# Reputation With Deterministic Stage Games

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November 6, 1991

**Abstract**

A single long-run player plays a fixed stage game (simultaneous or sequential move) against an infinite sequence of short-run opponents that play only once but can observe all the past realized actions. Assuming that the probability distributions over types of long-run and short-run players have full support, we compute a lower bound on the Nash equilibrium payoffs to the long-run player.

*The financial support of the National Science Foundation, Grant SES 90-7999, and of Consiglio Nazionale delle Ricerche is gratefully acknowledged. I wish to thank David Levine and Wolfgang Pesendorfer for useful discussions and suggestions. All remaining errors are of course mine.

# 1 Introduction

Kreps and Wilson (1982) and Milgrom and Roberts (1982) have provided an explanation of the chain store paradox, assuming that there is a "chance" that the incumbent is a commitment type: this fact can be exploited by a "sane" incumbent that can therefore build up a reputation for toughness.

Fudenberg and Levine (1989) (in the following FL) build up on these results to provide a lower bound on the NE payoffs of the long run player (LR). However their main result (Theorem 1) applies only to games in which the commitment strategy of the LR player is revealed regardless of the strategies the short run players (SR) choose: this is true in simultaneous move games, in sequential move games in which the LR player moves first, and in some sequential move games in which the SR player moves first: an example of this last class is the chain store game, in which the strategy the SR players choose before the reputation is established is exactly the one that reveals the strategy by which the LR player builds up his reputation.

FL also provide a generalization of Theorem 1 (Theorem 2) in which the Stackelberg payoff is redefined to keep into account the fact that the outcome of the stage game may not reveal the LR player's strategy; the new bound is computed making use of the fact that the observed outcome of the stage game in general restricts a subset of the strategy space to which the strategy chosen by the LR player must belong.

Unfortunately, in some games this result does not provide a higher lower bound than the minimum payoff for the LR player. For example, consider the quality game with extensive form as in Fig. 1. The SR player moves first and decides whether to buy a product or not: if he decides not to buy, the game ends and both players get 0; if he decides to buy, the LR player decides whether to produce a low quality product, thus making a larger profit and causing the SR player a loss, or a high quality product, in which case the profit is smaller but the SR player's payoff is positive. When this stage game is repeated an infinite number of times, the lower bound provided in Theorem 2 in FL is just 0: if the prior probability that the LR player is committed to high quality is less than .5, the SR player decides not to buy. Given that no information is revealed, the game at the following stage is the same, the SR players never buy, and the LR player payoff is 0 (cfr. FL, pp. 772-773).

The purpose of this paper is to provide a different generalization of Theorem 1 in FL, one that uses perturbations of the original game with the property that every information set is reached with positive probability in the stage game.

The idea is simply to assume that not only the type of the LR player is uncertain, but also the types of the SR players are, and that for each strategy there exists at least one type of SR player, that is selected with strictly positive probability, that has that strategy as a strictly dominant strategy.

As is shown in Example 1, this might require a substantial increase in the number of periods necessary to build up a reputation with respect to the sequential move game, but may nevertheless provide a significantly higher lower bound on the NE payoffs of the LR player.

As in FL we provide a lower bound on the Nash equilibrium payoffs to the LR player by computing a lower bound on the payoff to the so called *Stackelberg strategy* to be defined. This strategy need not be the optimal one for the LR player, but since it is always feasible, the optimal strategy has to yield at least as high a payoff.

Our result holds for any stage game (simultaneous move[1] or sequential move) in which the realized actions of the LR player are observed, and the probability distributions over types of LR and SR players have full support.

Fudenberg and Levine (1991) show that there is a lower bound on the Nash equilibrium payoffs to the LR player also when the public outcome of the stage game is a random variable that provides only stochastic information about the strategy the LR player chose.

Our model is a special case of theirs in that in sequential move stage games in which the SR player moves first the public outcome only reveals the action of the LR player and not his strategy. Restricting to this special

[1] For simultaneous move stage games our result coincides with that of FL.

class of games however lets us explicitly compute the lower bound, and thus narrow down the set of equilibrium payoffs to the LR player.

The range of applications of our result seems very wide. In the following we just want to mention a few applications of the quality game.

International loan contracts are many times not enforceable or very costly to enforce. They are therefore well described by the quality game: an international lender decides whether to give credit to a foreign agent and the latter then decides whether to repay the loan or renege on his debt. Even though repaying is suboptimal in the stage game, it is a way of establishing a reputation for repayment that in turn guarantees prolonged access to international loan markets.

