An Arbitrage Approach to Competitive Equilibrium in an Exchange Economy*

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Abstract

We formulate the notion of competitive behavior in a nonatomic exchange economy as exploiting arbitrage possibilities. We show that arbitrage is an alternative to the standard description of how equilibrium is achieved: competitive equilibrium can be regarded as the elimination of arbitrage opportunities rather than the elimination of excess demands. Characterization and existence theorems for the arbitrage version of competitive equilibrium are given under various assumptions, e.g., with and without convexity of preferences, and comparisons with Walrasian equilibrium and the core are made.

We also make some historical connections between the arbitrage approach adopted here and its antecedents in the work of Jevons and Edgeworth. Pointing out certain strengths and weaknesses in this earlier work, we reach a different position on just where the margin is that links 'marginalism' to the competitive theory of value.

1. INTRODUCTION

What we think happens outside of equilibrium is important in understanding the meaning of equilibrium. Walras' *tâtonnement*, or groping process, remains the standard way to think about the path to competitive equilibrium. Taking prices as given, participants register their utility-maximizing and profit-maximizing quantities, and prices adjust to eliminate excess demands. Not only does price-taking occur at equilibrium, but away from equilibrium as well. While there is no particular justification for this behavior away from equilibrium, it has nevertheless been quite influential in reinforcing the view that competitive behavior *means* price-taking.

Suppose that out of equilibrium trades take place at different prices among different individuals and that these differences create arbitrage opportunities which are then exploited. We show that *competitive equilibrium may be regarded as the elimination of arbitrage opportunities* rather than the elimination of excess demands.

There are two notable qualifications for this conclusion. The first is that there should be numerous buyers and sellers, literally a continuum; otherwise, arbitrage with small numbers generally has no definite implications for equilibrium. This qualification is more or less implicit in the traditional idea that thick markets are required for competitive equilibrium; therefore, it is only a qualification compared to the Walrasian definition of equilibrium, where the price-taking hypothesis acts as a substitute for thick markets. Note: While we rely on the continuum idealization of thick markets, we shall assume that an individual arbitrager can trade with only a *finite* number of other market participants.

The second qualification is more substantive: arbitrage is based on 'reservation prices' rather than market prices, i.e., an arbitrager can make offers to buy and sell which exploit differences in individual tastes. In Section 7, we shall distinguish this entrepreneurial form of arbitrage from the standard version which exploits only a monotonicity hypothesis on tastes. Until Section 7, however, the term 'arbitrage' will stand for 'entrepreneurial arbitrage'.

There is a distinctive and appealing geometry associated with the arbitrage approach to perfectly competitive equilibrium which may be usefully separated into two parts. The first is that arbitrage results in the formation of an opportunity set—the arbitrager's budget set—which is a convex *cone*. A similar condition characterizes (the standard version of) arbitrage in financial markets where traders are allowed unlimited short sales (e.g., Ross [1978], Harrison and Kreps [1979], Werner [1987]). In an exchange economy with a continuum of individuals, an arbitrager—whose scale is infinitesimal with respect to the economy as a whole—may be able to deliver much more of any commodity than his endowment would permit. The cone condition on trading opportunities underlies the distinctive features of the arbitrage approach to competitive equilibrium in contrast to Walrasian price-taking: in particular, prices emerge as the supporting hyperplane to a convex cone, rather than as the supporting hyperplane to a convex set.

Convex cones are flatter than convex sets, but they need not be flat, i.e., the cone may be pointed. The second feature of the geometry is that for the definition of *perfectly competitive* arbitrage equilibrium below, the arbitrage cone must be *flat* because it is only in this way that we obtain the 'emergence of the competitive budget line' as the arbitrager's opportunity set. (Note: the need for the flatness condition counters the presumption that in a continuum an individual will automatically be able to buy or sell any amount at the same terms of trade.)

To emphasize the integrity of the arbitrage approach to perfectly competitive equilibrium, we shall give a self-contained demonstration of the existence of arbitrage equilibrium. We say 'self-contained' because our formulation of arbitrage equilibrium will imply that it is a Walrasian equilibrium (plus some added restrictions), and we could therefore simply appeal to known results. But this would give the arbitrage approach the appearance of a mere adjunct to Walras' method of demand-and-supply, instead of the stand-alone alternative that it is.

Arbitrage is a story of what happens outside of equilibrium that fits the no-surplus characterization of perfectly competitive equilibrium (Ostroy [1980], Makowski [1980]). The no-surplus condition is related to other characterizations of competitive equilibrium having to do with marginal productivity theory (Makowski and Ostroy [1991a]) and efficient mechanism design (Makowski and Ostroy [1987, 1991b]), but the relation we want to emphasize in this paper is with the core.

Core bargaining is much closer to what one might mean by an arbitrage approach to competitive equilibrium than *tâtonnement*. With the core, prices are not taken as given. Away from equilibrium, groups of individuals seek, in effect, to find better terms of trade; and in a market with large numbers of individuals, this leads to the 'emergence of prices' (Edgeworth [1881], Shubik [1959], Debreu and Scarf [1963]). Nevertheless, with the exception of Mas-Colell [1982], the core has been more commonly regarded as *sui generis* than as related to arbitrage. One possible explanation is that the formation of coalitions for mutual gain used in the core does not directly suggest the individualistic, non-cooperative activity associated with the term 'arbitrage'.

Besides the fact that core bargaining is itself a form of arbitrage, other points of contact are: the equivalence of the core and Walrasian equilibrium only holds in general when there is a continuum of individuals (Aumann [1964]), and core equivalence in continuum models has been demonstrated for finite 'blocking coalitions' (Hammond, Kaneko and Wooders [1986]) much as an individual arbitrager is only permitted to deal with a finite number of other market participants.

That there is something special about the properties of the core in the Core Equivalence Theorem—in comparison to the way the core is obtained in economic models with small numbers—can be inferred from work characterizing the ubiquity and size of blocking coalitions (e.g., Schmeidler [1972], Grodal [1972], Vind [1972], Mas-Colell [1979]). The conclusions of this paper can be regarded as a continuation in this line, except that the notion of core bargaining has been streamlined and carried sufficiently far from its cooperative game-theoretic antecedents that it can be replaced by a non-cooperative model of individualistic arbitrage. In fact, our formulation of arbitrage equilibrium is, on the surface at least, closer in appearance to Dubey, Mas-Colell and Shubik [1980] on Cournot-Nash equilibrium than it is to the core.

What are the distinctions between the core and the arbitrage approaches to competitive equilibrium? At the conceptual level, we shall argue that the model of arbitrage adopted here is a more parsimonious description of the competitive process. Such a distinction would be relevant even when both approaches lead to the same conclusion about the competitivity of a particular economy. But conceptual differences are also reflected in formal distinctions: the 'emergence of prices' in the core does not have the same implications as the 'emergence of the competitive budget line' that takes place via arbitrage—core equivalence does not necessarily imply the flat cone condition underlying competitive arbitrage.

The arbitrage approach to equilibrium brings out certain issues that are tied to the origins of marginalism. It emphasizes the interplay between the (traditional) commodity margin and another infinitesimal margin, that of the arbitrager relative to the economy as a whole. Convexity of individual preferences implies that the boundary of an arbitrager's opportunity set is enlarged by making small trades with many individuals rather than large trades with a few. In the limiting ideal the (tiny) arbitrager trades only an infinitesimal amount with an arbitrarily large number of individuals at terms of trade reflecting individual marginal rates of substitution. This interaction between the two infinitesimal margins, (1) the infinitesimal arbitrager and (2) the infinitesimal quantities the arbitrager trades to exploit his position, leads to the emergence of the linear opportunity set. Note the new role the commodity margin plays in the arbitrage approach to equilibrium: Not only does the individual/arbitrager regulate his purchases based on his *own* marginal rate of substitution once equilibrium prices are determined—this is the emphasis of the price-taking approach,—but it is *others'* marginal rates of substitution that determine the trading opportunities open to any individual/arbitrager—this is its added role.

Rather surprisingly, the interaction between the two infinitesimal margins of analysis—the commodity and the individual—and their relation to arbitrage made an appearance at the very beginning of the marginalist era, in the work of Jevons [1879]. But Walras' theory of exchange, with its more exclusive emphasis on the commodity margin, took precedence and became the standard by which the contributions of his contemporaries were measured. Consequently, since Jevons' treatment of equilibrium in exchange does not rely on the Walrasian apparatus of demand and supply, it has been regarded as distinctly inferior.

Contemporary interest in the core has paired Edgeworth with Walras, but Edgeworth's *Mathematical Psychics* owes more to Jevons than to Walras. Both the core and the arbitrage approach described here may be viewed as alternative ways of filling out Jevons' theory of exchange, into a complete theory of perfect competition that does not depend on Walrasian demand-and-supply and price-taking behavior. This is documented in the concluding section of the paper which contains a discussion of Jevons' theory of exchange, its relation to the work of Edgeworth, and to the arbitrage approach adopted here. These remarks coupled with the results of this paper will provide the background for the following conclusion: (entrepreneurial) arbitrage establishes a different connection between 'marginalism' and the competitive theory of value than that laid out by Walras and maintained up to the present.

In Section 2 we describe the basic model of a nonatomic exchange economy consisting of a finite number of types of individuals having convex preferences. Section 3 gives the definition of arbitrage, the key Arbitrage Lemma and the definition of perfectly competitive arbitrage equilibrium. In Section 4, we compare arbitrage equilibrium with Walrasian equilibrium in the continuum and also make some comparisons for a model with a finite number of individuals. In Section 5 we prove the existence of perfectly competitive equilibrium for the model described in Section 2. In Section 6, the model is extended to include non-convex preferences and an infinite number of types. Similarly, we extend the Arbitrage Lemma, the definition of equilibrium and the existence theorem. Section 7 is devoted to historical and concluding remarks. The Appendix contains omitted proofs of theorems. For a first reading, Section 6 and the Appendix might be omitted.

2. THE MODEL

To focus on what is essential, we work in 'trade space' as opposed to 'consumption space'. The two are, of course, related since a person's consumption is just the sum of his endowment and his trade.

The basic building block is the characteristics of an individual of type i, which are concisely described by the extended real-valued function $v_i : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$. This function serves a double duty: Its effective domain gives the set of *feasible trades* for i, namely,

$$Z_i = \{z_i : v_i(z_i) > -\infty\};$$

and the values of v_i on Z_i describe i's preferences over his/her feasible trades. We convene that z_i will always represent a point in Z_i; however other elements such as y_i or z_i' need not be contained in Z_i. Our sign convention will be the usual one: positive components of z_i represent purchases and negative components, sales.

The following assumptions are made throughout:

(A.1) $Z_i = \mathbf{R}^\ell_+ - \{\omega_i\}$, where $\omega_i \in \mathbf{R}^\ell_{++}$.

(A.2) v_i is continuous on Z_i, with $v_i(0) = 0$ (a normalization).

(A.3) v_i is weakly monotone on Z_i: for any $z_i \in Z_i$, $z_i' \gg z_i$ implies $v_i(z_i') > v_i(z_i)$.

