

THE WOODEN HORSE THAT WOULDN'T ROCK:

RECONSIDERING FRISCH*

by

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Summary

The present work reconsiders the celebrated article "Propagation Problems and Impulse Problems" by Ragnar Frisch [1933].

The main conclusion is that the propagating mechanism, contrary to what currently accepted, oscillates only under very peculiar or 'non-economic' assumptions. Technically in Frisch's model the qualitative behaviour of the economy described by the first order differential equation is equivalent - contrary to what Frisch himself thought - to the behaviour described by the more cryptic and complicated mixed difference-differential equation.

Moreover the examples presented by Frisch himself in the original model are misleading because, in order for the macrodynamic variables to exhibit some sort of oscillating behaviour, it is necessary to postulate an unrealistic and inconsistent past historical evolution.

These intrinsic contradictions are here studied and presented.

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I. Introduction

The contribution of Ragnar Frisch to economics has been extensively and deeply analyzed. His specific work 'Propagation Problems and Impulse Problems in Dynamic Economics' (henceforth PPIP) has been carefully studied, since its publication, by several authors. In the beginning the attention was addressed toward general aspects (Cfr. For example Tinbergen [1935], Arrow [1960], Johansen [1969], Samuelson [1974], Blatt [1980]), while in recent years specific ones have been reconsidered. Velupillai [1987] has underlined the implications of the imposed linearity assumption, while Thalberg [1990] has considered different possible evolutions of the so-called 'primary' cycle and has modified some assumptions of the original PPIP. In the present paper the specific construction of the model presented in PPIP is analytically approached and worked out with computer simulations. The main conclusion is that PPIP is not a model of the cycle or, using the Wicksell-Frisch's metaphor, it is a wooden horse that wouldn't rock.

Before proceeding any further in discussing PPIP I would like to comment on the computing ability of Frisch and his collaborators. With the aid of my AT personal computer I have in fact worked out and reproduced (exactly) Frisch's numerical examples, and I have not found major mistakes in the computations, even when they were rather complex and laborious (for example, the derivation of the roots of the characteristic equations). It is probably due to these computational difficulties that some crucial aspects of PPIP were not analyzed before.

II. The Construction of the Model

It is well known and part of the folklore of the history of economic thought that the term macrodynamics was first introduced by Frisch himself in PPIP. As known, in PPIP Frisch suggested that in constructing a model of

the business cycles we have to separate the propagating mechanism (the swinging system) from the impulse mechanism. In what follows I will concentrate only upon the propagation problem.

Frisch's suggestion on how to construct a macrodynamic model can be summarized in the following passages in which he describes his basic strategy.

I shall commence by a system that represents, so to speak, the extreme limit of simplification, but which is, however, completely determinate in the sense that it contains the same variables as conditions. I shall then introduce little by little more complications into the picture, remembering, however, all the time to keep the system determinate. [...] Indeed, the most simplified cases are characterized by monotonic evolution without oscillations, and it is only by adding certain complications to the picture that we get systems where the theoretical movement will contain oscillations. It is interesting to note at what stage in this hierarchic order of theoretical set-ups the oscillatory movements come in. (PPIP, pp. 174-75, italics added).

In what follows the above program is analyzed and challenged. The main points of Frisch's own construction are catalogued in 12 steps.

1. The Simplified system without oscillations

Step 1. '[...] the yearly consumption is equal to the yearly production of consumers' goods (PPIP, p.175)'. No inventories.

Step 2. 'The depreciation on ... capital stock will be made up of two terms: a term expressing the depreciation caused by the use of capital goods in the production of consumers' goods, and a term caused by the use of capital goods in the production of other capital goods. h and k are the (constant) depreciation coefficients in the capital producing industry and in the consumer industry respectively (PPIP, p.175)'

Step 3. The rate of increase of the capital stock is given by:

$$\dot{z} = y - (hx + ky) \quad (1)$$

Z - capital stock;

x - consumption;

y - investment decision.

Step 4.

There is no capital accumulation. A stationary condition of $\dot{Z}=0$ is assumed. Therefore we have:

$$y = m x \quad (2)$$

where $m = h / (1 - k)$ represents the total depreciation of capital stock.

Step 5. Production of capital goods is made up of two factors, total depreciation as in step 4 and the increase in capital requirements determined by the increase of consumer goods production.

$$y = mx + \mu x \quad (3)$$

μ - size of capital stock that is needed directly and indirectly in order to produce one unit of consumption per year. The accelerator, equation (3), had already been the centre of a previous debate with J.M.Clark about causation (cf. Clark [1931, 1932]; Frisch [1931, 1932a,b]). Equations (3) may be solved only when one of the two variables of time, either y or x, is explicitly defined, i.e., only if we introduce another condition. Therefore 'in order to make the problem determinate we need to introduce an equation expressing the behavior of the consumers (PPIP, p.179)'. And this is done in the following way.

Step 6. 'The encaisse désirée, the need for cash on hand, is made up of two parts: cash needed for the transaction of consumer goods and producer goods respectively (PPIP, p. 179)'

$$\omega = rx + sy \quad (4)$$

ω - encaisse désirée

r, s - constants given by habits and by the nature of the given monetary institutions.

Step 7. (The approximation of a non-linear relation). '[...] the total stock of money, or money substitutes, cannot be expanded at infinitum under the present economic system. [...] it seems plausible to assume that the encaisse désirée w will enter into the picture as an important factor which, when increasing, will, after a certain point, tend to diminish the rate of increase of consumption.

Assuming as a first approximation the relationship to be linear, we have

$$\dot{x} = c - \lambda\omega = c - \lambda(rx + sy) \quad (5)$$

where c and λ are positive constants (PPIP, pp.180, italics added)'.

As pointed out in Velupillai [1987] this first approximation was never removed and one wonders why if the encaisse désirée is an 'important factor', the linear formulation of it should have been kept. If we combine (3), (4) and (6) we derive the following:

$$x(t) = X_0 e^{-\lambda \frac{(r+ms)}{(1+\lambda\mu)} t} + \frac{c}{\lambda(r+ms)} \quad (6)$$

where $X(0) = x(t_0) - c(1 + \lambda sm)/(\lambda r + \lambda ms)$.

The above is a first order linear equation that allows only for monotonic evolutions. For example, given certain numerical values (the same one used by Frisch in the sequel of the article: $\lambda = 0.05$, $r = 1$, $s = 1$, $m = 0.5$, $\mu = 10$, $c = 0.165$.) the evolution of a ten percent perturbation of the system has a damped shape like the one represented in Figure 1.

As pointed out by Frisch himself '[...] this means that the variables

will develop monotonically as exponential functions. In other words, we shall have a secular trend but no oscillations. The system considered above is thus too simple to be able to explain developments which we know from observation of the economic world (PPIP, p.180, italics added)'. Therefore, after having discussed different possibilities on how to complicate matters, Frisch's goes on constructing, in Section IV of PPIP, A Macro-Dynamic System Giving Rise to Oscillations, introducing the distinction of Aftalion [1927] between capital goods whose production is started and the activity needed in order to carry them to completion.

2. A Macro-Dynamic System Giving Rise to Oscillations.

Step 8. The amount of production activity to produce capital goods may be considered to be a function of past investment decisions. The so-called carry-on-activity is then defined by:

$$z_t = \int_{\tau}^{\infty} D_{\tau} y_{t-\tau} d\tau \quad (7)$$

where D_{τ} is the 'advancement function'.

The activity of the system at any point of time is therefore represented by the "consumption production" x and production of capital goods $z(t)$. (The sum of the two can be called, in today's notation, GNP, and I will do so in the sequel). Accordingly equations (4) and (5) have to be modified into:

$$\omega = rX + sz_t \quad (4')$$

$$\dot{x} = c - \lambda\omega = \lambda(rX + sz_t) \quad (5')$$

Step 9. In order to be able to solve the system in an explicit form, Frisch defines D_{τ} as a box-function:

$$D_{\tau} = \{D_{\tau}: D_{\tau} = \frac{1}{\epsilon} \text{ if } \tau \in]0, \epsilon[, D_{\tau} = 0 \text{ if } \tau \in]0, \epsilon[\} \quad (8)$$

'This is obviously a simplified assumption, but may perhaps be taken as a first approximation (PPIP, p.182)'.

Now, a determinate system, from now on called, **SYSTEM 1**, is represented by the three following equations:

$$\dot{x} = c - \lambda\omega = \lambda(rx + sz_c) \quad (5')$$

$$y = mx + \mu\dot{x} \quad (12)$$

$$z_t = \int_{\tau}^{\infty} D_{\tau} y_{t-\tau} d\tau \quad (7)$$

where D_t is given by (8).

Step 10. Differentiating (7) and (5'), 'to get rid of the constant term (PPIP, p.182)' and after a few manipulations we obtain a "second order differential first order difference" equation in x :

$$\epsilon x(t) + \lambda(\epsilon r + s\mu) \dot{x}(t) + \lambda sm x(t) - \lambda s\mu \dot{x}(t-\epsilon) - \lambda sm x(t-\epsilon) = 0 \quad (9)$$

The characteristic equation of (9) is a transcendental function of the form:

$$\epsilon \rho^2 + \lambda(\epsilon r + s\mu) \rho + \lambda sm - (\lambda s\mu \rho + \lambda sm) e^{-\epsilon \rho} = 0 \quad (10)$$

where $\rho = -\beta + i\alpha$ and $i^2 = -1$;

The characteristic equation (10) has a countable infinity of solutions (or zeros). Therefore the solution of (9) is given by:

$$x(t) = a_0 + \sum_k a_k e^{-\rho_k t} \quad (11)$$

where the a_k 's are determined by the initial conditions.

Step 11. The countable infinity of solutions may be ordered in some way.

And this is what Frisch does using the trigonometric transformation:

$$e^{p_k t} = e^{-\beta_k} (\cos(\alpha_k t) + i \sin(\alpha_k t)) \quad (12)$$

The ordering goes from low frequencies (lowest values of α) to high ones or, which is the same, from longer periods to shorter ones. The determination of the roots is therefore important in order to determine explicit solutions of the various components. But we should keep in mind that the whole exercise aims at constructing A Macro-Dynamic System Giving Rise to Oscillations.

Step 12. Once the parameters are defined numerically, numerical algorithms may be worked out so that the roots of (10) can be computed. Frisch's numerical values are the ones reported in Step 7 plus the definition of $e = 6$. The solutions are ordered for increasing values of α_k in the following way:

$$|\alpha_0| = 0 < |\alpha_1| < |\alpha_2| < |\alpha_3| < \dots < |\alpha_k| < \dots \quad (13)$$

Once the roots have been ordered Frisch goes on defining the trend as the evolution corresponding to α_0 , the primary cycle as the evolution corresponding to α_1 , the secondary cycle as the evolution corresponding to α_2 , the tertiary cycle as the evolution corresponding α_3 and so on. No economic theoretical or empirical reasons are given or discussed for these definitions.

III. Main Critical Points and Remarks

With the aid of the computer I have simulated Frisch's examples, recomputed the roots and reproduced the same values and graphs that Frisch presented in his original paper. The 'cycles' in the three components (primary, secondary, tertiary) are presented in Figure 2, 3, and 4 and they reproduce the same pictures presented in PPIP. The initial conditions for each cycle have been set to be $x_1 = x_2 = x_3 = 0$, $x_1' = x_2' = x_3' = 1/2$ and $x_0 = 0$.

The examples developed by Frisch are interesting and when we look upon the different components of the solutions, we see that oscillations do take place by construction. But do these cycles represent 'plausible' histories? I believe that a first curiosum arises when we recall that the evolutions of the cycles in the three variables x, y, z were generated imposing specific initial conditions. Given the importance of the carry on activity, z , and the fact that present conditions are a function of investment decisions that occurred during the last $-\epsilon$ years, the past history is essential for the determination of the evolution of the cycles.

In Figure 2a, 3a, 4a, the past histories consistent with Frisch's actual simulations are reported. A striking result is that the past history consistent with the assumed initial conditions is given by an oscillation that is highly unlikely, to say the least. If we compare the evolution that took place in the interval $[t-\epsilon, t_0]$ with the evolution in $[t_0, t_0+16]$ we see that the swings are of the order of magnitude of 1 to 100. When we put 'past' history and 'future' history in a graph with the same abscissa 'future' histories are hardly distinguishable from a horizontal straight line.

Of course it can be claimed that the three evolutions were presented by Frisch only as examples. The implausability of this objection will become clear when it is realized that such initial conditions are necessary, if any resemblance of a cycle has to be maintained in the model. Regardless of what has just been said it should also be underlined that Frisch's own argument is rather incomplete. He had correctly pointed out in an early stage of the construction, (see Step 7 and the section 'Simplified Systems without Oscillations' of the original PPIP) that the first order linear differential equation (6) is 'too simple' to give rise to oscillations because 'the variables will develop monotonically as exponential function' (PPIP, p.180). Subsequently he went on constructing 'A Macro-Dynamic System Giving Rise to Oscillations', which is summarized by the equations (5'), (3) and (7) above (**SYSTEM1**).

Consistent with the argument previously developed he should have confronted the qualitative behaviour of **SYSTEM 1** with the qualitative behaviour of equation (6). In other words, the single members appearing on the right-hand side of the general solution of **SYSTEM 1** (equation (11)) should have been summed so as to give the complete solution. In fact we know that the sum of harmonic (or trigonometric) functions may well give rise to monotonic behaviour, and this is a concrete possibility.

Consequently, in order to make an attempt to close Frisch's argument, I have summed the trend, primary, secondary and tertiary components of the original example developed by Frisch. In Figure 5, 6, 7 the result of this aggregation is presented and in Figure 5a, 6a, 7a, the past history consistent with the behaviour implicit in the original Frisch presentation are reported.