Illegal contracts are also not enforceable: nevertheless cocaine dealers or illegal lottery organizers can decide to sell high quality cocaine or to pay the prizes in order to establish a reputation for "honesty".

Importers usually get short term credit from their suppliers. In some less developed countries, however, trading houses do not enforce these contracts, so that the importer has an incentive to renege on it and, by backward induction, foreign traders refuse him credit. Also in this case, however, the importer can guarantee himself a higher discounted payoff in the repeated game by establishing a reputation for repayment.

In Section 2 we describe the game and introduce the notation. The result is derived in Section 3. Section 4 provides two examples of the quality game that give substance to the results of the previous section.

# 2 The Model

A long run player (player 1 or LR) plays a fixed stage game against an infinite sequence of short run players (player 2 or SR). The LR player chooses a strategy $s_1$ from a finite nonempty set $S_1$ and the SR player chooses an action $s_2$ from a finite nonempty set $S_2$. The corresponding mixed strategy spaces are denoted by $\Sigma_1$ and $\Sigma_2$.

The public outcome of the stage game is given by a mapping $y : S_1 \times S_2 \to Y$, and is to be interpreted as the revealed actions of the LR and the SR player. When the stage game is simultaneous move or sequential move with the LR player moving first, the action reveals the LR player's strategy. But when the stage game is sequential move and the SR player moves first the LR player's revealed action doesn't reveal what he would have done had the SR player chosen a different strategy.

The unperturbed stage game is described by the payoffs to the LR and the SR players, a mapping $g : Y \to \mathbf{R}^2$; with an abuse of notation we let $g(y(\sigma)) = (g_1(y(\sigma_1, \sigma_2)), g_2(y(\sigma_1, \sigma_2)))$ denote the expected payoff corresponding to the mixed strategy profile $\sigma$. In the unperturbed repeated game the LR player maximizes the normalized discounted value of expected payoffs

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t g_1^t \tag{1}$$

Each period's SR player maximizes that period's payoff, $g_2^t$.

Both LR and SR players can condition their play on the past history of the game. Let $H_t = Y^t$ denote the set of possible histories of the game; then mixed strategies are mappings $\sigma_1^t : H_{t-1} \to \Sigma_1$, and $\sigma_2^t : H_{t-1} \to \Sigma_2$.

Let $B : \Sigma_1 \to \Sigma_2$ be the correspondence that maps mixed strategies by the LR player in the stage game to the best responses of the SR player. Then we define the *Stackelberg payoff* $g_1^*$ as:

$$g_1^* = \max_{s_1 \in S_1} \min_{\sigma_2 \in B(s_1)} g_1(s_1, \sigma_2) \tag{2}$$

the *Stackelberg leader strategy* as the $s_1^*$ that solves

$$\max_{s_1 \in S_1} \min_{\sigma_2 \in B(s_1)} g_1(s_1, \sigma_2) \tag{3}$$

and the *Stackelberg follower strategy* as the $s_2^*$ that solves

$$\min_{\sigma_2 \in B(s_1^*)} g_1(s_1^*, \sigma_2) \tag{4}$$

(intuitively $s_2^*$ is the strategy of the SR player that the LR player wants to induce).

In the perturbed game the payoffs of the LR player, as well as those of the SR player, are made dependent on their types which are assumed to be private knowledge. For simplicity we assume that there are a countable number of types both of long and short run players:

$$\Omega_1 = \{\omega_0^1, \omega_1^1, \omega_2^1, \ldots\}, \Omega_2 = \{\omega_0^2, \omega_1^2, \omega_2^2, \ldots\} \tag{5}$$

The payoffs are therefore a mapping $g_i : \Sigma_1 \times \Sigma_2 \times \Omega_i \rightarrow R$, and the mixed strategies are mappings $\sigma_i^t : H_{t-1} \times \Omega_i \rightarrow \Sigma_i$. We let $\omega_1^0$ and $\omega_2^0$ be the rational players: in other words we assume that their payoffs are as in the unperturbed game: $g_i(\sigma_1, \sigma_2, \omega_i^0) = g_i(\sigma_1, \sigma_2)$, $i = 1, 2$.

The priors on the types are probability distributions $\mu_1 : \Omega_1 \rightarrow [0, 1]$ and $\mu_2 : \Omega_2 \rightarrow [0, 1]$ that are assumed to be common knowledge.

