Equilibrium Borrowing and Lending with Bankruptcy

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Abstract

In the long-term relationship with a borrower, the lender has a time inconsistency problem because the lender cannot make a binding commitment with respect to the credit limits. This paper establishes that credit rationing may result from the time inconsistency even in the absence of asymmetric information or the lack of repayment enforcing mechanism.

We construct a two-person game based on a model originally developed by Hellwig(1977). In the game we define the "credible credit limit" as the debt level beyond which there is no loan with a positive expected return to the lender. Using this definition we characterize the equilibrium as follows. First, if the lender has a negative expected return from the loan up to the credible credit limit, there exists a continuum of equilibria which all result in borrowing which is strictly less than the credible credit limit. This result indicates that credit rationing may exist due to time inconsistency apart from asymmetric information or the unavailability of enforcement mechanisms. Second, in the case where the lender has a positive expected return from the loan up to the credible credit limit, the subgame perfect equilibrium results in a unique outcome in which the lender always extends the loan up to the credible credit limit.

1 Introduction

Many economic models assume that an individual may borrow to smooth out the consumption path over time as far as it does not exceed one's total resources. Considering the pervasive presence of borrowing constraints in reality, however, the frictionless borrowing is obviously an extreme idealization. On the other hand, the literature on the borrowing problem usually seeks to explain the presence of borrowing constraints with asymmetric information or the unavailability of repayment enforcing mechanisms.[1] The present paper attempts to analyze the problem of borrowing and lending from a different perspective; in the long-term relationship with a borrower, a lender has a time inconsistency problem because the lender cannot make a binding commitment with respect to the credit limits. We explain the presence of borrowing constraints as a consequence of the time inconsistency problem of the lender.

Time inconsistency arises in the following way. Consider a borrower with uncertain future income. Specifically the borrower has no income initially but he may become rich at an unknown time in the future. At the beginning of the relationship with the borrower, the lender may set a credit limit which maximizes his expected return if the borrower consumes subject to the given credit limit. If the borrower is unable to repay until after his borrowing hits the given credit limit, however, it may be renewed to a bigger one; otherwise the borrower has to go bankrupt immediately and the lender gets nothing back from his lending. Taking account of the anticipated increase in the credit limit, the borrower consumes more at each moment of time than he would subject to the outstanding credit limit. A higher consumption path shortens the time during which the given credit limit is exhausted and lowers the probability of the borrower's becoming rich while consuming the given credit. Consequently, the lender's expected return from the loan may become negative. Notice that the lender has the incentive to renew the outstanding credit limit at the time the borrowing reaches it even if the borrower has consumed subject to the additional credit limit.

From the explanation, it is immediate that the uncontrollability of the consumption path by the lender is a necessary element of the problem. In-

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[1] See Bulow and Rogoff(1989a), Gale and Hellwig(1985), Green(1987), and Stiglitz and Weiss(1981), for such attempts.

comes implies difficulties in predicting the equilibrium play of the lender and the borrower. An interesting consequence of the multiplicity of equilibrium is that whenever credit rationing arises in the equilibrium, there is a continuum of equilibria with different amount of equilibrium loan. Consequently the degree of credit rationing cannot be determined by the theory.

Our result can explain interesting situations including the sovereign debt problem. If it is postulated that the borrowing countries try to exploit the time inconsistent nature of the international banks, the frequent rescheduling of the sovereign debt services may be explained as equilibrium phenomena where the renewal of the credit limit occurs due to the changing incentive of the latter.

Before the formal analysis of the model, we briefly review the literature on the borrowing problem. Hellwig(1977) raised the issue of the time inconsistency of the lender from a different perspective. His paper was concerned with the non-existence of time consistent courses of action. In contrast to his paper we focus on the characterization of time consistent courses of action when they exist. Moreover his result cannot be regarded as a theoretical justification of the presence of borrowing constraints because it predicts no existence of equilibrium.

One of the dominant explanations of credit rationing in the literature relies on the asymmetric information as in Jaffee and Russell(1976) and Stiglitz and Weiss(1981). For instance Stiglitz and Weiss attempted to explain the credit rationing with asymmetric information as to the characteristics of the borrower and the borrower's action choice which affects the prospect of the future income. They could explain why a borrower is either granted a loan or rejected totally. In contrast our model does not rely on the asymmetric information. In addition, we could explain partial restriction as well as total rejection of the loan.

Recently difficulties in enforcing the repayment were given much attention in the literature as the source of borrowing constraints. Hart and Moore(1989) analyzed the debt problem in a dynamic setting with renegotiation. The model assumes that enforcing repayment of debt is costly because the collateral is more valuable to the borrower than to the lender. Bulow and Rogoff(1989a) analyzed the sovereign debt problem focusing on the reputation effect of the borrower in the international capital market where the lender does not have enforcement mechanisms. These two models depart from the present one in that costly enforcement of repayment is the driving

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credit limit.

The borrower is forced into bankruptcy if he cannot pay the interest accrued on the outstanding debt. The borrower may pay the interest by borrowing from the lender. If the lender decides not to increase the credit limit when the debt equals the credit limit, the borrower is forced into bankruptcy because the borrower cannot make the interest payment even out of borrowing. Alternatively the borrower may choose to go bankrupt any time before he is rich.

When the borrower is bankrupt, he incurs bankruptcy penalty $p(k)$ which depends on the amount of debt k outstanding at the time of bankruptcy, whether he does it voluntarily or he is forced into it by the lender. When the borrower goes bankrupt, the lender gets nothing back from the lending.