Until Section 6, we shall also assume

(A.4) v_i is quasi-concave.[1]

DEFINITION: The at-least-as-good-as set for individual i with respect to *changes* from z_i is

$$S_i(z_i) = \{y_i : v_i(z_i + y_i) \geq v_i(z_i)\}.$$

The following construction gives a local characterization of the at-least-as-good-as set near its origin. The cone generated by $S_i(z_i)$ is

$$\text{ray } S_i(z_i) = \{y_i : \text{for some } \alpha > 0,\ \alpha y_i \in S_i(z_i)\}.$$

This is the set of *directions* in which it is possible to move from z_i while remaining in $S_i(z_i)$. Thus, $y_i \in \text{ray } S_i(z_i)$ implies that there is some $\alpha_0 > 0$ such that for all $\alpha \in [0, \alpha_0]$, $\alpha y_i \in S_i(z_i)$ (this follows from the convexity of $S_i(z_i)$).

[1] For any $\alpha \in [0,1]$ and any $z_i, z_i' \in \mathbf{R}^\ell$, $v_i(\alpha z_i + (1-\alpha) z_i') \geq \min\{v_i(z_i), v_i(z_i')\}$. This inequality applies whether or not $z_i, z_i' \in Z_i$.

For any set S, the smallest convex cone containing S is denoted 'cone S'. We will see that cone $S_i(z_i)$ is the set of directions in which it is possible for an individual arbitrager to move when dealing with many individuals of type i. The convexity of $S_i(z_i)$ guarantees that these directions coincide with ray $S_i(z_i)$:

- ray $S_i(z_i) =$ cone $S_i(z_i)$

(Rockafellar [1970], Corollary 2.6.3).

To assemble the building blocks v_i into the description of an economy, let I be a finite set of types. Denote by $\mathcal{E} = \{v_i\}_{i \in I}$ the economy consisting of one member of each type. Contrast this with the economy consisting of a unit mass of each type, denoted by $\mathbf{E} = \{v_i \cdot \mathbf{1}\}_{i \in I}$. Under the hypotheses of our model, both economies will have the same Walrasian equilibria, but the theorem to be established below holds only for $\mathbf{E}$ and not for $\mathcal{E}$.

Let $z = (z_i)$ be an allocation for $\mathcal{E}$. As a simplification to emphasize comparisons between $\mathcal{E}$ and $\mathbf{E}$, we shall assume that members of the same type will receive identical allocations. (This assumption is eliminated in Section 6.) Denote an allocation for the economy $\mathbf{E}$ by $\mathbf{z} = (\mathbf{z}_i)$ where $\mathbf{z}_i = z_i \cdot \mathbf{1}$. We use boldface to distinguish the fact that z_i, the allocation to a single member of type i, is a vector of infinitesimal magnitudes compared to $\mathbf{z}_i$, the allocation to all members of type i.

Definition: The set of feasible allocations in $\mathcal{E}$ is

$$Z = \{z = (z_i) : \forall i,\ z_i \in Z_i \ \&\ \sum z_i = 0\}.$$

Therefore the set of feasible (equal-treatment) allocations in $\mathbf{E}$ is

$$\mathbf{Z} = \{\mathbf{z} = (\mathbf{z}_i) : \forall i,\ \mathbf{z}_i = z_i \cdot \mathbf{1},\ z_i \in Z_i \ \&\ \sum z_i \cdot \mathbf{1} = 0\}.$$

Because each Z_i is closed, convex and bounded below, $\mathbf{Z}$ is compact and convex. Unless the contrary is explicitly stated, from now on we assume $\mathbf{z} \in \mathbf{Z}$.

There are three orders of commodity magnitudes in $\mathbf{E}$:

- the vector $\mathbf{z}_i$ which is on the scale of the economy as a whole;
- the vector z_i, the scale of an individual, which is infinitesimal with respect to the economy as a whole;
- the vector dy_i, an infinitesimal compared to z_i.

'Marginal analysis' is traditionally associated with the commodity margin, the third and smallest magnitude; however, the economy $\mathbf{E}$ introduces the possibility of making the second magnitude, i.e., the vector z_i associated with the individual, the infinitesimal margin of analysis. In the following section, we show that the individual margin can be combined with the commodity margin to yield an arbitrage approach to perfectly competitive equilibrium. Our method will be to approximate the commodity margin dy_i on the boundary of $S_i(z_i)$ by feasible directions in cone $S_i(z_i)$.

3. Arbitrage and Equilibrium

Arbitrage is usually associated with opportunities for trade due to differences in market prices. Here we consider arbitrage as an activity exploiting differences in reservation prices.

At the allocation $\mathbf{z}$ the opportunities for an individual arbitrager to make deals in the economy $\mathbf{E}$ are given by

$$K_{\mathbf{z}} = \{y : y = \sum_i \sum_k y_{i_k} \text{ and for each } i_k,\ -y_{i_k} \in S_i(z_i)\}.$$

The trade y is possible at $\mathbf{z}$ if it can be decomposed into a sum of trades among individuals each of whom would be no worse off than they are at $\mathbf{z}$. (Given our sign conventions on net trades, if an individual gives $-y_{i_k}$ to someone of type i then he gets y_{i_k} for himself.)

Assume that arbitragers are individuals who also trade on their own account. Then, an individual of type i will not be satisfied if the opportunities for arbitrage are such that there exists a y in $K_{\mathbf{z}}$ for which

$$v_i(z_i + y) > v_i(z_i).$$

Also assume that any arbitrager can *individually recontract*, i.e., he can drop his contracted trades z_i to return to his endowment (no-trade) position ω_i and begin arbitraging from there rather than from z_i. Then, i also will not be satisfied with z_i if there is a y in $K_{\mathbf{z}}$ such that

$$v_i(y) > v_i(z_i).$$

This leads to the following definition.

Definition: The allocation $\mathbf{z}$ is an *arbitrage equilibrium* for $\mathbf{E}$ if for each i,

$$v_i(z_i) \geq v_i(z_i') \quad \text{for all } z_i' \in K_{\mathbf{z}} \cup K_{\mathbf{z}} + \{z_i\}.$$

An arbitrage equilibrium is an allocation $\mathbf{z}$ which no individual arbitrager can improve upon. The definition is an amalgam of core-like and non-cooperative equilibrium concepts. In its emphasis on individual behavior, it is evidently non-cooperative; whereas in its out-of-equilibrium behavior, it is reminiscent of the core in the sense that the arbitrager is organizing an improving coalition. But this is not an improving coalition in the usual sense: for the core, an improving coalition with respect to $\mathbf{z}$ requires that its members be able to do better by withdrawing from the economy and arranging another trade using only their own resources. This may involve *group recontracting* since several or even all the members of the coalition may have to drop their contracts in $\mathbf{z}$ to improve upon it for themselves. By contrast, the individual arbitrager does not suppose his new trading partners could also recontract. Nevertheless, we shall see that individual arbitrage suffices for reaching perfectly competitive equilibrium in all environments for which such an equilibrium exists. This will be the theme of Section 4, after a discussion of the properties of agents' arbitrage possibilities.

REMARK 1: Formally, our restriction to individual recontracting amounts to a reduction in the number of possible improving coalitions, and therefore arbitrage equilibria are a superset of the core. To demonstrate, suppose $y \in K_{\mathbf{z}}$ and

$$(a) \qquad y = \sum_i \sum_{k=1}^{k_i} y_{i_k},$$

where $-y_{i_k} \in S_i(z_i)$ for each i_k. Let $r = \max_i\{k_i\}$. Note that, given equal-treatment, any $\mathbf{z}$ can be realized by trade within finite groups of individuals involving one of each type. Therefore,

$$(b) \qquad \sum_i \sum_{k=1}^{r} z_{i_k} = 0.$$

Set $y_{i_k} = 0$ for $k = k_{i+1}, \ldots, r$. Hence, since $0 \in S_{i_k}$, $v_i(z_{i_k} - y_{i_k}) \geq v_i(z_{i_k})$ for all i_k. Further, from (a) and (b), $y + \sum_i \sum_{k=1}^{r}(z_{i_k} - y_{i_k}) = 0$. Thus, individual recontracting with respect to $\mathbf{z}$ may be regarded as the formation of a coalition organized by a single arbitrager, say j, and at most r of all types. In core terminology, it is at least a weakly improving coalition if $v_j(y) > v_j(z_j)$. For more on the relation between arbitrage equilibrium and the core, see the discussion centering around Example 1A in Section 4.

The Properties of $K_{\mathbf{z}}$

In a model with a small number of individuals such as $\mathcal{E}$, the arbitrage opportunities associated with an allocation would reflect the properties of each $S_i(z_i)$; for example, since each $S_i(z_i)$ is not typically a cone, the set of arbitrage opportunities would not be a cone (see Section 4.2). However, in $\mathbf{E}$ the following result shows that the global curvature properties of each S_i are flattened out in the aggregate. This flattening phenomenon is the origin of arbitrage pricing.

3.1 Arbitrage Lemma with quasi-concavity and finite types

$K_{\mathbf{z}} = -\sum \text{cone } S_i(z_i)$, *a convex cone containing the origin.*

Proof: Suppose $y \in \sum \text{cone } S_i(z_i)$. Then $y = \sum y_i$, where each $y_i = r_i y_i'$, $y_i' \in S_i(z_i)$, $r_i > 0$. Since $S_i(z_i)$ is convex, without loss of generality we can assume each $r_i = n$, an integer. Letting $-y_{i_k} = y_i'$ for each i_k, $k = 1, \ldots, n$, shows $-y \in K_{\mathbf{z}}$.

Conversely, if $-y \in K_{\mathbf{z}}$ then $-y = \sum_i \sum_{k=1}^{n} -y_{i_k}$. Since each $y_{i_k} \in S_i(z_i)$, by convexity $\sum_{k=1}^{n} \frac{1}{n} y_{i_k} \in S_i(z_i)$; or $\sum_k y_{i_k} \in \text{cone } S_i(z_i)$. Hence $y = \sum_i \sum_k y_{i_k} \in \sum \text{cone } S_i(z_i)$.

Finally, since $K_{\mathbf{z}}$ is the sum of convex cones containing the origin, it is also such a cone. □

The Arbitrage Lemma says that the traditional margin of analysis does play an important role in the description of arbitrage opportunities in $\mathbf{E}$ in the sense that these opportunities depend on the local curvature properties of each $S_i(z_i)$. An explanation can be given in terms of (the ordinal version of) diminishing marginal utility. Suppose $-y_i \notin S_i(z_i)$ so that an arbitrager would find it impossible to make the trade y_i with an individual of type i. Nevertheless, from the convexity of the at-least-as-good-as set it may be the case that $-y_i/2 \in S_i(z_i)$, i.e., by trading half this amount with each of two members of the same type, an arbitrager may be able accomplish what he could not with one. Thus, the prospects for making $y \in K_{\mathbf{z}}$ are enhanced if $y = \sum_i \sum_k y_{i_k}$ is broken up so that $y_{i_k} = \alpha y_i$ for each i, where α is small. Evidently the traditional assumption of frictionless trade is essential for this idealization. However, the construction of $K_{\mathbf{z}}$ does not permit unlimited arbitrage possibilities—the arbitrager cannot trade infinitesimal quantities dy_{i_k} with a continuum of individuals. But the arbitrager can come arbitrarily close to this ideal since, by the Arbitrage Lemma, the closure of $K_{\mathbf{z}}$ includes $-\text{bdry } S_i(z_i)$ for each type i.

