsilon + V_{pm}(Y) - V_{cm}(Y)\}.$$

If $u_\epsilon + V_c(Y) - V_{cm}(Y) \le 0$, Bellman equations for the trading states imply $V_c(Y) > V_{cm}(Y)$. Contradiction. QED.

Now we will show that money types barter whenever they can:

Lemma 4 *If $Y \in MYT$, then*

$$u_\epsilon + V_{pc}(Y) - V_c(Y) > 0,$$

$$u_\epsilon + V_{pm}(Y) - V_{cm}(Y) > 0.$$

Proof. Lemma 3, by summing inequalities, implies

$$u_\epsilon + V_{pm}(Y) - V_{cm}(Y) - \epsilon > 0.$$

Hence the second inequality holds. The first one follows from the implication of Lemma 2: $V_{pc}(Y) - V_c(Y) \ge V_{pm}(Y) - V_{cm}(Y)$. QED.

In this model money types are the most productive ones of the economy. This stems from the fact that the gains from monetary exchange increase in the productivity of the type:

Proposition 5 *The utility gains from accepting money in exchange for goods,*

$$V_{pm}(Y) - V_c(Y) - \epsilon,$$

is increasing in Y.

Proof. See Appendix D.

The following establishes the Pareto superiority of monetary exchange:

Proposition 6 *If $Y \in MT_m$, then Y's average utility is bounded below by*

$$\frac{\alpha\beta(u_\epsilon - e(Y))}{\beta + \alpha M^2}.$$

Proof. Consider first the case $Y \notin MYT$. Consequently, this type will switch between states c and pc. In particular, we have

$$rV_{pc}(Y) = \alpha(V_c(Y) - V_{pc}(Y) - e(Y)),$$
$$rV_c(Y) = \beta M^{-2}(u_\epsilon + V_{pc}(Y) - V_c(Y)).$$

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From the second section of the Appendix we know that $\beta f_c = \alpha M^2 f_{pc}$, where f_j stands for the steady state proportion of traders in state j.[20] An straightforward manipulation yields

$$r\left(\frac{f_c V_c(Y) + f_{pc} V_{pc}(Y)}{f_c + f_{pc}}\right) = \frac{\alpha\beta(u_\epsilon - e(Y))}{\beta + \alpha M^2}.$$

Alternatively, if $Y \in MYT$, we must have

$$V_{pc}(Y) < V_c(Y) < V_{pm}(Y) < V_{cm}(Y).$$

By Lemma 4 we get

$$\begin{aligned} rV_c(Y) &= \beta M^{-2}(u_\epsilon + V_{pc}(Y) - V_c(Y)) + \\ &\quad \beta(1 - w_c)(M - 1)M^{-2}(V_{pm}(Y) - V_c(Y) - \epsilon). \end{aligned}$$

Operating in a similar fashion:

$$r\left(\frac{f_c V_c(Y) + f_{pc} V_{pc}(Y)}{f_c + f_{pc}}\right) > \frac{\alpha\beta(u_\epsilon - e(Y))}{\beta + \alpha M^2}.$$

It follows immediately that $f_{pc} V_{pc}(Y) + f_c V_c(Y) + f_{pm} V_{pm}(Y) + f_{cm} V_{cm}(Y)$ has, a fortiori, the same lower bound. QED.

Lemma 5 *For $Y \in MT_m : V_{pc}(Y) \geq V_p(Y)$.*

Proof. In the Appendix it was shown that $N_s = f_c/(f_c + f_{pc})$. Then Proposition 3 can be rewritten as follows:

$$N_s V_c(Y) + (1 - N_s)V_{pc}(Y) \geq N_s V_t(Y) + (1 - N_s)V_p(Y).$$

On the other hand, recalling Bellman equations for production states

$$\begin{aligned} (r + \alpha)V_p(Y) &= \alpha(V_t(Y) - e(Y)), \\ (r + \alpha)V_{pc}(Y) &= \alpha(V_c(Y) - e(Y)), \end{aligned}$$

we can eliminate $V_c(Y)$ and $V_t(Y)$ in the above inequality, and conclusion will come out. Notice that if, in addition, $Y \in MYT$, the inequality becomes strict. QED.