When the borrower becomes rich, he converts his risky debt into a riskless security because with certainty, his income is sufficient to pay the interest on the debt.[4] Hence after becoming rich, the borrower consumes the remainder of his income after the interest payment and the lender's return from the rich borrower is the discounted sum of riskless interest stream accrued to the debt outstanding at the moment the borrower becomes rich.

The borrower's income is common knowledge and there is an institution which forces the rich borrower to pay the interest on his debt. Therefore the rich borrower may not go bankrupt consuming his whole income without paying the interest.

We make assumptions on the borrower's preference, the stochastic process governing the timing of borrower's income increase, and the magnitude of various parameters.

Assumption 1 *The borrower has a utility function $u(c)$ which is continuously differentiable, strictly increasing, strictly concave, and bounded above and below and discounts the future by the discounting factor δ.*

Assumption 1 is standard for a risk averse utility function.

Assumption 2 *Income $y(t)$ is a random variable having values 0 and a, such that*

1. $y(0) = 0$,

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[4]This will be guaranteed by Assumption 4 later.

borrow in excess of the credit limit and that the initial debt is 0, while the third row implies that the borrower may not make a negative consumption.

Problem (B) can be reformulated using the fact that the problem after the borrower becomes rich is a optimal consumption problem under certainty because the borrower's income changes only once by Assumption 2. The following problem (R) is the rich borrower's optimal consumption problem whose income change occurs with the outstanding debt k_0.

(R)

$$\overline{V}(k_0) = \max_{c(t)} \int_0^\infty e^{-\delta t} u(c(t)) dt \tag{2}$$

$$s.t. \begin{cases} \dot{k}(t) = rk(t) + a - c(t), \\ k(t) \geq -A, \ k(0) = k_0, \\ c(t) \geq 0. \end{cases}$$

Now Problem (B) is reformulated using $\overline{V}(k_0)$ as the maximized value of Problem (R).

(B')

$$V(0, A) = \max_{c(t)} \int_0^T e^{-(\delta+\lambda)t} [u(c(t)) + \lambda \overline{V}(k(t))] dt + e^{-(\delta+\lambda)T} B(k(T)) \tag{3}$$

$$s.t. \begin{cases} \dot{k}(t) = Rk(t) - c(t), \\ k(t) \geq -A, \ k(0) = 0, \\ c(t) \geq 0 \end{cases}$$

where T is the time of bankruptcy.

Notice that $B(k(T))$ denotes the value of bankruptcy which happens at time T with the outstanding debt $k(T)$. Because the bankrupt borrower consumes nothing until he becomes rich and incurs bankruptcy penalty $p(k)$ depending on the debt he owes, the value of bankruptcy is written as:

$$B(k) = \int_0^\infty e^{-(\delta+\lambda)} [u(0) + \lambda \overline{V}(0) - p(k)] dt \tag{4}$$

$$= \frac{u(0) + \lambda \overline{V}(0) - p(k)}{\delta + \lambda}. \tag{5}$$

The following assumption is made on the bankruptcy penalty $p(k)$.

term is the consumption loan net of the interest at time t if he is not rich. The variables in the lender's objective function are implicitly functions of the credit limit A and thus the lender's objective function is maximized with respect to the credit limit, A.

We denote the lender's expected return from the loan with initial debt k_0 and the credit limit A as $P(k_0, A)$. The return $P(k_0, A)$ is computed using the solution to the borrower's problem (B') where the borrower maximizes subject to the credit limit A as if there is no further increase in the credit limit.

Finally we make an assumption on the credit limit which restricts the lender's strategy.

Assumption 5 $A \leq A_{\max} = \frac{a}{r}$.

By Assumption 5, the lender cannot set the credit limit in excess of the discounted sum of the rich borrower's income, a/r. Implicit in the assumption is that the lender is not allowed to forgive a part of the debt. If the debt grows bigger than $A_{\max}$, even the rich borrower cannot pay the interest out of his income. Because the borrower goes bankrupt if he cannot pay the interest out of his income or borrowing, the lender should forgive a part of the debt to avoid the bankruptcy of the rich borrower. Hence the assumption is equivalent to the no forgiveness of debt.[5]

3   The Game, Γ

Next we formulate the model as a two person game, Γ, allowing explicitly the lender to increase the credit limit when the borrowing reaches it. Consequently each agent makes more than one move. We use the agent normal form in which the same player making different moves are considered as different players; we call the lender making the nth move the nth lender and similarly for the borrower. Therefore the first lender moves at the beginning of the game and the second lender moves when the debt grows to the credit limit set before and so on, and similarly for the borrower.

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[5]Without the assumption, there is not a subgame perfect equilibrium in Markov strategies in the subgame after the debt reaches $A_{\max}$. See the discussion on the non-existence of equilibrium in the analysis.

paths from T_0 to T_1, T_1 to T_2, and so on. They denote the strategy profiles of the lender and the borrower, respectively.

Next we define the payoff functions of the players. The payoff functions have a recursive structure because the payoff at a certain moment of time is the sum of the payoffs from the contemporary play and from the future play. In the definition, we use the notation, $A_{-n} = (A_{n+1}, A_{n+2}, \ldots)$, and $c_{-n}(t) = (c_{n+1}(t), c_{n+2}(t), \ldots)$.