Consider the following alternatives: either

(A) $\quad K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} \neq \emptyset$ or

(NA) $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$.

Condition (A) says that by "buying low and selling high" it is possible for an arbitrager to make unlimited profits: If $y \gg 0$ is in $K_{\mathbf{z}}$ then so is ry for any $r > 0$ (recall $K_{\mathbf{z}}$ is a cone). Given the monotonicity of preferences, this implies i can achieve unbounded utility. Therefore, a necessary condition for arbitrage equilibrium is that

$\mathbf{z}$ must satisfy (NA).

DEFINITION: A *no arbitrage allocation* is a $\mathbf{z}$ such that $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$. $\mathbf{Z}_{\mathrm{NA}}$ denotes the set of all such allocations.

The alternatives (A) and (NA) hold no matter what the nature of the set $K_{\mathbf{z}}$. But the fact that arbitrage possibilities for an individual in $\mathbf{E}$ are given by a convex cone containing the origin has immediate implications for commodity pricing: they follow from the dual notion of the *polar* of $K_{\mathbf{z}}$ defined as

$$K^{\circ}_{\mathbf{z}} = \{p : pK_{\mathbf{z}} \leq 0\}.$$

(Note: this set always contains the origin.)

Evidently, when $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} \neq \emptyset$ and therefore arbitrage profits are possible, $K^{\circ}_{\mathbf{z}} = \{0\}$. The well known converse property for polars shows that *commodity prices emerge from the elimination of arbitrage possibilities.*

DEFINITION: An element $p \in K^{\circ}_{\mathbf{z}}\backslash\{0\}$ is a *no arbitrage price* vector.

The following is the basic separation theorem for convex cones.

Lemma 3.2 (Existence of no arbitrage prices) $K^{\circ}_{\mathbf{z}} \neq \{0\}$ *if and only if* $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$.

We have emphasized the flattening phenomenon associated with arbitrage as evidenced by the fact that $K_{\mathbf{z}}$ is a convex cone rather than merely a convex set. Convex cones differ from convex sets (with boundary point at the origin) in that any boundary point of the cone lies on one of its supporting hyperplanes through the origin. More formally, let $H_p = \{y : py = 0\}$ be the hyperplane perpendicular to p; then for any $y \in$ bdry $K_{\mathbf{z}}$, there is a $p \in K^{\circ}_{\mathbf{z}}\backslash\{0\}$ such that $y \in H_p$. Nevertheless, the cone $K_{\mathbf{z}}$ itself need not be flat.

DEFINITION: Among $\mathbf{z}$ in $\mathbf{Z}_{\mathrm{NA}}$, $K_{\mathbf{z}}$ is *flat* if bdry $K_{\mathbf{z}}$ is a hyperplane; otherwise, $K_{\mathbf{z}}$ is *pointed.*

The definition of flatness is equivalent to any of the following conditions: (i) bdry $K_{\mathbf{z}} = H_p$ for any $p \in K^{\circ}_{\mathbf{z}}\backslash\{0\}$, (ii) dim $K^{\circ}_{\mathbf{z}} = 1$, or (iii) the boundary of $K_{\mathbf{z}}$ is smooth at the origin. Because $K_{\mathbf{z}}$ is a convex cone, when it is smooth *its entire boundary coincides with the tangent hyperplane H_p.* Therefore, when $K_{\mathbf{z}}$ is flat, no arbitrage prices (an element of $K^{\circ}_{\mathbf{z}}\backslash\{0\}$) literally characterize trading opportunities. In contrast, when $K_{\mathbf{z}}$ is pointed, arbitrage prices lose this distinctive feature and play a role similar to that of a supporting hyperplane to a convex set whose boundary merely lies in one of its half-spaces, but not necessarily on the hyperplane.

The following is a characterization of no arbitrage allocations.

DEFINITION: The allocation $\mathbf{z}$ is *weakly Pareto efficient* if there is no $\mathbf{z}' \in \mathbf{Z}$ such that $v_i(z_i') > v_i(z_i)$ for all i.

3.3 No Arbitrage/Efficiency Lemma *$\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$ if and only if $\mathbf{z}$ is weakly Pareto efficient.*

Proof: If $\mathbf{z}$ is weakly efficient then $0 \in \text{bdry } \sum S_i(z_i)$, and hence there is a $p > 0$ such that $p \sum S_i(z_i) \geq 0$. Since 0 is in each $S_i(z_i)$ set, this can be strengthened to $pS_i(z_i) \geq 0$ for each i. Or, since cone $S_i(z_i)$ is contained in the half-space $\{y : py \geq 0\}$ (another convex cone), $p \cdot \text{cone } S_i(z_i) \geq 0$ for each i. Summing, $p \sum \text{cone } S_i(z_i) \geq 0$ or $pK_{\mathbf{z}} \leq 0$. That is, $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$.

Conversely, $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$ implies there is a $p > 0$ such that $-pK_{\mathbf{z}} = p \sum \text{cone } S_i(z_i) \geq 0$. Hence $p \sum S_i(z_i) \geq 0$ or $0 \in \text{bdry } \sum S_i(z_i)$. That is, $\mathbf{z}$ is weakly efficient. □

4. Perfectly Competitive and Walrasian Equilibrium

We will be interested in economies for which $K_{\mathbf{z}}$ is flat (for $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$). The flatness of $K_{\mathbf{z}}$ may be interpreted as a necessary and sufficient condition for any individual arbitrager to have a linear opportunity set or, equivalently, to face perfectly elastic demands and supplies (**PED**) at prices p, where $p \in K_{\mathbf{z}}^{\circ} \setminus \{0\}$. To see this observe that given flatness the arbitrager can buy or sell as much as he likes at p; i.e., for any trade y such that $py = 0$, the rest of the economy would be willing to trade y (or something arbitrarily close to y) with him. (More formally, given flatness, $y \in H_p$ implies $y \in \text{bdry } K_{\mathbf{z}}$; hence there are trades y_{i_k} such that $\sum_i \sum_k y_{i_k}$ is in $K_{\mathbf{z}}$ and is arbitrarily close to y.) Identifying perfect competition with perfectly elastic demands and supplies leads to the following definition.

DEFINITION: The allocation $\mathbf{z}$ is a *perfectly competitive (arbitrage) equilibrium* for $\mathbf{E}$ if $\mathbf{z}$ is an arbitrage equilibrium with $K_{\mathbf{z}}$ flat.

To compare perfectly competitive with Walrasian equilibrium, define the (excess) demand correspondence

$$D_i(p) = \{z_i : pz_i = 0 \quad \& \quad v_i(z_i) = \max_y \{v_i(y) : py = 0\}\}.$$

DEFINITION: A *Walrasian equilibrium* for $\mathbf{E}$ is a pair $(p, \mathbf{z})$ such that

- (price-taking utility maximization) $z_i \in D_i(p)$ for each i, and

- (market clearance) $\sum z_i \cdot \mathbf{1} = 0$.

In contrasting the two definitions, the main difference is the way in which price-taking acts in Walrasian equilibrium as a substitute for the background conditions of a perfectly competitive market, namely, the flatness of the arbitrage cone or, equivalently, the linearity of the arbitrager's opportunity set.

REMARK 2: It would have been possible to call the above a perfectly competitive equilibrium *allocation* rather than merely a perfectly competitive equilibrium because it is only quantities that are explicitly recognized. Nevertheless, and to emphasize the contrast with Walrasian price-taking, the explicit addition of prices would be redundant because equilibrium implies the elimination of arbitrage opportunities, and it is from the elimination of these opportunities that the determination of equilibrium prices immediately follows. The redundancy of prices here in comparison to their primary role in the Walrasian tradition is, of course, an obvious consequence of price-taking in the latter where there is no other means to adjust to disequilibrium.

Define $L_{\mathbf{z}}$ as the smallest linear subspace containing the set $\{z_i\}_{i\in I}$. Except for scale this is identical to the smallest linear subspace containing $\{\mathbf{z}_i\}_{i\in I}$, but $L_{\mathbf{z}}$ is to be thought of as on a scale comparable to $K_{\mathbf{z}}$, i.e., on a scale at which an individual trades, rather than on a scale comparable to the economy as a whole.

Some formal properties of perfectly competitive equilibrium are:

Theorem 4.1 (Characterization of perfectly competitive equilibrium) *The following are equivalent:*

(PCE.1) $\mathbf{z}$ *is a perfectly competitive equilibrium for* $\mathbf{E}$

(PCE.2) $(p, \mathbf{z})$ *is a Walrasian equilibrium for every* $p \in K_{\mathbf{z}}^{\circ}\backslash\{0\}$, *where* $\dim K_{\mathbf{z}}^{\circ} = 1$

(PCE.3) $L_{\mathbf{z}} \subset \text{bdry } K_{\mathbf{z}}$, *a hyperplane*

(PCE.4) $K_{\mathbf{z}}^{\circ}\backslash\{0\} \perp L_{\mathbf{z}}$, *where* $\dim K_{\mathbf{z}}^{\circ} = 1$.

Proof: (PCE.1) $\Rightarrow$ (PCE.2): First we verify that $pz_i = 0$ for each i. Since by feasibility $\sum z_i = 0$, if the contrary then $pz_i < 0$ for some i. Then there would exist $y \gg z_i$ with $py < 0$. But then, by monotonicity, $v_i(y) > v_i(z_i)$; and by flatness $y \in K_{\mathbf{z}}$. Contradiction.

Next we show $z_i \in D_i(p)$ for each i. If the contrary then there would exist an i and y such that $v_i(y) > v_i(z_i)$ with $py = 0$. Since v_i is continuous and $py \neq \min pZ_i$ (recall $\omega_i \gg 0$), there would then be a y' close to y satisfying $v_i(y') > v_i(z_i)$ and $py' < 0$ (so $y' \in K_{\mathbf{z}}$). Contradiction.

(PCE.2) $\Rightarrow$ (PCE.1): We need to prove that z is an arbitrage equilibrium. Since z is Walrasian, $v_i(z_i) \geq v_i(y)$ for all y such that $py = 0$, hence for all $y \in K_{\mathbf{z}}$ (using dim $K_{\mathbf{z}}^{\circ} = 1$ implies $K_{\mathbf{z}}$ is flat). To show $v_i(z_i) \geq v_i(z_i + y)$ for all $y \in K_{\mathbf{z}}$, observe that this statement follows from $v_i(z_i) \geq v_i(z_i + y)$ for all y such that $py \leq 0$ (using flatness). But this is just equivalent to $v_i(z_i) \geq v_i(y')$ for all y' such that $py' \leq 0$ (using $pz_i = 0$).