In the following we will make the following assumptions about the types of LR and SR players:

**Assumption 1:** For each $s_1 \in S_1$, there is a LR player that has that strategy as a dominant strategy in the repeated game and $\mu_1$ has full support, i.e. $\exists \bar{\mu}_1 \in \mathbf{R}_+ : \forall \omega_1 \in \Omega_1, \mu_1(\omega_1) > \bar{\mu}_1$.

**Assumption 2:** For each $s_2 \in S_2$ there is a type of SR player that has $s_2$ as a strictly dominant strategy, and $\mu_2$ has full support, i.e. $\exists \bar{\mu}_2 \in \mathbf{R}_+ : \forall \omega_2 \in \Omega_2, \mu_2(\omega_2) > \bar{\mu}_2$.

In the following we will call a *Stackelberg leader type* a LR player that has $s_1^*$ as a dominant strategy in the repeated game, and we denote by $\omega_1^*$ the event that the LR player is such type, and by $\bar{\omega}_1^*$ the event that the LR player is not such type. We will denote by $\omega_2^j$ the event that the SR player is the type that has $s_2^j$ as a dominant strategy.[2].

Let $H^*$ be the set of histories such that the play of the LR player is consistent with the description of the Stackelberg type for all $t$, and let $h^*$ denote the event $h \in H^*$. Finally, let $\pi_t^*$ be the random variable $\pi(s_1^t =$

[2]Less strong assumptions about the types of SR players can be made; see Section 4.

$s_1^*|h_{t-1})$ and let $n(\pi_t^* \leq \bar{\pi})$ be the random variable denoting the number (possibly infinite) of the random variables $\pi_t^*$ for which $\pi_t^* \leq \bar{\pi}$.

# 3 The Result

First we show that $\pi(\omega_1^*|h_t)$ is nondecreasing in $t$ when $h_t$ is the truncation of a history $h \in H^*$.

**Lemma 1** *For any infinite history $h \in H^*$ such that the truncated histories $h_t$ have positive probability, $\pi(\omega_1^*|h_t)$ is nondecreasing in $t$.*

**Proof:** We want to show that

$$\pi(\omega_1^*|h_t) = \pi(\omega_1^*|y(s_1^*, s_2), h_{t-1}) \quad (6)$$

$$= \frac{\pi(\omega_1^*|h_{t-1})\pi(y(s_1^*, s_2)|\omega_1^*)}{\pi(\omega_1^*|h_{t-1})\pi(y(s_1^*, s_2)|\omega_1^*) + (1 - \pi(\omega_1^*|h_{t-1}))\pi(y(s_1^*, s_2)|\bar{\omega}_1^*)} \quad (7)$$

$$\geq \pi(\omega_1^*|h_{t-1}) \quad (8)$$

Inequality (8) is equivalent to

$$\frac{\pi(y(s_1^*, s_2)|\omega_1^*)}{\pi(\omega_1^*|h_{t-1})\pi(y(s_1^*, s_2)|\omega_1^*) + (1 - \pi(\omega_1^*|h_{t-1}))\pi(y(s_1^*, s_2)|\bar{\omega}_1^*)} \geq 1 \quad (9)$$

which is in turn equivalent to

$$\pi(y(s_1^*, s_2)|\bar{\omega}_1^*) \leq \pi(y(s_1^*, s_2)|\omega_1^*). \quad (10)$$

which is trivially satisfied since $\pi(y(s_1^*, s_2)|\omega_1^*) = 1$. □

The following Lemma computes an upper bound on the probability that the probability that the LR player plays $s_1^*$ is less than a fixed probability $\bar{\pi}$ when the stage game is repeated a number of times, and is to be used to compute the lower bound on the NE payoffs to the LR player. In the following we will assume that the cardinality of $S_1$ is $N + 1$ and will denote by $[.]$ the operator integral part ($[x]$ is the greatest integer less than or equal to $x$).