Proposition 7 *Monetary exchange is statewise Pareto superior to barter:*

$$V_t(Y) \leq \min\{V_c(Y), V_{cm}(Y)\} \text{ and } V_p(Y) \leq \min\{V_{pc}(Y), V_{pm}(Y)\}.$$

Proof. This is an almost immediate corollary from the previous Proposition and the last Lemma.

Proposition 8 *The set of market types under the monetary economy is at least as large as that under the barter economy, namely, $MT_b \subseteq MT_m$.*

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[20]We refer to Appendix C for an extended discussion about these proportions.

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Proof. This follows from Lemma 5 and the fact that $V_{pc}(Y)$ and $V_c(Y)$ are non-decreasing in Y. [21]

Proposition 9 *If $MT_b \subset MT_m$ and $MT_b \neq MT_m$, then $MYT = MT_m$.*

Proof. In Appendix A.4 it is shown that the gains from accepting money in exchange for goods, $V_{pm}(Y) - V_c(Y) - \epsilon$, increases with Y, that is, with the productivity of the agent. Let us prove the contrapositive of this corolary. Thus suppose $MYT \subset MT_m$ and $MYT \neq MT_m$. The set $MT_m - MYT$ comprises the most inefficient types which only barter. By Proposition 5 they get exactly the same payoff as in the barter economy. Thus, if they are in the market in the monetary economy, they produced for the market under barter. Therefore, both economies have the same set of marginal types and then $MT_b = MT_m$. Contradiction. QED.

9.4 A generalized monetary model

We state without proof a sequence of propositions and corollaries which parallel the ones given in the previous section. Indeed, proofs are extensions of the ones already given and only involve simple finite inductions.

Lemma 6 *For all types Y and $j' < j$: $V_{pj'}(Y) \leq V_{pj}(Y)$.*

Corollary 2 *For all types Y and $j' < j$: $V_{cj'}(Y) \leq V_{cj}(Y)$ and*

$$V_{pj'}(Y) - V_{cj'}(Y) \geq V_{pj}(Y) - V_{cj}(Y).$$

Lemma 7 *In a monetary equilibrium: $Y \in MYT_j$ implies implies $u_\epsilon + V_{c(j'-1)}(Y) - V_{cj'}(Y) > 0$ for $j' \leq j$.*

Lemma 8 *If $Y \in MYT_j$, then $u_\epsilon + V_{pj'}(Y) - V_{cj'}(Y) > 0$ for $j' \leq j$.*

Proposition 10 *The monetary economy of order k is Pareto superior to barter. If $Y \in MT_k$, then his average utility is bounded below by*

$$\frac{\alpha\beta(u_\epsilon - e(Y))}{\beta + \alpha M^2}.$$

Lemma 9 *For $Y \in MT_k$ and $k \geq 1$: $V_{p0}(Y) \geq V_p(Y)$.*

Now we can restate our main result in this more general setup:

Proposition 11 *The set of market types is non-decreasing in the order of the economy k, namely, $MT_{k'} \subseteq MT_k$.*

Proposition 12 *If $MT_{k'} \subset MT_k$, $k' < k$ and $MT_{k'} \neq MT_k$, then $MYT_k = MT_k$.*

In this model, for a given type Y, the marginal utility of holding money is non-increasing in the level of inventory j:

Proposition 13 *Let type $Y \in (MYT_i - MYT_{i+1})$ and $k \geq 1$. For $j' < j$: $\Delta V_{cj'}(Y) \geq \Delta V_{cj}(Y)$.*

---

[21]See Appendix D for explicit expressions for $\Delta V_j(Y)/\Delta Y$.

