Financial Innovation in a General Equilibrium Model*

Wolfgang Pesendorfer
UCLA

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Abstract

We develop a model of financial innovation, in which intermediaries can issue new financial securities against collateral in the form of standard securities. Intermediaries have to market their innovations and marketing is costly. We show that the equilibrium asset structure may exhibit redundancies as frequently observed on financial markets. We give conditions for efficiency of financial innovation and show that with small innovation costs the indeterminacy of equilibrium allocations has small utility consequences. A possible interpretation of this result is that with small innovation costs the real allocation is almost independent of the exact specification of the political and economic process determining state contingent inflation.

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1 Introduction

The past two decades have witnessed an unprecedented wave of financial innovations in the U.S. capital markets. Stripped and collateralized mortgage obligations, putable convertibles, straps, swaps, primes and scores are only a small sample of the products developed in this period.

The purpose of this paper is to model financial innovation in a general equilibrium setting and to contrast the results with findings in models of financial markets, as pioneered by Arrow (1964) and Radner (1972); (see also Geanakoplos (1990) for a recent survey). In particular we will examine the effect of financial innovation on efficiency and determinacy of equilibrium allocations.

It is generally believed (cf. Merton (1990)) that the driving forces behind financial innovation fall into 3 categories:

(1) A demand for opportunities for risk-sharing, risk-pooling, hedging and intertemporal or spatial transfers of wealth that are currently not available.

(2) The lowering of transaction costs.

(3) Reductions in "agency" costs caused by either asymmetric information between trading parties or principals' incomplete monitoring of their agents performances.

Of the three forces behind financial innovation quoted above, we will only consider the first two and ignore the third. Issues of enforceability will, however, enter the picture when we impose constraints on what financial innovators can do.

To incorporate financial innovation into the model we assume a simple dichotomy on financial markets (see Ross (1989) for a similar structure). First there is a set of standard securities which the agents take as given. Buying and selling of these well-known, uniform securities is unrestricted for all agents. Standard securities are traded on institutionalized markets. Those market institutions provide enforcement technologies that allow households to make credible payment promises.

We assume that there is no cost of trading in standard securities. This is a fairly good approximation if we think of treasury bonds as one kind of standard securities. In other cases, e.g. mortgage loans or lease contracts, this assumption is less justified. It should be emphasized, however, that an explicit consideration of transaction costs for standard securities would not significantly alter the analysis of this paper. Thus, to keep things simple, we omit those trading costs.

Second, there are securities that are less well known and less frequently traded. These securities are created by financial intermediaries whose job it is to convert standard securities into custom-tailored instruments for their clients.

A financial intermediary can purchase a portfolio of standard securities and issue a collection of financial products against the returns of this portfolio. The intermediary must observe the following restrictions:

- All future payment promises of the intermediary must be covered by returns of the purchased portfolio.
- Intermediated securities have to pay off non-negative amounts in any future state.

For intermediated securities there are no established market institutions and therefore the traders of these securities have to be concerned with enforcement issues. Since intermediaries are forced to fully collateralize their products, they can be trusted to carry out the payment promises once a specific state of nature occurs. In addition to the two restrictions above, we prohibit short sales of intermediated assets. Thus by trading in intermediated products, households cannot make future payment promises. Together these restrictions imply that markets for intermediated securities do not need institutions to enforce future payment promises.

This simple dichotomy is clearly a crude way of dividing financial instruments. In reality most financial instruments are somewhere in between these two classes. In particular, successful innovations after some period of time succeed to climb up the hierarchy and become standard instruments. In this paper we abstract from these complications.

Financial innovation is a resource consuming activity. In particular there are marketing costs for intermediated securities. Intermediaries must look for prospective buyers, find out their demands, explain their new products, etc. We assume that there are two types of marketing costs for a new product, (1) a fixed cost *per customer* and (2) a cost proportional to the number of units sold.

We do *not* include a fixed-cost *per innovated security* in the cost structure. Tufano (1989) states that investment bankers estimate the costs of developing a new financial product to vary between $50,000 and $5 million. It is unclear, however, what proportion of these costs are actually fixed costs. Including fixed costs would force us to leave the perfectly competitive environment and hence would make the analysis considerably more difficult. Moreover, Tufano's (1989) empirical findings on pricing behavior of financial innovators are consistent with the hypothesis of competitive innovation: Investment banks that create new products do not charge higher prices in the brief period of monopoly before imitative products appear (cf. Tufano, 1989, pg.214).

Allen and Gale (1989, 1991) consider a similar process of financial innovation. In their framework firms can issue different financial structures against a given income stream. Financial innovation there is part of the corporate policy to maximize profits whereas in our framework financial intermediaries purchase standard assets to create new financial products. The innovation cost in Allen and Gale's framework is a fixed cost for every collection of securities a firm issues.

Other models of financial innovation can be found in Duffie and Jackson (1989), Cuny (1989) and Ross (1989); (see also Duffie (1990) for a recent survey).

1.1 Some Examples of Financial Innovation

In the following a few empirical examples of financial innovations of the kind discussed in this paper will be presented.

1.1.1 Mortgage Backed Securities

This type of innovation includes all securities that use bundles of mortgages as collateral. *Collateralized Mortgage Obligations (CMO's)*, for example, are bonds that are collateralized by whole loan mortgages.[1] The CMO innovation utilizes cash flows of long maturity, monthly-pay collateral to create securities with short-, intermediate- and long maturities and expected lives (Parseghian (1991)). This is done by using all the principal payment in the early periods to repay the short term security. After the short term security is retired, all the principal payment goes to the intermediate term security, and so on.[2]

The short term securities are more appealing than the underlying collateral for investors seeking low exposure to interest risk. On the other hand, since the short-term securities (short-tranches) have to be retired before longer tranches, the longer tranches possess a form of call protection. This feature appeals to investors who want less call and reinvestment risk than the original mortgage loans carry. CMO's were introduced in 1983 by First Boston and Salomon Brothers.

Stripped Mortgage-Pass-Through Securities (see Carlson and Sears (1991)) also use mortgage-pass-throughs as collateral. There are two types of stripped mortgage-pass-through securities. One that pays interest only (IO-security) and one that pays principal only (PO-security).

The purpose of stripping mortgages is to provide a way to trade the prepayment risk inherent in mortgage loans and mortgage pass through securi-

[1]Or mortgage pass-through securities which is a bond created by a large number of mortgages.

[2]CMO's exist in many variations. For a thorough description of the CMO innovation see Parseghian (1991).

ties. The holder of the PO-security benefits from the homeowners decision to prepay, while the IO-holder gains from the decision not to prepay.[3] Since the prepayment rate is negatively correlated with interest rates, the IO-security is an asset whose return is strongly positively correlated with interest rates. Hence the IO-security moves in the opposite direction from most fixed income securities which makes it valuable for investors who want to hedge against interest risk.

1.1.2 Asset Backed Securities

Like the innovations based on mortgage loans these innovations utilize bundles of nontraded claims to create securities that produce a regular cash flow. The underlying assets include auto loans, leases, credit card receivables and commercial loans.

1.1.3 Common Equity Innovations

One of the common equity innovations is the Americus Trust. The Americus Trust offers common stockholders of a company the opportunity to strip each of their common shares into a PRIME component, which carries full dividend and voting rights and limited capital appreciation rights and a SCORE component which carries full capital appreciation rights above a threshold price. Thus shares of a company are divided into two subassets that are sold separately. The first Americus Trust was offered to owners of AT&T stock in 1983 (Finnerty, 1988).

1.1.4 Options, Futures and other Derivative Assets

Options, futures and other derivative securities belong to the most widely recognized financial innovations of the past 20 years (Miller (1986)). In our

[3] Note that interest on an outstanding mortgage is computed on the principal the homeowner still owes.

framework financial intermediaries can create securities that include options and futures.

Markets for Options and Futures are highly institutionalized and allow short positions. Hence for this example our restriction that buyers cannot go short in innovated assets and intermediaries are not allowed to sell the new security "naked", i.e. without the backing of a standard instrument is too strong.

1.1.5 Stripped Treasury Securities

In 1982 Merill Lynch and Salomon Brothers created zero coupon bonds. The two investment firms did this by purchasing long-term Treasury bonds and depositing them in a bank custodian account. They then issued receipts representing an ownership interest in each coupon payment on the underlying Treasury bond in the account and a receipt on the underlying Treasury bond's maturity value (see Fabozzi and Fabozzi (1991)). The purpose of this innovation was to create a variety of instruments without reinvestment risk.

1.1.6 Consumer Type Financial Instruments

On a less institutionalized level investment firms supply custom-tailored packages to their clients and thereby in less spectacular manner qualify for the label innovator. From Brokers Cash Management Accounts, Equity Access Accounts, to Variable life insurance policies these new instruments repackage existing assets so that specific consumer needs are met. Variable life insurance policies, for example, are really families of mutual funds embedded in a life insurance contract. Current yield and redemption value are tied directly to the particular mutual fund that is embedded in the contract.

This type of innovation combines *several* existing assets and restructures the payoffs in a marketable way. In the model to be presented we allow financial intermediaries to use only *one* standard securities as "raw material" for financial innovation. This is done purely for expositional simplification. It is straightforward to extend the analysis to allow financial intermediaries to use portfolios of several standard securities as collateral for financial innovation.

1.2 The Issues

Many commonly traded instruments like mutual funds and other combinations of existing assets are redundant in the sense that without them the agents could achieve the same wealth patterns, i.e. they do not increase the span of the existing assets. Nevertheless their existence seems to considerably enhance the agent's utility. The set of intermediated securities in our model exhibits the same phenomenon. In section 2 we give an example that demonstrates that innovation can be profitable even if the span of the supplied assets is unaffected by the innovation. Clearly this result depends on the assumption of marketing costs for intermediated securities.

Section 3 of the paper analyzes whether intermediaries choose an efficient[4] level of financial innovation. First we give an example to demonstrate that an efficient set of intermediated securities may not be realized in equilibrium. Financial innovation can get stuck at an inefficient level if products that are complements cannot be issued by the same intermediary. This will be the case if innovation technologies are decentralized in the sense that each intermediary is operating on a few markets only. The reason for this inefficiency is that although a simultaneous issue of all complementary products may be profitable, a single issue of one new product is not. Since intermediaries cannot coordinate their activities the innovation will not be undertaken.

For this coordination problem to arise the assumption of a fixed cost per customer is crucial. If those fixed costs are zero then intermediaries can supply very small amounts of new securities to the households with the largest reservation values. For small purchases the contributions of two securities to a households utility will be approximately the sum of the contributions of each

[4]By efficient level of innovation we mean that a planner cannot find a set of financial innovations and feasible trading plans at current prices of consumptions that would make every household better off.

security, i.e. there are no complementarities for small purchases (Makowski and Ostroy (1991)). With individual set up costs, it only pays to supply relatively large amounts of an innovation to each customer and hence this argument brakes down.

An alternative interpretation of the question analyzed in this section is whether financial intermediaries should be confined to certain subsets of markets. We show by examples that these regulations may actually hinder the process of innovation due to the fact that complementary innovations that are available on different markets will not be undertaken. We show (Theorem 2) that if some financial intermediaries are able to access all markets, i.e. use every available standard security for their financial innovation, then there will be an efficient level of financial innovation.