Definition 1 *The lender's payoff function, $\{P_n\}_{n=1}^{\infty}$, and the borrower's pay-off function, $\{V_n\}_{n=1}^{\infty}$, are defined recursively by*

$$P_n(A_n, A_{-n}; c(t)) = \int_{T_{n-1}}^{T_n} e^{-(r+\lambda)t}(\lambda(-k(t)) - c_n(t))dt$$
$$+ e^{-(r+\lambda)(T_n - T_{n-1})} P_{n+1}(A_{n+1}, A_{-(n+1)}; c(t)) \quad (7)$$

and

$$V_n(A; c_n(t), c_{-n}(t)) = \int_{T_{n-1}}^{T_n} e^{-(\delta+\lambda)t}(u(c_n(t) + \lambda \overline{V}(k(t)))dt$$
$$+ e^{-(\delta+\lambda)(T_n - T_{n-1})} V_{n+1}(A; c_{n+1}(t), c_{-(n+1)}(t)) \quad (8)$$

$$s.t. \begin{cases} \dot{k}(t) = Rk(t) - c_n(t), \\ k(t) \geq -A_n, \quad k(T_{n-1}) = -A_{n-1}, \\ c_n(t) \geq 0, \end{cases}$$

where $k(0) = A_0 = 0$, and

$$\text{if } A_{n-1} = A_n, \text{ then } \begin{cases} P_i(A_i, A_{-i}; c(t)) = 0 \text{ for all } i \geq n, \\ V_n(A; c_n(t), c_{-n}(t)) = B(-A_n), \\ V_i(A; c_i(t), c_{-i}(t)) = 0 \text{ for all } i \geq n+1. \end{cases} \quad (9)$$

In the definition above, the present payoffs, $P_n(\cdot; \cdot)$ and $V_n(\cdot; \cdot)$, depend on the future play via the future payoffs, $P_{n+1}(\cdot; \cdot)$ and $V_{n+1}(\cdot; \cdot)$, but not vice versa. The equations in (9) represent the players' payoffs when the nth lender decides not to increase the credit limit. We do not consider the payoffs after

additional loan with positive expected return to the lender. Standing by itself such a credit limit provides the lender enough incentive to stop increasing credit limit. We define this credit limit as the *credible credit limit.*[8]

Definition 4 *The <u>credible credit limit</u>, $\overline{A}$, is the smallest real number satisfying,*

$$\overline{A} = A_{max},$$

or

$$P(-\overline{A}, A) < 0, \quad \text{for all } A \in (\overline{A}, A_{max}].$$

Recall that $P(k_0, A)$ is the expected return to the lender from a loan up to A starting with the initial debt k_0. The significance of the initial debt k_0 in $P(k_0, A)$ lies in that if the credit limit is increased to a bigger one A when the outstanding debt equals the present credit limit k_0, the lender prevents the borrower from going bankrupt immediately, that is, it has the effect of keeping the debt alive until the next decision moment. Therefore a bigger outstanding debt tends to make the expected return from the additional loan positive because increasing the credit limit at a bigger outstanding debt prevents an immediate default of a bigger outstanding debt. The credible credit limit $\overline{A}$ is the amount of debt at which the benefit of keeping the debt alive is dominated by its cost. By definition, for all $A < \overline{A}$, there always exits an additional credit limit $A' \in (A, \overline{A}]$ with positive expected return and $\overline{A}$ is unique.

To avoid the difficulties of non-existence of equilibrium later, we discuss the source of the non-existence and subsequently rule it out by making proper assumption. The non-existence arises in two ways: the one noted by Hellwig and the one arising when debt can be forgiven. Hellwig's case arises when there is no credible credit limit which has a positive expected return standing by itself. The definition of the credible credit limit is distinctly asymmetric and consequently it is possible that the credible credit limit does not yield a positive expected return starting from any debt level, that is, there may not exist any $A < \overline{A}$ such that $P(-A, \overline{A}) \geq 0$, that is, for any $A < \overline{A}$, there exists $A' \in (A, \overline{A})$ such that $P(-A, A') \geq 0$ but $P(-A, \overline{A}) < 0$. If this holds true for the credible credit limit, then there is no subgame perfect equilibrium;

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[8]Our definition of the credible credit limit corresponds to that of the "naive cut-off point", A^N, in Hellwig's model.

Lemma 1(b) implies that given the same amount of debt, the consumption before becoming rich is smaller than after becoming rich. By Lemma 1(c), bankruptcy occurs only if the lender refuses to renew the credit limit or the debt grows to $A_{\max}$ in which case the income of the rich borrower is not enough to pay the interest.

Given the characterization of the borrowing pattern, we can characterize the lender's expected return $P(k_0, A)$. All proofs are relegated to the appendix.

Lemma 2 *Given* $A \geq 0$, $P(k_0, A)$ *is continuous in* $k_0 \in [-A, 0]$.

Lemma 3 *If* $-\frac{cu''(c)}{u'(c)} \geq \frac{R-\delta}{R+\lambda}$ *for* $c(t)$ *such that* $-\lambda k(t) - c(t) = 0$, *then* $P(k_0, A)$ *is decreasing in* k_0 *when* $P(k_0, A) \leq 0$.

Lemma 2 is obvious and Lemma 3 indicates that the lender's expected return for a fixed credit limit crosses zero downward at most once. The two lemmas indicate that the graph of $P(k_0, A)$ as a function of the initial debt k_0 can be drawn either as in Figure 1 or Figure 2. The implication of the diagram is that if the expected return from a loan up to a credit limit starting with a certain level of initial debt is positive, then it is positive for all loans up to the same credit limit starting with a bigger initial debt. Hence once the lender has lent to k_0 for which $P(k_0, A)$ is positive, he cannot stop lending until the credit limit A.