**Lemma 2** *Let $0 \leq \bar{\pi} < 1$. Suppose that $(\sigma_1^t, \sigma_2^t)$ are such that $\pi(h^*|\omega_1^*) = 1$. Let $K_1 = [\log \bar{\mu}_1 / \log(1 - (1 - \bar{\pi})/N)] + 1$, and $\forall \epsilon > 0$ let $K_2(\epsilon) = [\log(1 - (1 - \epsilon)^{1/K_1})/\log(1 - \bar{\mu}_2)] + 1$. Then, $\forall \epsilon > 0$,*

$$\pi(n(\pi_t^* \leq \bar{\pi}) > K_1 \cdot K_2(\epsilon)|h^*) \leq \epsilon. \tag{11}$$

**Remark 1.** The purpose of Lemma 2 is to provide an upper bound on the probability that the probability that the LR player plays $s_1^*$ is less than a given $\bar{\pi} \in [0, 1)$ after the stage games has been played a given number of times, and to make this upper bound dependent on $\bar{\mu}_1$, $\bar{\mu}_2$ (the lower bounds on $\mu_1$ and $\mu_2$) and $\bar{\pi}$ only, and otherwise independent of $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$. To do this we argue that whenever $\pi_t^* = \pi(s_1^t = s_1^*|h_{t-1})$ is low, if $s_1^*$ is played, there is a strictly positive probability that $\pi(\omega_1^*|h_t)$ increases by a nontrivial amount. Since $\pi(\omega_1^*|h_t)$ has to be less than or equal to 1, this cannot happen too often, so that the probability that $\pi_t^*$ is low in many periods has to be low.

**Proof:** By Bayes's law we have

$$\begin{aligned} \pi(\omega_1^*|h_t) &= \pi(\omega_1^*|y(s_1^*, s_2), h_{t-1}) & (12) \\ &= \frac{\pi(\omega_1^*|h_{t-1})\pi(y(s_1^*, s_2)|\omega_1^*)}{\pi(\omega_1^*|h_{t-1})\pi(y(s_1^*, s_2)|\omega_1^*) + (1 - \pi(\omega_1^*|h_{t-1}))\pi(y(s_1^*, s_2)|\bar{\omega}_1^*)} & (13) \end{aligned}$$

Substituting $\pi(y(s_1^*, s_2)|\omega_1^*) = 1$ in the numerator of the fraction in (13) and recognizing that the denominator is equal to $\pi(y(s_1^*, s_2)|h_{t-1}, s_2^*)$ we have

$$\pi(\omega_1^*|h_t) = \frac{\pi(\omega_1^*|h_{t-1})}{\pi(y(s_1^*, s_2)|h_{t-1})} \tag{14}$$

where $\pi(\omega_1^*|h_1) = \mu_1(\omega_1^*) \geq \bar{\mu}_1$.

$\pi(y(s_1^*, s_2)|h_{t-1})$ is the probability that $y(s_1^*, s_2)$ is observed, which is equal to the probability that $s_1^*$ is being played plus the probability that other strategies observationally equivalent to $s_1^*$ for $s_2$ are being played. Define $S_1^*(s_2)$ as the set of strategies of the LR player different from $s_1^*$ that are observationally equivalent to $s_1^*$ when the SR player plays $s_2$, i.e. $S_1^*(s_2) =$

$\{s_1 \neq s_1^* : y(s_1, s_2) = y(s_1^*, s_2)\}$. With this notation (13) can be rewritten as

$$\pi(\omega_1^*|h_t) = \frac{\pi(\omega_1^*|h_{t-1})}{\pi(s_1^t = s_1^*|h_{t-1}) + \sum_{s_1 \in S_1^*(s_2)} \pi(s_1^t = s_1|h_{t-1})} \tag{15}$$

Saying that $\pi(s_1^t = s_1^*|h_{t-1}) > \bar{\pi}$ is equivalent to saying that

$$\sum_{s_1 \in S_1^*(s_2)} \pi(s_1^t = s_1|h_{t-1}) < 1 - \bar{\pi}. \tag{16}$$

Given that the cardinality of $S_1$ is $N + 1$, a sufficient condition for (16) to be satisfied is $\pi(s_1^t = s_1|h_{t-1}) < \tilde{\pi}$ for all $s_1 \neq s_1^*$, where $\tilde{\pi} = (1 - \bar{\pi})/N$.