Throughout this paper we consider only securities that pay off in units of account. This construction should not be taken literally. Clearly, there is a difference between a specific currency which has the backing of government institutions and serves as a medium of exchange and a mere unit of account. Modelling security payoffs in terms of units of account should be seen as a shortcut which leaves out the complex political and economic structure that determines the value of money in every contingency. Therefore, the question we address is: What can we say about the model when leaving the value of money *indeterminate*, i.e. what can we can say for all possible state contingent valuations of the unit of account?

For example, think of the states of nature as exogenous productivity shocks. Then we can imagine many different monetary theories together with policy rules that give rise to specific state contingent inflation rates. Here we want to ignore issues of monetary economics and rather ask what we can say irrespective of how we decided to model that part of the economy.

In a framework where there is no financial innovation it has been shown (Balasko and Cass (1989), Geanakoplos and Mas-Colell (1989) that if securities pay off in units of account and markets are incomplete, then there is a severe indeterminacy of equilibrium allocations. This indeterminacy stems from the fact that the value of the unit of account ("money") in each of the states is indeterminate and this nominal indeterminacy has real consequences when markets are incomplete. This is intuitively plausible since with incomplete markets it matters which financial instruments are available and by changing the value of money in the different states of nature we change the real characteristics of the assets traded. However, when markets are complete the nominal indeterminacy does not translate into real indeterminacy since changing the state contingent value of money does not alter the set of wealth transfers that can be achieved by trading in securities.

In a framework with financial instruments given, the asset structure cannot respond even when inflation and inflation expectations vary over a wide range. This is contrary to recent experience on financial markets where larger volatility in inflation and interest rates is a frequently quoted source of financial innovation; (see e.g. van Horne (1986), Silber (1986), Merton (1990)).

Here we allow financial markets to adjust to changes in the volatility of inflation. Thus to every price level in the future states of nature there corresponds a possibly different set of equilibrium innovations. This flexibility of the equilibrium set of innovations *reduces* the indeterminacy of the equilibrium allocation of consumption. In particular it is shown (see section 4, Theorem 4) that if innovation costs are "small" then equilibrium allocations are almost efficient and the utility consequences of the indeterminacy of equilibrium allocations are "small".

Going back to the above interpretation of the nominal indeterminacy as resulting from an incomplete model, we can conclude that the sensitivity of an economy's allocation of consumption goods with respect to the exact specification of the political and economic process determining inflation depends on the size of the innovation costs. With small innovation costs, changes in this process affect commodity allocations little but may have a large impact on the financial structure of an economy. The introduction of financial innovation substantially changes the view on the importance of indeterminacy: The size of the innovation costs determines whether an economy can successfully counteract changes in inflation or whether the agents will be subject to a severe indeterminacy in their utility levels.

1.3 Outline of the paper

Section 2 introduces the model and provides an existence theorem (Theorem 1). We also give an example to illustrate that the equilibrium set of financial products may exhibit redundancies as frequently observed on financial markets (section 2.5.), i.e. innovated securities may be linear combinations of already existing securities but nevertheless be utility improving for households.

Section 3 examines under what conditions financial innovation will be efficient. We give an example to demonstrate how the financial structure can fail to be efficient in equilibrium. It is shown that if some intermediaries can issue all possible financial products or if the fixed cost per customer is zero, there will be an efficient financial structure. (Theorem 2).

Section 4 analyzes the indeterminacy of equilibrium allocations. We show that if innovation costs are "small", then the utility consequences of the indeterminacy of equilibrium allocation will be "small" (Theorem 4). Section 5 presents concluding remarks.

2 The Model

In the following we present a 2-period model with financial innovation.

There is one physical commodity x and 2 dates, $t = 0, 1$. In the second period one of S states of nature occurs. Let $I = [0, H]$ be the set of households with $i \in I$. I can be partitioned into H subintervals $[h-1, h)$ that correspond to the H types of households, $h = 1, ..., H$. $(I, \mathcal{B}, \mu)$ is the measure space of agents, where $\mathcal{B}$ denotes the Borel sets on $[0, H]$ and μ is Lebesgue measure.

A is the set of standard securities, $A = \{1, ..., A\}$[5] with generic element a. R^a is the payoff of security a in the second period in terms of units of account ("money"). Rz is the payoff of a portfolio $z \in \mathbf{R}^A$. (R has A columns and S

[5]In abuse of notation we will use the same letter to denote the set and the last element.

rows.) We impose the following simplifying restrictions on R:

- $R^a \geq 0$ for all a.
- The vectors R^a are linearly independent and $A < S$, i.e. the set of standard securities does not span all of $\mathbf{R}^S$.

Let $q \in \mathbf{R}^{\mathbf{A}}$ denote the price of the standard securities. Denote by $p = (p_0, p_1, ..., p_S)$ the price of consumption in period 0 and in the states $(1, ..., S)$ of period 1. Normalize $p_0 = 1$; hence we can identify p with a vector in $\mathbf{R}^S_+$. Let

$$P = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & p_S \end{pmatrix}$$

be the diagonal matrix with period 1 prices on the main diagonal.

2.1 Intermediaries

The business of intermediaries is the production of new financial products using standard securities as collateral. We will describe the activities of financial intermediaries analogously to standard production theory with the additional complication that the set of possible products is infinite.

The possible financial products can be identified with vectors in $\mathbf{R}^S$. Since assets that are multiples of each other are qualitatively the same product we will use a normalization rule to characterize the set of potential financial products. Let J be the unit simplex in $\mathbf{R}^S$ i.e.

$$J = \{d \in \mathbf{R}^S_+ : \sum_s d_s = 1\}$$

$J \cup \{0\}$ is the set of possible financial products. We restrict every financial product to pay off non-negative amounts so there are no issues of enforceability on the side of the buyers of financial innovation.[6]

[6]Note that we will not allow households to hold short positions in financial products

A *financial innovation on market a* consists of 3 components:

(i) A collection of financial products $Q^a = (Q^a_1 ..., Q^a_K)$; $Q^a_k \in J \cup \{0\}$.[7]

(ii) Quantities $y^a = (y^a_1, ..., y^a_K)$ at which the financial products are supplied.

(iii) Standard securities $z^a \in \mathbf{R}^A_+$ which are used as collateral for the financial products. We assume that for innovation on market a only the returns of standard security a can be used as collateral. Hence z^a_a, the a-th component of the portfolio z^a has to generate sufficient cash-flow in the second period to pay for all the produced financial products; i.e.

$$\sum_{k=1}^{K} Q^a_k y^a_k \leq R^a z^a_a$$

Thus the set of feasible innovations on market a can be summarized by:

$$Y^a = \{(Q^a, y^a, z^a) : Q^a \in (J \cup \{0\})^K, y^a \in \mathbf{R}^K_+, z^a \in \mathbf{R}^A_+, \sum Q^a_k y^a_k = R^a z^a_a\}$$

There are M types of financial intermediaries, $M = \{1, ..., M\}$. To each $m \in M$ there corresponds an $A^m \subset A$, where A^m denotes the set of markets (of standard securities) on which the financial firm m is operating. We assume that the collection of sets $\{A^m\}_{m=1}^M$ forms a partition of A, so that for each market a there is exactly one type of financial intermediary that can operate on it. In particular we will be interested in two cases:

- $M = A$. Each intermediary is confined to exactly one market.
- $M = 1$. There is one type of intermediary that can access all markets.

Intermediaries not only create financial innovations, they also have to market them. Marketing is a costly activity.

created by intermediaries. We will, however, allow households to hold short positions in standard securities.

[7]The number K will be chosen so that no intermediary is constrained by it.

For each new security $Q_k^a \in J$ that is created on market $a \in A$ there is a proportional cost $c^a(Q_k^a, p)$, which depends on the payoff vector of the created asset and on the price level in the second period. Hence if the intermediary creates y units of innovation Q_k^a the cost will be $c^a(Q_k^a, p)y$. The idea is that there are operational and marketing costs that are proportional to the size of operation. Moreover these costs depend on the type of security that is created on a specific market.

Assumption 1 *$c^a : J \times \mathbf{R}_{++}^S \to \mathbf{R}_+$ is continuous on $J \times \mathbf{R}_{++}^S$ and $c^a(Q_k^a, \lambda p) = (1/\lambda)c^a(Q_k^a, p)$*

Assumption 1 implies that changes in the value of money across all states in the second period do not affect the proportional innovation cost. I.e., if all payoffs were to be measured in cents instead of dollars then the costs per unit would decrease by a factor of one hundred, leaving real costs unchanged.

Example: Suppose there is one (or more) "natural splits" of the asset a. Suppose, for example, that a is a treasury bond. Selling the coupons for one or more periods[8] as a separate asset ("zero-coupon bonds") is relatively cheap since the new securities utilize the payment structure of the underlying collateral. Note that annual coupon payments are well defined and well understood even before zero-coupon bonds are created. Let Z be this collection of possible zero coupon bonds.

$$c^a(Q_k^a, p) = c_0 \sum_s Q_{k,s}^a / p_s + c_1 \inf_{Q_z^a \in Z} (P^{-1} |Q_k^a - Q_z^a|)$$

In this example the second part of the cost function grows with the distance from the set of zero coupon bonds. □

In addition to the proportional cost there is a *marketing cost* b^a per individual, per security. Think of this cost as the cost of locating potential customers and the cost of setting up a business relation with a client; (explaining the properties of the new security and convincing the customer of its reliability, etc.)

[8] Note that we can interpret the S states as being divided into T different periods.

All costs are measured in units of period 0 consumption. The set-up cost implies that there is a non-convexity in the innovation technology. Hence selling 10 units of an intermediated security to a single customer will be less costly per unit than selling only 1 unit.

Intermediaries face perfect competition. We assume that there are financial intermediaries of each type that are willing to enter the market if there are profit opportunities.

2.2 Households

Each type of household is characterized by a utility function u^h and an endowment $\omega^h = (\omega_0^h, \omega_1^h)$ where the subscript denotes the period.

Assumption 2 $\omega^h >> 0$, *for all* h.

Assumption 3 *For all* $h \in H$, u^h *is differentiable, strictly quasiconcave with indifference curves not touching the axes.*

In the following we will index households by $i \in I$ rather than their type h. Note that we cannot impose equal treatment for all members of one type since we have non-convexities in the cost structure. Thus households of the same type may chose different trade plans. We will write u^i and $\omega(i)$ instead of u^h and ω^h.

Before we can consider the households decision problem we need to introduce some notation. Suppose for the following the financial products $Q = (Q^1, ..., Q^A)$ are given. To distinguish total supplies and input demands of intermediaries from trades of households we will use the following notation:

- y^a, z^a denote total supplies and input demands of the financial intermediary on market a.
- $y_a(i), z(i), x_0(i), x_1(i)$ denote trades of household $i \in I$. $y_a(i) \in \mathbf{R}_+^K$, $z(i) \in \mathbf{R}^A, x_0(i) \in \mathbf{R}_+, x_1(i) \in \mathbf{R}_+^S$.