The condition of Lemma 3 requires that the relative risk aversion is greater than a certain number which depends on the parameters of the problem. The number would be fairly small in most cases so that most of well-behaved utility functions satisfy the condition. Under the condition the consumption does not increase too fast relative to the debt. In the following, we assume that the condition of Lemma 3 holds.

Figure 1 and Figure 2 imply that the lender's expected return from the loan up to the credible credit limit starting from zero debt is either positive or strictly negative. The sign of $P(0, \overline{A})$ which is the expected return from the loan up to the credible credit limit starting from zero debt is crucial in determining whether credit rationing arises in the equilibrium.

The result of Lemma 5 can be visualized as in Figure 3. The sequence $\overline{A}_i$ in Figure 3 represents the debt level at which the lender is indifferent between the renewal and the refusal of additional loan. In the proof of Proposition 1 and Proposition 2, we use the fact that once the debt level hits $-\overline{A}_i$, the lender is indifferent between the refusal and renewal of the loan up to $-\overline{A}_{i-1}$ so that he can stop lending or use mixed strategy. Proposition 1 characterizes the subgame perfect equilibrium where the lender stops lending when he is indifferent between the renewal and the refusal of a loan.

Proposition 1 *If $P(0, \overline{A}) < 0$, there exists a subgame perfect equilibrium with positive borrowing less than the credible credit limit in which the lender always refuses to grant a loan with zero expected return.*

The subgame perfect equilibrium in Proposition 1 implies the presence of credit rationing because the equilibrium borrowing is less than the credible credit limit. The credit rationing in Proposition 1 is different from that in Stiglitz and Weiss(1981) where the borrower is either granted the total amount of the loan which he wants or rejected totally; the agent in our model is given a loan less than his expected wealth can support.

On the other hand, it seems intuitively appealing that the lender may not grant any loan initially in the case $P(0, \overline{A}) < 0$ for the following reason. If the borrower is not sure that the lender stops lending when the lender is indifferent between the renewal and the refusal of a loan with zero expected return, the borrower may consume subject to the credible credit limit leaving the lender a negative expected return. Indeed the intuition is proved to be true in the next proposition.

Proposition 2 *If $P(0, \overline{A}) < 0$, there exists a subgame perfect equilibrium with no positive borrowing in which off the equilibrium path the lender randomizes between the renewal and refusal of a loan with zero expected return.*

The proof of Proposition 2 relies on mixed strategy off the equilibrium path although only pure strategy is used along the equilibrium path. The intuition of the proof can be explained as follows. Consider the lender's strategy of not lending at all. It is obviously a Nash equilibrium because the borrower cannot affect the outcome and the lender can guarantee himself zero return. However, it is not a subgame perfect equilibrium, because the

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Proposition 3 *If $P(0, \overline{A}) < 0$, there exists a continuum of equilibria whose outcomes all result in the borrowing less than the credible credit limit.*

The continuity of equilibria arises because the equilibrium with no more borrowing can be attached in the subgame after a certain amount of borrowing. The multiplicity of equilibrium implies that diverse observations of lending and borrowing behavior may indeed result from equilibrium plays. It is interesting to notice that whenever credit rationing arises in the equilibrium, there is a continuum of equilibria and consequently the degree of credit rationing cannot be determined by the theory.

4.2 Equilibrium without Credit Rationing

In this section we consider the case $P(0, \overline{A}) \geq 0$. The equilibrium characterization of this case is simple, yet has some interesting feature; although the lender sets the credit limit and decides whether to make loans, the lender has no control over the borrower in the equilibrium. We start with a lemma which characterizes $P(k_0, \overline{A})$. The lemma implies the expected return of the lender as drawn in Figure 1.

Lemma 6 *Given any $A \geq 0$, if $P(0, A) \geq 0$, then $P(k_0, A) \geq 0$, for all $k_0 \in [-A, 0]$.*

Lemma 6 implies that if $P(0, \overline{A}) \geq 0$, the expected return to the lender is positive throughout when the borrower consumes subject to the credible credit limit. Therefore the lender always extends the loan up to the credible credit limit.

Proposition 4 *If $P(0, \overline{A}) \geq 0$, $(A^*; c^*(t))$ is the subgame perfect equilibrium in pure strategies, if $A^* = (A_1^*, A_2^*, \ldots)$ is any increasing sequence such that*

$$\sup_{n} \{A_n^*\} = \overline{A},$$

and strictly increasing up to $\overline{A}$, and $c^*(t) = (c_1^*(t), c_2^*(t), \ldots)$ is the solution to the problem $V(0, \overline{A})$ satisfying

$$\int_{T_{n-1}}^{T_n} e^{-Rt} c_n(t) dt = e^{-R(T_n - T_{n-1})} A_n - A_{n-1}.$$

Moreover, all the subgame perfect equilibria support a unique outcome.

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$P(0,\frac{1}{r})$	$\frac{1}{\lambda}$ (expected time to be rich)		
$r = 3.6\%$	12 yrs	10 yrs	8 yrs
	4.8%	-6.414	-2.107	-.043
R	6.6%	-4.318	.202	2.373
	8.4%	-2.790	1.909	4.223

Table 1: Lender's Expected Return

it is necessary to check that the maximum credit limit is locally profitable which holds true for all parameter values in the example.

After solving for the consumption path, we can compute the creditor's expected return from lending up to the credible credit limit. We fix the riskless rate at .003 and compute the expected return for various combinations of the parameter values of R and λ. The riskless rate of .003 is approximately the monthly interest rate when the yearly rate is 3.6%. It should be noted, however, that only the relative magnitude of the riskless rate in comparison with other parameters, R and λ, is important.