In this very specific example some cyclical resemblance in the

interval $[t_0, t_0+16]$ is maintained, but it implies a rather unlikely oscillating evolution in the interval $[t-\epsilon, t_0]$. It should by now be clear that the way in which Frisch constructs the argument, when he shows the 'propagating' evolution of the primary, secondary and tertiary cycles, is rather arbitrary because full information, with respect to the position and the speed of the single components as given at point of time t_0 , is assumed. This is not proper because there can easily be other components with higher frequencies and also higher amplitudes and therefore the approximating error could be high. This certainly occurs in Frisch's own examples. If we take, for instance, the behaviour of consumption, Fig. 2a, in the interval $[-6, -4]$ the tertiary cycle exhibits wider amplitudes than the primary.

The proper procedure would be to decompose in harmonics the evolution of the aggregated magnitudes, x, y, z that took place in the past, interval $[t-\epsilon, t_0]$. This would allow the researcher to compute the values of the a_k 's in (11). In so doing it should be possible to derive an explicit solution for **SYSTEM 1**. But even when this has been done one needs to investigate the qualitative behaviour of the system. The explicit solution may, of course, exhibit monotonic or non cyclical behaviour.

The procedure suggested above is rather laborious and requires a certain computational effort. There is a much simpler way to tackle the problem, which preserves all the qualitative and quantitative properties of the system: solve **SYSTEM 1** in a recursive manner. The linearity of the problem and the absence of singularities in the functions, guarantees that the actual system may be numerically approximated as closely as we like (in my case as closely as my computer allows me to do).

Therefore, to keep also the recursive-numerical analysis at a high

level of simplicity, equation (7) is numerically integrated using a simple trapezoidal algorithmic rule, while the differential equations (5') and (3) are computed using a simple Euler approximation.

Following the trapezoid algorithm we have that (7) is written in the form:

$$z(t) = \sum_{j=0}^{e/h} \frac{y(t-jh) h \operatorname{tr}_j}{e} \quad (14)$$

where:

j is a positive integer;

$\operatorname{tr}(j)=1$ for j belonging to $]0, e/h[$ and $\operatorname{tr}(j) = .5$ at the boundaries;

h is the step size.

Following the Euler approximating procedure we have that (5') may be written as:

$$x_{(t_h+1)} = (c - \lambda (rx_{t_h} + sz_{t_h})) h + x_{t_h} \quad (15)$$

and (3) as:

$$y(t_h) = m x(t_h) + \mu \frac{(x(t_{h+1}) - x(t_h))}{h} \quad (16)$$

where t_h represents the 'discretization' of time in intervals of size h . As long as the investment decision function, $y(t)$, is defined in the interval $[t_0 - \epsilon, t_0]$ and $x(t_0)$ is also given, the recursive computation of the system represented by equations (14), (15) and (16) (from now on **SYSTEM 2**) is straightforward and simple.

The type of information necessary to solve in explicit form **SYSTEM 1**, is equivalent to the information required to solve **SYSTEM 2**. Moreover, for all practical purposes and apart from very specific cases, the computation

of the general solution must be carried on resorting also to numerical approximations. In other words it can be concluded that to solve **SYSTEM 1** in an explicit form, as the one represented by (11), is equivalent to the recursive computation of **SYSTEM 2**. Moreover, if we consider that (11) has infinite elements and that truncation of the series is almost always unavoidable, one could claim that the study of **SYSTEM 2** gives faster and better results.

By way of example let us compute with the direct recursive method the evolution of **SYSTEM 2**, in which the behaviour of the system from $[t_0 - \epsilon, t_0]$ is assumed to be the one implicit in the example developed by Frisch himself. In Figures 5, 6, and 7 the continuous line represents the solution achieved by a direct computational method, while the dashed line is the evolution of the system when we sum the four components considered by Frisch (trend, primary, secondary and tertiary cycle). It is obvious that the qualitative behaviour is the same, the difference between the two curves is negligible and is due to the numerical approximation algorithm. In decreasing the step size or using a slightly more sophisticated rule for the integration procedure the approximating error decreases. In the estimated evolutions the step size was kept fixed to $h = 1/6$, i.e., to the same magnitude chosen by Frisch in his numerical examples. When decreasing the step size the reduction of the error is noticeable.

The simple system represented by equations (3) and (5), that has the explicit solution (6), is homogeneous. It is, of course, well known that adding a forcing term, will give rise to oscillations. The point stressed by Frisch himself (see Step 7) is that system (3)-(5), when displaced outside equilibrium, 'will develop monotonically'. The example worked out by Frisch, when constructing 'A Macro-Dynamic System Giving Rise to Oscilla-

tions', is not a very good one because it is equivalent to the imposition of a forcing term into the system. The forcing term being past investment decisions that will impose a very specific evolution to the present value of the carry on activity (z).

This fact should be clear if we consider the type of oscillations that were consistent, in the interval $[t_0 - \epsilon, t_0]$, with the evolution presented in the original paper and aggregated in Figure 5,6,7. A proper procedure to analyze the behaviour of a 'swinging' system is represented by the perturbation method. We should try to answer the question: what is the behaviour of a system when 'displaced' from its equilibrium position? Or using Frisch's own analogy: what is the behaviour of the rocking horse when displaced from its resting position? To repeat, this is the type of question that Frisch should have asked and answered, especially because he wanted to show that his 'enlarged' **SYSTEM 1** was capable 'of giving rise to oscillations', while the system captured by equations (3) and (5), with the explicit solution (6), was not.

For the purposes of the rest of the discussion it is appropriate now to give a "definition" of cyclical behaviour. I think that for our purposes it can be agreed that a system shows cyclical behaviour when the variables, as function of time, will "bounce" above and below the equilibrium position. It is implicit from this very general definition that non-monotonic behaviour does not imply a cyclical one. The need for this clarification will become obvious in the sequel.

Starting from a position of rest, in which the variables are assumed to have been in equilibrium up to time t_0 , the behaviour of the system has been studied as a response to a positive ten percent shock, let's say, in consumption (see Figures, 8, 9, 10). A first result is that not only that

the behaviour of consumption is not cyclical, but it is monotonic. Moreover it is almost equal to the behaviour of the simple system (3)-(6), when it is exposed to the same type of shock. In Figure 1, the behaviour of consumption derived from the analysis of the two different systems does practically overlap.

When we observe the behaviour of the production starting and that of the carry on activity we see that they are not monotonic, but also not at all cyclical, i.e., there is no oscillation around the equilibrium values. It is therefore possible to conclude that the system described by Frisch, with the reported values of the parameters it is not a cycle model. Given the intrinsic linear structure of the model the results presented above would not change with respect to the intensity and the nature of the original shock. It can easily be shown, in fact, that a positive shock in the decision of investment would propagate in a way similar and symmetrical to what shown in Figures 8, 9, 10.

It can be argued that in the above analysis the behaviour of the total activity, considered as the sum of consumption (x) and carry on activity (z) - from now on called also GNP -, has been disregarded. In Figure 11 it is shown that also when we consider the behaviour of GNP the model will not exhibit oscillations around equilibrium positions.

Finally it is consequential to carry on parameter analysis and to try to find out under which conditions cycles would be generated. In Figure 12, 13, 14, and 15 the evolution of the variables, after a ten per cent shock occurred at time t_0 , is represented as a function of λ (ranging from .05 to .6), while the other parameters are kept at the original numerical values of PPIP. As λ changes also the equilibrium values would change, therefore all the evolutions have been normalized to 100. The evolution of con-

sumption (x) is still non-cyclical and monotonic for all the values of λ considered, while the evolution of the production starting (y) and of the carry on activity are non-cyclical, but also non-monotonic. It is only gnp, a variable not explicitly considered by Frisch, that would exhibit some cyclical behaviour.

To discover the properties of the system it is appropriate to scan the behaviour of the variables according to different values of the parameters. I have constructed an index that gives the following symbolic values:

- 0 if the behaviour of the variable is convergent and non-cyclical;
- 1 if the behaviour of the variable is convergent but in a cyclical fashion (as defined above);
- 2 if the behaviour of the variable is divergent in a cyclical fashion;
- 3 if the behaviour of the variable is explosive, but in a non-cyclical way, or the variable is not defined at all because of the existence of critical points (for example when $l=0$, the equilibrium value is not defined).

Having done this, the parametric analysis of the system has been conducted with all the parameters fixed and one parameter changing, in turn, inside a preassigned range. For each of the variables, x,y,z and GNP, the above index numbers have been assigned.

In Table 1 the result of this inquiry is reported. Information about the behaviour of the whole system, as a function of a given set of parameters, is captured by a four digit number. A series of four 0's would express the fact that all the four variables considered would behave non-cyclically.

TABLE 1.

λ	xyzg	r	xyzg	s	xyzg	m	xyzg	μ	xyzg	ϵ	xyzg	c	xyzg
0	3333	-1	3003	-1	1111	0	0000	0	0000	0	2333	-.165	0110
.05	0000	-0.5	3333	-0.5	0001	0.1	0000	2	0000	3	0000	0	0000
.10	0001	0	0000	0	0000	0.2	0000	4	0000	6	0000	0.165	0000
.15	0001	0.5	0000	.5	0000	0.3	0000	6	0000	9	0000	0.330	0000
.20	0101	1	0000	1	0000	0.4	0000	8	0000	12	0000	0.495	0000
.25	0101	1.5	0000	1.5	0000	0.5	0000	10	0000	15	0000	0.660	0000
.30	0101	2	0000	2	0000	0.6	0000	12	0000	18	0000	0.825	0000
.35	0101	2.5	0000	2.5	0000	0.7	0000	14	0000	21	0000	0.990	0000
.40	0101	3	0001	3	0000	0.8	0000	16	0000	24	0000	1.155	0000
.45	0101	3.5	0001	3.5	0000	0.9	0000	18	0001	27	0000	1.320	0000
.50	0101	4	0001	4	0000	1	0000	20	0001	30	0000	1.485	0000

The values under the column xyzg represent a condensed indicator of the evolution of the system. The first digit (from the left) is the index number associated with the behaviour of consumption, x; the second digit the index number associated with investment decisions, y; the third the index number associated with the carry on activity, z and the fourth to GNP. For example, the value 0101 summarizes the fact that x and z have convergent behaviour, while y and g(np) exhibit both cyclical and convergent behaviour.

From information summarized in Table 1 we can derive the following conclusions with respect to the four variables considered:

x - does not exhibit, independently of the set of parameters considered, cyclical behaviour;

y - exhibits cyclical behaviour only for very high values of λ and ϵ ;

z - does not exhibit, independently of the set of parameters considered, cyclical behaviour;

gnp - exhibits cyclical behaviour only for very high values of λ , r, μ , and ϵ .

IV. CONCLUSION

As shown above, there is a logical fault in the argument developed in PPIP. The whole construction implies, in fact, that once the individual components of the general solution have been identified, the closure of the model is achieved only when the single components are summed together, i.e., when the analysis is conducted on the general solution (11). In the attempt to complete Frisch own presentation and to show that his model is a cyclical one the opposite has been discovered. The main result of the paper is that the model presented in PPIP, the so-called propagation mechanism, is not intrinsically cyclical. Therefore it is not 'A Macro-Dynamic Model Giving Rise to Oscillations', as claimed by Frisch. When the system is perturbed from equilibrium, i.e., is subject to an external shock, it evolves back to the equilibrium position in a non-cyclical manner. Using Frisch own metaphor, we have the paradoxical result that the rocking horse is not rocking: or 'a wooden horse that wouldn't rock'.

Oscillations are possible only in the forcing form. It implies that the qualitative behaviour of the 'propagating mechanism' is equivalent to the behaviour exhibited by a (forced) first order linear differential system like (6).

The use of mixed difference-differential equations in economics was (presumably) first introduced by both Frisch and Kalecki at the meeting of the econometric society held in Leyden in 1933 (cf. Tinbergen [1935], p. 269). Frisch presented a version of PPIP, while Kalecki discussed his 'Proba Teoriy Konjungtury'. A slightly different version of Kalecki model was then published in *Econometrica* in 1935 with the title 'A Macrodynamic Theory of Business Cycles' and a version of 'Proba Teoriy Konjungtury' appeared in an English edition in Kalecki [1939] with the title 'Outline of a Theory of the

Business Cycle'. It is well known that Frisch (together with Holme) strongly criticized the 'improper' use of difference - differential equations made by Kalecki in his 1935 article, in which self repeating cycles were generated under the 'non-economic' assumption of a parameter fixed to a preassigned value. Given an infinitesimal change in the value of that crucial parameter, it was underlined that the system would have evolved exhibiting damped or explosive oscillations, and not endogenous ones. In modern terminology, a structurally unstable model.

Frisch's criticism turned out to be correct. But, in light of what I have discussed above, a 'counter-critique' could have been addressed toward PPIP. That is, if Kalecki's argument relied on very specific values of the parameters, Frisch's construction in turn did not - quite independently from the values of the parameters - account for cyclical behaviour.

Why such a paradoxical result found in PPIP went unnoticed for almost sixty years is an intriguing question that, in my opinion, should be of interest to scholars of mathematical economics, business cycle theory and history of economic thought.