$r_k^a : \mathbf{R}_+ \to \mathbf{R}_+$ is a price function that tells one household (for a given set of financial products) what he has to pay for y units of intermediated asset k on market a. Note that whereas prices of standard securities are linear prices of financial innovations need not be linear. To allow non-linear pricing is necessary in the presence of personalized non-convexities (Makowski (1979)). Hence a household that buys 10 units of a financial innovation will typically pay less per unit than a household that buys 1 unit of the same innovation.

Let $r^a : \mathbf{R}_+^{\mathbf{K}} \to \mathbf{R}_+$ be the price of a portfolio of innovations on market a, i.e. $r^a(y^a) = \sum_k r_k^a(y_k^a)$ and let $r : \mathbf{R}_+^{\mathbf{AK}} \to \mathbf{R}_+$ be the price of any portfolio of innovations, i.e. $r(y) = \sum_a r^a(y^a)$; we restrict the price functions for intermediated securities so that the average price is non-increasing. I.e.

$$\frac{r_k^a(y_k^a)}{y_k^a}$$

is non-increasing.

Households are not allowed to sell intermediated securities short. In the present context this is a natural restriction since short selling of intermediated assets would be equivalent to supplying those assets *without paying the marketing costs.* Clearly households can have no such cost advantage over intermediaries. Households are allowed to sell standard securities short, however.

Now we can describe the households decision problem. Let (Q, r, p, q) be given and note that Q is a $S \times AK$ matrix. Given household i's trades in standard securities $z(i)$ and his purchases of intermediated financial products $y(i)$ i's period 0 consumption is

$$x_0(i) = \omega_0(i) - qz(i) - r(y(i))$$

His period 1 consumption is

$$x_1(i) = \omega_1(i) + P^{-1}Rz(i) + P^{-1}Qy(i)$$

Note that $P^{-1}R$ is the return matrix of the standard securities in terms of consumption and $P^{-1}Q$ is the return matrix of the intermediated securities in terms of consumption.

The household thus solves

$$\max u^i(x_0, x_1)$$

subject to

$$x_0 = \omega_0(i) - qz - r(y)$$
$$x_1 = \omega_1(i) + P^{-1}Rz + P^{-1}Qy$$
$$x_0 \geq 0, x_1 \geq 0, y \geq 0$$

Let $v^i(r, p, q, \omega(i))$ be the value of this maximization problem as a function of prices and the individual's endowment. Clearly for this maximization problem to make sense we have to restrict prices in such a way that a no-arbitrage condition is satisfied.

Condition 1 *(No Arbitrage) There does not exist a pair (z, y) with $y \geq 0$ such that*

$$\begin{pmatrix} -qz \\ Rz \end{pmatrix} + \begin{pmatrix} -r(y) \\ Qy \end{pmatrix} \geq 0$$

This condition is a generalization of the usual No-Arbitrage condition. Note that since short sales of intermediated securities are prohibited a strict inequality may hold.

Remark: Makowski (1983) and Allen and Gale (1991) consider the effect of allowing short sales when the only innovation cost is a fixed cost per innovated security. Clearly in this case every firm that issues a new security creates a positive externality since it opens new opportunities for costless trading among agents. Because the innovator cannot appropriate these additional gains from trade stemming from its innovation, an inefficient level of financial innovation may result.

In our framework the issue of short sales appears in a different light: The only sensible way to incorporate short sales is to allow intermediaries to market two sets of assets, one that pays off positive amounts and one that pays off the corresponding negative amounts. For both types of assets marketing costs have to be paid. Allowing short sales in this sense would not substantially alter our model nor our results. If, however, we were to change the cost structure in such a way that there is only a fixed cost per security then we would have globally increasing returns to scale and this would create problems similar to the ones encountered in Makowski's and Allen and Gale's work. In this sense short sales are not the source of the problem. They are just the vehicle through which the increasing returns to scale implicit in the cost structure of Allen and Gale's model are realized.

2.3 Equilibrium for a given set of financial products

In this section we keep the set of intermediated products Q fixed and define an equilibrium with repect to a given Q. A *trading plan* for an economy consists of (1) total supplies and input demands of intermediaries, $(y^a, z^a)_{a \in A}$, (2) trades of households $y : I \to \mathbf{R}^{AK}, z : I \to \mathbf{R}^A, x_0 : I \to \mathbf{R}_+, x_1 : I \to \mathbf{R}^S_+$.

An equilibrium for a given financial structure consists of a trading plan together with prices so that (1) all markets clear (2) every household's trades are optimal (3) every intermediary's trades are optimal. The first two requirements are standard. The conditions for optimality for intermediaries need some explanations:

First we require that each intermediary gets zero profits in equilibrium. Remember that there is a large number of potential intermediaries willing to enter the market if entry is profitable. Moreover costs are proportional to the size of operations.[9] Thus we can pretend that only one intermediary of each type is operating.

Second we require that no feasible production plan can be sold with positive profits. Remember that the price function r is non-linear and therefore to evaluate revenues an intermediary needs to consider the distribution of its output. Therefore for any production plan and for any distribution of output, profits must be non-positive.

[9] An intermediary supplying to 1/10 of the total mass of households of each type incurs 1/10 of the total marketing costs on each market it operates.

Consider a financial innovation (Q^a, y^a, z^a) and trades for households y_a then the total marketing costs $C^a(Q^a, y^a, y_a)$ are:

$$C^a(Q^a, y^a, y_a) = \sum_k c^a(Q_k^a, p)y_k^a + \sum_k b^a \mu(support(y_{aj}))$$

where the first term on the right hand side captures the proportional cost and the second term describes the set-up costs per individual that are incurred by the selling plan. These set-up costs are given by the measure of the support of the selling plan of each innovated security times the fixed cost per individual b^a. Thus the profits of intermediary a are:

$$\int_I r(y_a(i))di - qz^a - C^a(Q^a, y^a, y_a)$$

where the first term is the revenue from the innovation, the second term is the cost of the standard securities that are used as collateral and the third term is the marketing costs.

Definition 1 *$((y^a, z^a), x, y, z, r, p, q)$ is an equilibrium for Q if*

(i) x, y and z are measurable functions and

$$\int_I z(i)di + \sum_a z^a = 0; \int_I y_a(i)di = y^a, \forall a \in A$$

$$\int_I (\omega(i) - x_0(i))di = \sum_a C^a(Q^a, y^a, y_a)$$

(ii) $x(i), y(i), z(i)$ maximizes household i's utility given (Q, r, p, q).

(iii) For all $a \in A, (Q^a, y^a, z^a) \in Y^a$ and

$$\int_I r^a(y_a(i))di - qz^a - C^a(Q^a, y^a, y_a) = 0$$

and there does not exist a feasible financial innovation $(Q^a, \hat{y}^a, \hat{z}^a) \in Y^a$ together with a measurable function $\hat{y}_a : I \to \mathbf{R}^K, \int_I \hat{y}_a(i)di = \hat{y}^a$ such that

$$\int_I \hat{r}(\hat{y}(i))di - q\hat{z}^a - C^a(Q^a, \hat{y}^a, \hat{y}_a) > 0$$

2.4 Equilibrium with Financial Innovation

Intermediaries do not have to stick to the existing set of intermediated securities. If they find it profitable they can create new assets as long as their innovations are feasible, i.e. lie in the sets $Y^a, a = 1, ..., A$

But how can intermediaries calculate profits for alternative innovations?

The idea is the following: When an intermediary contemplates an alternative innovation it calculates what the households would be willing to pay for that innovation under the assumption that prices of consumption, standard securities, and all the other innovations remain constant. Thus the firm computes the household's reservation values assuming that the alternative innovation it supplies does not affect prices. With this formulation we follow Hart (1980) and Makowski (1980) in their work on general commodity innovation.

Remember that there is a continuum of households of each type and also note that the innovation costs are proportional to the size of the group to which an intermediary supplies; i.e. if we multiply the mass of each type of household by a positive number and leave the quantities supplied to each household constant then the costs will multiply by the same number. Hence we can imagine the intermediary supplying his alternative innovation at a very small scale and hence leaving the overall state of the economy unchanged. We treat intermediaries as "large" agents as compared to individual households. Nevertheless intermediaries face a perfectly competitive situation on each market due to free entry.

Suppose (Q, r, p, q) are given. We call measurable functions $(\hat{y}_a)_{a\in A^m}$, $\hat{y}_a : I \to \mathbf{R}_+^K$ and $\rho, \rho : I \to \mathbf{R}_+$, *individually rational (IR) trades* for the innovation $(\hat{Q}^a, \hat{y}^a, \hat{z}^a)_{a\in A^m}$ if for all $a \in A^m$ the following two conditions hold:

$$\int_I \hat{y}_a di = \hat{y}^a, \forall a \in A^m \tag{1}$$

$$v^i(r, p, q, \hat{x}_0(i), \hat{x}_1(i)) \geq v^i(r, p, q, \omega(i)) \tag{2}$$

where

$$\hat{x}_0(i) = \omega_0(i) - \rho(i)$$

and

$$\hat{x}_1(i) = \omega_1(i) + P^{-1} \sum_{a \in A^m} \hat{Q}^a \hat{y}_a(i).$$

Individually rational trades do not make any household worse off, given that the rest of the environment remains unchanged.

An intermediary is behaving optimally if its trades are optimal with respect to the existing (and priced) securities and if it cannot find an innovation together with individually rational trades that would guarantee larger profits than it currently gets.

Definition 2 *$(Q, (y^a, z^a)_{a \in A}, x, y, z, r, p, q)$ is a competitive equilibrium with financial innovation (CEI) if*

(i) $((y^a, z^a)_{a \in A}, y, r, p, q)$ is an equilibrium for Q.

(ii) There does not exist a feasible financial innovation $(\hat{Q}^a, \hat{y}^a, \hat{z}^a)_{a \in A^m}$ together with IR-trades $(\hat{y}_a,)_{a \in A^m}, \rho$ such that

$$\int_I \rho(i)di - \sum_{a \in A^m} [q\hat{z}^a - C^a(\hat{Q}^a, \hat{y}^a, \hat{y}_a)] > 0$$

In order to guarantee existence of equilibrium we need to assume that each type of intermediary can create sufficiently many financial products so that all the profit opportunities on its market can be exhausted. Let $\bar{K} = A^2(S+2)^2 H$. We will require that $K \geq \bar{K}$ to insure that the number of assets that each intermediary can issue on every market will not be a restriction. Note that since 0 is a possible asset, there is no loss of generality in forcing each intermediary to create exactly K financial products on every market.

Furthermore we assume that each individual's demand for innovations is bounded by some positive constant. This is needed for technical reasons and we believe it could be dispensed with.

Theorem 1 *Suppose Assumptions 1 and 2 and 3 hold. Suppose* $K \geq \bar{K}$ *and* $y_k^a(i) < L$, *for some* $L < \infty$ *for all* $i \in I$ *and for all* (a, k). *Then there exists a Competitive Equilibrium with Financial Innovation.*

The proof of Theorem 1 will be given in the appendix. In order to establish existence we will show existence of a standard equilibrium of a transformed economy with a large number of commodities. We will interpret each possible quantity of every possible innovation as a different commodity and restrict households to purchase integer-amounts of the innovations. Hence we replace the personalized set-up costs by indivisibilities, a technique first used by Makowski (1979) (see also Ellickson (1979)).