Table 1 shows the lender's expected return for various parameter values. It is checked that for all the parameter values, the maximum credit limit is locally profitable so that it is the credible credit limit. In the table, the higher R and λ, the bigger the expected return. The cases with negative expected return correspond to equilibrium with credit rationing and the ones with positive expected return to the one without credit rationing, respectively.

It is interesting to notice that there exists a low λ for which raising R does not make $P(0, \overline{A})$ positive. Hence if the expected time to be rich is long, the credit rationing may not disappear even if the lender raises the risky interest rate R.

5 Conclusion

This paper establishes that credit rationing may arise solely due to the time inconsistency problem of the lender. The equilibrium may be characterized as a total rejection or a partial award of a loan which is less than the amount of expected future wealth. Because of the multiplicity of the equilibrium, the

Appendix

Lemma 2 *Given $A \geq 0$, $P(k_0, A)$ is continuous in $k_0 \in [-A, 0]$.*

Proof: Given $A \geq 0$, the optimal consumption path, $\bar{c}(t)$, is uniquely determined by solving the problem $V(0, A)$, and is continuous in t. The debt level, $\bar{k}(t)$ is also continuous in t and, moreover, strictly decreasing in t. Therefore the inverse of $\bar{k}(t)$, denoted as $\bar{k}^{-1}(\cdot)$, is continuous and decreasing in the debt level k. Fix a debt level $k_0 = \bar{k}(\tau)$, and equivalently the time at which the debt level reaches k_0 when consuming according to $\bar{c}(t)$ from 0 debt level, that is, $\tau = \bar{k}^{-1}(k_0)$. Because the consumption path from any initial debt level greater than 0 up to the same credit limit is identical by the principle of optimality, the lender's expected return from setting a credit limit A at the initial debt level of k_0 is written as the following:

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\begin{aligned}
P(k_0, A) &= \int_0^{\bar{T}} e^{-(r+\lambda)t} [\lambda(-\bar{k}(t)) - \bar{c}(t)] dt \\
&= \int_{\bar{k}^{-1}(k_0)}^{T} e^{-(r+\lambda)(t - \bar{k}^{-1}(k_0))} [\lambda(-\bar{k}(t)) - \bar{c}(t)] dt \quad (13)
\end{aligned}
$$

where T is the time of bankruptcy for the problem $V(0, A)$.

Obviously $P(k_0, A)$ is continuous in the lower limit of the integration, $\bar{k}^{-1}(k_0)$. Because $\bar{k}^{-1}(k_0)$ is continuous in k_0, $P(k_0, A)$ is continuous in k_0. ∎

Lemma 3 *If $-\frac{cu''(c)}{u'(c)} \geq \frac{R-\delta}{R+\lambda}$ for $c(t)$ such that $-\lambda k(t) - c(t) = 0$, then $P(k_0, A)$ is decreasing in k_0 when $P(k_0, A) \leq 0$.*

Proof: We denote optimal solution of the borrower's problem by bar over the variable as in the proof of Lemma 2. First notice that if the integrand of $P(k_0, A)$, $-\lambda \bar{k}(t) - \bar{c}(t)$, is increasing in t everywhere, then we are done because to the left of t for which $-\lambda \bar{k}(t) - \bar{c}(t) = 0$, $P(k_0, A)$ is decreasing in k_0.

To get the lemma, we only need to guarantee that $-\lambda \bar{k}(t) - \bar{c}(t)$ is increasing at t for which $-\lambda \bar{k}(t) - \bar{c}(t) = 0$, because in that case, $-\lambda \bar{k}(t) - \bar{c}(t)$

Therefore we have $\hat{t}$ such that $-\lambda \overline{k}(t) - \overline{c}(t) \geq 0$ for all $t \geq \hat{t}$ and vice versa. Because $\overline{k}^{-1}(k_0)$ is a decreasing function of k_0, $-\lambda \overline{k}(t) - \overline{c}(t)$ is always negative as k_0 increases from $k(\hat{t})$. It follows that

$$P(k_0, A) = \int_{\overline{k}^{-1}(k_0)}^{T} e^{-(r+\lambda)(t-\overline{k}^{-1}(k_0))}[\lambda(-\overline{k}(t)) - \overline{c}(t)]dt$$

is decreasing for all $k_0 \geq k(\hat{t})$, and it is decreasing when $P(k_0, A) \leq 0$, a fortiori. ∎

Lemma 4 If $P(0, A) < 0$, and $P(k_0, A)$ increasing at $k_0 = -A$, there exists a unique $\overline{k} \in (-A, 0)$ such that $P(\overline{k}, A) = 0$.

Proof: Because $P(k_0, A)$ is continuous in k_0 by Lemma 2 and $P(k_0, A)$ is increasing at $k_0 = -A$, we have $\overline{k} \in (-A, 0)$ at which $P(\overline{k}, A) = 0$ by the intermediate value theorem.

The uniqueness follows because by Lemma 3, $P(k_0, A)$ remains negative for all $k_0 \in [\overline{k}, 0]$ once it becomes negative at $\overline{k}$. ∎

Lemma 5 If $P(0, \overline{A}) < 0$, there exists a finite integer, $i \geq 1$, such that $P(0, \overline{A}_i) \geq 0$.