Now suppose $\exists s_1 \neq s_1^*$ such that $\pi(s_1^t = s_1|h_{t-1}) > \tilde{\pi}$. Since $s_1 \neq s_1^*$, there exists an $s_2$ such that $s_1$ is not observationally equivalent to $s_1^*$, $s_1 \notin S_1^*(s_2)$ (in other words, $y(s_1, s_2) \neq y(s_1^*, s_2)$). If the SR player plays such an $s_2$ (an event that, by Assumption 2, happens with probability at least $\bar{\mu}_2$), then by (14) we have that

$$\pi(\omega_1^*|h_t) \geq \frac{\pi(\omega_1^*|h_{t-1})}{1 - \tilde{\pi}} \tag{17}$$

since the denominator of (14) is less than or equal to $1 - \tilde{\pi}$. In the following we will call such an $s_2$ an *information revealing strategy.*

If the stage game is repeated $K$ times and every time an information revealing $s_2$ is selected, then

$$\pi(\omega_1^*|h_t) \geq \frac{\bar{\mu}_1}{(1 - \tilde{\pi})^K}. \tag{18}$$

However, since

$$\pi(\omega_1^*|h_t) \leq 1 \tag{19}$$

if

$$\frac{\bar{\mu}_1}{(1 - \tilde{\pi})^K} > 1 \tag{20}$$

inequality (19) is violated and a contradiction to the hypothesis that $\pi(s_1^t = s_1|h_{t-1}) > \tilde{\pi}$, any $s_1 \neq s_1^*$, is obtained. Taking the log of (20) and substituting $\tilde{\pi} = 1 - (1 - \bar{\pi})/N$ the condition becomes

$$K > \frac{\log \bar{\mu}_1}{\log(1 - (1 - \bar{\pi})/N)}. \tag{21}$$

Defining $K_1 = [\log \mu_1^* / \log(1 - (1 - \bar{\pi})/N)] + 1$ provides the first part of the result.

Finally we want to find an upper bound on the number of times the stage game is played and the probability that $\pi_t^* < \bar{\pi}$ is less than a given $\epsilon > 0$, when the LR player plays $s_1^*$, i.e. we want to find the smallest integer $K_2(\epsilon)$ such that

$$\pi(n(\pi_t^* \leq \bar{\pi}) > K_1 \cdot K_2(\epsilon)|h^*) \leq \epsilon. \tag{22}$$

The probability on the LHS of inequality (21) is less than or equal to the probability that information revealing $s_2$ are played less than $K_1$ times when the stage game is repeated $K_1 \cdot K_2(\epsilon)$ times.

Suppose that the stage game is played $K_2(\epsilon)$ times; then the probability that no information revealing $s_2$ is played is

$$\eta = (1 - \bar{\mu}_2)^{K_2(\epsilon)} \tag{23}$$

and $1 - \eta$ is the probability that at least one information revealing $s_2$ is played.

If the stage game is played $K_1 \cdot K_2(\epsilon)$ times, i.e. if the experiment of playing the stage game $K_2(\epsilon)$ times is repeated $K_1$ times, the probability that at least $K_1$ information revealing $s_2$ are played is greater than $(1 - \eta)^{K_1}$. Therefore a sufficient condition for the probability that less than $K_1$ information revealing $s_2$ are played when the stage game is repeated $K_1 \cdot K_2(\epsilon)$ times to be less than $\epsilon$ is

$$(1 - \eta)^{K_1} \geq 1 - \epsilon \tag{24}$$

whence

$$\eta \leq 1 - (1 - \epsilon)^{1/K_1}. \tag{25}$$

Substituting (20) in (22) and rearranging provides

$$K_2(\epsilon) \geq \frac{\log(1 - (1 - \epsilon)^{1/K_1})}{\log(1 - \mu_2^*)}. \tag{26}$$

Defining $K_2(\epsilon) = [\log(1 - (1 - \epsilon)^{1/K_1})/\log(1 - \mu_2^*)] + 1$ concludes the proof. □

**Remark 2:** The lower bound on $\mu_2$, $\bar{\mu}_2$, is to be interpreted as a lower bound on the probability that information revealing $s_2$ are played. In simultaneous move stage games and in simultaneous move stage games in which the LR player moves first all $s_2$ are information revealing because the strategy of the LR player is observed. For this class of game our result coincides with the one of FL.