The standard competitive equilibrium obtained from this procedure will have prices for all possible innovations, not only for the supplied innovations as in the definition of a competitive equilibrium with financial innovation. For the characterization of equilibria and their economic interpretation we will insist on the weaker definition of equilibrium used in the text, since it does not seem to be an economically meaningful requirement of equilibrium to insist on having a market price for financial innovations that are not available.

Financial innovation will not necessarily stop when supplied securities span R^S. Due to the costs of innovation, financial intermediaries will try to create instruments that enable households to economize on the number of securities traded and on the volume of trade in each security. This implies that we will frequently observe "redundancies" in the financial structure, i.e. there will be innovations that do not change the span of the available instruments but rather make it cheaper for certain types of households to make the desired transactions.

Note that there are many examples of "redundant" products on financial markets, e.g. mutual funds or other combinations of existing assets.

The following example illustrates the potential redundancies of financial innovation in our model.

2.5 An Example of "Redundant" Financial Innovations

There are 6 types of households and 3 states of nature in the second period. Households have identical utility functions:

$$u^h = x_0^h + \sum_{s=1}^{3}(x_s^h - 1/3x_s^{h2})$$

In period 0 every household has an endowment of 2 units.

$$\omega_1^1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \omega_1^2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \omega_1^3 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\omega_1^4 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}; \omega_1^5 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}; \omega_1^6 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix};$$

There is one standard security and the proportional innovation costs are zero.

$$R = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, b^a = .10, c^a = 0.$$

Equilibrium with financial innovation: For each price vector p with $p_1 = 1$ we will have a different set of innovations. The allocation of consumption will be independent of p and each household will consume 2 units of the consumption good in every state of the second period. (The determinacy of equilibrium allocations depends on the fact that proportional innovation costs are zero in this example.)

Suppose $p_s = 1$ for all s. Then the equilibrium set of innovations will be:

$$Q_1 = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}; Q_2 = \begin{pmatrix} 1/3 \\ 0 \\ 2/3 \end{pmatrix}; Q_3 = \begin{pmatrix} 0 \\ 2/3 \\ 1/3 \end{pmatrix};$$

$$Q_4 = \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix}; Q_5 = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \end{pmatrix}; Q_6 = \begin{pmatrix} 2/3 \\ 0 \\ 1/3 \end{pmatrix};$$

If any of the innovations in this example were not supplied agents could still make the same trades. It would be more costly for some households, however.

Since the allocation is independent of prices it is easy to compute equilibrium financial structures at different prices. Also note that since the proportional innovation cost is zero there are other innovations that could also serve as equilibrium innovations.[10] The ones shown above minimize the volume of trade.

This example shows that our model of financial innovation replicates a feature that is common on financial markets: Even if a new asset does not increase the span of existing assets it may be supplied in equilibrium and actually improve the utility of households.

3 Efficiency and Complementarities

In this section we discuss the efficiency of competitive innovation. The appropriate notion of efficiency is, however, not obvious.

First note that since financial innovation consumes resources, overall Pareto-efficiency of the allocation cannot be achieved by CEI-allocations. Thus every notion of efficiency that has some chance of being satisfied by innovational equilibria must respect the constraints the model imposes on wealth transfers. In other words: A planner who tries to improve every households utility should be restricted to the use of standard securities and assets that are producible via the existing innovation technology. Furthermore if the planner uses innovated assets he should be subject to the same

[10] It is easily checked that innovations of the form $1/(3\lambda+1)(Q_i+\lambda R)$ are also equilibrium innovations.

costs that intermediaries have to pay when they create a new security.

We know that the corresponding economy without financial innovation possesses a plethora of equilibria that will frequently be Pareto-ranked (see Balasko and Cass (1989), Geanakoplos and Mas-Colell (1989)). For each period 1 state price vector ($p \in \mathbf{R}^{\mathbf{S}+1}$) with $p_0 = p_1 = 1$ we will typically get a different real allocation.

This indeterminacy carries over in part to the model with financial innovation (this will be the subject of section 4). To illustrate this point, think of a model with very large innovation costs. Equilibria will be the same as in the corresponding economy without financial innovation. Hence equilibria will be typically indeterminate and frequently Pareto-ranked. Thus if we allow the planner to chose any price vector for consumption in period 1, then there is no chance for the model to satisfy the corresponding efficiency criterion: It will be easy for a planner to beat CEI-allocations in the Pareto sense.

To avoid these problems in this section (we deal with them at great length in the following section) we fix the prices of consumption $p \in \mathbf{R}^{\mathbf{S}}$. By fixing those prices, we actually transform the "nominal-securities" into "real-securities", i.e. the payoffs of the securities can be expressed in units of the consumption good.

To summarize: We will restrict the hypothetical planner in our definition of *constrained innovation efficiency* in two ways:

(i) The prices of consumption remain fixed at some predetermined p.

(ii) All the wealth transfers must be performed with financial instruments. The costs of creating new financial instruments are the same as for the intermediaries in the model.

Definition 3 *A trading scheme* $\{(Q^a, y^a, z^a)_{a\in A}, x, y, z, \}$ *is constrained innovation efficient with respect to* p *if there does not exist an innovation* $(\hat{Q}^a, \hat{y}^a, \hat{z}^a)_{a\in A}$ *and trades* $(\hat{y}, \hat{z})$ *together with period* 0 *consumption* $\hat{x}_0 : I \to \mathbf{R}_+$ *such that:*

(i) For all $a \in A$ $(\hat{Q}^a, \hat{y}^a, \hat{z}^a) \in Y^a$ and

$$\int \hat{y}_a(i) = \hat{y}^a, \int \hat{z}_a(i) + \hat{z}^a = 0$$

$$\int_I (\omega_0(i) - \hat{x}(i))di = \sum_{a \in A} C^a(\hat{Q}^a, \hat{y}^a, \hat{y}_a)$$

(ii)

$$u(\hat{x}_0(i), \hat{x}_1(i)) > u(x(i))$$

where

$$\hat{x}_1(i) = \omega(i) + P^{-1}R\hat{z}(i) + P^{-1} \sum_{a \in A} Q^a \hat{y}_a(i).$$

The definition of constrained innovation efficiency resembles the familiar notion of constrained efficiency due to Diamond (1967). By fixing the price level we transform the nominal securities into real securities and if agents could trade only in standard securities then the well known Diamond result holds: With one physical commodity[11] equilibrium allocations with incomplete financial markets are constrained efficient.

The new aspect of this definition is the inclusion of financial innovation; (see also Allen and Gale (1988) for a similar notion of constrained efficiency). When financial innovation is taken into account, equilibrium allocations need no longer be constrained efficient. This inefficiency is similar to the inefficiencies in general models of commodity innovation first analyzed by Hart (1980) and Makowski (1980). There is one important difference, however. The examples of inefficiencies in Hart (1980) and Makowski (1980) stem either from kinks in the indifference curves or technologies or from endowments on the boundary of the consumption set. If utility functions are smooth and endowments are in the interior then it can be shown that innovation will be efficient (Makowski and Ostroy (1991)).

[11]This is a necessary restriction, see Geanakoplos and Polemarchakis (1986) for a result on generic constrained inefficiency of equilibrium allocation with many physical commodities in each state.

This is very different in the present model. Personalized set-up costs make it possible that even with smooth utility functions and endowments in the interior of the consumption set[12] complementarities among innovations may prevent an efficient allocation from being realized in equilibrium.

The following example shows that CEI allocations are in general not constrained innovation efficient.

3.1 An Example of Inefficient Innovation

Suppose there are 8 states of nature in the second period. There are 2 types of agents:

$$u^1(x_0, x_1) = x_0 + \sum_{s=1}^{8}(x_{1s} - 1/4x_{1s}^2)$$

$$u^2(x_0, x_1) = x_0 + 1/4\sum_{s=1}^{8} x_{1s}$$

Suppose $c^a = 0$, for all a. There are 3 assets and 3 types of innovators.

$$R^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} R^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} R^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The endowments in period 1 are given by:

$$\omega_1^1 = (3/2, 1, 2, 1, 1, 2, 1, 3/2), \omega_1^2 = (2, 2, 2, 2, 2, 2, 2, 2)$$

[12] Note that in the present model the endowments of the households are in the interior with respect to every potential innovation.

Suppose $p_1 = (1, ..., 1)$; $q_3 = q_1 = .75; q_2 = 1$. Suppose b^2 is large so that innovation using the second asset does not pay. Let $b^1 = b^3 = b$ and we will show that for appropriate choices of b there are 2 types of equilibria:

- either intermediaries of type 2 and 3 innovate simultaneously or
- no innovation occurs.

First suppose innovation takes place on markets 1 and 3. The corresponding allocations will be:

$$x_1^1 = (3/2, 3/2, 3/2, 3/2, 3/2, 3/2, 3/2, 3/2)$$

$$x_1^2 = (2, 3/2, 5/2, 3/2, 3/2, 5/2, 3/2, 2)$$

Clearly only household 1 is willing to pay for the innovation. Household 1's gain in utility is:

$$(12 - 4.5 - .25) - (11 - 4.125) = .375$$

Hence his willingness to pay for 2 innovations simultaneous is .375\$. If $4b <$ .375 then the household is willing to purchase both innovations.

Now suppose that only one intermediary innovates. Since the situation is symmetric for both asset 1 and 3 suppose financial intermediary 1 considers an innovation. The corresponding allocation will be:

$$x_1^1 = (3/2, 3/2, 3/2, 1 + 3/16, 1 + 3/16, 2 + 2/16, 1 - 1/16, 3/2 - 1/16)$$

$$x_1^2 = (2, 3/2, 5/2, 2 - 3/16, 2 - 3/16, 2 - 2/16, 2 + 1/16, 2 + 1/16)$$

Again only household 1 is willing to pay and his utility gain is:

$$(11.375 - .09375 - 4.2578125) - (11 - 4.125) = .1484375$$

In order for this innovation to be profitable we need to have that $2b <$ .1484375.

Now suppose

$$b > .1484375/2$$

$$b < .375/4$$

i.e. $b \in (.07421875, .09375)$. Then although in terms of efficiency both innovations should be undertaken the competitive equilibrium might get stuck in a no-innovation situation. Thus the initial endowment and no innovation constitutes a competitive equilibrium with financial innovation.

Remark: For computational simplicity we chose a risk neutral type of household and a household with quasilinear preferences. These simplifying assumptions (which contradict Assumption 3) are however not essential for the existence of complementarities. Crucial however are

- The set up costs per individual ($b > 0$).
- No intermediary has access to all standard security markets. This implies that the economy lacks a mechanism to coordinate innovations.

3.2 A Theorem for constrained efficient innovation

Theorem 2 *Suppose that (i) $M = 1$ or (ii) $b^a = 0$, for all $a \in A$. Then each CEI is constrained innovation efficient.*

Proof: First suppose $M = 1$. Suppose we are in a CEI and there exists an innovation $(\hat{Q}^a, \hat{y}^a, \hat{z}^a)_{a \in A}$ and trades $(\hat{y}, \hat{z})$ together with period 0 consumption $\hat{x}_0$ such that (i) and (ii) of Definition 3 are satisfied.