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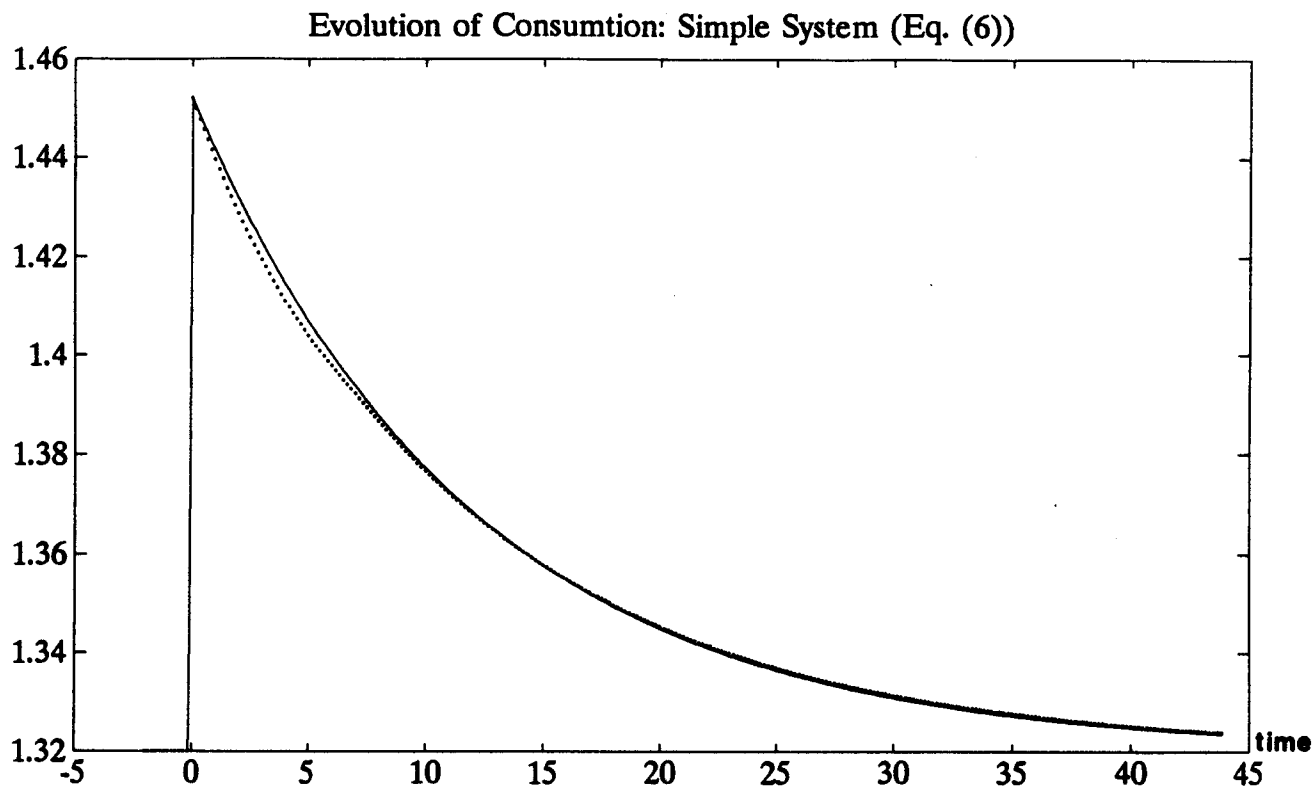


Figure 1.

Legend: _____ Simple case. First order differential equation.
 Evolution of consumption subsequent to a ten percent disturbance.
 Mixed difference and differential equation.
 Evolution of consumption subsequent to a ten percent disturbance.

Comment: The two evolutions above have been computed referring to the same values for the structural parameters used in the original paper by Frisch. A visual comparison between the two curves indicate that there is not a substantial difference between the evolution of the simple system and the more 'complex' one.

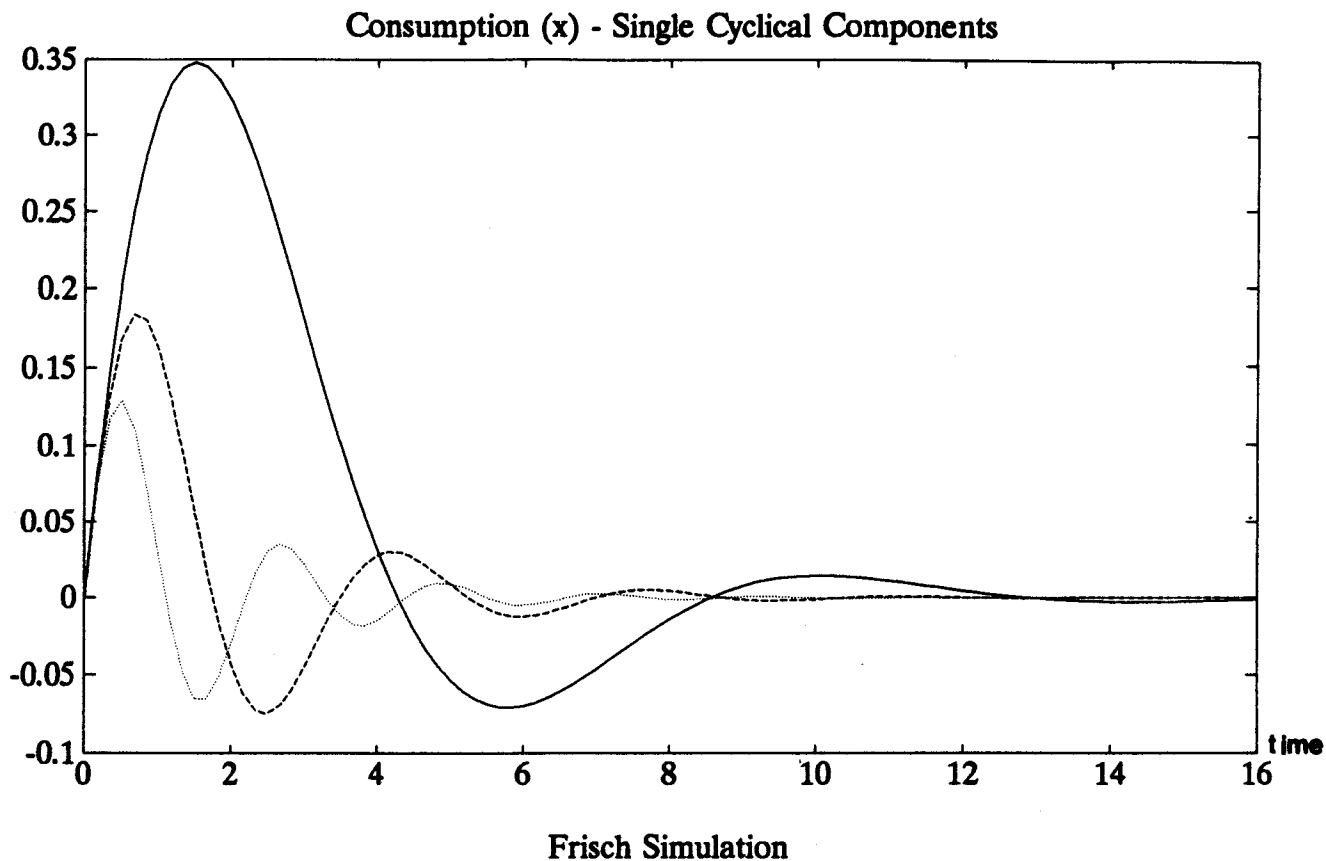


Figure 2.

Legend: — Primary Cycle. Consumption.
 - - - - Secondary Cycle. Consumption.
 Tertiary cycle. Consumption.

Comment: The curves above are the one reported by Frisch. The original conditions for each cycle are set to be $x_1(0) = x_2(0) = x_3(0) = 0$ and $dx_1/dt = dx_2/dt = dx_3/dt = 1/2$ and $x_0 = 0$. Of course also the values of the parameters are the same one used by Frisch (see text above).

Consumption (x) - Past History, Single Cyclical Components

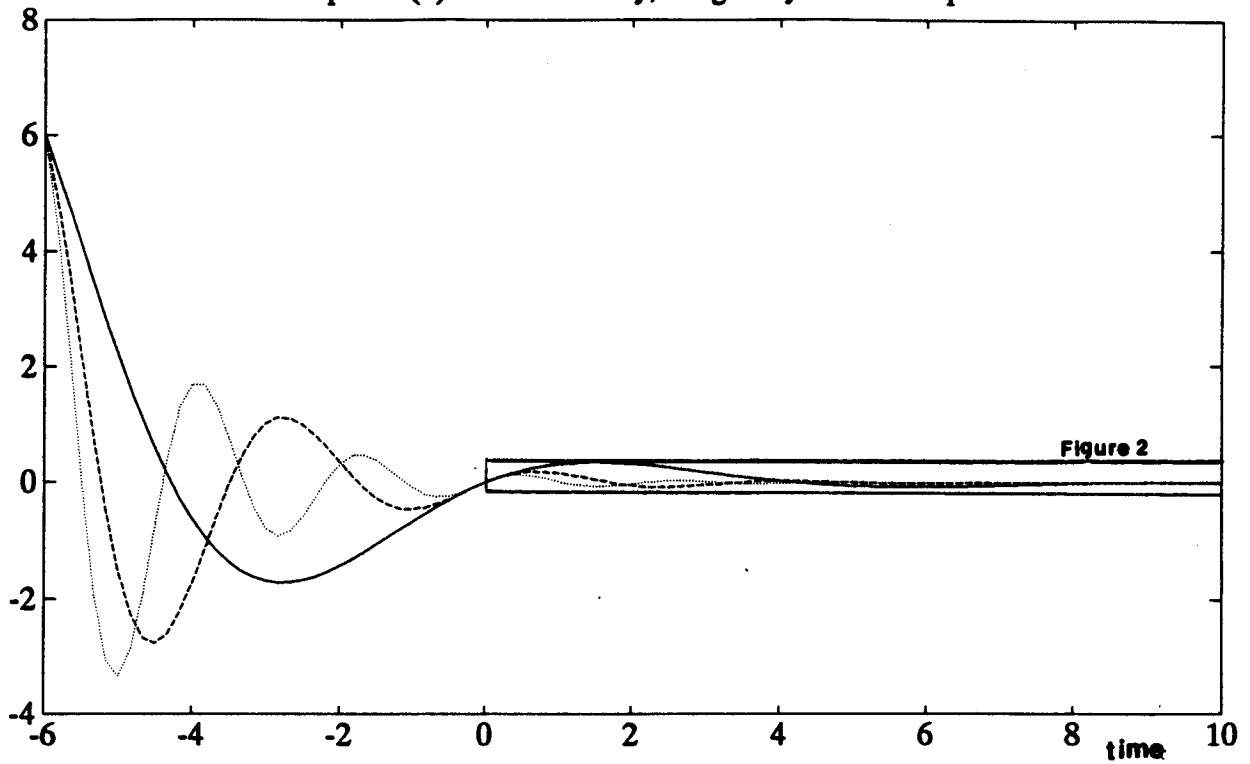


Figure 2a.

Legend: — Primary Cycle. Consumption.
----- Secondary Cycle. Consumption.
..... Tertiary cycle. Consumption.

Comment: The curves above show the implicit past history consistent with the evolution reported in PPIP. The small box on the right is Figure 2.

Production Starting (y) - Cyclical Components

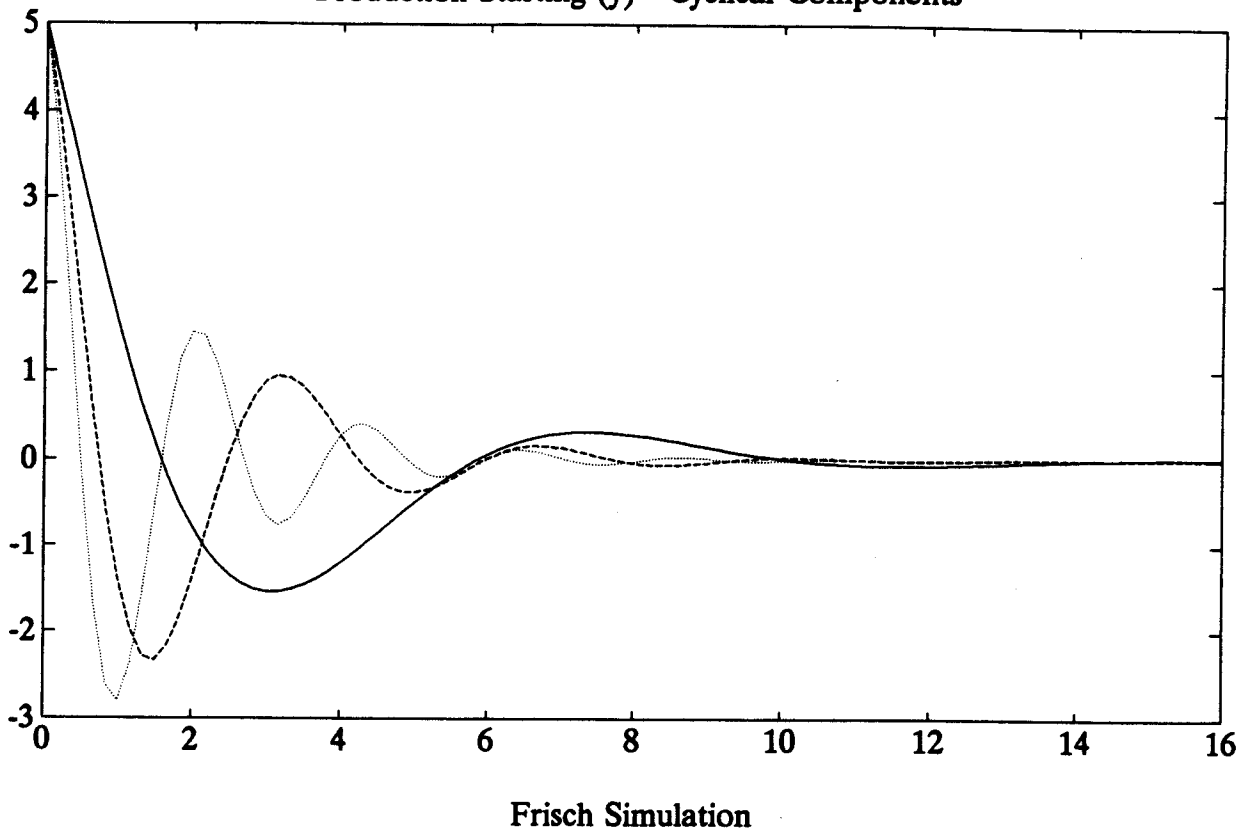


Figure 3.