Proof. By hypothesis, this type accumulates up to i units of money, where i is not greater than the order of the economy k. Let us substract the Bellman equation for $V_{c(i-1)}(Y)$ from the equation for $V_{ci}(Y)$. After a few manipulations we get:

$$\beta\left(\sum_{h=0}^{h=k-1} w_h \theta_{h+1}\right)(M-1)\Delta V_{c(i-2)}(Y) =$$

$$\beta(1-w_0)(M-1)(V_{pi}(Y) - V_{c(i-1)}(Y) - \epsilon) +$$

$$\left(rM^2 + \beta r/(r+\alpha) + \beta\left(\sum_{h=0}^{h=k-1} w_h \theta_{h+1}\right)(M-1)\right)\Delta V_{c(i-1)}(Y).$$

Notice that, in a monetary economy, $1 - w_0 > 0$; furthermore, $Y \in MYT_i$ means that $V_{pi}(Y) - V_{c(i-1)}(Y) - \epsilon > 0$. Therefore,

$$LHS > \left(rM^2 + \beta r/(r+\alpha) + \beta\left(\sum_{h=0}^{h=k-1} w_h \theta_{h+1}\right)(M-1)\right)\Delta V_{c(i-1)}(Y).$$

On the other hand, the coefficient of $\Delta V_{c(i-1)}(Y)$ is strictly greater than that of $\Delta V_{c(i-2)}(Y)$. Further, we know that $\Delta V_{cj}(Y) \geq 0$ for $j < i$. Hence, we must have

$$\Delta V_{c(i-2)}(Y) \geq \Delta V_{c(i-1)}(Y).$$

Now substract the equation for $V_{c(i-2)}(Y)$ from the one for $V_{c(i-1)}(Y)$. After similar manipulations, we obtain:

$$\left(rM^2 + \beta r/(r+\alpha) + \beta\left(1 - w_0 + \sum_{h=0}^{h=k-1} w_h \theta_{h+1}\right)(M-1)\right)\Delta V_{c(i-2)}(Y) =$$

$$\beta(1-w_0)(M-1)(\alpha/(r+\alpha))\Delta V_{c(i-1)}(Y) + \beta\left(\sum_{h=0}^{h=k-1} w_h \theta_{h+1}\right)(M-1)$$

$$\Delta V_{c(i-3)}(Y).$$

It is easy to see that, if $\Delta V_{c(i-3)}(Y) < \Delta V_{c(i-2)}(Y)$, then the LHS of the equation would exceed the RHS by

$$(rM^2 + \beta r/(r+\alpha))\Delta V_{c(i-2)}(Y),$$

since we know that

$$\Delta V_{c(i-2)}(Y) \geq \Delta V_{c(i-1)}(Y).$$

Consequently, $\Delta V_{c(i-3)}(Y) \geq \Delta V_{c(i-2)}(Y)$. A finite repetition of this backwards induction proves the proposition. QED.

Proposition 14 *If $Y \in (MYT_i - MYT_{i+1})$ and $j' < j \leq i$, then*

$$V_{pj}(Y) - V_{c(j-1)}(Y) \leq V_{pj'}(Y) - V_{c(j'-1)}(Y).$$

Proof. From the Bellman equation for production states we conclude that

$$\Delta V_{pj}(Y) = (\alpha/(r+\alpha))\Delta V_{cj}(Y).$$

Suppose that the proposition is false and $j' = j - 1$, then

$$\Delta V_{c(j-1)}(Y) > \Delta V_{p(j-1)}(Y) > \Delta V_{c(j-2)}(Y),$$

which implies increasing marginal utility of money. Contradiction. QED.