Now we want to show that a profitable innovation exists for the intermediary and obtain a contradiction.

Set $\rho(i) = \omega(i) - (\hat{x}_0(i) - q\hat{z}(i))$. Then

$$\int_I (\omega(i) - \hat{x}_0(i)) - q \int_I \hat{z}(i))$$

is the total revenue collected from the households.

By hypothesis ((i) in Definition 3) $\int_I(\omega(i) - \hat{x}_0(i))$ covers the innovation cost (both the proportional and the set up cost per individual) and $-q\int_I \hat{z}(i)$ covers the cost for the collateral.

Thus the financial intermediary can supply this innovation at prices (q, p) at no loss. Since every household benefits strictly the intermediary can make a profit by charging

$$\hat{\rho} = \rho + \epsilon(i), \epsilon(i) > 0$$

for some appropriately chosen $\epsilon : I \to R_+$.

Now suppose $b^a = 0$, for all a and $M > 1$. It is sufficient to show that an innovation on several markets is profitable if and only if an innovation on one market alone is profitable.

Let $MRS_0(i)$ be the vector of marginal rates of substitution between consumption in the S-states of the second period and period 0 consumption. A necessary and sufficient condition for $(\hat{Q}^a, \hat{y}^a, \hat{z}^a)_{a\in A}$ to be profitable is that there exists a measurable function $\hat{y}_a : I \to R^K, \int_I \hat{y}_a(i) = \hat{y}^a$ for all a such that:

$$\sum_{a\in A}\int_I \hat{Q}^a\hat{y}_a(i)MRS_0(i) > q\sum_a \hat{z}^a + \sum_a\sum_k c^a(\hat{Q}^a_k, p)\hat{y}^a_k$$

But this inequality can only hold if it holds for at least one $a \in A$, which proves the theorem.

Corollary 1 *There always exists a CEI which is constrained innovation efficient.*

Proof: Consider the equilibria corresponding to $M = 1$. These constitute also equilibria for any other partition (A^m), since zero-profits on each market is satisfied and since there is no profitable alternative innovation.

Remark: The issue of this section can be rephrased as follows: Is decentralized financial innovation efficient? It was shown that the answer is in general no and since personalized set up costs are an important cost factor for financial intermediaries the efficiency result for $b^a = 0$ should be discounted in terms of its economic relevance.

One implication of this result is that limiting financial firms to certain markets via regulation may be harmful for the innovation process. The reason is that if firms are forced to operate in some given subset of markets only they may not exploit complementarities among financial innovations. Since each firm separately is only considering the effect of its financial innovation alone, the potential additional benefit of simultaneous innovation cannot be realized. In other words: Intermediaries cannot appropriate the additional profit opportunities which their innovation may create for other intermediaries. This *pecuniary externality* (Ostroy and Makowski (1991)) prevents the economy from realizing an efficient level of innovation.

4 Small Innovation Costs.

In this section we will consider the issue of indeterminacy of equilibrium allocations. Cass (1984), Balasko and Cass (1989), Geanakoplos and Mas-Collel (1989) have shown that for the model presented here *without financial innovation*, generically equilibrium allocations form an $S-1$ dimensional manifold (if some restrictions on the return matrix R are satisfied).

It is easy to see that there is a large *nominal* indeterminacy in this model since the price level in each of the S-states in the second period can be chosen arbitrarily. If markets are complete (i.e. if R has rank S) then this nominal indeterminacy does not have any real consequences. If markets are incomplete, however, then with a different price normalization the agents will typically face different *real* opportunities to transfer wealth and hence the equilibrium allocation will change.

The introduction of financial innovation "reduces" the problem of indeterminacy but does not eliminate it. The difference is that with innovation the financial structure now has a chance to react to changes in the price levels. The financial intermediaries in a sense reduce the impact of changes in the future price levels but since innovation is costly, equilibrium allocation will still be indeterminate .

The following example illustrates this point.

4.1 Example of a Competitive Equilibrium with Financial Innovation

There are 2 types of households, $S = 2$, there is one financial intermediary and one standard security. The innovation costs are given by:

$$c^a(Q,p) = c \cdot (Q_1/p_1 + Q_2/p_2), b^a = 0.$$

The utility functions are:

$$u^1(x_0, x_1, x_2) = u^2(x_0, x_1, x_2) = x_0 + x_1 - 1/4x_1^2 + x_2 - 1/4x_2^2.$$

The endowments in the first and second period are

$$\omega_0^1 = \omega_0^2 = 2, \omega_1^1 = (1,1), \omega_1^2 = (2,2)$$

The payoff of the standard security in terms of "money" is given by:

$$R = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let $p_0 = 1$, $p_1 = p$, $p_2 = 1 - p$. The equilibrium innovations in this simple example will be:

$$Q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If $p = 1 - p = 1/2$ then in this example the Pareto-optimal allocation $x_1^1 = x_1^2 = 3/2; x_2^1 = x_2^2 = 3/2$ will be achieved as an equilibrium without any innovation. Hence there is only room for innovation as the inflation is different for the two states.

Since the situation is symmetric for $p > 1/2$ and $p < 1/2$ we assume in the following that $p > 1/2$:

Using Theorem 2 and the fact that utility functions are quasi-linear the equilibrium allocation can be calculated by the following maximization problem:

$$\max_{x,z,y} \sum_h \sum_s (x_s^h - 1/4x_s^{h2}) + c(y/p + y/(1-p))$$

subject to

$$x_s^1 + x_s^2 = 3, s = 1, 2$$
$$x_1^1 = 1 + z^1/p + y/p; x_2^1 = 1 + z^1/(1-p)$$
$$x_1^2 = 2 - z^2/p; x_2^2 = 2 - z^2/(1-p) + y/(1-p)$$
$$z^1 + y = z^2$$

Solving this maximization problem (assuming an interior solution) we get:

$$x_1^1 = 3/2 - c/p(1-p); x_2^1 = 3/2 + c/p(1-p)$$
$$x_1^2 = 3/2 + c/p(1-p); x_2^2 = 3/2 - c/p(1-p)$$

and the amount of innovation bought by each household will be:

$$y = (p - 1/2) - c\frac{p^2 + (1-p)^2}{p(1-p)}$$

This solution is valid for $1/2 \leq p - c/p(1-p)$. (Note that the problem is symmetric, so for $p \leq 1/2$ we get a similar inequality.) Outside this price range we will have a no-innovation equilibrium.

The competitive equilibrium allocation without financial innovation can easily be shown to be:

$$x_1^1 = 2 - p; x_2^1 = 1 + p$$
$$x_1^2 = 1 + p; x_2^2 = 2 - p$$

(This will also be the CEI if p is outside the range given above.)

Suppose, for concreteness, that $c = 1/100$ then for $p \in [.021, .46]$ and for $p \in [.54, .979]$ there will be financial innovation. For p outside these 2 intervals we will have an equilibrium without financial innovation.

To compare indeterminacy with and without financial innovation suppose that $p \in [.1, .9]$. Then for the equilibrium allocation with financial innovation we have:

$$x_s^h \in [1.4, 1.6]$$

And for the corresponding economy without financial innovation we get:

$$x_s^h \in [1.1, 1.9]$$

The following story can be told to illustrate this example: Suppose, as in the example, the two states in the second period are identical as for endowments and utility functions of the individuals. Hence the uncertainty has no real content but is merely a sunspot. The standard security in this case is built for equal prices, maybe stemming from a situation where these 2 states were not even distinguished. Hence if prices are equal, then a first best equilibrium can be achieved.

If prices are close to $1/2$ then the allocation will be almost Pareto-efficient without innovation and hence it does not pay to innovate.

If the variability in prices gets "large" however, then innovation becomes profitable, and agents are able to reduce the real effects of the "inflation-variability" by purchasing those innovations.

If the differences in inflation gets "extreme", then innovation again ceases to be profitable. To understand this effect, consider a sequence of prices $p_k, (1-p_k)$ with $p_k \to 0$. The corresponding real payoff vector of the standard security is

$$\begin{pmatrix} 1 \\ \frac{p_k}{1-p_k} \end{pmatrix}.$$

As p_k gets smaller, innovation gets more difficult. The portfolio needed to back up a given real payment promise in state 2 commands more and more consumption in state 1. Thus the innovator has to issue some innovation with a very large (real) payment promise in the first state. The proportional cost of issuing this innovation may get very large, as e.g. in this example. In other words: Creating an innovation that protects agents against inflation gets very expensive as inflation gets very volatile.

4.2 A Theorem for Small Innovation Costs

In the following we consider what happens when innovation costs are "small". The result we want to obtain is that for small innovation costs the CEI-allocations are "almost" efficient and the utility consequences of indeterminacy are "small". As indicated in the example above, an important qualification is needed to obtain this result. We have to assume that all relative prices stay bounded away from zero. Otherwise, with relative prices converging to zero even small innovation costs will not be sufficient to induce financial intermediaries to supply innovations for inflationary states.

Definition 4 *Let WE be the set of allocations x^* such that there is a p^* satisfying*

(i) $u^i(x^*(i)) \geq u^i(\hat{x})$ *for all* $\hat{x}$ *with* $p^*\hat{x} \leq p^*\omega(i), p^*x^*(i) \leq p^*\omega(i)$.

(ii) $\int_I x(i) = \int_I \omega(i)$

WE is the set of complete markets equilibria for the economy.

Assumption 4 *There exists an* $\bar{a} \in A$ *such that* $R_s^{\bar{a}} > 0$ *for all* s.

Let

$$P_\eta^{S-1} = \{p \in \mathbf{R}_+^\mathbf{S} : \sum p_s^2 = 1, p_s \geq \eta > 0\}$$

We want to consider economies with small innovation costs. To do this we fix a set of cost functions (c^a) and personalized set-up costs (b^a) and multiply $(c^a), (b^a)$ with $\delta > 0$ to obtain the new cost structure $(\delta c^a), (\delta b^a)$. If δ is small then also innovation costs will be small.

Theorem 3 *Suppose Assumptions 1-4 hold. Let* $p_0 = 1, p_1 \in P_\eta^{S-1}$ *for some* $\eta > 0$. *Then for all* $\epsilon > 0, \exists \bar{\delta} > 0$ *such that if we replace the original cost structure by* $(\delta c^a), (\delta b^a), \delta < \bar{\delta}$ *we have* $\inf_{x^* \in W_E} \|x - x^*\| < \epsilon$ *for all* x *in the set of competitive equilibrium allocations with financial innovation.*

The Theorem says that if innovation costs are small, then every equilibrium allocation will be close to some allocation in a corresponding (fictitious) complete markets economy. This clearly also implies that for small innovation costs the equilibrium allocation will be near-efficient.