We are now ready to state the main result. Let $\underline{V}_1(\delta, \bar{\mu}_1, \bar{\mu}_2, \omega_1^0)$ be the least NE payoff to a LR player of type $\omega_1^0$, with payoffs as in the unperturbed game, when the discount factor is $\delta$. Then

**Theorem 1** *Let Assumptions 1 and 2 be satisfied, and let $1 - \bar{\mu}_2$ be the probability that the SR player is the rational type. Then for all $\epsilon > 0$, there exists a $K(\bar{\mu}_1, \bar{\mu}_2, \epsilon) = K^*$ otherwise independent of $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$ such that*

$$\underline{V}_1(\delta, \bar{\mu}_1, \bar{\mu}_2, \omega_1^0) \geq (1-\epsilon)(1-\bar{\mu}_2)\delta^{K^*} g_1^* + (1-(1-\epsilon)(1-\bar{\mu}_2)\delta^{K^*}) \min g_1 \quad (27)$$

**Proof:** Suppose the LR player always plays the Stackelberg strategy. Since the best response correspondence[3] $B(\sigma_1^t)$ is upper hemi-continuous, each element of $B(\sigma_1^t)$ is near to an element of $B(s_1^*)$ when $\pi_t^*$ is sufficiently near to one. Since $s_2$ is finite, if $\sigma_2$ is near to an element of $B(s_1^*)$, then it must place probability close to one on $s_2^*$. Since the rational SR player has to be indifferent between all strategies that he is willing to assign positive probability, there is a probability $\bar{\pi} < 1$ such that $B(s_1^t) \subseteq B(s_1^*)$ whenever $\pi_t^* > \bar{\pi}$.

Set $K^* = K^*(\epsilon) = K^*(\epsilon, \bar{\mu}_1, \bar{\mu}_2, \bar{\pi}) = K_1 \cdot K_2(\epsilon)$. If the LR player always plays $s_1^*$, then from Lemma 2 it follows that the probability that there are more than $K^*(\epsilon)$ occasions where the rational SR player plays outside of $B(s_1^*)$ (corresponding to the events $\pi_t^* \geq \bar{\pi}$) is less than $\epsilon$. In the worst case these events occur at the beginning of the game where the payoffs are discounted the least. Recalling that only a fraction $1 - \bar{\mu}_2$ of SR players is

[3]Recall that $B(.)$ is the best response correspondence of the rational SR player.

rational provides the RHS of (27). Since the Stackelberg strategy is always feasible for the LR player, the RHS is a lower bound on any NE payoff. □

**Remark 3:** As said in Remark 2, in the case in which the stage game is simultaneous move or sequential move with the LR player moving first, $\bar{\mu}_2 = 1$ and the lower bound in Theorem 1 coincides with the lower bound in Theorem 1 in FL. The same is true for simultaneous move stage games in which the SR player moves first in which the SR players choose an information revealing $s_2$ when $\pi_t^* \leq \bar{\pi}$, such as the chain store game.

# 4 The Quality Game

In the following we want to discuss an important application of our results, the quality game. The analysis will turn out to be simpler than in the previous section given the simple structure of the game. In particular $S_1$ has only 2 elements, therefore $N = 1$ and $1 - (1 - \bar{\pi})/N = \bar{\pi}$.

EXAMPLE 1. Consider the version of the quality game whose extensive form is described in Fig. 2. When $a = 0$, $b = 1$, and $c = -1$, as argued in the introduction, provided that $\mu_1(\omega_1^*)$ is not too high, the lower bound for the LR player NE payoffs given by Theorem 2 in FL is just $\min g_1 = 0$.

Now suppose that there are two types of SR player, the rational player, $\omega_2^0$, with payoffs as given above, and a second one, $\omega_2^*$, with payoffs such that he always buys. Suppose that these payoffs are $a = 1$, $b = 1/2$, and $c = 0$. The rational player $\omega_2^0$ on the other hand buys only if $\pi_t^* \geq \bar{\pi} = 1/2$.

In this example we only have one LR player commitment type $(\omega_1^*)$ and one SR player commitment type $(\omega_2^*)$. In the following we will therefore replace $\bar{\mu}_1$ and $\bar{\mu}_2$ with $\mu_1^* = \mu_1(\omega_1^*)$ and $\mu_2^* = \mu_2(\omega_2^*)$. Finally, notice that since type $\omega_2^*$ always buys, we can disregard the term $(1 - \bar{\mu}_2)$ in (27), since buying is a best response to producing high quality.

Let $\mu_1^* = .1$. Then $K_1 = \lfloor \log \bar{\mu}_1 / \log \bar{\pi} \rfloor + 1 = 4$.