(PCE.3) $\iff$ (PCE.4): This readily follows from the fact that bdry $K_{\mathbf{z}}$ is a hyperplane H_p if and only if dim $K_{\mathbf{z}}^{\circ} = 1$ and $p \in K_{\mathbf{z}}^{\circ} \backslash \{0\}$.

(PCE.4) $\iff$ (PCE.2): It suffices to prove that $(p, \mathbf{z})$ is Walrasian if and only if $p \perp L_{\mathbf{z}}$, where $p \in K_{\mathbf{z}}^{\circ} \backslash \{0\}$.

If $(p, \mathbf{z})$ is Walrasian then $pS_i(z_i) \geq$ for all i. Hence $p \cdot \text{cone } S_i(z_i) \geq 0$ for all i; or summing, $pK_{\mathbf{z}} \leq 0$. We conclude that $p \in K_{\mathbf{z}}^{\circ}$. Since $pz_i = 0$ for each i, $p \perp L_{\mathbf{z}}$ is obvious. Conversely, $p \in K_{\mathbf{z}}^{\circ} \backslash \{0\}$ implies $p \cdot \text{cone } S_i(z_i) \geq 0$ for each i, or $pS_i(z_i) \geq 0$ for each i. Combining with $p \perp L_{\mathbf{z}}$ shows

$$pz_i = 0 \leq pS_i(z_i) \quad \text{for each } i.$$

Since $pz_i \neq \min pZ_i$ (recall $\omega_i \gg 0$), a standard argument now shows that $z_i \in D_i(p)$ for each i. □

The characterization of Walrasian equilibrium is somewhat different.

Theorem 4.2 (Characterization of Walrasian equilibrium) *The following are equivalent:*

(WE.1) $(p, \mathbf{z})$ *is a Walrasian equilibrium for* $\mathbf{E}$

(WE.2) $p \perp L_{\mathbf{z}}$ *for some* $p \in K_{\mathbf{z}}^{\circ} \backslash \{0\}$.

Proof: See proof that (PCE.4) $\iff$ (PCE.2), above. □

The contrast is revealed in the comparison between (PCE.4) and (WE.2). The latter says that at least one nonzero element of the polar is orthogonal to $L_{\mathbf{z}}$, while the former adds the requirement that there is no other linearly independent element in the polar. Since the uniqueness condition, dim $K_{\mathbf{z}}^{\circ} = 1$, will often hold when $\mathbf{z}$ satisfies (NA), we can say that there is often no difference between perfectly competitive and Walrasian equilibrium in the class of economies such as $\mathbf{E}$ (see, below, for a more extensive discussion). Nevertheless, when there is a discrepancy it occurs because Walrasian equilibrium does not exhibit "perfectly elastic exchange opportunities". The following example illustrates this point.

EXAMPLE 1. Consider the 2-good, 2-type economy in Figure 1. The Edgeworth box

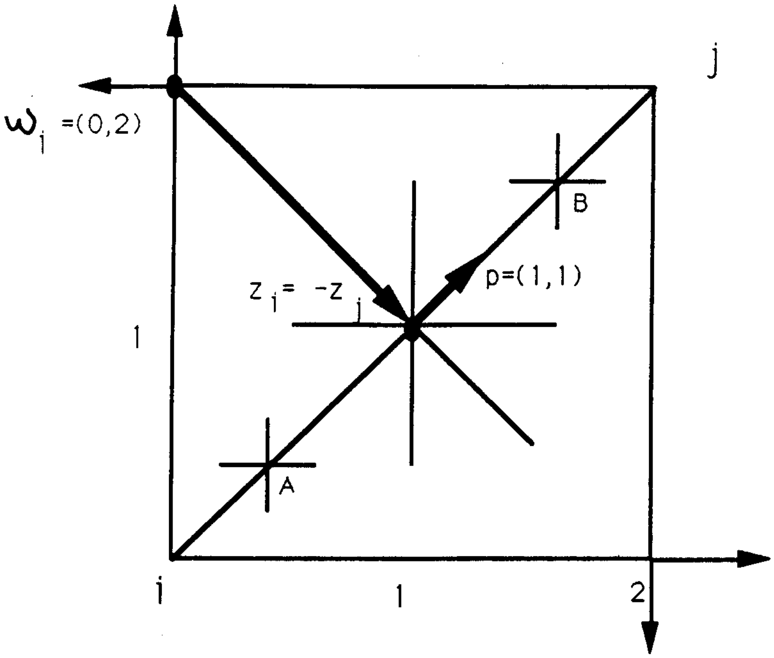

Figure 1: A Walrasian equilibrium that is not perfectly competitive

should be thought of as representing the Walrasian equilibrium trading between a typical pair of individuals of types i and j. Since cone $S_i(z_i) = S_i(z_i) = \mathbf{R}^2_+$ and cone $S_j(z_j) = S_j(z_j) = \mathbf{R}^2_+$, in this example $K_{\mathbf{z}} = \mathbf{R}^2_-$ and $K^\circ_{\mathbf{z}} = \mathbf{R}^2_+$. Hence the Walrasian equilibrium illustrated is *not* a perfectly competitive equilibrium. (Recall from (PCE.1–2) that a perfectly competitive equilibrium is a Walrasian equilibrium with dim $K^\circ_{\mathbf{z}} = 1$.)

As our discussion would suggest, this Walrasian equilibrium fails to satisfy **PED**. If any individual of type i were to try to switch from z_i to any trade $y \neq z_i$ on his budget line H_p, no other individuals would be willing to accommodate him since such a trade would make them worse off than at $\mathbf{z}$ (only $y \in K_{\mathbf{z}} = \mathbf{R}^2_-$ would be acceptable to others). Similarly if any individual of type j tried to switch. Hence individual arbitragers do not face linear opportunity sets.

Often failures of **PED** are associated with the implausibility of price-taking behavior. This also is illustrated by the example. Continuum economies are idealizations of economies with a large but finite number of individuals; similarly, a nonatomic individual is an idealization of an individual with arbitrarily small but *positive* mass, i.e., an 'infinitesimal individual'. In this spirit observe that if any infinitesimal (nonnull) mass of type i traders were to sell anything less than one unit of commodity 2 then market clearing prices would jump to $p' = (0, 1)$—the relative price of commodity 2 would jump to infinity. (*Any* price vector in the unit simplex is market clearing when each type i individual sells exactly one unit. Thus any decrease in the supply of commodity 2 would create an excess demand unless its price jumps as indicated.) Similarly, if any infinitesimal mass of type j traders were to sell anything less than one unit of commodity 1, its relative price would jump to infinity. This illustrates that even with a large number of buyers and sellers, a single (infinitesimal) seller may be able to profitably influence market clearing prices—when $K_{\mathbf{z}}$ is not flat,—and further motivates our interest in large economies with flat arbitrage cones.

Like the core, arbitrage equilibrium is a stability condition on $\mathbf{z}$ under which there would be no further departures based on arbitrage opportunities alone. But, as already noted in Remark 1, arbitrage equilibrium is a weaker stability condition. Specifically, since $0 \in K_{\mathbf{Z}}$, the definition of arbitrage equilibrium implies that for any arbitrager of any type i, $v_i(z_i) = \sup v_i(K_{\mathbf{z}} + \{z_i\})$. But in the absence of a flat cone, it remains possible that $v_i(z_i) > \sup v_i(K_{\mathbf{z}})$; that is, z_i may be strictly better than anything i can realize starting from ω_i via arbitrage. Indeed, in economies without flat cones, arbitrage equilibria need not even exhibit a Law of One Price, much less be in the core. This is illustrated by the following variant of Example 1.

EXAMPLE 1A. Divide the two types into four equal-sized types called i_a, i_b and j_a, j_b. Give the members of i_a and j_a the allocation at A in Figure 1 and the members of i_b and j_b the allocation at B, so that i_a is treated worse than the twin i_b and j_b worse than j_a. Call this allocation $\tilde{\mathbf{z}}$. Note that $K_{\tilde{\mathbf{Z}}} = K_{\mathbf{Z}} = \mathbf{R}^2_-$. Therefore, $v_{i_a}(\tilde{z}_{i_a}) = \sup v_{i_a}(K_{\tilde{\mathbf{Z}}} + \tilde{z}_{i_a}) > \sup v_{i_a}(K_{\tilde{\mathbf{Z}}}) = 0$. Similarly, $\tilde{\mathbf{z}}$ satisafies the conditions for arbitrage equilibrium for each i_b, j_a, and j_b. To improve upon $\tilde{\mathbf{z}}$ requires *both* a type i_a and j_b to drop their contracts in $\tilde{\mathbf{z}}$; in the terminology of Remark 1, it requires *group recontracting*, not just *individual recontracting.*

The example illustrates the importance of the flat cone condition for the concept of arbitrage equilibrium. In environments with flat cones—heuristically, ones in which everyone truly faces perfectly elastic demands and supplies—we have seen that individual arbitrage suffices to ensure not just Walrasian equilibrium, but perfectly competitive equilibrium. The conclusion is that *the remarkable implications of arbitrage come from its application to a particular family of environments.*

It is interesting to observe that if preferences were smooth rather than kinked at $\mathbf{z}$ in Figure 1 then $K_{\mathbf{z}}$ would be flat with boundary H_p. Hence, each individual arbitrager *would* face a linear opportunity set. This follows from the following characterization. For any set S with $0 \in S$, denote by $N(S)$ the *normal cone* to S at zero, i.e., $N(S) = \{p : pS \leq 0\}$. Of course, if S is a cone then $N(S) = S^\circ$. Let $S_{\mathbf{z}}$ denote the aggregate at-least-as-good-as $\mathbf{z}$ set for the economy $\mathbf{E}$, i.e., $S_{\mathbf{z}} = \sum S_i(z_i) \cdot \mathbf{1}$. This is a convex set with zero on its boundary if $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$. An element of $-N(S_{\mathbf{z}}) \backslash \{0\}$ is frequently referred to as *efficiency prices.*

4.3 No Arbitrage/Efficiency Price Lemma $K^\circ_{\mathbf{z}} = -N(S_{\mathbf{z}})$.

Proof: The Arbitrage Lemma 3.1 implies that $p \in K_{\mathbf{z}}^{\circ}$ if and only if $p \sum \text{cone } S_i(z_i) \geq 0$. In turn, the latter implies $p \sum S_i(z_i) \geq 0$ or $pS_{\mathbf{z}} \geq 0$.

Conversely, $pS_{\mathbf{z}} \geq 0$ implies $pS_i(z_i) \geq 0$ for each i (since 0 is in each $S_i(z_i)$ set); hence $p \cdot \text{cone } S_i(z_i) \geq 0$ for each i. Summing, $p \sum \text{cone } S_i(z_i) \geq 0$ or $pK_{\mathbf{z}} \leq 0$. That is, $p \in K_{\mathbf{z}}^{\circ}$. □

Hence, it immediately follows that:

Corollary 4.4 *$K_{\mathbf{z}}$ is flat if and only if $S_{\mathbf{z}}$ is smooth at the origin.*

REMARK 3: Even though $K_{\mathbf{z}}^{0}$ and $-N(S_{\mathbf{z}})$ are equivalent, it is worth emphasizing that they represent support theorems for different kinds of convex sets. The contrast is sharpest in environments where no arbitrage prices represent the tangent hyperplane to a convex cone that is smooth because it is here that the supporting hyperplane and the boundary of the set being supported coincide. In this case, arbitrage leads to the "emergence of the budget line".