Proof:

Step 1: First note that if x is the equilibrium allocation and $(\delta c^a), (\delta b^a)$ is the cost structure, then for all $\delta \geq 0$

$$\|x\|_\infty \leq K < \infty$$

for some K. This follows from the fact that every intermediated security and every standard security must have a price bounded away from zero and the No-Arbitrage condition holds. Hence

$$\eta \leq x_s(i) \leq K, a.e. i \in I$$

Let $MRS_{0,s}(i)$ denote the marginal rate of substitution of agent i between consumption in period 0 and state s. Suppose there are sets U and $V \subset I$ with $\mu(U) > 0, \mu(V) > 0$ and $MRS_{0,s}(i) \geq MRS_{0,s}(j) - \epsilon$ for $i \in U$, $j \in V$. Chose a market a with $R_s^a > 0$ and the following innovations:

$$\hat{Q}_{1s'}^a = \begin{cases} 1 & \text{if } s' = s \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{Q}_{2s'}^a = \begin{cases} 0 & \text{if } s' = s \\ R_s^a / (\sum_{s' \neq s} R_{s'}^a) & \text{if } s' \neq s \end{cases}$$

Since $x_s(i)$ is in a compact set for all i and all s there is a $\beta > 0$ such that

$$\hat{y}_1(i) = \frac{\beta R_s^a}{\sum R_s^a}, \hat{y}_2(i) = 0; \hat{y}_1(j) = 0, \hat{y}_2(j) = \frac{\beta \sum_{s' \neq s} R_s^a}{\sum R_s^a},$$

together with

$$\hat{r}(i) = \hat{y}(i)\frac{\hat{Q}^a_{s1}}{p_s}MRS_{0,s}(j) + \frac{3\epsilon}{4}\hat{y}(i)\frac{\hat{Q}^a_{s1}}{p_s}$$

$$\hat{r}(j) = \hat{y}(j)(\sum_s MRS_{0,s'}\frac{\hat{Q}^a_{2s}}{p_s}) - \frac{\epsilon}{4}\hat{y}(i)\frac{\hat{Q}^a_{s1}}{p_s}$$

is a feasible selling plan.

$$\hat{r}(i) + \hat{r}(j) = \frac{\beta}{\sum R^a_s}q^a + \frac{\beta R^a_s/p_s}{\sum R^a_s}\epsilon/2$$

since

$$q^a = \sum_s \frac{R^a_s}{p_s}MRS_{0,s}(j)$$

Note that c^a is uniformly bounded on $P^{S-1}_\eta \times J$. Hence $c^a \leq \bar{c}$ for some $\bar{c} < \infty$. Now chose $\bar{\delta}$ such that

$$\frac{\beta R^a_s/p_s}{\sum R^a_s}\epsilon/2 > \bar{\delta}\bar{c}\frac{\beta}{\sum R^a_s} + 2\bar{\delta}b^a$$

Then the innovation is profitable for $\delta < \bar{\delta}$. Hence we have established that

$$|MRS_{0,s}(i) - MRS_{0,s}(j)| < \epsilon$$

for $\delta < \bar{\delta}$.

Step 2. The existence of market $\bar{a}$ (Assumption 3) guarantees that the resources used for innovation are bounded by δB, for some $B < \infty$. Intermediary $\bar{a}$ can supply an innovation such that $x(i) = x^h = \int_{[h-1,h)} x(i)di$ for $i \in [h-1, h)$, for all $h = 1, ..., H$. Let every agent chose $z^h_{\bar{a}}$ so that $\omega^h_1 + P^{-1}R^{\bar{a}}z^h_{\bar{a}} \leq 0$. Then let $(P^{-1}\hat{Q}^{\bar{a}}_{h'})\hat{y}^h = x^h - \omega^h + P^{-1}R^{\bar{a}}z^h_{\bar{a}}$. This defines H innovations with innovation costs bounded by $\delta\bar{c}B$ for some B. Total willingness to pay for this innovation is greater or equal to the total willingness to pay for the original innovations since everybody is at least as well off, thus the total ressources spent on innovation costs cannot exceed δB. Step 3. We know that if the difference in the marginal rate of substitution is small then agents of the same type have to consume almost the same amount in each state. (This follows from strict concavity of the utility functions). Hence for every $\eta > 0$ there is a δ such that $|x(i) - x^h| < \eta$ if $i \in [h-1, h)$.

Let $\overline{MRS}_{0,s} = 1/H \int_I MRS_{0,s}(i)di$ and let $p^* = (1, \overline{MRS}_{0,1}, ..., \overline{MRS}_{0,s})$. For δ sufficiently small $(p^*, (x^h)_{h=1}^H)$, where x^h is an ϵ-equilibrium in the sense that

(i) $\sum \omega_0^h - \epsilon \leq \sum x_0^h \leq \sum \omega_0^h$ since innovation costs are smaller than ϵ.

(ii) $\sum \omega_s^h = \sum x_s^h, s = 1, ..., S$.

(iii) $p^* \omega^h - \epsilon \leq p^* x^h \leq p^* \omega^h + \epsilon$

(iv) $u^h(x^h) \geq u^h(x) - \epsilon$ for all x with $p^* x \leq p^* \omega^h$.

To see (iii) note that $p^* x^h \leq p^* \omega^h + \epsilon'$ since average costs of purchasing an intermediated security are non increasing and optimality of $x(i)$ implies

$$(P^{-1} Q_k^a) p^* \geq \lim_{\epsilon \to 0} [\frac{r_k^a(y_k^a(i) + \epsilon) - r_k^a(y_k^a(i))}{\epsilon}] - \eta$$

Since the total innovation costs are bounded by δB and since (i) holds we have $p^* x^h \geq p^* \omega^h - \delta B - (H-1)\epsilon'$, where ϵ' can be made arbitrarily small for small δ. Hence (iii) follows.

Item (iv) follows from the fact that the marginal rate of substitution at $x(i)$ is almost equal to p^* by step 1 and the distance between x^h and $x(i)$ is arbitrarily small for $i \in [h-1, h)$ and sufficiently small δ.

Now consider the correspondence $\xi(\epsilon) : \epsilon \to \{(x^h)_{h=1}^H : ((x^h)_{h=1}^H, p^*)$ is an ϵ-equilibrium$\}$. We know:

(i) $\xi(\epsilon)$ is uhc.

(ii) $\xi(0) = WE$.

The definition of uhc now gives the result.□

Remark: In Theorem 3 we assumed that prices stay bounded away from zero. This assumption is necessary in a setting where financial innovation can only be done using securities that pay off in units of account (nominal securities). Clearly, in the case of extreme differences in the change of the value of money between states it could be expected that financial intermediaries would use real assets (i.e. equity, commodity futures, etc.) as "raw-material" for financial innovation. If we introduced real securities in the model, the assumption on prices being bounded away from zero would be unnecessary. Financial intermediaries could switch to using real securities as collateral when differences in inflation make nominal securities an expensive raw material for financial innovation.

Theorem 4 *Suppose Assumptions 1-4 hold. For a generic set of endowments and for any $\epsilon > 0$ there exists a $\delta > 0$ such that for $\delta < \bar{\delta}$ we have: The set of equilibrium allocations is contained in a finite number of neighborhoods N_ϵ, where for any $x, x' \in N_\epsilon$ we have for a.e. $i \in I$*

$$|u^i(x(i) - u^i(x'(i))| < \epsilon$$

Proof: We know that for a generic set of endowments the set of complete markets equilibria is finite. Theorem 2 implies that for small costs if innovation every CEI allocation is close to a complete markets equilibrium. Continuity of the utility function then gives the result. □

Although equilibrium allocations will still be indeterminate, for small innovation costs Theorem 3 provides us with a substitute for determinacy: The utility consequences of the indeterminacy of equilibrium allocations are small if innovation costs are small and prices stay bounded away from zero.

The financial structure of the economy on the other hand may change dramatically across different equilibria: Different price normalizations will be accompanied by a different set of innovations and hence there may be a large indeterminacy of the financial structure of an economy even if innovation costs are small. The "instability" of the financial sector "stabilizes" the allocation of consumption in the sense that changing innovations counteract the real consequences of varying inflation and inflation expectations.

5 Conclusions

We develop a model of financial innovation in which intermediaries can issue new securities against collateral in the form of standard securities. Examples for this type of financial innovation are Collateralized and Stripped Mortgage Obligations, Primes and Scores, Stripped Treasury Securities and many other financial instruments created in the past two decades.

Under the assumption of proportional innovation costs and set up costs per individual it was shown that the model has the following characteristics:

- The equilibrium asset structure may exhibit "redundancies" as frequently observed on financial markets; i.e. new securities may be linear combinations of other securities but nevertheless be utility improving for households.

- Complementarities among innovations available to different intermediaries may lead to an inefficient level of innovation. There will be an efficient financial structure either if some intermediaries have access to all markets or if the personalized set up costs are zero. Therefore forcing intermediaries (via regulation) to specialize in a subset of the potential financial products can be harmful for the process of financial innovation.

- The equilibrium price level in every state of nature is indeterminate. To every price level in the different states of nature there may be different equilibrium innovations. This corresponds to the empirical fact that the financial products supplied change when future price levels (inflation and inflation volatility) change. The flexibility of the supplied set of securities reduces the indeterminacy of equilibrium consumption of households as compared to a model with a fixed set of securities. We show that if innovation costs are "small" then the utility consequences of the indeterminacy of equilibrium allocations is "small".

Financial innovation is modeled similar to general *competitive* commodity innovation by Hart (1980) and Makowski (1980). As a topic for future research we plan to examine financial innovation in an imperfectly competitive environment. There it should be possible to consider set-up costs for each newly created security. It remains to be seen how imperfect competition would affect the results of this paper.

6 Appendix

To prove the existence of a competitive equilibrium with financial innovation we will transform the economy outlined in section 2 into a standard economy with "many" commodities and indivisibilities. The connection between personalized set up costs and indivisibilities was first explored in Makowski (1979).

Note that the nonconvexity implied by the personalized set up costs can be rewritten as an indivisibility in the consumption set. If we interpret different amounts of a financial innovation as different commodities and then restrict agents to buy integer amounts, then we can use a standard (linear) price system to clear the markets. Note, however, that a large number of individuals is crucial for this procedure.

6.1 The Commodity Space

We will prove existence of an equilibrium for a given (arbitrary) set of period 1 prices $p_1 \in \mathbf{R}^{\mathbf{S}}_{++}$. Hence for the rest of the appendix p_1 is fixed. $P^{-1}R^a$ is the return of a standard security, $P^{-1}Q$ is the return of an innovation.

Let

$$J^n = \{d \in \mathbf{R}^S_+ : \sum d_s = 1; d_s \in [0, 1/n, 2/n, ..., 1]\}$$

Let $C^n = \{1, ..., m(n)\}$ be a labelling for the commodities in J^n. Let

$$D^n = \{1/n, 2/n, ..., T\}, D = [0, T]$$

denote the different "packages" at which each commodity in C can be offered. The set of commodities the consists of:

- $C^n \times D^n$ is the set of intermediated securities with generic element (j,t).
- A is the set of standard securities.
- x_0 is consumption in period zero.

There are $l = A + 1 + n \cdot T \cdot m(n)$ commodities. Hence the commodity space is $\mathbf{R}^l$. $w \in \mathbf{R}^l_+$ denote (standard) prices for those commodities.