Legend: — Primary Cycle. Production Starting.
----- Secondary Cycle. Production Starting.
..... Tertiary cycle. Production Starting.

Comment: The curves above are the same one reported by Frisch. The original conditions for each cycle have been set to be $x_1(0) = x_2(0) = x_3(0) = 0$ and $dx_1/dt = dx_2/dt = dx_3/dt = 1/2$ and $x_0 = 0$. Of course also the values of the parameters are the same one used by Frisch (see text above).

Production Starting (y) - Past History, Single Cyclical Components

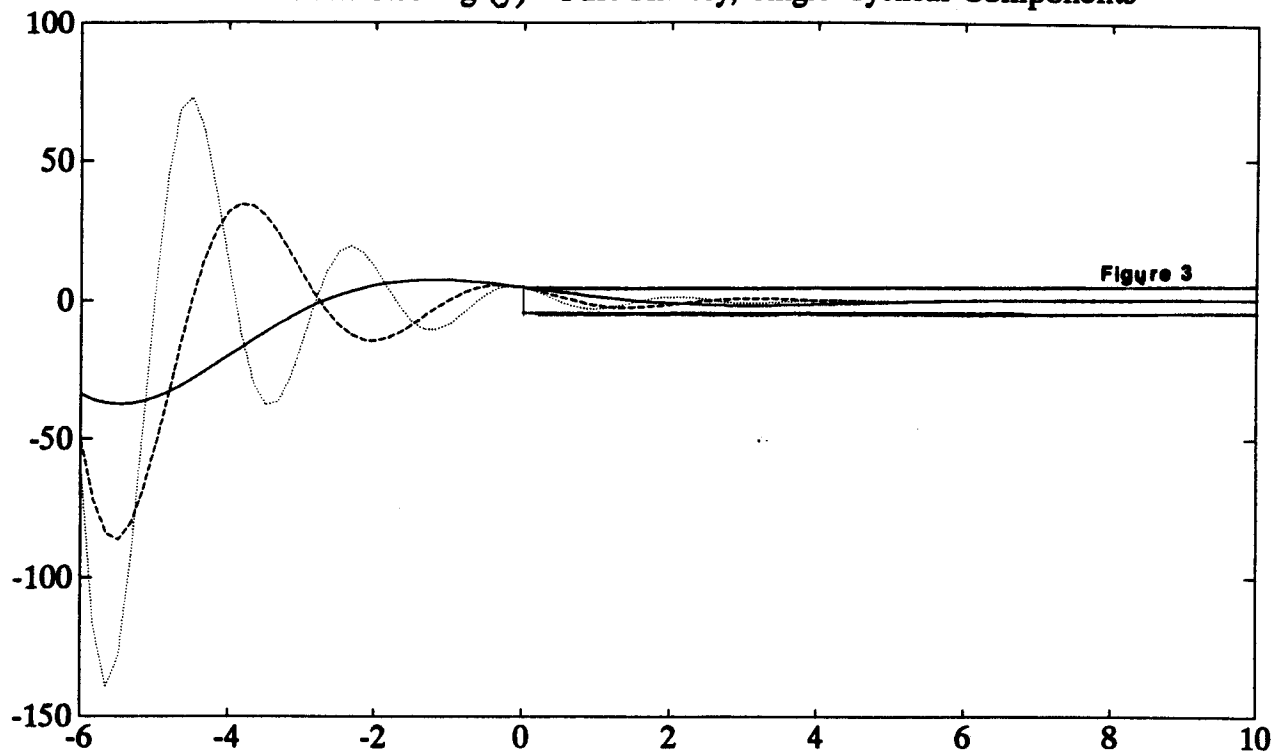


Figure 3a.

Legend: — Primary Cycle. Production Starting.
----- Secondary Cycle. Production Starting.
..... Tertiary cycle. Production Starting.

Comment: The curves above show the implicit past history consistent with the evolution reported in PPIP. The small box on the right is Figure 3.

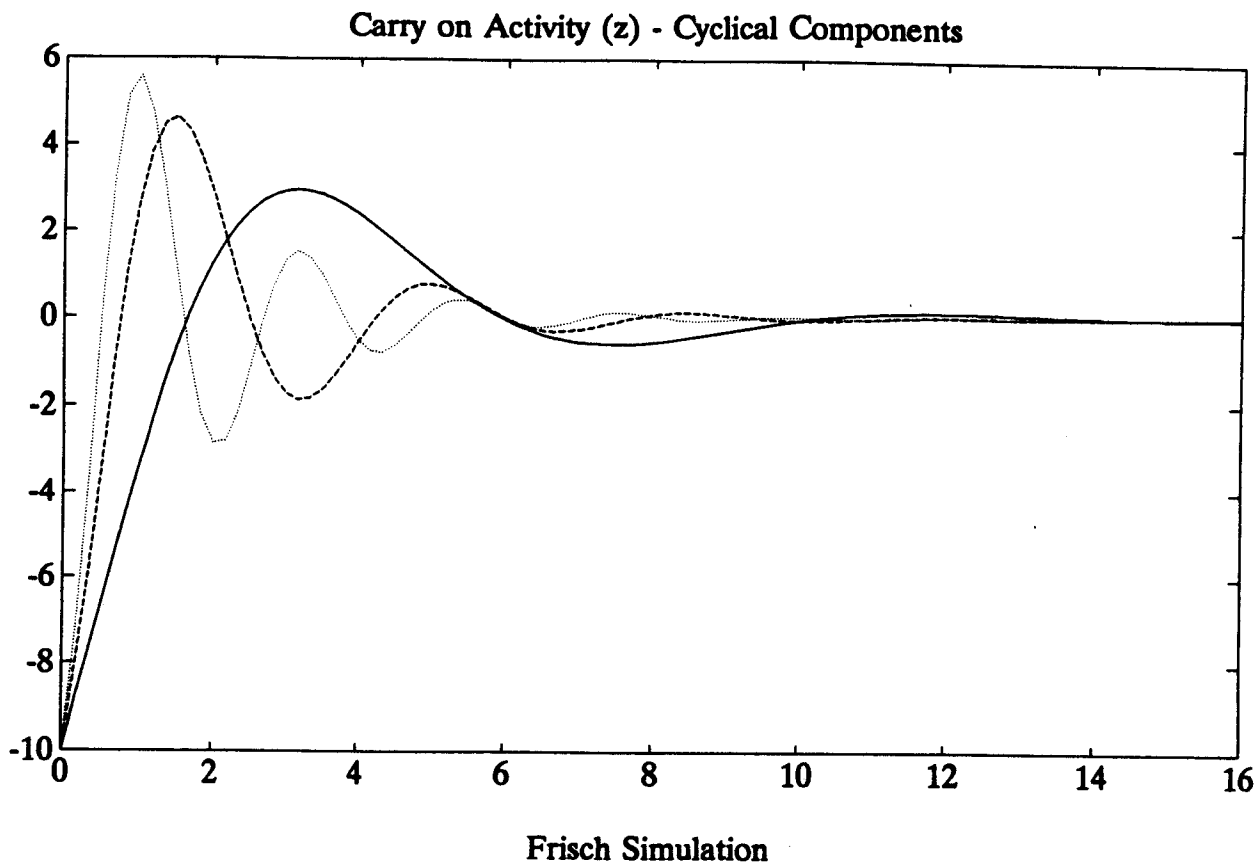


Figure 4.

Legend: — Primary Cycle. Carry on Activity.
 - - - - Secondary Cycle. Carry on Activity.
 Tertiary cycle. Carry on Activity.

Comment: The curves above are the same one reported by Frisch. The original conditions for each cycle have been set to be $x_1(0) = x_2(0) = x_3(0) = 0$ and $dx_1/dt = dx_2/dt = dx_3/dt = 1/2$ and $x_0 = 0$. Of course also the values of the parameters are the same one used by Frisch (see text above).

Carry on Activity (z) - Past History, Single Cyclical Components

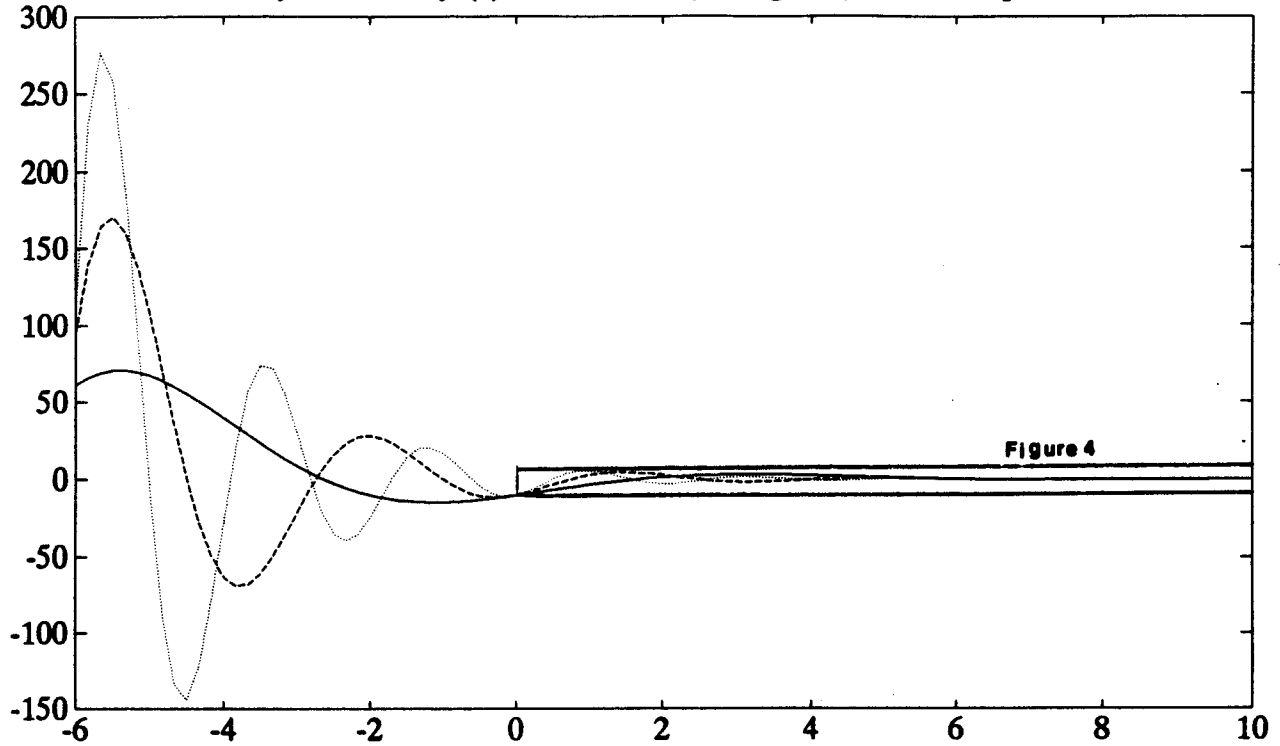


Figure 4a.

Legend: — Primary Cycle. Carry on Activity.
----- Secondary Cycle. Carry on Activity.
..... Tertiary cycle. Carry on Activity.

Comment: The curves above show the implicit past history consistent with the evolution reported in PPIP. The small box on the right is Figure 4.

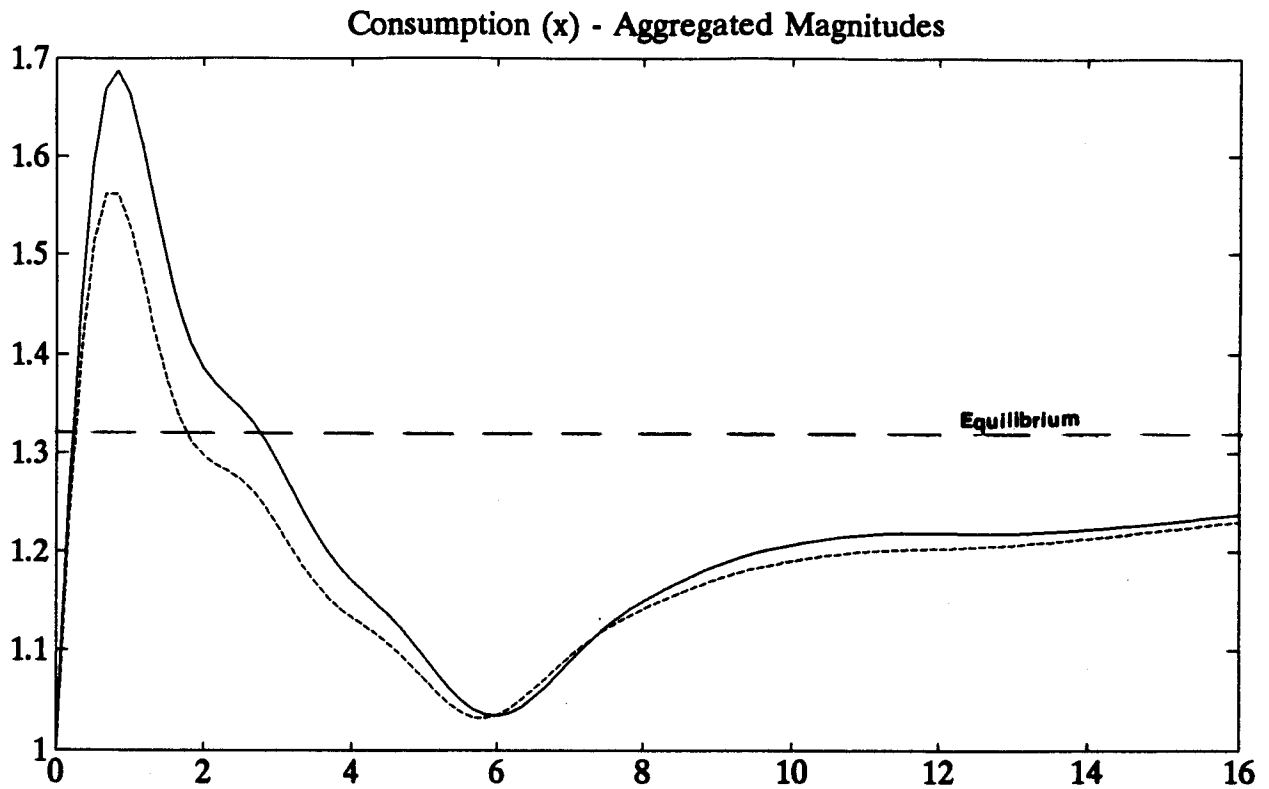


Figure 5.