The geometric intuition behind the "emergence of the budget line" is that when the aggregate at-least-as-good-as $\mathbf{z}$ set is smooth at the origin then it is seen as a linear opportunity set H_p by each individual, whose size is infinitesimal relative to $S_{\mathbf{z}}$ and who is operating around the origin of the latter set. The construction of the arbitrage possibilities $K_{\mathbf{z}}$ amounts to a blow-up or magnification of the trading opportunities implied by $S_{\mathbf{z}}$ as seen from the individual's perspective. See Figure 2. This picture is

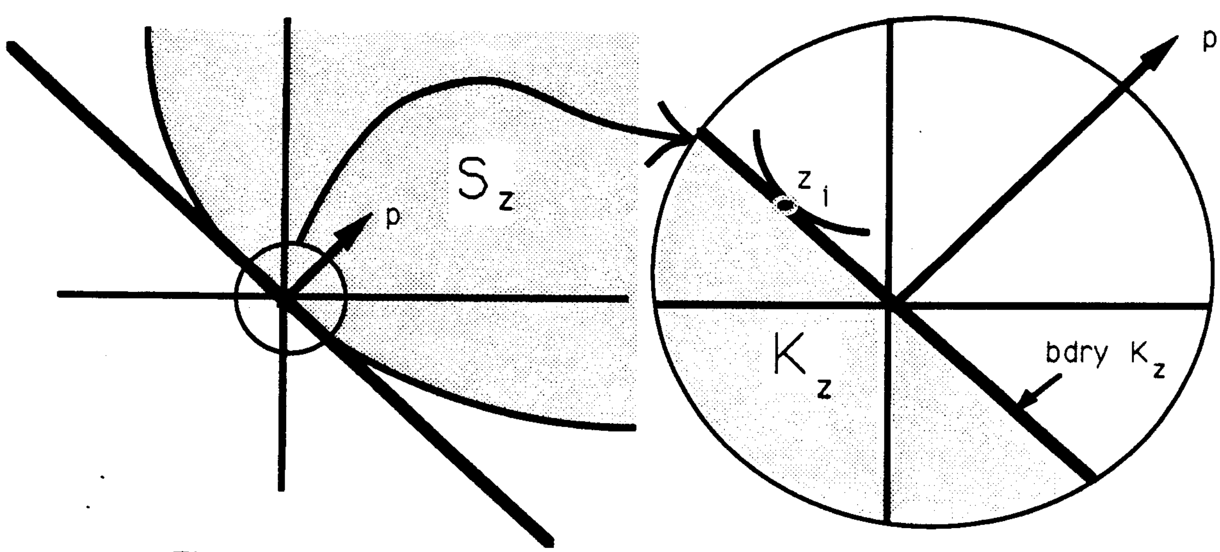

Figure 2: The "emergence of the budget line" under perfect competition

offered as the arbitrage alternative (or at least complement) to the familiar, partial equilibrium Marshallian picture of a perfectly competitive market. For the latter, the left panel of Figure 2 would be replaced by a downward-sloping market demand and an upward-sloping market supply curve; and the right panel, by a perfectly elastic demand (or supply) curve with height equal to the market-clearing price derived from the left panel. In either version, the left panel illustrates the market perspective while the right illustrates the (infinitesimal) individual's perspective.

4.1 The Arbitrage Path to Equilibrium

Let

$$\mathbf{Z}_{\mathrm{IR}} = \{\mathbf{z} \in \mathbf{Z} : v_i(z_i) \geq 0 \text{ for all } i\}$$

represent the subset of feasible allocations that are individually rational, i.e., that leave each individual at least as well off as with his endowment/no trade allocation.

Definition: Call $\mathbf{E}$ a *perfectly competitive economy* if it satisfies the flatness condition

(F) $K_{\mathbf{z}}$ is flat for any $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$.

Like core bargaining, arbitrage gives a story of how the economy gets to perfectly competitive equilibrium. First, it leads to a position on the locus of efficient, individually rational allocations: If $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} \neq \emptyset$ then individual arbitrage can generate unbounded profits in moving away from $\mathbf{z}$. Further, since agents will always individually recontract away from any $\mathbf{z} \notin \mathbf{Z}_{\mathrm{IR}}$, a necessary condition for arbitrage equilibrium is that $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$. Second, once a position on the locus of efficient, individually rational allocations is reached, individuals will only settle for a perfectly competitive equilibrium allocation: For any $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$, there are unique prices p defining each individual's remaining arbitrage possibilities ($p \in K^{\circ}_{\mathbf{z}} \backslash \{0\}$). Unless $pz_i = 0$ for each individual, $pz_i < 0$ for some individuals (recall feasibility implies $\sum z_i = 0$, hence $\sum pz_i = 0$). But since i has linear arbitrage possibilities at z_i, if $pz_i < 0$ then he will be able to reach a more preferred trade y with $py = 0$ by individually recontracting away from z_i. Hence, unless $pz_i = 0$ for all i, again $\mathbf{z}$ will not be an arbitrage equilibrium. Indeed, even if $pz_i = 0$ for all individuals, since i has linear arbitrage possibilities H_p at z_i, he will not be satisfied with z_i unless it maximizes his utility on H_p. Thus the economy reaches a price-taking equilibrium, but through arbitrage not through Walrasian *tâtonnement.*

The arbitrage approach differs from the core in that prices ($K^{\circ}_{\mathbf{z}}$) and the linearity/flatness of individuals' opportunity sets (**PED**) play an important role both in and out of equilibrium. Thus the arbitrage story is designed for perfectly competitive economies, to generate the equivalence between arbitrage equilibria and perfectly competitive equilibria. By contrast, the core story (only) generates the equivalence between core and Walrasian allocations—even when such allocations do not satisfy

PED and price-taking behavior is implausible, as in Example 1. Insofar as the logic of perfect competition is intimately connected with the logic of **PED**, the arbitrage story serves as a useful supplement to the core story of how equilibrium can be reached. And, insofar as the logic of perfect competititon derives from the behavior of individuals, arbitrage can serve as a replacement for the core.

4.2 Arbitrage in $\mathcal{E}$

A remarkable feature of Walrasian equilibrium is that its definition is precisely the same in $\mathcal{E}$ as in **E**, while with arbitrage a literal transcription makes no sense because the definition presupposes the existence of a continuum of individuals. In this section we exhibit a set of modifications to describe arbitrage possibilities in $\mathcal{E}$, show why its implications are distinct from those in **E**, and illustrate that arbitrage opportunities in **E** are the limiting case of an increasing sequence of replicas of $\mathcal{E}$.

Assume throughout that $z = (z_i) \in Z$, a feasible allocation for $\mathcal{E}$. Then, arbitrage opportunities available to individual i in $\mathcal{E}$ are

$$K_z^i = -\sum_{j \neq i} S_j(z_j),$$

in contrast to $K_{\mathbf{Z}} = -\sum \text{cone } S_i(z_i)$.

The key distinctions between the finite and continuum versions of arbitrage are

- K_z^i depends on i; $K_{\mathbf{Z}}$ does not
- K_z^i is a convex set but not a cone; $K_{\mathbf{Z}}$ is.

Nevertheless, we want to emphasize that the contrast we have drawn between the finite and continuum models does not point to a discontinuity at infinity. Letting z^r be the r-fold replica of z, define

$$K_{z^r}^i = -r \sum_{j \neq i} S_j(z_j) - (r-1) S_i(z_i).$$

Evidently, $K_{z^r}^i \subset K_{z^{r+1}}^i \subset K_{\mathbf{Z}}$. Further, $\lim K_{z^r}^i = K_{\mathbf{Z}}$. In fact, for any bounded set $B \subset \mathbf{R}^\ell$, it can be shown that $K_{z^r}^i \cap B$ is converging to $K_{\mathbf{Z}} \cap B$ more rapidly than r^{-1} is converging to zero.

5. Existence of Perfectly Competitive Equilibrium

In this section, we prove a theorem on the existence of perfectly competitive equilibrium. The key assumption is the flatness condition (F) on the arbitrage cones. Hence our existence theorem will only apply to perfectly competitive economies. The fact that a perfectly competitive equilibrium is necessarily a special case of Walrasian equilibrium means that this result is hardly new; indeed (PCE.1–2) imply that under the flatness condition, perfectly competitive and Walrasian equilibrium are equivalent. Thus, our result amounts to a demonstration of Walrasian equilibrium in a special case; and we could refer to the results of several authors (see for example Mas-Colell [1985]) on the existence of Walrasian equilibrium for an economy $\mathcal{E}$, translate into an existence theorem for the economy **E**, and then impose the requisite smoothness conditions to obtain the existence of a perfectly competitive arbitrage equilibrium for **E**. However, to emphasize the integrity of the arbitrage approach as a self-contained alternative to Walras' method of demand-and-supply, we prefer to provide a demonstration that is based directly on arbitrage rather than on price-taking behavior. The proof is broken up into two parts: first, we demonstrate that perfectly competitive equilibrium exists for the economy **E** when it satisfies (F); then we give sufficient conditions on the underlying characteristics $\{v_i\}_{i \in I}$ which imply the flatness condition.

5.1 Existence Theorem with quasi-concavity and finite types

In a perfectly competitive economy **E** *(i.e., one satisfying (F)), there exists a perfectly competitive equilibrium.*

To prove the result, suppose first that individuals' endowment/no-trade allocation happens to be efficient, i.e., $\mathbf{0} \in \mathbf{Z}_{\mathrm{NA}}$. Then, given convexity (A.4), there exists a $p \neq 0$ such that $0 \leq p \sum S_i(0)$. This p is in $K^{\circ}_{\mathbf{z}} \setminus \{0\}$ (recall the No Arbitrage/Efficiency Price Lemma 4.3). Hence, given (F), $\mathbf{z} = \mathbf{0}$ would satisfy (PCE.4) and we would be done.

If instead $\mathbf{0} \notin \mathbf{Z}_{\mathrm{NA}}$, then we will need to appeal to a fixed point mapping. We shall use an adaptation of a mapping due to Arrow and Hahn [1971, p.114–116]. The mapping is more in the spirit of the arbitrage approach to equilibrium in that it involves a search on $\mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$ (the locus of efficient, individually rational allocations), instead of a Walrasian excess demand correspondence. Whenever $pz_i < 0$ for some i, the mapping rejects the point $\mathbf{z}$ in $\mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$ and continues searching for a fixed point. Arrow-Hahn did not intend their mapping to prove the existence of an arbitrage equilibrium but rather of a Walrasian demand-and-supply equilibrium; so we give their mapping a new interpretation. Also, our restriction to perfectly competitive economies—ones satisfying (F)—allows for a simplification of their mapping.