Proof: The sequence $\{\overline{A}_i\}$ exists by Lemma 4 and is strictly decreasing by Definition 5. The difference between any two consecutive $\overline{A}_i$'s is bounded away from zero. Denote the minimum difference by d. Then there exists an integer N such that $\overline{A} - nd \leq 0$ for all $n \geq N$, i.e., the sequence $\{\overline{A}_i\}$ eventually reaches 0. Because there exists a credit limit which gives the lender a positive expected return at $k_0 = 0$, we have an integer, $1 \leq i \leq N$, such that $P(0, \overline{A}_i) \geq 0$. ∎

Proposition 1 If $P(0, \overline{A}) < 0$, there exists a subgame perfect equilibrium with positive borrowing less than the credible credit limit in which the lender always refuses to grant a loan with zero expected return.

Proof: Suppose $P(0, \overline{A}_1) \geq 0$, followed by a subgame with zero expected return to the lender. It is easy to show that after a history $A_n > \overline{A}_1$, the lender's unique subgame perfect equilibrium strategy is renewing the credit

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As in Proposition 1, we first suppose that $P(0, \overline{A}_1) \geq 0$, followed by a subgame with zero expected return to the creditor.

Consider $(A^*, c^*(t))$ such that $A_1^* = 0$ and the sequence $A_n^*, n \geq 2$, is a strictly increasing sequence up to $\overline{A}$ and $c^*(t)$ solves problem $V(0, \overline{A})$. Such a strategy pair is a Nash equilibrium because given his own strategy for $n \geq 2$ and the borrower's strategy, the lender cannot initially deviate with positive expected return and given the lender's strategy the borrower's deviation does not affect the play of the game.

The above strategy characterizes the equilibrium only along the equilibrium path; For the above strategy to be a subgame perfect equilibrium, we need to check the Nash equilibrium in any subgame after deviation. There are three types of deviations possible, $A_1 > \overline{A}_1$, $A_1 = \overline{A}_1$, and $A_1 < \overline{A}_1$. We analyze the play after each type of deviation.

Case 1:

Suppose that the lender deviates to $A_1 > \overline{A}_1$. After the deviation $c^*(t)$ as above is the borrower's best response because $P(-A_1, \overline{A}) > 0$ so that by Corollary 1 $\max\{A_n\} = \overline{A}$ is the unique Nash equilibrium in the subgame $\Gamma_1^b | A_1$. The deviation makes the lender worse off because $P(0, \overline{A}) < 0$ and $P_1(A_{-1}, A_{-1}^*; c^*(t)) < 0 = P_1(A^*; c^*(t))$.

Case 2:

Consider a deviation by the lender such that $A_1 = \overline{A}_1$. After the deviation the lender is indifferent between the renewal and the refusal of any additional loan beyond it because $P(-\overline{A}_1, \overline{A}) = 0$. The lender randomizes so that the borrower is indifferent between consuming subject to $\overline{A}_1$ and $\overline{A}$, i.e.,

$$\pi V(0, \overline{A}_1) = (1 - \pi) V(0, \overline{A}), \tag{17}$$

where π is the probability of refusal at $\overline{A}_1$. Given the mixed strategy of the lender, the borrower randomizes between consuming subject to $\overline{A}_1$ and $\overline{A}$ so that the lender is not better off after the deviation, i.e.,

$$\mu P(0, \overline{A}_1) + (1 - \mu) P(0, \overline{A}) \leq 0, \tag{18}$$

where μ is the probability that the borrower consumes subject to $\overline{A}_1$.

Because $P(0, \overline{A}_1) \geq 0$ and $P(0, \overline{A}) < 0$, we can make the lender worse off by choosing small μ. The strategy after the deviation constitutes a Nash

where μ_1 is the probability to consume subject to A_1, the lender gets smaller expected return from deviation and $(A^*, c^*(t))$ is a subgame perfect equilibrium.

However it is possible that there is no μ_0 satisfying equation (20) and equation(22) simultaneously, i.e., any μ_0 satisfying

$$\mu_0 P(-A_1, \overline{A}_1) + (1 - \mu_0) P(-A_1, \overline{A}) = 0, \qquad (23)$$

gives strictly positive expected return to the lender such that

$$\mu_1 P(0, A_1) + (1 - \mu_1)[\mu_0 P(0, \overline{A}_1) + (1 - \mu_0) P(0, \overline{A})] > 0. \qquad (24)$$

because equation (23) does not necessarily imply that the term in the square brackets in equation (24) is negative. If this term is positive, equation (24) holds for any μ_1 because $P(0, A_1) \geq 0$. Therefore given the mixed strategy of the borrower the lender gets better off by deviation. To prevent profitable deviation by the lender in this case, the borrower randomizes at $k(t) = \overline{A}_1$ so that

$$\mu_0 P(-A_1, \overline{A}_1) + (1 - \mu_0) P(-A_1, \overline{A}) \geq 0, \qquad (25)$$

and

$$\mu_0 P(0, \overline{A}_1) + (1 - \mu_0) P(0, \overline{A}) \leq 0. \qquad (26)$$

This is identical to the deviation seen in step 2 above.

Any additional deviation in the subgame after the initial deviation belongs to one of the above three cases and can be handled accordingly. It completes the proof of the theorem when $P(0, \overline{A}_1) \geq 0$.

Step 2:
When $P(0, \overline{A}_i) \geq 0$, for $i \geq 2$, without loss of generality we can assume that $P(0, \overline{A}_2) \geq 0$.