Legend: ——— Direct recursive method. Aggregate Consumption.
 ----- Sum of individual components. Aggregate Consumption.

Comment: The aggregated evolution of consumption implicit in Frisch's own examples is obtained in two ways. The first requires recursive computation of **SYSTEM 2**, while the second requires the summation of the individual components. The result above is consistent with $x_1(0) = x_2(0) = x_3(0) = 0$ and $dx_1/dt = dx_2/dt = dx_3/dt = 1/2$ and $x_0 = 0$ and the values for the parameters given by Frisch.

Consumption (x) - Past History

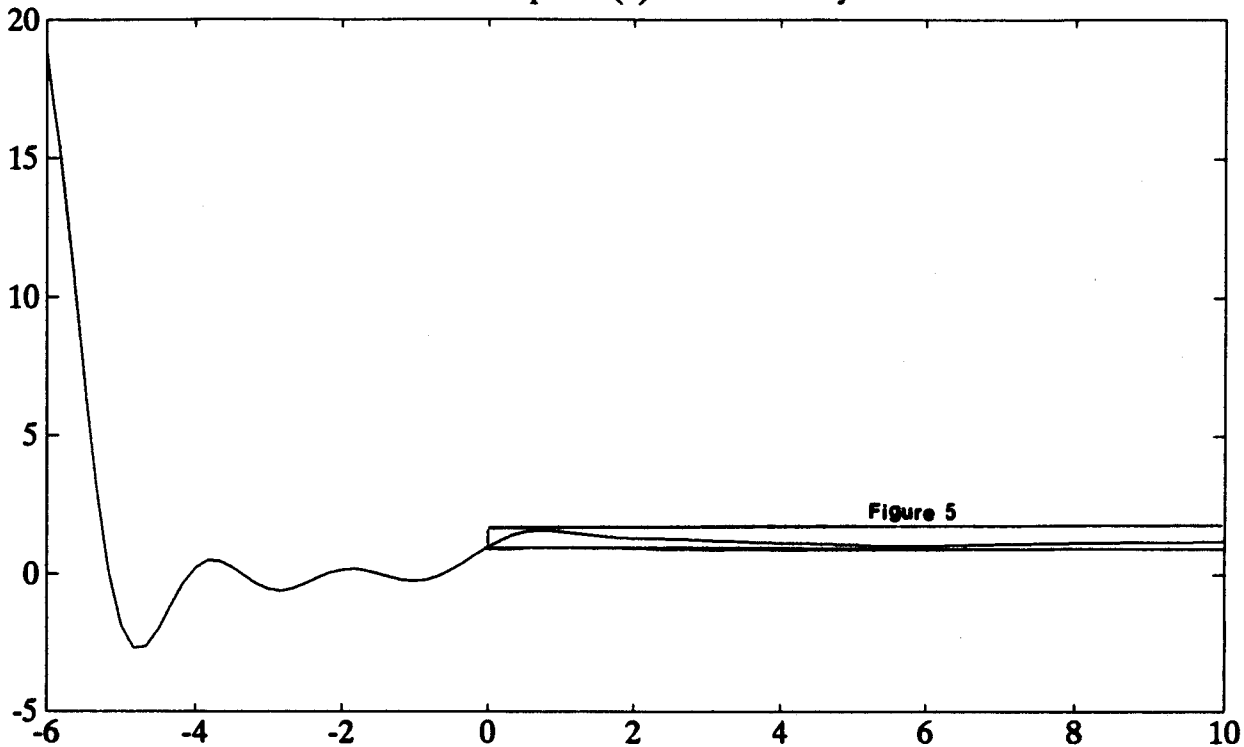


Figure 5a.

Legend: — Sum of individual components. Aggregate Consumption.

Comment: The curves above show the implicit past history consistent with the evolution reported in PPIP. The small box on the right is Figure 5 appropriately scaled.

Production Starting (y) - Aggregated Magnitudes

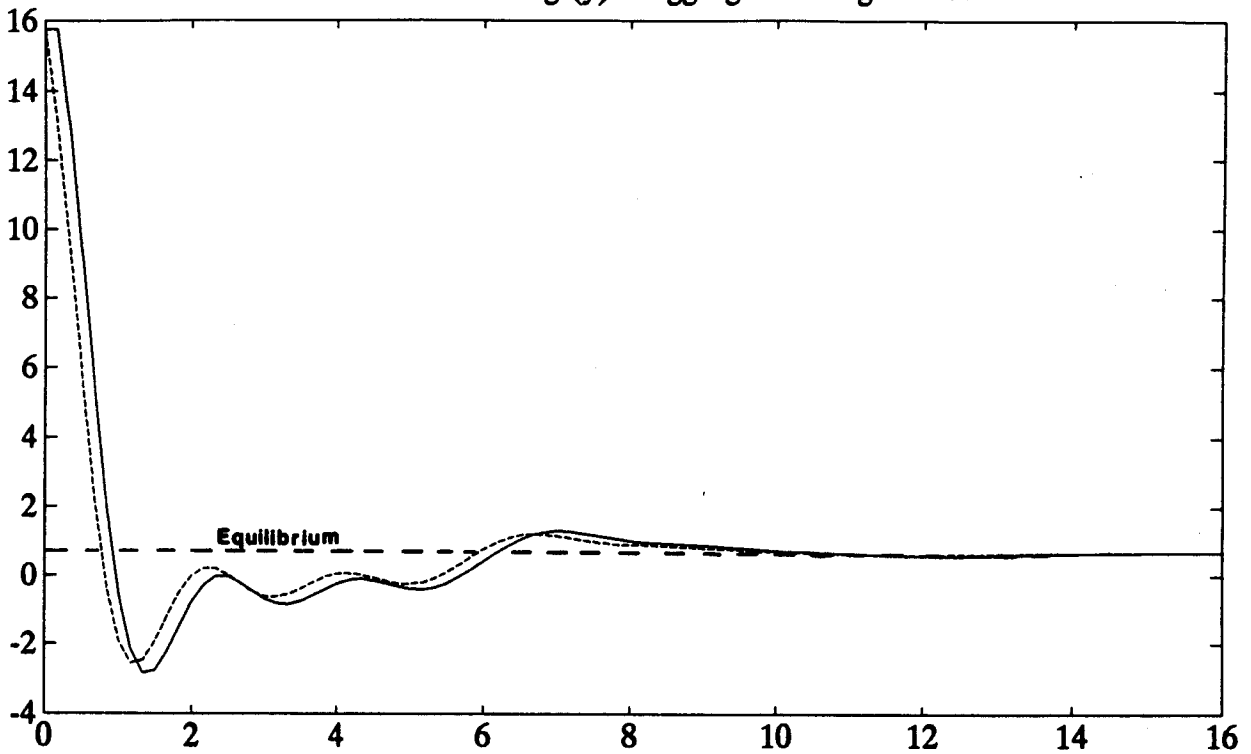


Figure 6.

Legend: ——— Direct recursive method. Production Starting.
----- Sum of individual components. Production Starting.

Comment: The aggregated evolution of production starting, i.e., investment decisions, implicit in Frisch's own examples is obtained in two ways. The first requires recursive computation of **SYSTEM 2**, while the second requires the summation of the individual components. The result above is consistent with $x_1(0) = x_2(0) = x_3(0) = 0$ and $dx_1/dt = dx_2/dt = dx_3/dt = 1/2$ and $x_0 = 0$ and the values for the parameters given by Frisch.

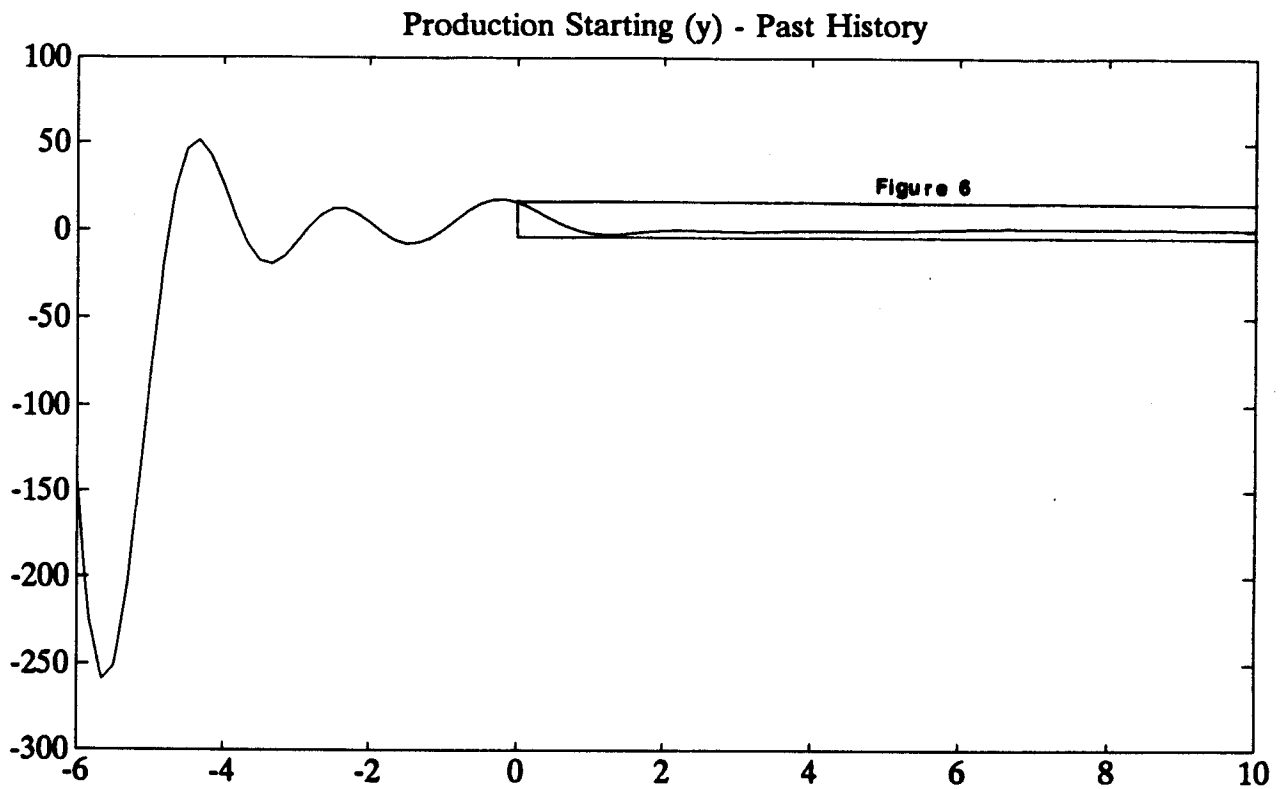


Figure 6a.

Legend: — Sum of individual components. Production Starting, i.e., Investment Decisions.

Comment: The curves above show the implicit past history consistent with the evolution reported in PPIP. The small box on the right is Figure 6 appropriately scaled.