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Appendix

A   Mappings from types to utilities under barter

For market types Y, straightforward computations yield

$$rV_p(Y) = \alpha \left(\frac{\beta u_\epsilon - (rM^2 + \beta)e(Y)}{\beta + (r+\alpha)M^2} \right)$$

$$rV_t(Y) = \beta \left(\frac{(r+\alpha)u_\epsilon - \alpha e(Y)}{\beta + (r+\alpha)M^2} \right)$$

$$r((1-N_t)V_p(Y) + N_t V_t(Y)) = \frac{\alpha\beta}{\beta + \alpha M^2}(u_\epsilon - e(Y))$$

$$V_t(Y) - V_p(Y) = \frac{\beta u_\epsilon + \alpha M^2 e(Y)}{\beta + (r+\alpha)M^2}$$

As can be seen from the foregoing equations, all RHS expressions are affine transformations of the function $e(\cdot)$. Simple inspection reveals that the value functions of both states, and its weighted average, are increasing in the productivity of the individual. On the other hand, the marginal benefit of one commodity produced for the market, $V_t(Y) - V_p(Y)$, decreases with the productivity of the agent type. However, since $V_p(Y)$ goes up with Y, we obtain

$$\mid \Delta(V_t(Y) - V_p(Y)) \mid \leq \mid \Delta e(Y) \mid .$$

Hence, the return of producing for the market,

$$V_t(Y) - V_p(Y) - e(Y),$$

is increasing in the type Y.

B   Bellman equations in discrete time

$V_p(Y) = e^{-rh}\{(1 - \alpha h)V_p(Y)+$
    $\alpha h \max\{u - e_Y(0) + V_p(Y), V_c(Y) - e(Y)\}\} + o(h),$
$V_{pm}(Y) = e^{-rh}\{(1 - \alpha h)V_{pm}(Y)+$
    $\alpha h \max\{u - e_Y(0) + V_{pm}(Y), V_{cm}(Y) - e(Y)\}\} + o(h),$
$V_c(Y) = e^{-rh}\{(1 - \beta h M^{-1})V_c(Y)+$
    $\beta h M^{-1}[w_c(M^{-1}\max\{V_c(Y), u_\epsilon + V_{pc}(Y)\} + ((M-1)/M)V_c(Y)) +$
    $(1 - w_c)(M^{-1}\max\{V_c(Y), u_\epsilon + V_{pc}(Y)\} +$
    $(M - 1/M)\max\{V_c(Y), V_{pm}(Y) - \epsilon\})]\} + o(h),$
$V_{cm}(Y) = e^{-rh}\{(1 - \beta h(w_c + (1 - w_c)M^{-1})V_{cm}(Y)+$
    $\beta h(1 - w_c)M^{-1}[M^{-1}\max\{V_{cm}(Y), u_\epsilon + V_{pm}(Y)\} + ((M-1)/M)V_{cm}(Y)] +$

$$\beta h w_c [M^{-1}(M^{-1}\max\{V_{cm}(Y), u_\epsilon + V_{pm}(Y)\} + ((M-1)/M)V_{cm}(Y)) +$$
$$\theta_M((M-1)/M)(M^{-1}\max\{V_{cm}(Y), u_\epsilon + V_c(Y)\} + ((M-1)/M)V_{cm}(Y))]\}$$
$$+o(h).$$

C   Distribution of trading population over states

Let us denote by f_j with $j = pc, pm, c$, and cm the fraction of market type people in state j at a point in time. They satisfy the following system of equations determining their law of motion:

$$\begin{aligned}
\dot{f}_{pc} &= \beta M^{-2} f_c - \alpha f_{pc} \\
\dot{f}_{pm} &= \beta(1 - w_c)(M - 1)M^{-2} f_c + \beta M^{-2} f_{cm} - \alpha f_{pm} \\
\dot{f}_c &= \beta w_c(M - 1)M^{-2} f_{cm} + \alpha f_{pc} - \\
&\quad \beta M^{-2}(1 + (1 - w_c)(M - 1))f_c.
\end{aligned}$$