Before we consider the play after a deviation, we provide a result which narrows the range of deviations we have to examine. It can be easily shown that if $P(0, \overline{A}_1) < 0$, then $P(0, A_1) < 0$ for any $A_1 \geq \overline{A}_1$. Hence any deviation by the lender such that $A_1 \geq \overline{A}_1$ definitely makes the lender worse off because even if the borrower consumes subject to A_1, the creditor gets negative expected return. From the observation we can confine ourselves to the deviation A_1 such that $P(0, A_1) \geq 0$ and $A_1 < \overline{A}_1$. Furthermore, if it is a Nash equilibrium in the subgame after the deviation that the lender renews

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Proof: Aiming at contradiction, suppose that there exists $\overline{k} \in [-A, 0]$ such that $P(k_0, A) < 0$. Then in the neighborhood to the right of $\overline{k}$, $P(k_0, A)$ is decreasing in k_0 by Lemma 3, and $P(k_0, A) < 0$ for all $k_0 > \overline{k}$. In particular, $P(0, A) < 0$ which is a contradiction to our hypothesis. ∎

Proposition 4 *If $P(0, \overline{A}) \geq 0$, $(A^*; c^*(t))$ is the subgame perfect equilibrium in pure strategy, if $A^* = (A_1^*, A_2^*, \ldots)$ is any increasing sequence such that*

$$\sup_n \{A_n^*\} = \overline{A},$$

and strictly increasing up to $\overline{A}$, and $c^*(t) = (c_1^*(t), c_2^*(t), \ldots)$ is the solution to the problem $V(0, \overline{A})$ satisfying

$$\int_{T_{n-1}}^{T_n} e^{-Rt} c_n(t) dt = e^{-R(T_n - T_{n-1})} A_n - A_{n-1}.$$

Moreover, all the subgame perfect equilibria support a unique outcome.

Proof: Given A^* as in the proposition, it is obvious that $c^*(t)$ is the best response of the borrower. Note that A^* is a strictly increasing sequence up to $\overline{A}$ because the game ends if $A_{n-1} = A_n$.

To show that A^* is the best response to $c^*(t)$, notice that if $\sup_n \{A_n^*\} < \overline{A}$, there exists another strategy, A', such that

$$\sup_n \{A_n'\} \in (\sup_n \{A_n^*\}, \overline{A}],$$

and

$$P_n(A^*; c^*(t)) \leq P_n(A', ; c^*(t))$$

for all n, with strict inequality for all $n \leq N$ for some N.

This is possible because there exits an additional credit limit with positive expected return for any credit limit less than the credible credit limit. Therefore contradiction.

Next suppose that $\sup_n \{A_n^*\} > \overline{A}$. Without loss of generality, assume $A_1^* = \overline{A}$, and $A_2^* > \overline{A}$. Then

$$P_2(A_2^*, A_{-2}^*; c^*(t)) < P_2(A_2 = \overline{A}, A_{-2}^*; c^*(t)) = 0.$$