Since $\mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$ need not be convex, the mapping will instead be on a convex set whose points can be associated with points in $\mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$. Specifically, suppose $\#I = m$ and let

$$U = \{u \in \mathbf{R}_+^m : u = (v_i(z_i)) \text{ for some } \mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}\}.$$

If $\mathbf{0} \notin \mathbf{Z}_{\mathrm{NA}}$, then this set is topologically an $(m-1)$ unit simplex (see Figue 3). Specifically, letting Δ denote this simplex, we have

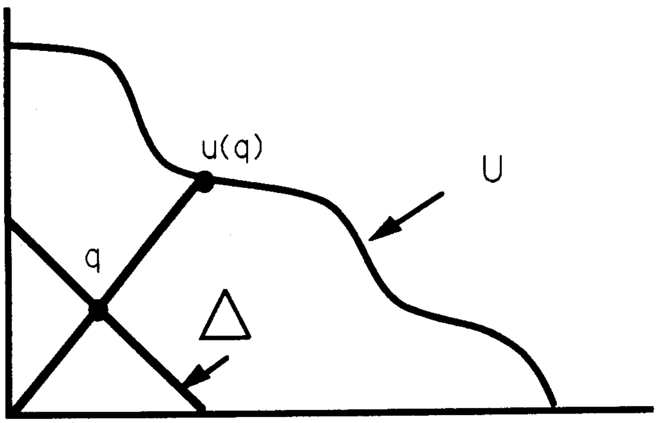

Figure 3: U is topologically an $(m-1)$ simplex

Lemma 5.2 *If $\mathbf{0} \notin \mathbf{Z}_{\mathrm{NA}}$, then there is a one-to-one continuous function from Δ to U such that for any i and any $q \in \Delta$, $u_i(q) = 0$ if and only if $q_i = 0$.*

(For a proof see Arrow and Hahn [1971], Chapter 5, Lemma 3 or Mas-Colell [1985], Proposition 4.6.1.)

Let $\mathbf{Z}(q) = \{\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}} : (v_i(z_i)) = u(q)\}$; let $\mathbf{z}(q)$ be an arbitrary selection from $\mathbf{Z}(q)$; and let $p_q = K^{\circ}_{\mathbf{z}(q)} \cap \{p : \| p \| = 1\}$. Consider the correspondence $\psi : \Delta \to \Delta$ defined by

$$\psi(q) = \Delta \cap \{\overline{q} : \overline{q}_i = 0 \text{ if } p_q z_i(q) > 0\}.$$

Note $\overline{q}_i = 0$ implies $v_i(z_i(\overline{q})) = 0$ by Lemma 5.2. Hence the mapping has the following natural interpretation. If $pz_i > 0$ for any individual i, where $\mathbf{z} = \mathbf{z}(q)$ and $p = p_q$, then some of i's trading partners j must have $pz_j < 0$ (recall Section 4.1). They will be able to improve on z_j by individually recontracting away from i; hence i will not remain with the 'subsidy' $pz_i > 0$. The mapping punishes such i's, recognizing that this $\mathbf{z}$ is not an equilibrium position.

Lemma 5.3 *ψ is a non-empty, upper hemicontinuous, convex valued correspondence.*

For a proof, see Arrow and Hahn, Chapter 5. The Arrow-Hahn mapping is on the Cartesian product of Δ, the price simplex, and Z—not just on Δ.[2] The flatness condition, (F), allows for the simplification. Given (F), there is no need to search on the price simplex since supporting prices are unique. Relatedly, there is no need to search on $\mathbf{Z}$ since all allocations in $\mathbf{Z}(q)$ are essentially equivalent; hence an arbitrary selection suffices. Specifically, let $\mathbf{z} \equiv \mathbf{z}(q)$ and $p \equiv p_q$. Then for *any* $\mathbf{z}' \in \mathbf{Z}(q)$ it will be the case that $pz'_i = pz_i$ for each i; hence $p \in K^\circ_{\mathbf{z}'}$. (Proof: $pK_{\mathbf{z}} \leq 0$ implies $pS_i(z_i) \geq 0$ for each i; or equivalently, $pz_i \leq p[S_i(z_i) + \{z_i\}]$. Hence $pz_i \leq pz'_i$ for each i. Summing, $p\sum z_i \leq p\sum z'_i$. But feasibility implies $p\sum z_i = p\sum z'_i = 0$; hence $pz_i = pz'_i$ for each i. That $p \in K^\circ_{\mathbf{z}'}$ now follows from Lemma 4.3 since $pS_{\mathbf{z}} = pS_{\mathbf{z}'}$.)

Given Lemma 5.3, we can apply Kakutani's fixed-point theorem and assert that there is a $q \in \Delta$ such that $q \in \psi(q)$.

Claim: For the fixed-point q, $pz_i = 0$ for all i, where $p \equiv p_q$ and $\mathbf{z} \equiv \mathbf{z}(q)$.

To verify, observe that if $pz_i > 0$ then $q_i = 0$, hence $v_i(z_i) = 0$; i.e., by monotonicity, $v_i(z_i) < v_i(z'_i)$ for all $z'_i \in \mathbf{R}^\ell_{++}$. But if $pz_i > 0$ then for some $\epsilon > 0$, $p\epsilon e < pz_i$ where e is the unit vector in $\mathbf{R}^\ell$, yet $v_i(\epsilon e) > v_i(z_i)$; contradicting $p \in K^\circ_{\mathbf{z}}$. (The details are $p \in K^\circ_{\mathbf{z}}$ implies $pK_{\mathbf{z}} \leq 0$; hence $p\sum \text{cone } S_i(z_i) \geq 0$, or $p \cdot \text{cone } S_i(z_i) \geq 0$ for each i (recall zero is in each $S_i(z_i)$ set). But $S_i(z_i) \subset \text{cone } S_i(z_i)$, hence $pS_i(z_i) \geq 0$ or $pz_i \leq p[S_i(z_i) + \{z_i\}]$.) We conclude that $pz_i \leq 0$ for all i. Hence, since $\sum z_i = 0$, $pz_i = 0$ for all i, as claimed.

In view of (PCE.4), the claim immediately implies that there exists a perfectly competitive equilibrium for $\mathbf{E}$, as was to be shown.

To translate the Existence Theorem into a statement involving only conditions on individual characteristics, we place restrictions on each v_i which imply that in the aggregate (F) will be satisfied. Since there may be only one type in the economy, there is little opportunity to exploit the smoothing effects of aggregation and the hypothesis for each v_i will have to be essentially the same as (F).

By Corollary 4.4, the flatness of $K_{\mathbf{z}}$ is equivalent to the uniqueness of efficiency prices (up to a normalization). Hence, if at least one type i has C^1 preferences and any $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$ gives i an allocation z_i in the interior of Z_i, then (F) will hold. That is, if $S_i(z_i)$ has a unique support for one type i then $\sum S_i(z_i)$ will have a unique support, hence $K_{\mathbf{z}}$ will have a unique support. This suggests the following individual smoothness/differentiability condition.

(S) for some type i, v_i is C^1 on the interior of Z_i and satisfies the boundary condition $v_i(y') > v_i(y)$ whenever $y' \in \text{int } Z_i$ and $y \in \text{bdry } Z_i$.

[2] We have written Z here instead of $\mathbf{Z}$ since the Arrow and Hahn proof is for a finite economy $\mathcal{E}$, not a continuum economy $\mathbf{E}$. But this difference is inconsequential given the assumption of equal-treatment.

The boundary condition ensures that z_i will be in the interior of Z_i for any individually rational allocation (recall $\omega_i \gg 0$). It may be interpreted as an indispensability condition, that some amount of each good is required for subsistence. While quite restrictive, it is not central to the theory being developed. (For applications, other boundary conditions could be substituted.)

Corollary 5.4 (Sufficiency condition for existence) *Assume (S). Then* **E** *satisfies (F) and there exists a perfectly competitive equilibrium.*

6. Extension of the Model

In this section we extend the model, characterizations, and existence results of the previous sections to exchange economies in which (1) there may be more than a finite number of types and (2) individuals do not necessarily have convex preferences.

Let V be a set of functions $v : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ satisfying (A.1-3) but not necessarily the quasi-concavity assumption (A.4). Instead of describing a type as v_i, where $i \in I$, a type is a $v \in V$. Consistent with the change, define $Z_v = \mathbf{R}^\ell_+ - \{\omega_v\}$ as the feasible trading set for an individual of type v.

We assume

- V is a compact metric space.

Specifically, we metrize V in the usual way: For any $v \in V$ let v' represent that type's preferences over consumption bundles rather than over net trades; i.e., $v' : \mathbf{R}^\ell_+ \to \mathbf{R}$ is defined by $v'(\omega_v + z) = v(z)$. Then, $v_n \to v$ means $\omega_{v_n} \to \omega_v$ and $v'_n \to v'$ uniformly on compacta (see Mas-Colell [1985]).

To allow for a continuum of possible types, instead of **E**, now describe an economy by a (Borel measurable) function $\mathbf{v} : [0,1] \to V$, where [0,1] is the set of traders in the economy.

A feasible allocation for this economy is a (Borel measurable) function $\mathbf{z} : [0,1] \to \mathbf{R}^\ell$ satisfying $\mathbf{z}(t) \in Z_t \equiv Z_{\mathbf{v}(t)}$ for a.e. t and $\int \mathbf{z}(t)\, d\lambda(t) = 0$, where λ is the Lebesgue measure, the population measure on the set of traders. Henceforth we will typically write v_t and z_t for $\mathbf{v}(t)$ and $\mathbf{z}(t)$, respectively.

Call a pair $(v, z) \in V \times \mathbf{R}^\ell$ a *subtype*. Notice $\mathbf{z}$ need not involve equal-treatment: $v_t = v_{t'} \not\Rightarrow z_t = z_{t'}$; i.e., t and t' may be of different subtypes. The distribution of subtypes implied by $\mathbf{z}$ is given by the product measure

$$\mu_{\mathbf{z}} = \lambda \circ \mathbf{v}^{-1} \times \lambda \circ \mathbf{z}^{-1}$$

on the Borel subsets of $V \times \mathbf{R}^\ell$. The number $\mu_{\mathbf{z}}(A \times B)$ gives the mass of individuals having types $v \in A$ and allocations $z \in B$. Denote by supp $\mu_{\mathbf{z}}$ the support of $\mu_{\mathbf{z}}$, i.e., the smallest closed subset of $V \times \mathbf{R}^{\ell}$ with unit mass; and call t *representative* if $(v_t, z_t) \in \text{supp } \mu_{\mathbf{z}}$.

To describe each trader's arbitrage opportunities if the status quo is $\mathbf{z}$, let G be any *finite* group of individuals selected from [0,1], with the proviso that $t \in G$ only if $(v_t, z_t) \in \text{supp } \mu_{\mathbf{z}}$; $\mathcal{G}$ denotes the set of all such G. Then,

$$K_{\mathbf{z}} = \{y : \text{ for some } G \in \mathcal{G} \quad y = \sum_{t \in G} y_t \text{ and } -y_t \in S_t(z_t) \text{ for each } t\}.$$

The definition of arbitrage equilibrium is unaltered, modulo the extension of $K_{\mathbf{z}}$.