For the steady state proportions we set $\dot{f}_j = 0$. We are not interested in the detail of the solution; however, we point out that the system implies:

$$\begin{aligned}
\beta f_c &= \alpha M^2 f_{pc}, \\
(1 - w_c)f_c &= w_c f_{cm}.
\end{aligned}$$

Interesting enough, the steady state proportion between c and pc is the same as that in barter between p and t. Hence

$$N_s = \frac{f_c}{f_c + f_{pc}} = \frac{\alpha M^2}{\beta + \alpha M^2}.$$

Regarding the second equation, w_c is an exogenous variable which reflects the real cash balances per capita. The only restriction we impose on is that $0 < w_c < 1$.

D   Behavior of the mappings from types to utilities $V_j(\cdot)$

The main goal of this section is to establish that the gains from accepting money in exchange for goods, $V_{pm}(Y) - V_c(Y) - \epsilon$, are are increasing in Y. In other words, money types are always the most productive ones.

If $Y \in (MT_m - MYT)$, then the states pc and c are the only relevant ones. In this case $V_{pc}(Y) = V_p(Y)$ and $V_c(Y) = V_t(Y)$, and explicit expressions appear in the section of the Appendix.

Now consider $Y \in MYT$. It is useful to define

$$\begin{aligned}
x(Y) &= V_c(Y) - V_{pc}(Y) \\
z(Y) &= V_{cm}(Y) - V_{pm}(Y).
\end{aligned}$$

Our computations yield:

$$a_0 = \frac{\Delta x(Y)}{\Delta e(Y)} = \frac{\alpha M^2}{M^2(r + \alpha) + \beta(1 - \gamma)(1 - w_c)(M - 1)}$$

$$a_1 = \frac{\Delta z(Y)}{\Delta e(Y)} = \frac{(r\gamma + \alpha)M^2}{M^2(r + \alpha) + \beta(1 - \gamma)(1 - w_c)(M - 1)}$$

$$\frac{z(Y)}{x(Y)} = \frac{r\gamma + \alpha}{\alpha}$$

where

$$\gamma = \frac{\beta(1 - w_c)(M - 1)(\alpha/r)}{(r + \alpha)M^2 + \beta(1 + (M - 1)\alpha r^{-1} + w_c(M - 1))}$$

is a constant.

Clearly,

$$0 < \gamma < 1, 0 < a_0 < 1, 0 < a_1 < 1.$$

Since $(\Delta e(Y)/\Delta Y) < 0$,

$$\frac{\Delta V_{pc}(Y)}{\Delta Y} = \alpha\left(\frac{\Delta x(Y)}{\Delta Y} - \frac{\Delta e(Y)}{\Delta Y}\right) = \alpha(a_0 - 1)\frac{\Delta e(Y)}{\Delta Y} > 0$$

$$\frac{\Delta V_{pm}(Y)}{\Delta Y} = \alpha(a_1 - 1)\frac{\Delta e(Y)}{\Delta Y} > 0$$

$$\frac{\Delta V_c(Y)}{\Delta Y} = \frac{\alpha + r}{\alpha}\frac{\Delta V_{pc}(Y)}{\Delta Y} > 0$$

$$\frac{\Delta V_{cm}(Y)}{\Delta Y} = \frac{\alpha + r}{\alpha}\frac{\Delta V_{pm}(Y)}{\Delta Y} > 0.$$

Now we can turn to the aim of this section. Define the function g:

$$\begin{aligned} g(Y) &= V_{pm}(Y) - V_c(Y) - \epsilon \\ &= (\alpha/r)(z(Y) - x(Y)) - (x(Y) + \epsilon) \\ &= -(1 - \gamma)(x(Y) + \epsilon) + constant. \end{aligned}$$

Hence,

$$\frac{\Delta g(Y)}{\Delta Y} = -(1 - \gamma)a_0\frac{\Delta e(Y)}{\Delta Y} > 0.$$

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