6.2 Households

Each type of household will be characterized by a utility function $U^h : \mathbf{R}^l \to \mathbf{R}_+$ and a set of net feasible net trades X^h.

$$U^h(x_0, z, y) = u^h(\omega_0^h + x_0, \omega_1^h + P^{-1}Rz + P^{-1}\sum_{j,t} Q_j \cdot t \cdot y_{j,t})$$

Clearly this transformation of the utility function preserves continuity and strict concavity. Hence U^h is strictly concave and continuous for all types of households $h \in H$.

$$\begin{aligned} X^h &= \{(x_0, z, y) : L^1 \geq x_0 \geq -\omega_0; |z^a| \leq L^1; \sum_{j,t} t \cdot y_{j,t} \leq T; \\ &\quad y_{j,t} \in Z = \{0, 1, ..., Z\}; \omega_1^h + P^{-1}Rz + P^{-1}\sum_{j,t} Q_j \cdot t \cdot y_{j,t} \geq 0\} \end{aligned}$$

X^h is the consumption set with short sales bounded by L^1. Note that households are forced to purchase integer amounts of any intermediated product.

6.3 Firms

There is one firm for each market. The technology of firm a is given by

$$\begin{aligned} Y^a &= \{(x_0^a, z^a, y^a) : R^a z_a^a \leq -\sum_{j,t} Q_j \cdot t \cdot y_{j,t}^a ; z^a \leq 0, |z^a| < L^2; \\ & x_0^a \leq -b^a \sum_{j,t} y_{j,t}^a - \sum_{j,t} c_j^a \cdot t \cdot y_{j,t}^a ; x_0^a \geq -\sum_h \omega_0^h \} \end{aligned}$$

where $c_j^a > 0$ is the proportional cost of intermediated asset j at the given prices p. Clearly L^2 can be chosen in such a way that the firm is not constrained since the total amount of inputs x_0 is bounded above. The bound on x_0 is innocuous since in equilibrium the firm has to use strictly less of period zero consumption than totally available.

Furthermore note that there is no indivisibility for the firm and its technology exhibits constant returns to scale. Hence equilibrium profits of the firm will be zero and we do not need to worry about distributing profits.

6.4 Equilibrium

An equilibrium of this economy will be defined as a standard Walrasian equilibrium.

Definition 5 *A measurable function $(x_0, z, y) : I \to \mathbf{R}^l$, production plans $(x_0^a, z^a, y^a)_{a \in A}$ and a price $w \in \mathbf{R}^l$ is a Standard-Equilibrium if*

(i) For $i \in [h-1, h)$, $x_0(i), z(i), y(i)$ maximizes U^h on the budget set $B^h(w) = \{(x_0', z', y') \in X^h : w \cdot (x_0, z, y) \leq 0\}$

(ii) For all $a \in A$, (x_0^a, z^a, y^a) maximizes $w \cdot (x_0', z', y')$ on Y^a.

(iii)

$$\sum_a x_0^a - \int_I x_0(i) di \geq 0; \sum_a z^a - \int_I z(i) di \geq 0; \sum_a y^a - \int_I y(i) di \geq 0$$

Theorem 5 *Suppose Assumptions 1-3 hold. Then there exists a standard equilibrium for this economy.*

Proof: To prove of this theorem we can use the existence theorem in Makowski (1979). All the assumptions imposed on the economy there are satisfied in this special case.

Note that production sets are compact and consumption sets are closed and bounded for indivisible commodities. Furthermore the divisible commodity x_0 is *essential* as defined in Makowski (1979), i.e. no agent ever wants to trade to the lower x_0-boundary of the consumption set (Assumption 2). Utility functions are quasi-concave and all endowments are in the interior with respect to the divisible commodities. Since utility functions are strictly monotonic with respect to the divisible commodities, prices for divisible commodities have to stay bounded away from zero. □

6.5 Equilibrium with a Large Commodity Space

We have shown existence of a standard equilibrium for a finite set of commodities. In our original economy we had a continuum of potential innovations and hence we want to extend this existence theorem for a larger commodity space. (We also limited short sales of households by L^1, a restriction we will relax at the end of this proof.)

We will use the following notation: w_0 is the price of x_0, w_a is the price of standard security a and $w_{j,t}$ is the price of commodity $(j,t) \in J^n \times C^n$. Normalize

$$w_0 + \sum_{a \in A} w_a = 1$$

Thus the first $A+1$ prices sum to one. Let

$$\underline{R} = \{\min R^a_s | R^a_s > 0\}$$

Then we can restrict all prices of innovations to be below $T/\underline{R}$, i.e.

$$w_{j,t} \leq T/\underline{R}$$

since by buying $T/\underline{R}$ units of each standard security the agent is better off than by buying one unit of (j,t).

Let $\Gamma^n = J^n \times D^n$ be the set of commodities, $\Gamma = J \times D$. An equilibrium price w^n can be interpreted as a mapping

$$w^n : \{0,1,...,A\} \cup \Gamma^n \to [0, T/\underline{R}]$$

Let W^n be the set of equilibrium prices when the set of commodities is $\{0,1,...,A\} \cup \Gamma^n$ and for which $w_{j,t}/t \geq w_{j,t'}/t'$ if $t' \geq t$. Note that due to the cost structure for financial innovation this restriction will always be satisfied for some equilibrium prices, since for $w_{j,t}/t < w_{j,t'}/t', t' > t$, commodity (j,t) will be in zero supply and hence $w_{j,t}$ can be increased without changing excess demand. Let

$$W^{*n} = \{w^n \in W^n | \sum_{\Gamma^n} w^n_{j,t} \leq \sum_{\Gamma^n} w'^n_{j,t}, \forall w'^n \in W^n\}$$

Note that $W^{*n} \neq \emptyset$ since the demand and supply correspondences are uhc and the feasible sets are closed.

Lemma 1 *Let $\{w^{*n}\}$ be a sequence of equilibrium prices $w^{*n} \in W^{*n}$. Then there exists a sequence of extensions $\{\bar{w}^{*n}\}$ such that*

$$\bar{w}^{*n} : \{0,1,...,A\} \cup \Gamma \to [0, T/\underline{R}]$$

$$\bar{w}^{*n} | \Gamma^n = w^{*n} | \Gamma^n$$

and $\{w^{*n}\}$ is an equicontinuous family.

Proof: *Claim 1*: For each $\epsilon > 0$ there is a $\delta > 0$ such that if $w_{j,t} > w_{j',t'} + \epsilon$ then $|(Q_j, t) - (Q_{j'}, t')| < \delta$ implies $y_{j,t}(i) = 0$ for all $i \in I$ if i is behaving optimally. Moreover δ is independent of i and of (Q_j, t) and $(Q_{j'}, t')$.

Pf: Continuity of u^h implies that

$$|u^h(x_0, x_1 + P^{-1}Q_j \cdot t) - u^h(x_0, x_1 + P^{-1}Q_{j'} \cdot t')| \to 0$$

as $|Q_j \cdot t - Q_{j'} \cdot t'| \to 0$. Note that for all optimal consumption plans (x_0^h, x_1^h) is bounded below by some $\eta > 0$ and bounded above by some $C < \infty$ in each component. (This follows from Assumptions 1 and 2 and compactness of X^h.) Let y' be identical to y in every component apart form the (j,t), and the (j',t')-component.

$$y'_{j,t} = 0; y'_{j',t'} = y_{j,t} + y_{j',t'}$$

Then uniformly over all types of households and all (x_0^h, z^h, y^h) with

$$\eta \le \omega_0^h + x_0 \le C$$

$$\eta \le \omega^h + P^{-1}Rz^h + P^{-1}\sum_{j,t} Q_j \cdot t \cdot y_{j,t}^h \le C$$

we have for any $\epsilon > 0$

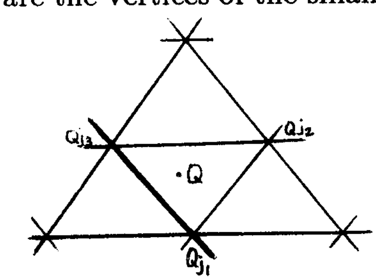
$$(U^h(x_0 + \epsilon, z, y) - U^h(x_0, z, y)) + (U^h(x_0, z, y) - U^h(x_0, z, y')) \to$$
$$U^h(x_0 + \epsilon, z, y) - U^h(x_0, z, y) > 0$$

as $|Q_j \cdot t - Q_{j'} \cdot t'| \to 0$ which proves claim 1.

Let $\bar{w}^{*n} : \{0, 1, ..., A\} \times \Gamma \to [0, T/\underline{R}]$ be the following extension of w^{*n}: Given $t \in D^n$

$$\bar{w}^{*n}(Q,t) = \sum_{i=1}^{S} \lambda^i(Q) w_{j_i,t}^{*n}; Q = \sum_{i=1}^{S} \lambda^i(Q) Q_{j_i}$$

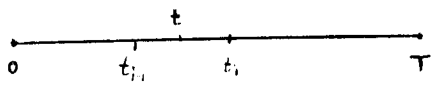
where $\{Q_{j_i}\} \in \Gamma^n$ are the vertices of the smallest subsimplex which contains Q.



and for $t \notin D^n$

$$\bar{w}^{*n}(Q,t) = \lambda(t) w^{*n}(Q, t_{i-1}) + (1 - \lambda(t)) w^{*n}(Q, t_i)$$

where $t = \lambda(t)t_{i-1} + (1-\lambda(t))t_i$, $t_{i-1}, t_i \in D^n$.



Hence $w^{*n} : \Gamma \to [0, T/\underline{R}]$ is a (uniformly) continuous function.

Claim 2: $\{\bar{w}^{*n}\}$ is an equicontinuous family of functions.

Pf: Given $2\epsilon > 0$, then if $|(Q,t) - (Q',t')| < \delta(\epsilon)/2$ (where $\delta(\epsilon)$ is taken from claim 1 above) then $|w^{*n}(Q,t) - w^{*,n}(Q',t')| \leq 2\epsilon$ if $n \geq \bar{n}$ for some $\bar{n}$. Suppose the contrary is true. Take $\bar{n}$ so that $1/\bar{n} < \delta(\epsilon)/2$. Then if

$$\bar{w}^{*n}(Q,t) > \bar{w}^{*n}(Q',t') + 2\epsilon$$

then there is a $(Q_j,t),(Q_{j'},t') \in \Gamma^n$ with $|(Q_j,t) - (Q_{j'},t| < \delta(\epsilon)$ and

$$\bar{w}^{*n}_{j,t} - \bar{w}^{*n}_{j',t'} > 2\epsilon$$

Since for $w_{j,t} \geq w_{j',t'} + \epsilon$, (Q_j,t) will never be chosen by any household we can decrease $\bar{w}^{*n}_{j,t}$ without affecting demand. (Since (Q_j,t) is in zero supply, decreasing the price will also leave the supply unaffected.) Hence we have a contradiction to $\bar{w}^{*n} \in \bar{W}^{*n}$.