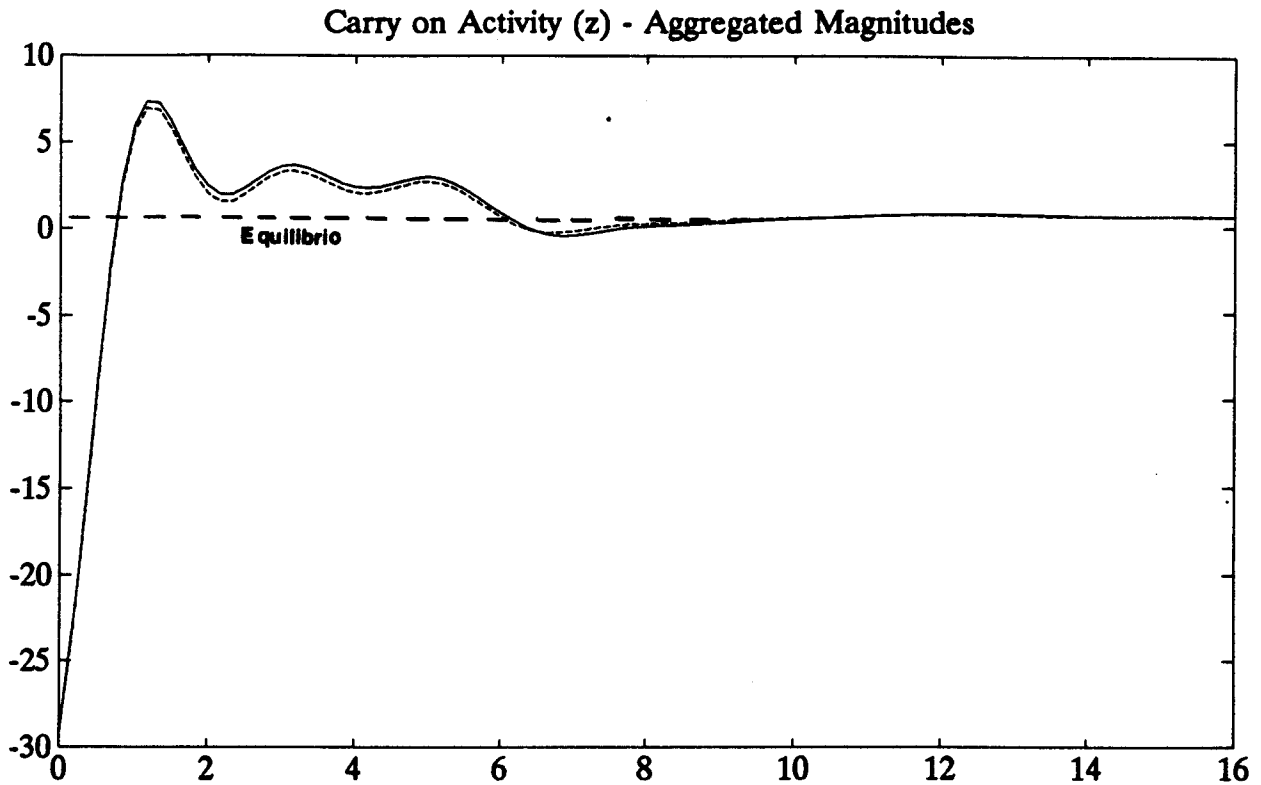


Figure 7.

Legend: Direct recursive method. Carry on Activity.
 ----- Sum of individual components. Carry on Activity.

Comment: The aggregated evolution of the carry on activity implicit in Frisch's own examples is obtained in two ways. The first requires recursive computation of **SYSTEM 2**, while the second requires the summation of the individual components. The result above is consistent with $x_1(0) = x_2(0) = x_3(0) = 0$ and $dx_1/dt = dx_2/dt = dx_3/dt = 1/2$ and $x_0 = 0$ and the values for the parameters given by Frisch.

Carry on Activity (z) - Past History

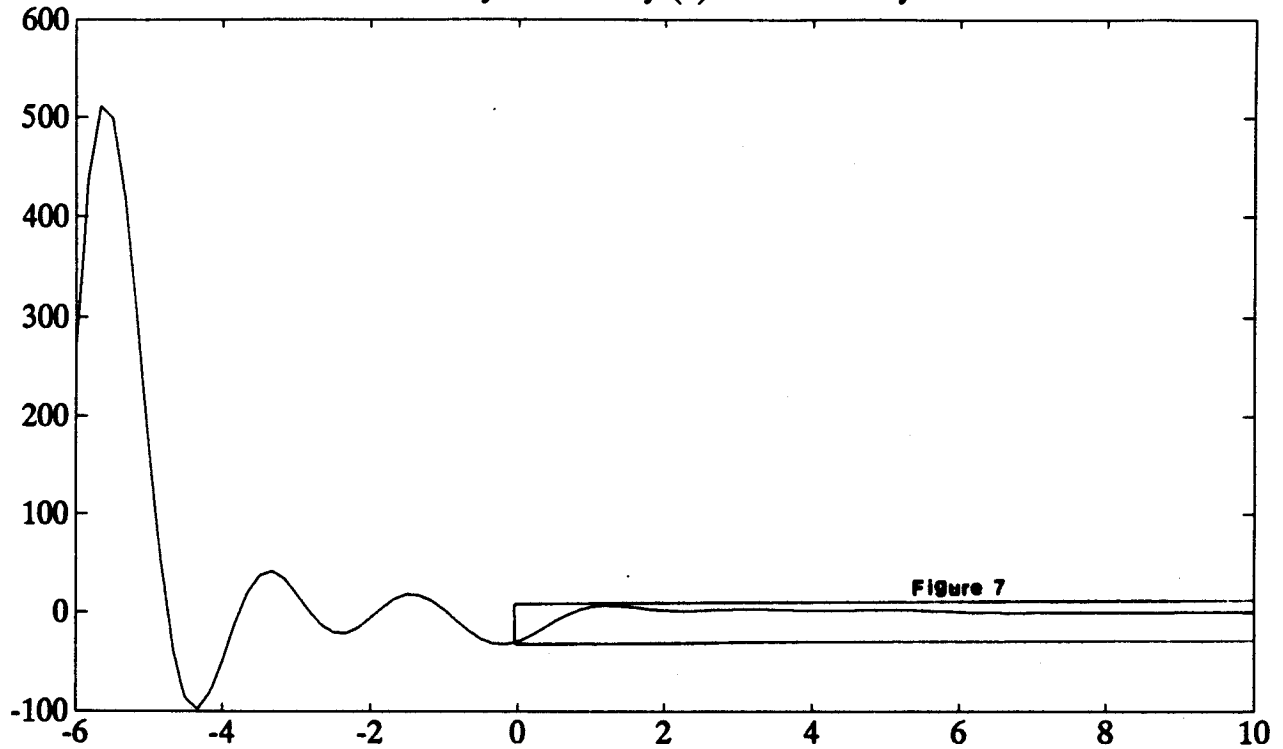


Figure 7a.

Legend: — Sum of individual components. Carry on Activity.

Comment: The curves above show the implicit past history consistent with the evolution reported in PPIP. The small box on the right is Figure 7 appropriately scaled.

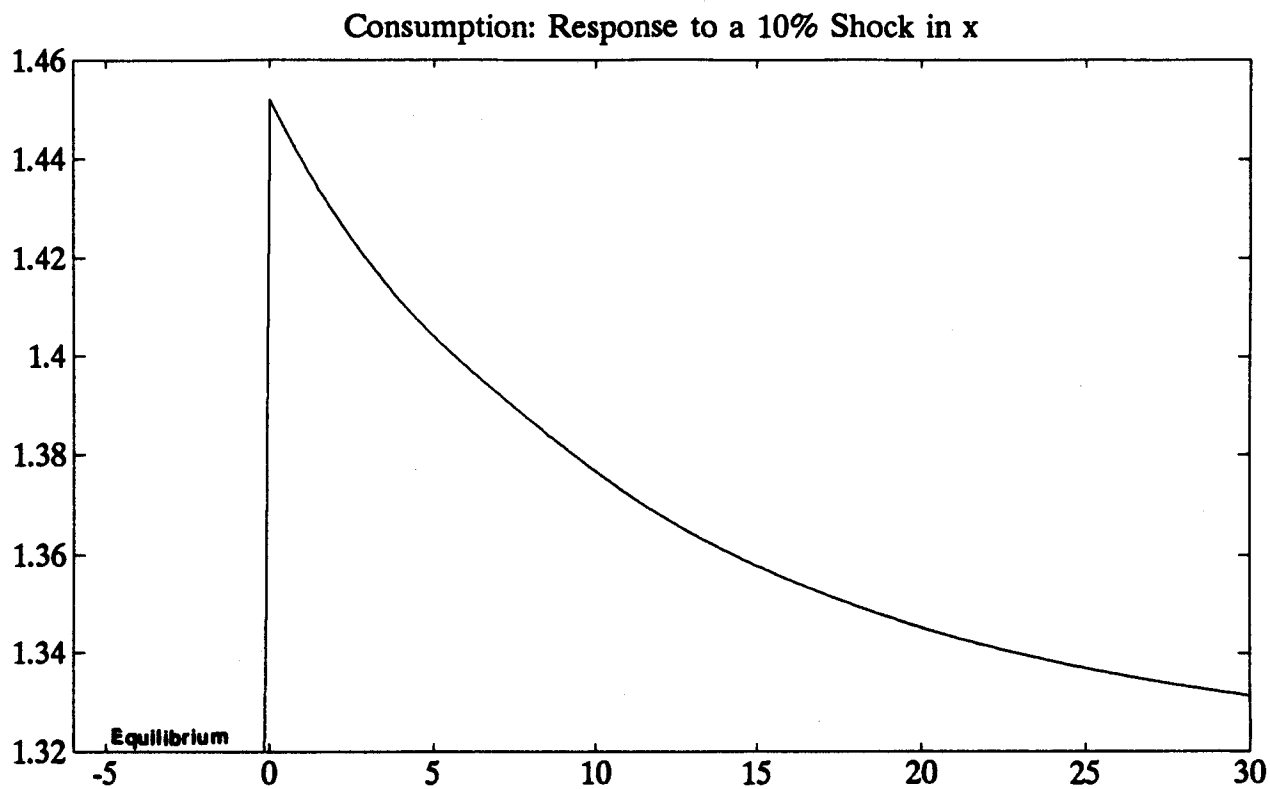


Figure 8.

Legend: — Consumption.

Comment: Evolution of consumption as a response to a ten percent shock in consumption that perturbrates the system away from an equilibrium condition. The computation is conducted on the approximating **SYSTEM 2**, using the same parameters values as in PPIP.

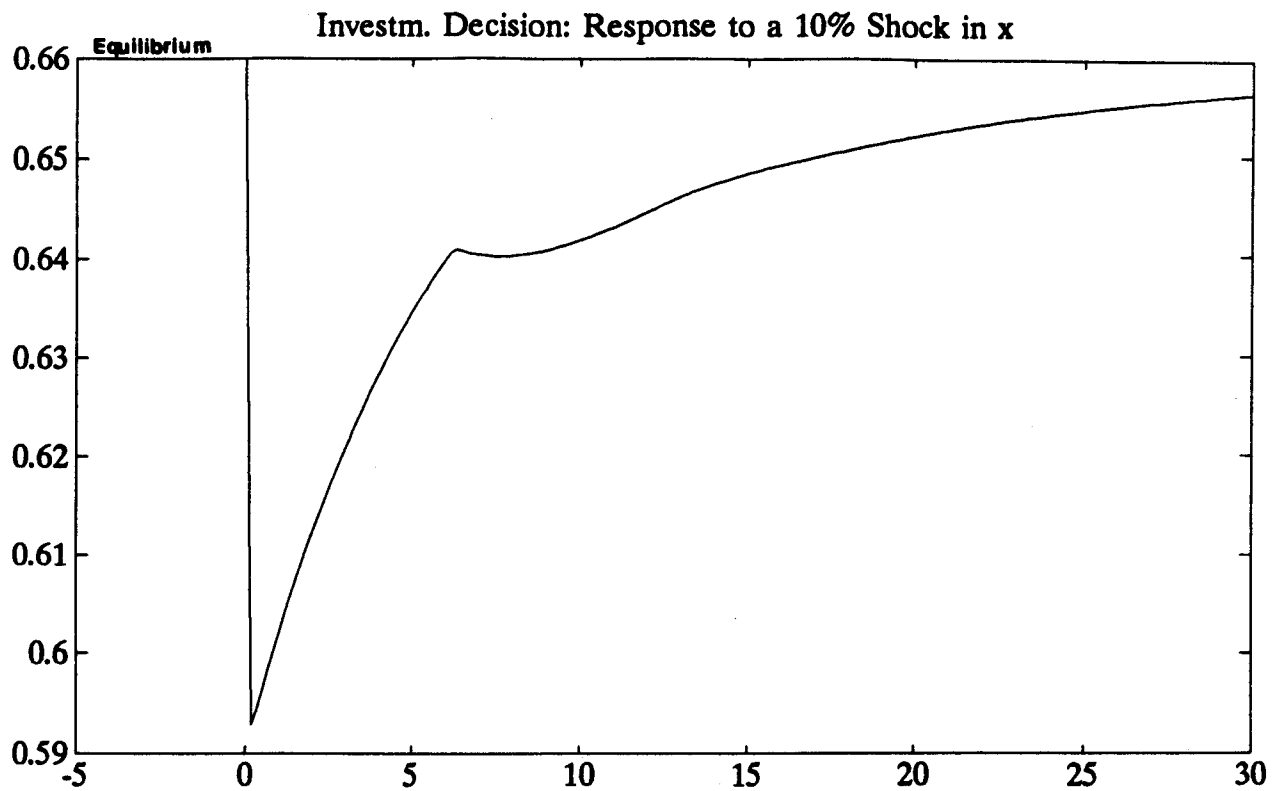


Figure 9.

Legend: — Production Starting, i.e., Investment Decisions.

Comment: Evolution of the decisions of investment as a response to a ten percent shock in consumption that perturbrates the system away from an equilibrium condition. The computation is conducted on the approximating **SYSTEM 2**, using the same parameters values as in PPIP.