DEFINITION: The allocation $\mathbf{z}$ is an arbitrage equilibrium for $\mathbf{v}$ if for each representative t

$$v_t(z_t) \geq v_t(z_t') \quad \text{for all } z_t' \in K_{\mathbf{z}} \cup K_{\mathbf{z}} + \{z_t\}.$$

6.1 Extension of the Characterizations

The following extension of the Arbitrage Lemma shows that if V only includes types with convex preferences then the closure of $K_{\mathbf{z}}$ remains a convex cone—whether or not there are a continuum of types in $\mathbf{v}$ (property (ii) below). But if V includes types with nonconvex preferences then $K_{\mathbf{z}}$ may be neither convex nor a cone, whether or not supp $\mu_{\mathbf{z}}$ is finite. Nevertheless, whenever $K_{\mathbf{z}}$ is not a convex cone, its structure will be related to such a cone. Specifically, define the *reduction* of $K_{\mathbf{z}}$ as

$$\hat{K}_{\mathbf{z}} = \text{cl}\,\{y : y = \alpha y' \text{ for some } y' \in K_{\mathbf{z}} \text{ and some } \alpha \in [0,1]\}.$$

$\hat{K}_{\mathbf{z}}$ is the smallest closed set that contains both $K_{\mathbf{z}}$ and all the 'scaled down' elements of $K_{\mathbf{z}}$. Note if cl $K_{\mathbf{z}}$ is a convex cone, $\hat{K}_{\mathbf{z}}=$ cl $K_{\mathbf{z}}$. But even when the latter is not a convex cone, the former will be (property (i) below).

6.1 Extended Arbitrage Lemma

(i) $\hat{K}_{\mathbf{z}} = \text{cl (cone } K_{\mathbf{z}}) = -\text{cl}\,(\cup_{G \in \mathcal{G}} [\sum_{t \in G} \text{cone } S_t(z_t)])$, *a convex cone containing the origin.*[3]

Further,

(ii) $\hat{K}_{\mathbf{z}} = \text{cl } K_{\mathbf{z}}$ *if all $v \in V$ are quasi-concave.*

[3]When there are a finite number of subtypes, i.e., when supp $\mu_{\mathbf{z}}$ is finite, then the third equality reduces to $-\text{cl}\,[\sum \text{cone } S_t(z_t)]$, where the sum is taken over all t such that $(v_t, z_t) \in \text{supp } \mu_{\mathbf{z}}$.

Proof: See Appendix A. □

Some further useful properties of $K_{\mathbf{z}}$ can be deduced as a corollary. Let $\hat{K}_{\mathbf{z}}^{\circ} = \{p : p\hat{K}_{\mathbf{z}} \leq 0\}$, the polar of $\hat{K}_{\mathbf{z}}$. Also recall that for any set S with $0 \in S$, $N(S)$ denotes the normal cone to S at zero, i.e., $N(S) = \{p : pS \leq 0\}$. As in the finite types model, denote the aggregate at-least-as-good-as $\mathbf{z}$ set by $S_{\mathbf{z}} = \int S_t(z_t)\, d\lambda(t)$. (See Hildenbrand [1974] for the integral of a correspondence.)

Corollary 6.2

(i) $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$ *if and only if* $\hat{K}_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$.

(ii) $N(K_{\mathbf{z}}) = \hat{K}_{\mathbf{z}}^{\circ} = -N(S_{\mathbf{z}})$.

(iii) $\hat{K}_{\mathbf{z}} = -\mathrm{cl}\,(\mathrm{cone}\; S_{\mathbf{z}})$.[4]

Proof: See Appendix A. □

Now reconsider the alternatives (A) or (NA). Define no arbitrage allocations as before; and to accommodate the possibility that $K_{\mathbf{z}}$ may not be a cone, define a no arbitrage price vector as a $p \in N(K_{\mathbf{z}})\backslash\{0\} = \hat{K}_{\mathbf{z}}^{\circ}\backslash\{0\}$ (by (ii) of Corollary 6.2). As in the finite types model we have:

Corollary 6.3 (Existence of no arbitrage prices) $\hat{K}_{\mathbf{z}}^{\circ} \neq \{0\}$ *if and only if* $z \in \mathbf{Z}_{\mathrm{NA}}$.

Proof: This follows immediately from (i) of Corollary 6.2 and the basic separation theorem for convex cones. □

The characterization of no arbitrage allocations also remains intact. Call $\mathbf{z}$ *weakly Pareto efficient* if there is no other feasible allocation $\mathbf{z}'$ such that $v_t(z'_t) > v_t(z_t)$ for a.e. t.

6.4 Extension of No Arbitrage/Efficiency Lemma $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$ *if and only if* $\mathbf{z}$ *is weakly Pareto efficient.*

Proof: If $\mathbf{z}$ is weakly efficient then $0 \in \mathrm{bdry}\; S_{\mathbf{z}}$. Hence there is a $p > 0$ such that $pS_{\mathbf{z}} \geq 0$. We conclude, by (ii) of Corollary 6.2, that $pK_{\mathbf{z}} \leq 0$; i.e., $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$. Conversely, $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$ implies there is a $p > 0$ such that $pK_{\mathbf{z}} \leq 0$. So $pS_{\mathbf{z}} \geq 0$, again by (ii) of Corollary 6.2. That is, $0 \in \mathrm{bdry}\; S_{\mathbf{z}}$; hence $\mathbf{z}$ is weakly efficient. □

[4] To emphasize the symmetry with (i) of the Extended Arbitrage Lemma, it can also be shown that $\mathrm{cl} \int \mathrm{cone}\; S_t(z_t)\, d\lambda(t) = \mathrm{cl}\,(\mathrm{cone}\; S_{\mathbf{z}})$.

Our primary interest continues to be economies for which $K_{\mathbf{z}}$ is flat—i.e., the boundary of $K_{\mathbf{z}}$ is a hyperplane—for $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$ because, as in the finite types case, in such economies any individual arbitrager has a linear opportunity set or, equivalently, faces perfectly elastic demands and supplies at prices p, where $p \in N(K_{\mathbf{z}})\backslash\{0\}$. Thus it is important to observe that even if preferences are not quasi-concave, $K_{\mathbf{z}}$ will often be flat. Just a little smoothness on the individual level will suffice. The analogue of the individual smoothness/differentiability condition in the finite types model is

(S) for some representative individual t, v_t is C^1 on the interior of Z_t and satisfies the boundary condition $v_t(y') > v_t(y)$ whenever $y' \in \text{int } Z_t$ and $y \in \text{bdry } Z_t$.

Lemma 6.5 (Extended sufficiency condition for flatness) *Let* $\mathbf{z}$ *be any allocation for* $\mathbf{v}$ *satisfying (S). The* $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$ *implies* $K_{\mathbf{z}}$ *is flat.*

Proof: Since $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$, Corollary 6.3 and (ii) of Corollary 6.2 imply there is a $p \in N(K_{\mathbf{z}})\backslash\{0\}$. Hence it suffices to show that for any y such that $py = 0$, $y \in \text{cl } K_{\mathbf{z}}$; that is, bdry $K_{\mathbf{z}} = H_p$. By (ii) of Corollary 6.2, $0 \leq pS_{\mathbf{z}}$. Hence for all representative individuals t, $0 \leq pS_t(z_t)$. Specifically, for the t satisfying (S), p must be colinear with the gradient $\partial v_t(z_t)$ since $\mathbf{z} \in \mathbf{Z}_{\mathrm{IR}}$ implies $z_t \in \text{int } Z_t$. This permits the following construction. For any $\delta > 0$ let $y(\delta) = y - \delta e$. Since $py = 0$, $p(-y(\delta)) > 0$. Hence, by Taylor's formula, for the t satisfying (S) there is a sufficiently large positive integer n such that

$$v_t(z_t - \frac{1}{n}y(\delta)) > v_t(z_t).$$

Further, there is an $\epsilon > 0$ such that for all subtypes $(v, z) \in B_\epsilon(v_t, z_t)$

$$v(z - \frac{1}{n}y(\delta)) > v(z).$$

Since t is representative implies that $\mathbf{z}$ includes a nonnull set of subtypes in this ϵ-ball, we conclude that $n(\frac{1}{n}y(\delta)) = y(\delta) \in K_{\mathbf{z}}$ for all $\delta > 0$. Letting $\delta \to 0$ shows $y \in \text{cl } K_{\mathbf{z}}$. □

The intuition for the proposition is that if $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$ then any $p \in N(K_{\mathbf{z}})\backslash\{0\}$ supports $\mathbf{z}$. Specifically, for the t satisfying (S), his at-least-as-good-as set $S_t(z_t)$ is tangent to the hyperplane H_p; hence it must be *locally* smooth and convex for trades in a neighborhood of the origin (even if it globally exhibits nonconvexities). This suffices to yield a *flat* arbitrage cone, much like in the finite types case, since there are many subtypes (v, z) close to (v_t, z_t). Essentially, the closure of $K_{\mathbf{z}}$ becomes $-$cone $S_t(z_t)$ (see Figure 4).

If $K_{\mathbf{z}}$ *is* flat then, as in the quasi-concave case covered by the Extended Arbitrage Lemma 6.1, the closure of $K_{\mathbf{z}}$ just equals its reduction.

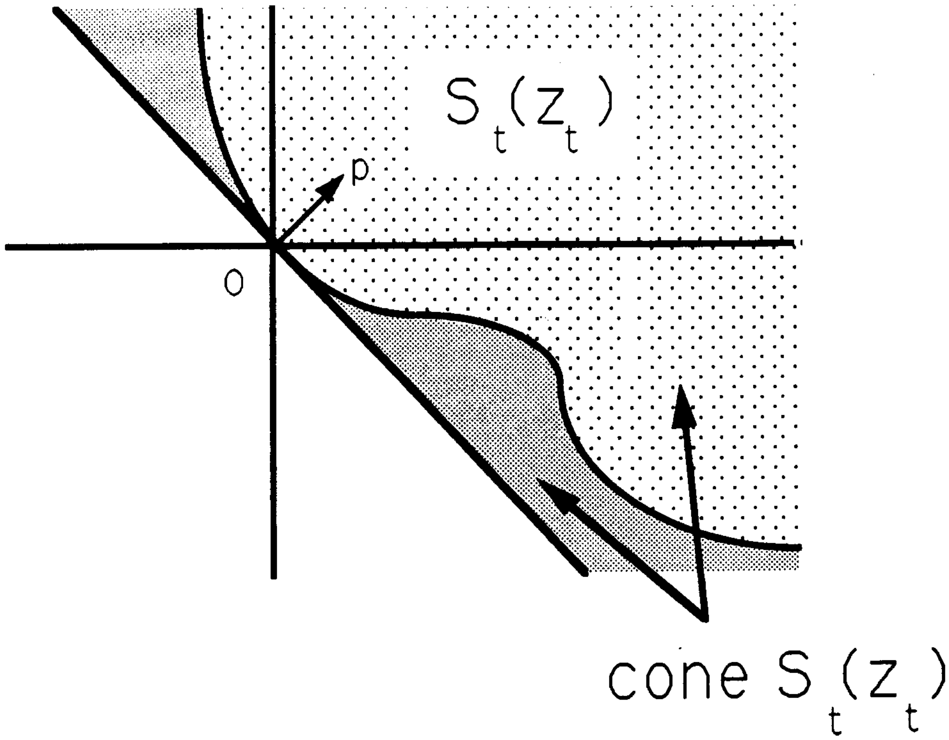

Figure 4: $S_t(z_t)$ is smooth and convex around the origin. Hence cone $S_t(z_t)$ is flat.