For $n \leq \bar{n}$ let $\delta(\epsilon) = \epsilon/n(T/\underline{R})$. □

Lemma 2 *For every n and every equilibrium price w there exists an equilibrium allocation such that for every $i \in I$ $y_{j,t} > 0$ for at most $N_1 = A(S+1)$ different $(j,t) \in \Gamma^n$.*

Proof Consider the mapping $\xi^{an} : \Gamma^n \to \mathbf{R}_+$ defined by $\xi^{an}(Q_j,t) = tc^a_j$. Consider the lower convex envelope of this mapping. Since $\Gamma^n \subset \mathbf{R}^S$ every point on the lower envelope is a convex combination of at most $(S+1)$ distinct points in Γ^n. This implies that we can restrict each household to purchase at most $A(S+1)$ different commodities in Γ^n. □

Lemma 3 *For every n and for every equilibrium price w there exists an equilibrium allocation such that agents of one type use at most $N_2 = (S+2)A+1$ distinct trade plans.*

Proof Consider equilibrium trade plans (x, y, z), $(x_0^a, y^a, z^a)_{a\in A}$. Let $y_a : I \to \mathbf{R}^{m(n)}$ be such that $\int_I y_a(i)di = y^a$. Define

$$d_1^a(i) = \sum Q_j \cdot t \cdot y_{a,j,t}(i)$$
$$d_2^a(i) = w \cdot y_a(i) - \sum_{j,t} y_{a,j,t}[c_j^a \cdot t + b^a]$$
$$d_3(i) = z(i)$$

Let $d^a(i) = (d_1^a(i), d_2^a(i), d_3(i))$ and $d^{ah} = \int_{[h-1,h)} d^a(i)di$. Note that any trade plan that gives rise to d^{ah} for all $a \in A$ and for all $h \in H$ will be profit maximizing for intermediaries and will clear the market. Now define the set

$$B^h = \{(b^1(i), ..., b^A(i)) \in \mathbf{R}^{(S+2)A} : i \in [h-1, h)\}$$

B^h is a subset of $\mathbf{R}^{(S+2)A}$, hence by Caratheodory's Theorem every point in the convex hull of B^h can be expressed as a convex combination of $A(S+2)+1$ not necessarily distinct points in B^h. Thus for every equilibrium price we find an equilibrium allocation with the claimed properties. □

We have shown that for every equilibrium price there exists an equilibrium allocation for which the total number of commodities supplied by intermediaries is less that or equal to $N = H \cdot N_1 \cdot N_2$.

Let $C = [-L, L], L = \max[L^1, L^2]$. Using Lemmata 2 and 3 there is an equilibrium trade plan of firm a that can be expressed as a vector

$$\pi^a = (x_0^a, z^a, (Q_1^a, t_1^a, y_1^a), ..., (Q_N^a, t_N^a, y_N^a)) \in C^{A+1} \times [\Gamma \times C]^N$$

$\pi = (\pi^1, ..., \pi^A)$.

Similarly there is an equilibrium trade plan such that the trades of households of any type can be expressed as a vector

$$(\mu^h, \psi^h) = ((\mu_1^h, \psi_1^h), ..., (\mu_{N_2}^y, \psi_{N_2}^h)) \in \{[0,1] \times C^{A+1} \times [\Gamma \times Z]^{N_1}\}^{N_2}$$

where the first component denotes the fractions of households of type h using a specific trade plan;[13] $\mu^{h,j} \in [0,1]$ and

$$\psi^{h,j} = (x_0^{h,j}, z^{h,j}, (Q_1^{h,j}, t_1^{h,j}, y_1^{h,j}), .., (Q_{N_1}^{h,j}, t_{N_1}^{h,j}, y_{N_1}^{h,j})) \in C^{A+1} \times [\Gamma \times Z]^{N_1}$$

[13]Remember that we always find an equilibrium allocation where the members of any type use no more than N_2 distinct trade plans.

Let $(\mu, \psi) = ((\mu^1, \psi^1), ..., (\mu^H, \psi^H))$.

Thus $(\pi; (\mu, \psi))$ describes an equilibrium allocation. Note that $(\pi; (\mu, \psi))$ lies in a compact set. Now consider a sequence of equilibrium allocation for increasing n, $\{(\pi^n; (\mu^n, \psi^n))\}_{n=1}^{\infty}$ together with a sequence of equilibrium prices $\{\bar{w}^n\}_{n=1}^{\infty}$ with $w^n \in W^{*n}$. This sequence has a converging subsequence for which $w^n \to w$ uniformly and $(\pi^n; (\mu^n, \psi^n)) \to (\pi; (\mu, \psi))$.

We want to show that this limit allocation $(\pi; (\mu, \psi))$ and the limit prices w form a standard equilibrium. To do this we first have to define an equilibrium for the limit economy.

A trade plan for intermediated securities of firm a can be interpreted as a positive measure $\bar{y}^a$ on the set Γ and the set of feasible innovations is:

$$
\begin{aligned}
\bar{Y}^{an} \;=\; & \{(x_0^a, z^a, \bar{y}^a) : R^a z_a^a \leq -\int_\Gamma Q \cdot t d\bar{y}^a; 0 \geq z_{a'}^a \geq -L^2; \\
& x_0^a \leq -b^a \int_\Gamma d\bar{y}^a - \int_\Gamma c^a(Q) \cdot t d\bar{y}^a; x_0^a \geq -\sum_h \omega_0^h, support(\bar{y}) \in \Gamma^n\}
\end{aligned}
$$

where n refers to the grid on the commodity space; if n is omitted we refer to the limiting case where all of Γ could be produced.

Similarly a trade plan of a household of intermediated securities can be interpreted as a positive measure y on Γ with the restriction that y can be expressed as the sum of Dirac measures. $d \in \mathcal{D}$ implies $d = \sum \beta(Q,t)\delta_{(Q,t)}$ where $\beta(Q,t) \in Z$ and $\delta_{(Q,t)}$ is the Dirac measure at (Q,t). The set of feasible net trades then is:

$$
\begin{aligned}
\bar{X}^{hn} \;=\; & \{(x_0, z, \bar{y}) : L^1 \geq x_0 \geq -\omega_0; |z^a| \leq L^1, \bar{y} \in \mathcal{D}, \int_\Gamma t \cdot d\bar{y} \leq T, \\
& \omega_1^h + P^{-1}Rz + P^{-1}\int_\Gamma Q \cdot t d\bar{y} \geq 0, support(\bar{y}) \in \Gamma^n\}
\end{aligned}
$$

We also define

$$
\bar{U}^h(x_0, z, \bar{y}) = u^h(x_0, \omega^h + P^{-1}Ry + \int_\Gamma Q \cdot t \cdot d\bar{y})
$$

and

$$
w \cdot (x_0, z, \bar{y}) = w_0 \cdot x_0 + \sum_A w_a z_a + \int_\Gamma w \cdot d\bar{y}
$$

Note that to π^a there corresponds a $\bar{\pi}^a = (x_0^a, z^a, \bar{y}^a) \in Y^a$, where

$$\bar{y}^a = \sum_{n=1}^{N} y_n^a \delta_{(Q_n^a, t_n^a)}$$

And to $\psi^{h,j}$ there corresponds an $\bar{\psi}^{h,j} = (x_0^{h,j}, z^{h,j}, \bar{y}^{h,j}) \in \bar{X}^h$ where

$$\bar{y}^{h,j} = \sum_{n=1}^{N_2} y_n^{h,j} \delta_{(Q_n^{h,j}, t_n^{h,j})}$$

Using the Dirac measure we can convert the vectors of purchases of intermediated securities into measures. Let $(\pi, (\mu, \psi))$ be the limit of a converging subsequence for increasing n. Let $(\bar{\pi}, (\mu, \bar{\psi}))$ be the corresponding measures.

Definition 6 *$(\bar{\pi}, (\mu, \bar{\psi}), w)$ is a standard equilibrium if*

(i) $\bar{w}$ is a (uniformly) continuous function on $\{0, 1, ..., A\} \cup \Gamma$.

(ii) For all h and for all $j = 1, ..., N_2$ we have

$$U^h(x_0, z, \bar{y}) \leq U^h(\bar{\psi}^{h,j})$$

for all $(x_0, z, \bar{y}) \in \bar{X}^h$ with $w(x_0, z, \bar{y}) \leq 0$.

(ii) $\bar{\pi}^a \in \bar{Y}^a$ and for all $(x_0', z'^a, \bar{y}'^a) \in \bar{Y}^a$ we have $\bar{w} \cdot \bar{\pi}^a \geq \bar{w}_0 \cdot (x_0'^a, z'^a, \bar{y}'^a)$

(iii) All markets clear if

$$\sum_{a=1}^{A} \bar{\pi}^a + \sum_{h=1}^{H} \sum_{j=1}^{N_2} \mu^{h,j} \cdot \bar{\psi}^{h,j} = 0$$

Theorem 6 *Suppose Assumptions 1-3 hold. Let $(\pi, (\mu, \psi), w)$ be the limit of a sequence of equilibria $(\pi^n, (\mu^n, \psi^n), \bar{w}^n)$ as $n \to \infty$. Then $(\bar{\pi}, (\mu, \bar{\psi}), w)$ is a standard equilibrium.*

Proof: (i) Market clearing and feasibility are obvious. $\bar{w}^n$ is a uniformly convergent sequence and hence w is continuous.

(ii) For every π^a there exists a sequence $\pi^{an} \to \pi^a$ and $\bar{w}^n \bar{\pi}^{an} \to w\bar{\pi}^a$. Suppose that $\bar{\pi}^a$ was not optimal for firm a, then there is a $\bar{\pi}'^a \in \bar{Y}^a$ with $w\bar{\pi}'^a > w\bar{\pi}^a$. $\bar{\pi}^a = (x_0'^a, z'^a, \bar{y}'^a)$. There exists an approximation of $\bar{\pi}^a$ with $\bar{\pi}'^{an} \in Y^{an}$ such that $\bar{w}^n \bar{\pi}'^{an} > w\bar{\pi}$ for large n and hence $w^n \bar{\pi}'^{an} > \bar{w}^n \bar{\pi}^{an}$ contradicting optimality of $\bar{\pi}^{an}$. Hence $\bar{\pi}^a$ is optimal at prices w.

(iii) Suppose for some (h, j), $\bar{\psi}^{h,j}$ is not optimal,then there exists an $\psi'^{h,j} \in \bar{X}^h$ with $w\bar{\psi}'^{h,j} \leq 0$ and $U^h(\bar{\psi}'^{h,j}) > U^h(\bar{\psi}'^{h,j})$. Similar to part (ii) we can approximate the improving trade plan for large n and obtain a contradiction. □

Finally we have to get rid of the bound L^1 in the consumption set $\bar{X}^h$. We know that for each L^1 there exists a standard equilibrium. Take a sequence of increasing L^1; if some types of consumers are constrained by L^1 for each element of the sequence then the optimal demand for some asset or period zero consumption goes to infinity.

This implies (since the payoffs of standard securities are linearly independent) that $x_s(i) \to \infty$ or $x_0(i) \to \infty$ where $x_s(i)$ is the optimal consumption in state s and $x_0(i)$ is the optimal consumption in period 0 respectively. Note that since endowments are bounded and consumption is nonnegative in each state, this implies that the agent can assemble a portfolio that pays off 1 unit in state s and a nonnegative amount otherwise and the price of which goes to zero. Since all agents utilities are strictly monotonic in this portfolio this violates market clearing. Hence for large L^1 no consumer is constrained by the bound on consumption of z and x_0.

This proves that every standard equilibrium is a competitive equilibrium with financial innovation and hence the existence of a CEI is established.

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