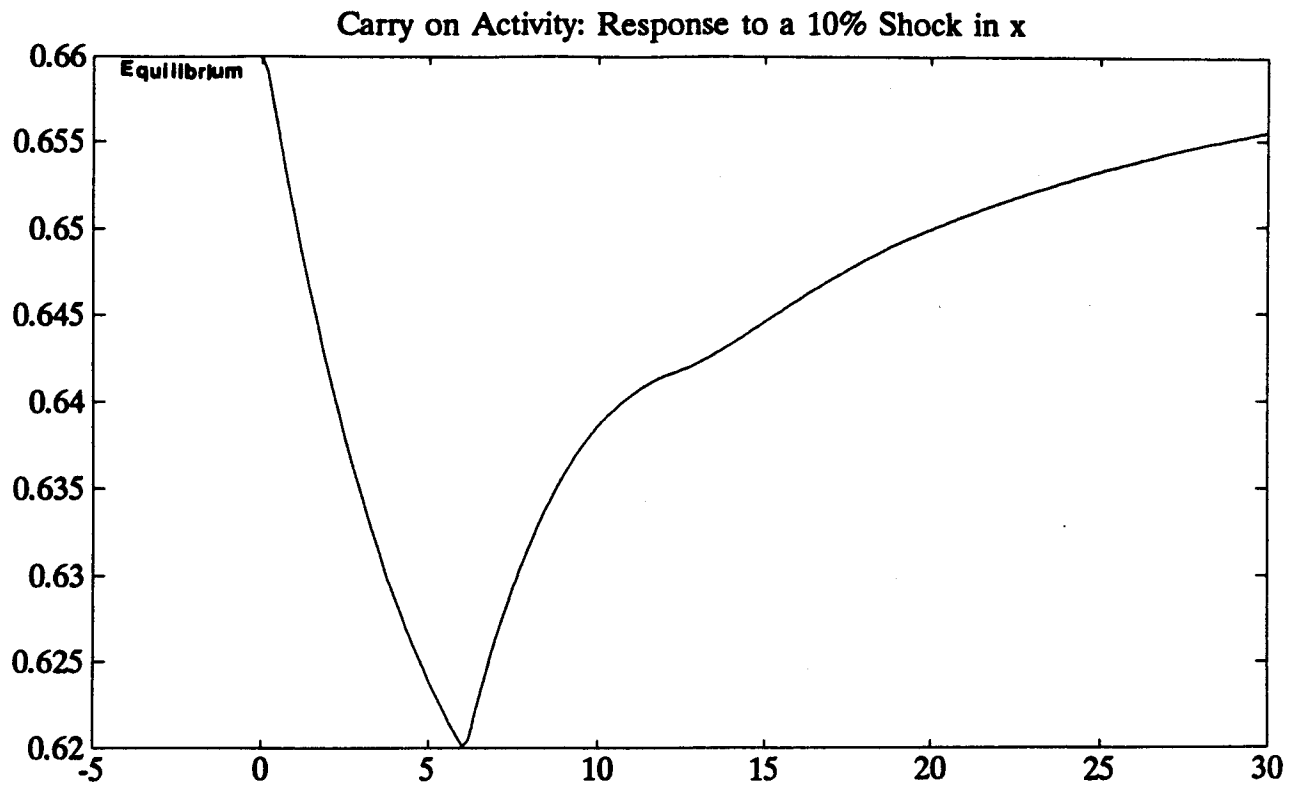


Figure 10.

Legend: — Carry on Activity.

Comment: Evolution of the carry on activity as a response to a ten percent shock in consumption that perturbrates the system away from an equilibrium condition. The computation is conducted on the approximating **SYSTEM 2**, using the same parameters values as in PPIP.

G.N.P: Response to a 10% Shock in x (Consumption)

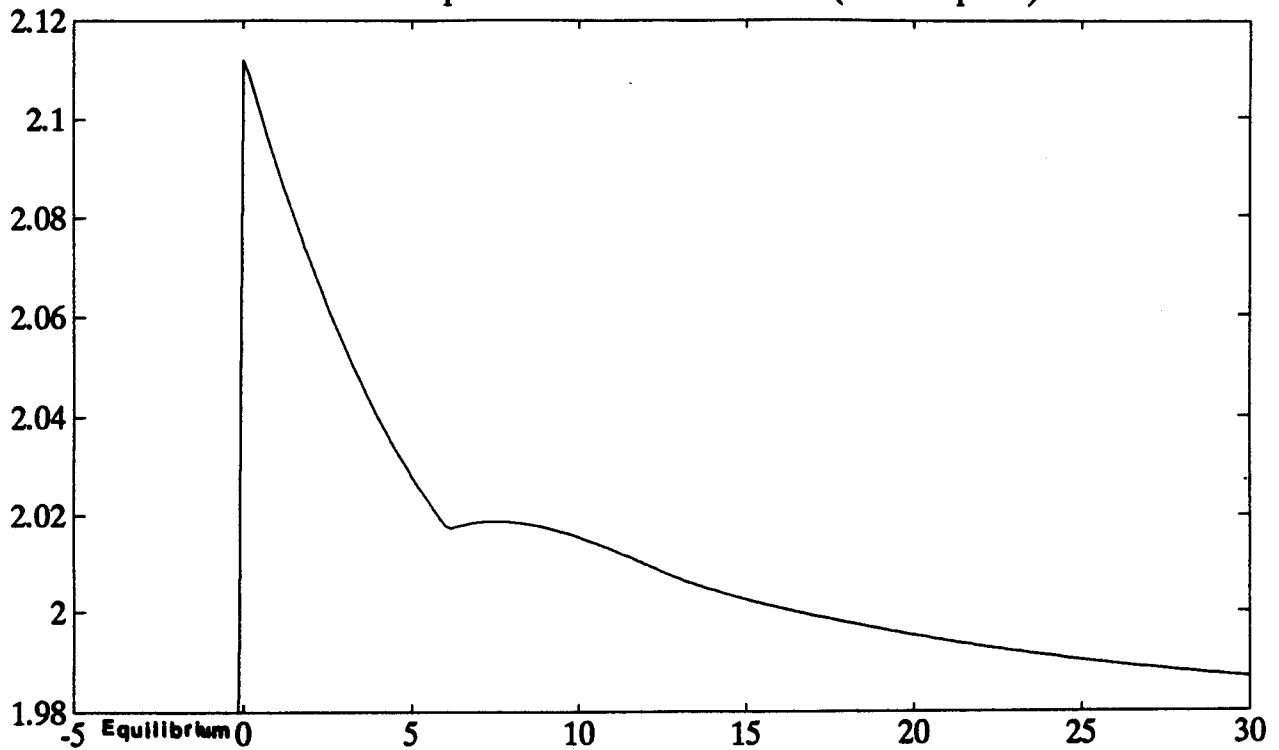


Figure 11.

Legend: — G.N.P.

Comment: Evolution of the gross national product, i.e., sum of consumption and carry on activity, as a response to a ten percent shock in consumption that perturbs the system away from an equilibrium condition. The computation is conducted on the approximating **SYSTEM 2**, using the same parameters values as in PPIP.

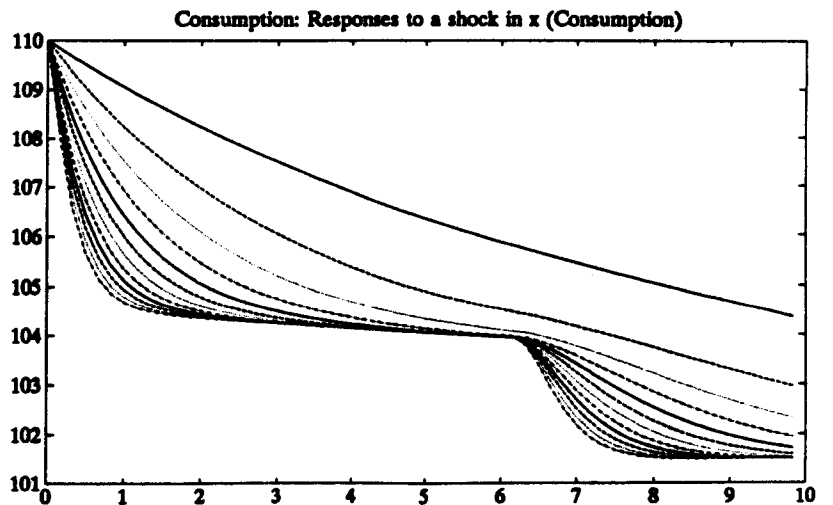
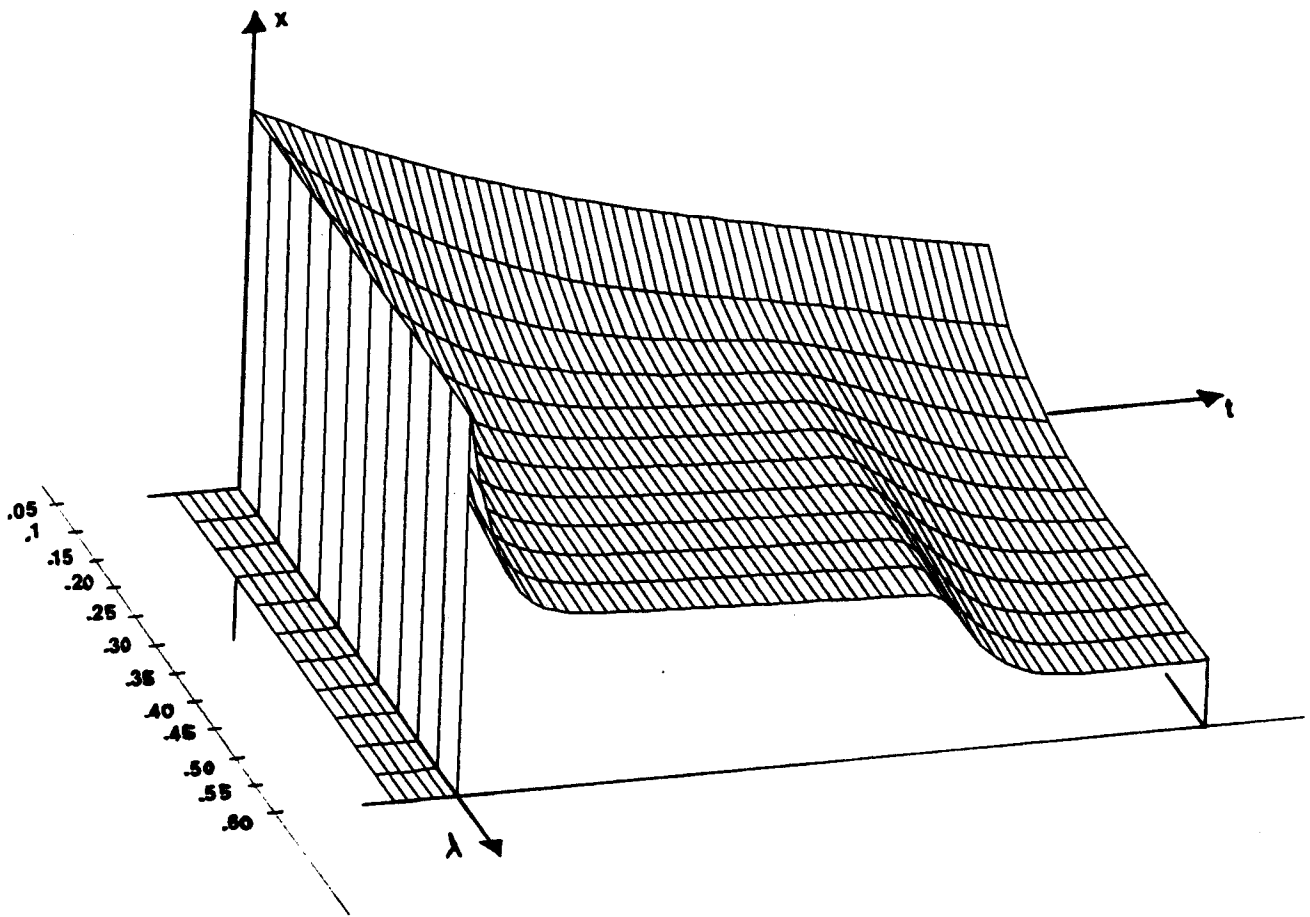


Figure 12.

Legend: — Consumption.

Comment: Evolution of consumption as a function of the structural parameter lambda (the starting impulse is a ten percent shock of the equilibrium value of consumption). The computation is conducted on the approximating **SYSTEM 2** and the equilibrium value of consumption is normalized to 100.