Therefore A^* is not a best response. This proves that $(A^*, c^*(t))$ is a Nash equilibrium.
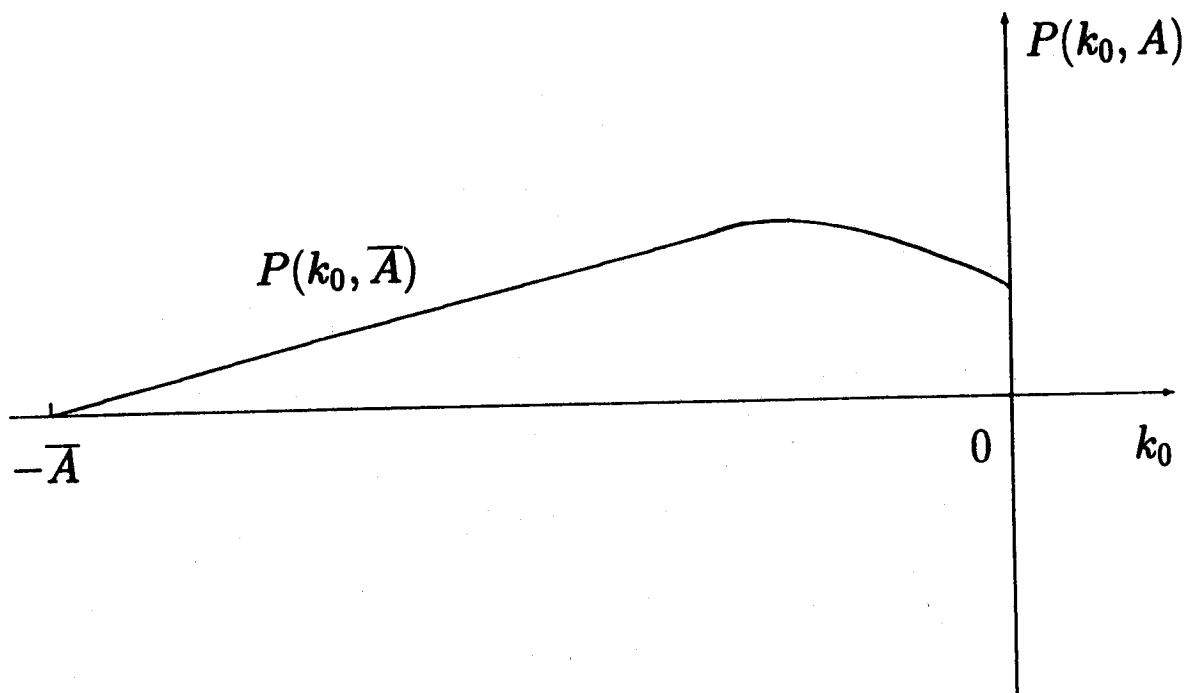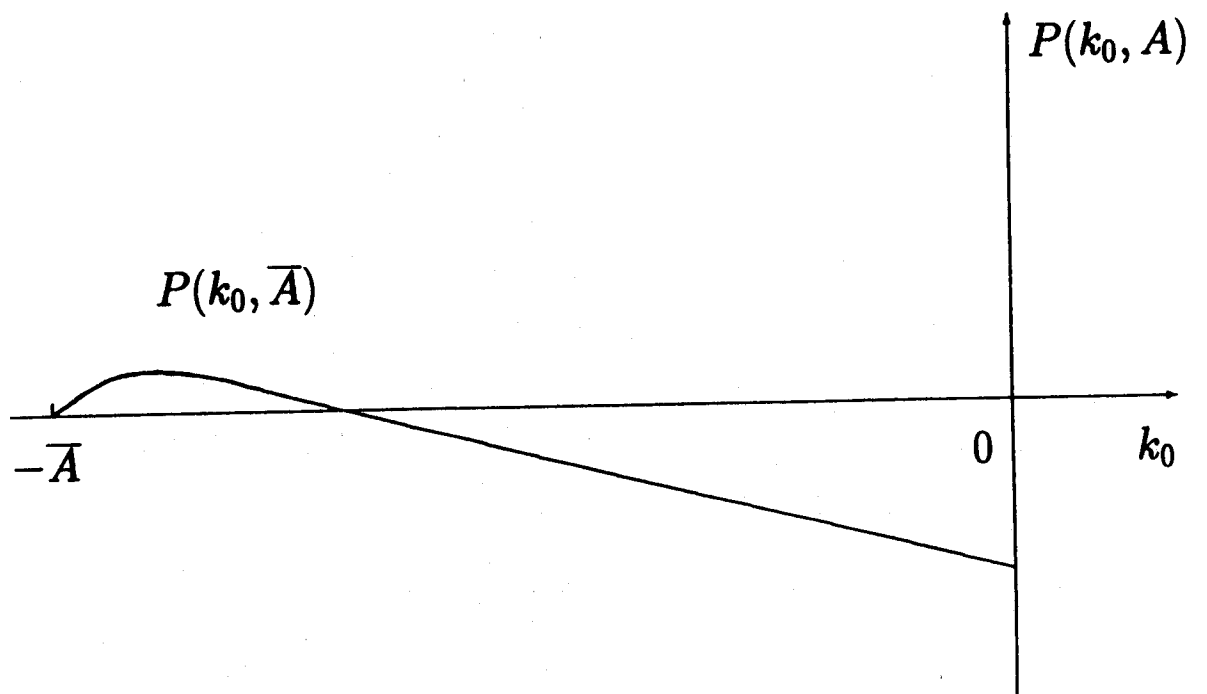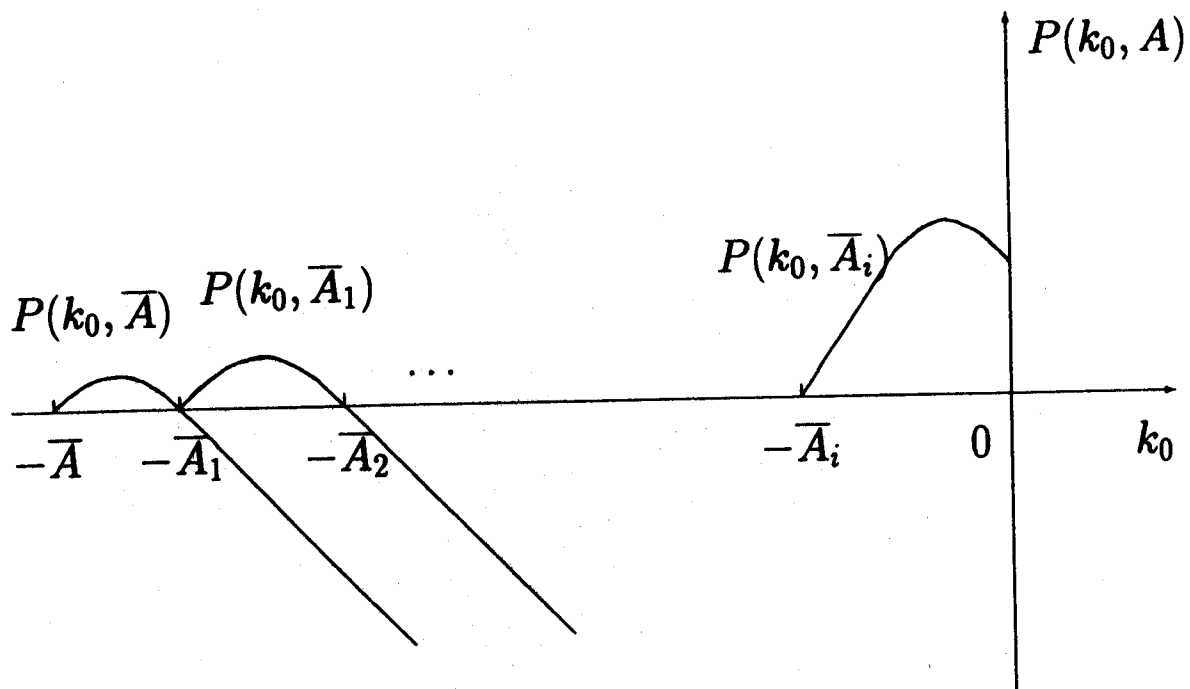
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Figure 1.

Figure 2.

Figure 3.

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