Suppose $\bar{\mu}_2 = .3$, and let $\delta = .99$. Then we have:

$$\underline{V}_1(\delta, \mu_1^* \bar{\mu}_2, \omega_1^0) \geq \max_{\epsilon}(1 - \epsilon) \cdot .99^{K_1 \cdot K_2(\epsilon)} = .60 > 0 \tag{28}$$

which is obtained maximizing with respect to $\epsilon$ the RHS of inequality (27) in Theorem 1. The $\epsilon$ that maximizes that expression turns out to be .11, which implies that $K_2(\epsilon) = 10$.

As claimed above, the introduction of uncertainty on the side of the SR players improves substantially the lower bound on the LR player NE payoffs.

The purpose of the next example is to assess the sharpness of the lower bound on NE payoffs $\underline{V}_1$ that is computed using only $\mu_1^*$ and $\mu_2^*$. We will show that, while the use of additional information relative to the distribution of the SR player type does provide a better bound, the induced improvement is far from dramatic.

EXAMPLE 2. Suppose we introduce another SR player, $\omega_2^1$, with payoffs $a = 1$, $b = -1/3$, and $c = 0$, and whose prior is $\mu(\omega_2^1) = \mu_2^1 = .2$; SR players of type $\omega_2^1$ buy if $\pi_t^* = 1/4$, thus increasing the probability that the LR player's action be revealed. In this case, as suggested above, $\underline{V}_1$ turns out to be larger.

In such a case $K^* = K' + K''$ where $K' = K^*(\epsilon, \mu_1^*, \mu_2^*, \pi') = K_1(\mu_1^*, \pi') \cdot K_2(\epsilon, \mu_2^*)$ and $K'' = K^*(\epsilon, \pi(\omega_1^*|h_{K'}), \mu_2^*+\mu_2^1, \bar{\pi}) = K_1(\pi(\omega_1^*|h_{K'}), \bar{\pi}) \cdot K_2(\epsilon, \mu_2^*+\mu_2^1)$. To see this assume that at least one type $\omega_2^*$ SR player is selected when the stage game is repeated $K'$ times. Then we have that $\pi(\omega_1^*|h_{K'}) \geq \mu_1^*/\pi' = .4$, since $\pi_t^* \leq \pi' = 1/4$. In this case $K'' \leq K^*(\epsilon, \mu_1^*/\pi', \mu_2^* + \mu_2^1, \bar{\pi})$. Since in computing $K''$ we have assumed that an event had happened whose probability is $1 - (1 - \mu_2^*)^{K'}$, the probability that $\pi_t^* \geq \bar{\pi} = .5$ is equal to $(1 - \epsilon) \cdot (1 - (1 - \mu_2^*)^{K'})$ and therefore

$$\underline{V}_1(\delta, \mu_1^*, \mu_2^*, \mu_2^1, \omega_1^0) \geq \max_{\epsilon}(1 - \epsilon) \cdot (1 - (1 - \mu_2^*)^{K'}).99^{K'+K''} = .69 \qquad (29)$$

In the previous examples we have made the assumption that a type of SR player exists with strictly positive probability that had $s_2^*$, an information revealing strategy as a strictly dominant strategy, which implies that that type of SR player will play $s_2^*$ regardless of the LR player he believes to face.

Another assumption that is perfectly consistent with the structure of the model is the following:

**Assumption 3:** A type of SR player exists with strictly positive probability that plays $s_2^*$ provided that the probability that the LR player is the Stackelberg leader type is greater than or equal to $\mu_1^*$, the prior probability that he is of that type.

In the quality game studied above Assumption 3 means that

$$\mu_1^* a + (1 - \mu_1^*) b \geq c \tag{30}$$

whereas Assumption 2 was equivalent to

$$a \geq c \, , \qquad b \geq c \tag{31}$$

As is clear Assumption 2 is stronger than Assumption 3 in that (31) implies (30) but not viceversa: (30) might hold also when $a \geq c$ but $b < c$. In the context of the quality game this means that the SR player commitment types don't prefer purchase to no purchase independently of the quality; it just means that given their preferences they are more willing to take the risk of buying than the rational SR player.

Consider again the game of Fig. 2, and suppose that type $\omega_2^*$ has payoffs $a = 1$, $b = -1/9$, $c = 0$. If we assume, as in Example 1, that $\mu_1^* = .1$, we then have $\mu_1^* a + (1 - \mu_1^*) b = .1 \cdot 1 + .9 \cdot (-1/9) = c = 0$. Assumption 3 is satisfied and our results follow.