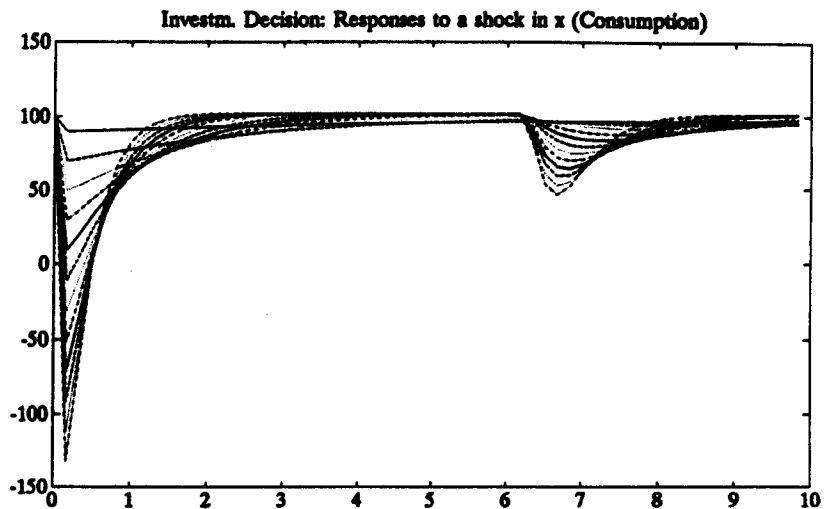
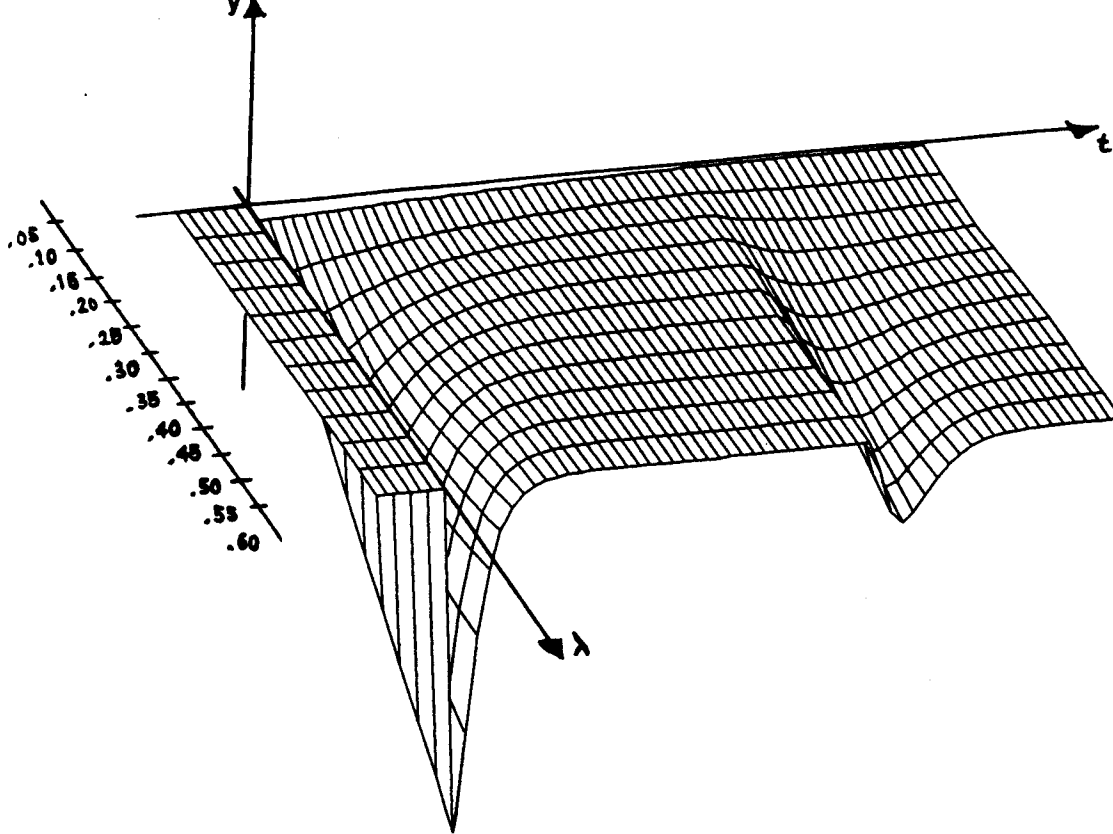


Figure 13.

Legend: _____ Production Starting, i.e., Investment Decisions.

Comment: Evolution of production starting as a function of the structural parameter lambda (the starting impulse is a ten percent shock of the equilibrium value of consumption). The computation is conducted on the approximating **SYSTEM 2** and the equilibrium value of the investment decision is normalized to 100.

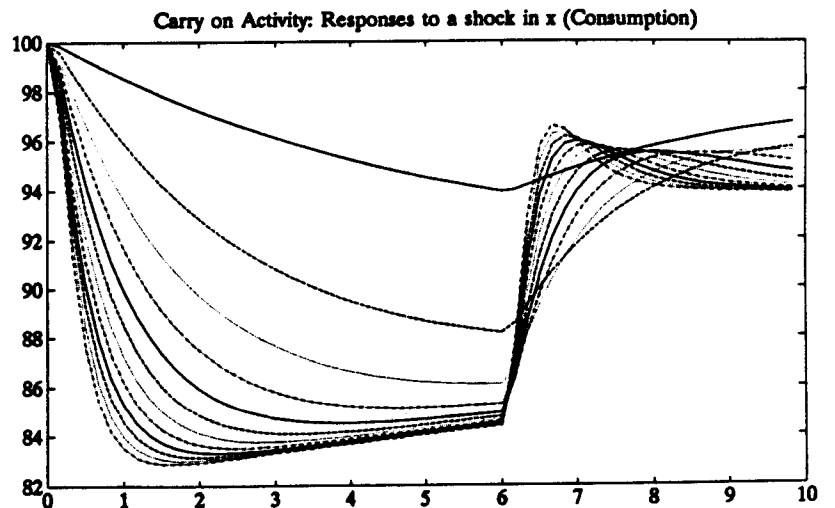
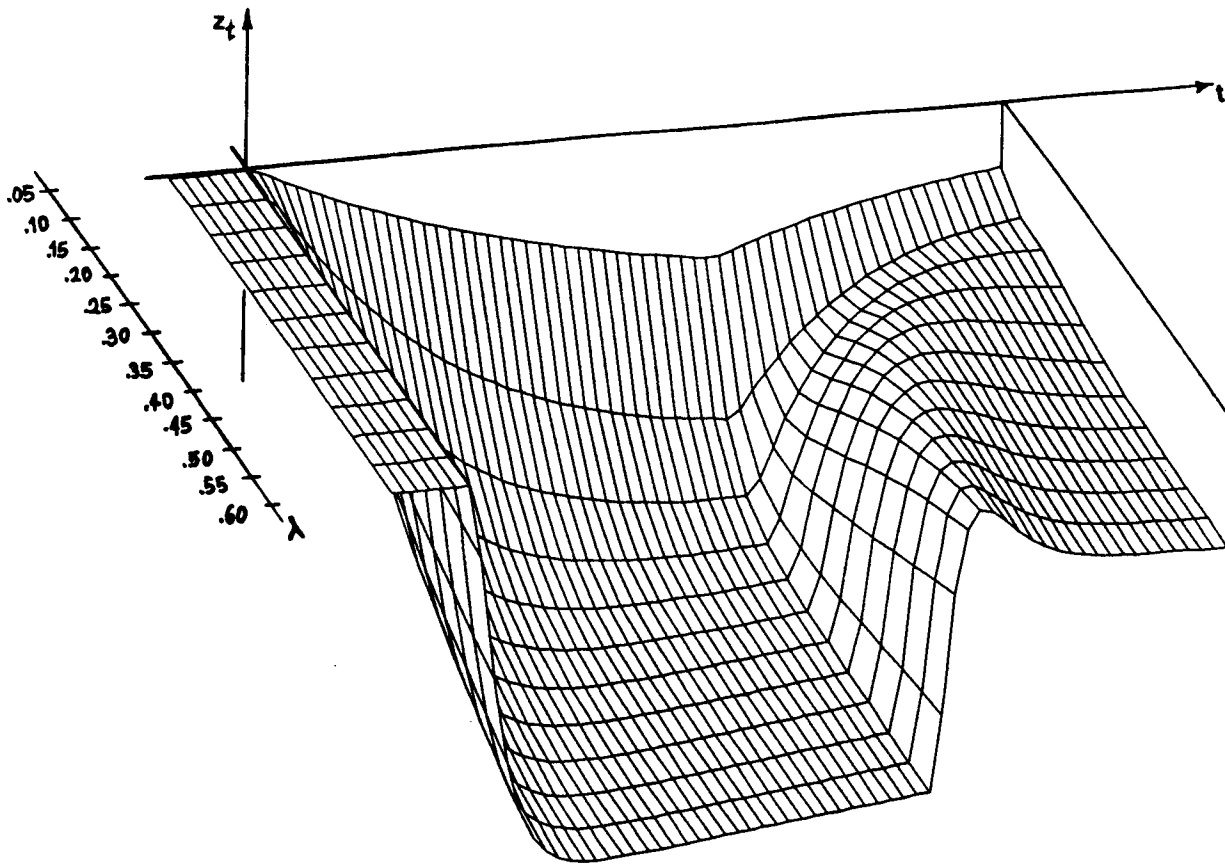


Figure 14.

Legend: — Carry on Activity.

Comment: Evolution of the carry on activity as a function of the structural parameter lambda (the starting impulse is a ten percent shock of the equilibrium value of consumption). The computation is conducted on the approximating **SYSTEM 2** and the equilibrium value of the carry on activity is normalized to 100.

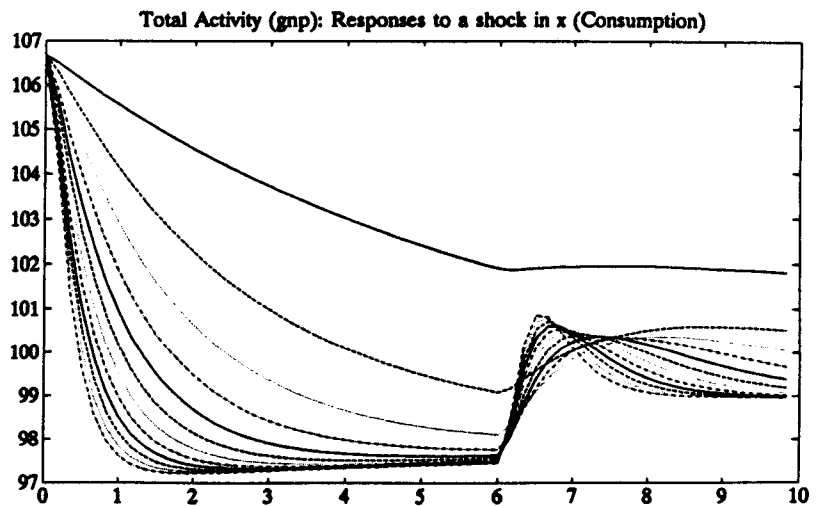
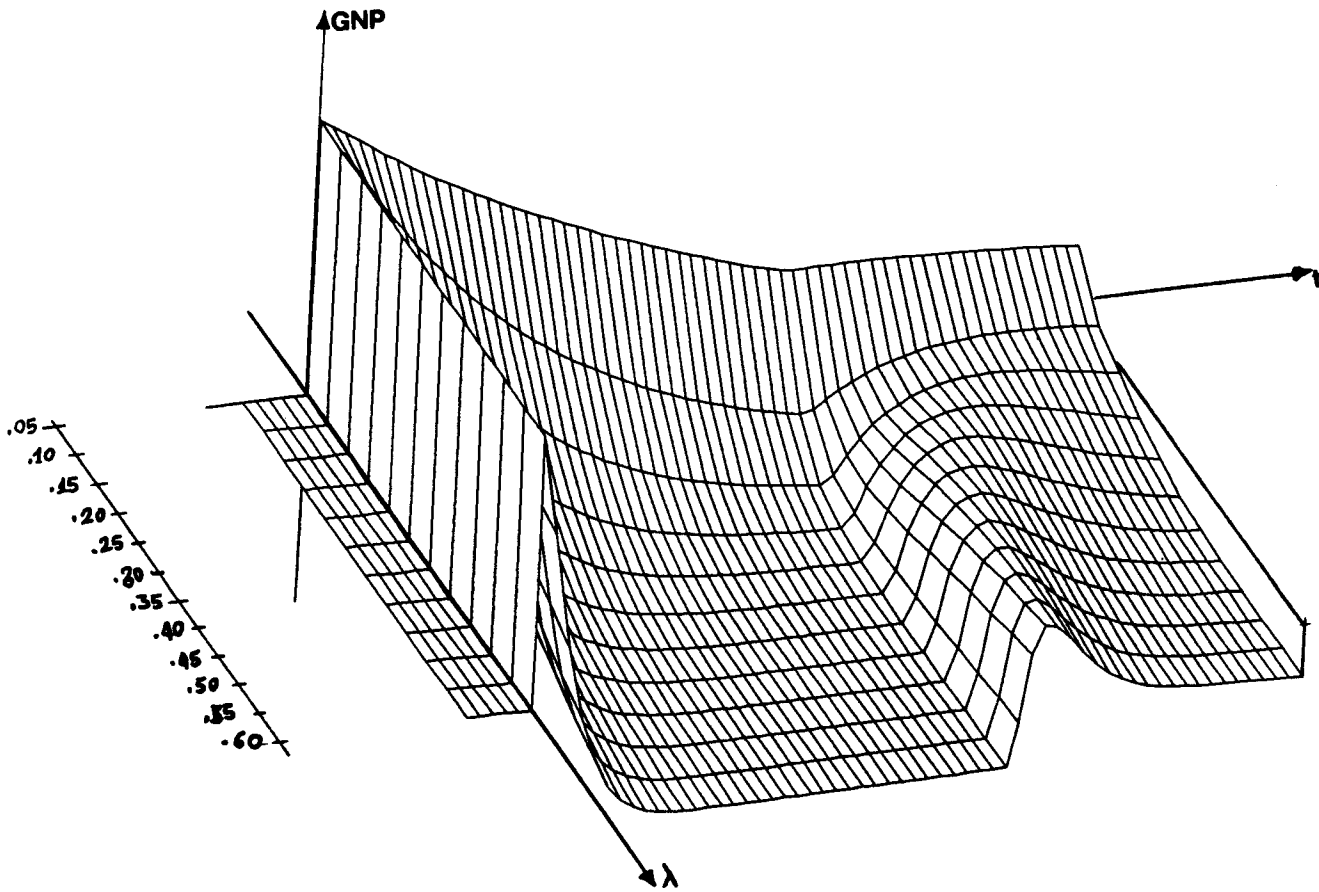


Figure 15.

Legend: — G.N.P.

Comment: Evolution of the gross national product as a function of the structural parameter lambda (the starting impulse is a ten percent shock of the equilibrium value of consumption). The computation is conducted on the approximating **SYSTEM 2** and the equilibrium value of gross national product is normalized to 100.