A major difference between Assumptions 2 and 3 however exists. Suppose that in the game of Fig. 2 the payoff to the LR player when he produces low quality is 4 rather than 3/2. If we make Assumption 2, and $\mu_2^* = .3$ it turns out that the Stackelberg leader strategy is to produce low quality, since in this case his expected payoff is $.3 \cdot 4 = 1.2$. In other words if enough SR players exist that always buy and the difference between the payoff to the LR player when he produces low and high quality is large enough, it might be better for him to exploit the SR commitment types rather than building a reputation for honesty. If we make Assumption 3, on the other hand, and we assume that $b < c$, the same result does not hold: after the first time the LR player produces low quality he is revealed to be the rational type ($\pi(\omega_1^*) = 0$) and no other SR player is guaranteed to ever buy in the future, not even the commitment types.

While we think that the two assumptions we have been discussing can be appropriate for different games, we also believe that Assumption 2 is interesting in that it highlights that reputation doesn't always work.

Another important point to make is that the impact of $\bar{\mu}_1$ and $\bar{\mu}_2$ on the lower bound on NE payoffs is rather different. From the proof of Lemma 2 it is clear that the smaller $\bar{\mu}_1$ is, the larger will be the increase in $\pi(\omega_1^*)$ when the LR player plays according to the description of $\omega_1^*$, so that $K_1$ does not increase by much and $\underline{V}_1$ does not decrease significantly. The same argument is not true for $\bar{\mu}_2$: if $\bar{\mu}_2$ is very low, the time required for $\pi(\omega_1^*)$ to get greater than or equal to $\bar{\pi}$ can be long and the decrease in $\underline{V}_1$ is nonnegligible. In Example 1 if $\mu_2^* = .2$, $\underline{V}_1 \geq .48$, and if $\mu_2^* = .1$, $\underline{V}_1 \geq .27$.

In the examples we have presented so far we have chosen a discount factor that is not too large: if the reference period is one month, $\delta = .99$ translates to a yearly interest rate of 12.8%. We have chosen to do so to stress the fact that the result doesn't hold only for very patient LR players. However in many economic examples the relevant reference period can be shorter: if the relevant period is for example one week, a weekly discount factor $\delta = .999$ would translate to a yearly interest rate of 5.3%, and in this case $\underline{V}_1$ in Example 1 would be larger than .93.

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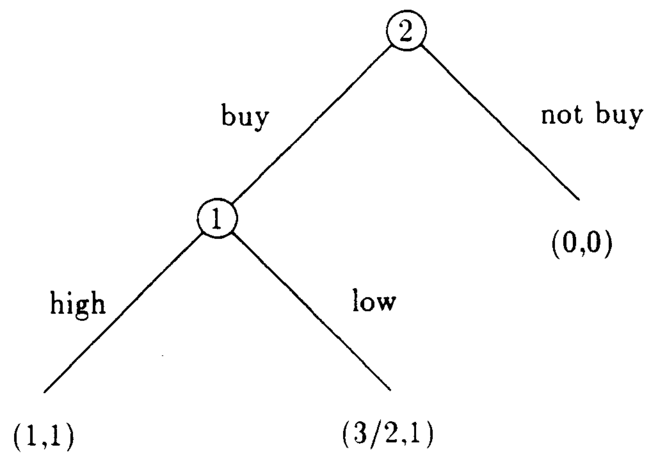


Fig. 1: Quality game

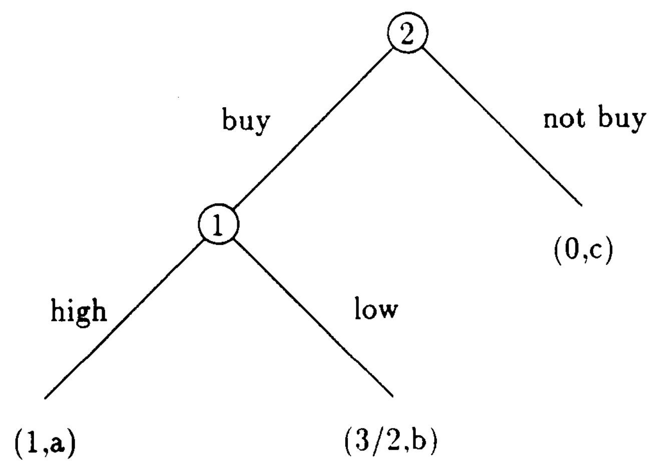


Fig. 1: Quality game with unspecified payoffs for the SR player