Lemma 6.6 (Consequences of flatness) *If $K_{\mathbf{z}}$ is flat then* cl $K_{\mathbf{z}} = \hat{K}_{\mathbf{z}} = \{y : py \leq 0\}$ *for any nonzero price vector p in the polar of $K_{\mathbf{z}}$.*

Proof: If $K_{\mathbf{z}}$ is flat then bdry $K_{\mathbf{z}} = \{y : py = 0\}$ for any $p \in N(K_{\mathbf{z}})\backslash\{0\}$. Since by (ii) of Corollary 6.2, $pS_{\mathbf{z}} \geq 0$, monotonicity implies $p > 0$. Hence since $\mathbf{R}^{\ell}_{-} \subset K_{\mathbf{z}}$, cl $K_{\mathbf{z}} = \{y : py \leq 0\}$. The latter is a closed convex cone; hence, by the construction of the reduction, $\hat{K}_{\mathbf{z}} =$ cl $K_{\mathbf{z}}$. □

The characterization of perfectly competitive equilibria and Walrasian equilibria remain basically unaltered. The definition of Walrasian equilibrium needs the obvious modification:

DEFINITION: A *Walrasian equilibrium* for $\mathbf{v}$ is a pair $(p, \mathbf{z})$ such that $z_t \in D_t(p)$ for every representative t.

Let $L_{\mathbf{z}}$ be the smallest linear space containing $\{z : (v, z) \in \text{supp } \mu_{\mathbf{z}}\}$. Then

Theorem 6.7 (Extended characterization of perfectly competitive equilibrium) *The following are equivalent:*

(PCE.1) $\mathbf{z}$ *is a perfectly competitive equilibrium for* $\mathbf{v}$

(PCE.2) $(p, \mathbf{z})$ *is a Walrasian equilibrium for every* $p \in N(K_{\mathbf{z}})\backslash\{0\}$*, where* $K_{\mathbf{z}}$ *is flat*

(PCE.3) $L_{\mathbf{z}} \subset \text{bdry } K_{\mathbf{z}}$*, a hyperplane*

(PCE.4) $N(K_{\mathbf{z}})\backslash\{0\} \perp L_{\mathbf{z}}$*, where* $K_{\mathbf{z}}$ *is flat.*

Compared to the finite types characterization, the only modifications are that $N(K_{\mathbf{z}})$ has replaced $K_{\mathbf{z}}^{\circ}$ and "$K_{\mathbf{z}}$ is flat" has replaced "dim $K_{\mathbf{z}}^{\circ} = 1$". Regarding the latter change note that, since $K_{\mathbf{z}}$ need not be convex in the absence of quasi-concave preferences, the requirement dim $N(K_{\mathbf{z}}) = 1$ would not suffice to guarantee $K_{\mathbf{z}}$ is flat.

Proof: The proof that (PCE.1 and 2) are equivalent is basically unchanged from the proof in the finite types case, with a representative individual t now playing the role that an individual of type i played before. Further, the equivalence of (PCE.3 and 4) follows readily from Lemma 6.6.

To show the equivalence between (PCE.2 and 4), it suffices to prove that $(p, \mathbf{z})$ is Walrasian if and only if $p \perp L_{\mathbf{z}}$ and $p \in N(K_{\mathbf{z}})\backslash\{0\}$. If $(p, \mathbf{z})$ is Walrasian then clearly $p \perp L_{\mathbf{z}}$. Further, $pS_{\mathbf{z}} \geq 0$; hence, by (ii) of 6.2, $pK_{\mathbf{z}} \leq 0$. That is, $p \in N(K_{\mathbf{z}})\backslash\{0\}$. Conversely, $pK_{\mathbf{z}} \leq 0$ implies $pS_{\mathbf{z}} \geq 0$, again by (ii) of Corollary 6.2. Hence $pS_t(z_t) \geq 0$ for all representative t. Now proceed as in the finite types proof to show that $(p, \mathbf{z})$ is Walrasian. □

Theorem 6.8 (Extended characterization of Walrasian equilibrium) *The following are equivalent:*

(WE.1) $(p, \mathbf{z})$ *is a Walrasian equilibrium for* $\mathbf{v}$

(WE.2) $p \perp L_{\mathbf{z}}$ *for some* $p \in N(K_{\mathbf{z}})\backslash\{0\}$.

Proof: See proof of equivalence between (PCE.2 and 4) above. □

Again the contrast between the two concepts of equilbrium is revealed in comparing (PCE.4) and (WE.2).

6.2 Extension of the Existence Theorem

Let $\mathcal{V}$ represent the set of all possible economies, i.e., the set of all (Borel measurable) functions $\mathbf{v} : [0,1] \to V$. To extend the existence theorem to economies with a continuum of types, we endow $\mathcal{V}$ with a concept of nearness (i.e., a topology). For any given economy $\mathbf{v} \in \mathcal{V}$, the measure

$$\mu_{\mathbf{v}} = \lambda \circ \mathbf{v}^{-1}$$

on the Borel sets of V describes the distribution of types in $\mathbf{v}$. We say that $\mathbf{v}_n \to \mathbf{v}$ if

- $\mu_{\mathbf{v}_n} \to \mu_{\mathbf{v}}$ weakly, and
- supp $\mu_{\mathbf{v}_n} \to$ supp $\mu_{\mathbf{v}}$ (in the Hausdorf distance).

Under this topology, the set of finite economies (i.e., the ones in which the support of $\mu_{\mathbf{v}}$ is finite) is dense in $\mathcal{V}$. (See Mas-Colell [1985], E.3.3 and Chapter 5.8.)

Let $\Pi(\mathbf{v})$ denote the set of Walrasian equilibrium prices for $\mathbf{v}$ normalized so that $\| p \| = 1$. This price mapping is often closed, i.e.,

(C) If $\mathbf{v}, \mathbf{v}_n \in \mathcal{V}$, $p_n \in \Pi(\mathbf{v}_n)$, $\mathbf{v}_n \to \mathbf{v}$, and $p_n \to p$ then $p \in \Pi(\mathbf{v})$.

Sufficient conditions for closedness are given at the end of this section. With the aid of (C), we first extend the finite type existence result to economies with a continuum of types, maintaining the assumption that individuals have quasi-concave preferences.

Assume all economies in $\mathcal{V}$ are perfectly competitive in the sense of satisfying the flatness condition (F):

(F′) For any given economy $\mathbf{v} \in \mathcal{V}$, if $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}} \cap \mathbf{Z}_{\mathrm{IR}}$ then $K_{\mathbf{z}}$ is flat.

Theorem 6.9 (Extended existence with quasi-concavity)
Assume (F′), (C), and that every $v \in V$ is quasi-concave. Then for any economy $\mathbf{v}$ *in $\mathcal{V}$, there exists a perfectly competitive equilibrium.*

Proof: Suppose first that supp $\mu_{\mathbf{v}}$ is finite and $\mu_{\mathbf{v}}(v)$ is a rational number for all $v \in$ supp $\mu_{\mathbf{v}}$. Then there is a positive integer m such that $m\mu_{\mathbf{v}}(v)$ is an integer for each $v \in$ supp $\mu_{\mathbf{v}}$. Hence there is a finite types economy $\mathbf{E} = \{v_i \cdot \mathbf{1}\}_{i=1,\dots,m}$ such that $\#\{i : v_i = v\} = m\mu(v)$ for each $v \in$ supp $\mu_{\mathbf{v}}$. Clearly, the existence of a perfectly competitive equilibrium for $\mathbf{E}$ (ensured by Theorem 5.1) implies the existence of a perfectly competitive equilibrium for $\mathbf{v}$.

Now if supp $\mu_{\mathbf{v}}$ has a continuum of types, observe that the economies with finite support and a rational number of each type are dense on all economies $\mathbf{v} : [0,1] \to V$. Hence there is a sequence $\mathbf{v}_n \to \mathbf{v}$ with $p_n \in \Pi(\mathbf{v}_n)$. Let $p_n \to p$ on a subsequence. Then, by (C), $p \in \Pi(\mathbf{v})$; i.e., $\mathbf{v}$ has a perfectly competitive equilibrium.□

To extend the finite existence theorem to economies with nonconvex preferences, a difficulty must be solved. Economies $\mathbf{E}$ have equal treatment built in, more specifically, *equal-trade treatment*: individuals of the same type are given the same allocation. But with nonconvexities, such equal treatment may be inconsistent with efficiency (e.g., imagine a two good, one type economy in which all individuals have quarter circular indifference curves and endowments along the 45° line). Consequently, such equal treatment may be inconsistent with arbitrage equilibrium since the arbitrager may want to take advantage of the convexifying effects of large numbers to generate middleman profits. Note however that any arbitrage equilibrium will still satisfy *equal-utility treatment*; i.e., individuals of the same type will obtain equal utilities in equilibrium. This follows from their facing the same linear arbitrage possibilities.

To solve the difficulty, it will suffice to restrict our attention to allocations without 'too much' unequal trade treatment. By Carathéodory's Theorem (Rockafeller [1970, Theorem 17.1]) for any allocation satisfying

(1) weak Pareto efficiency and

(2) equal-utility treatment,

we can find another feasible, utility-equivalent allocation satisfying (1), (2), and

(3) different individuals of any given type engage in at most $\ell + 1$ different trades.

The restriction to trades satisfying (3) suffices to completely 'convexify preferences'.

With this background, our method of proving existence with nonconvexities is to replace **v** by an economy with convexified preferences. Then we show a perfectly competitive equilibrium for the latter implies a perfectly competitive equilibrium for the former. We convexify preferences in the usual way (e.g., see Arrow and Hahn [1971], Chapter 7); but for our method of proof, we require a numerical representation of these convexified preferences. The Representation Theorem proved below, which is of some independent interest, first will be formalized in consumption space. Then, as a corollary, we translate it to trade space.

Let $u : \mathbf{R}^\ell_+ \to \mathbf{R}$ be any continuous and weakly monotone (but not quasi-concave) utility funtion.[5] For any $x \in \mathbf{R}^\ell_+$, let $A(x)$ represent the at-least-as-good-as x set for preferences u, i.e., $A(x) = \{x' : u(x') \geq u(x)\}$. The function u^*, defined below, gives a numerical representation of the convexified preferences. For any $x \in \mathbf{R}^\ell_+$ let

$$\begin{aligned} R(x) &= \{r \in \mathbf{R}_+ : x \in \text{conv } A(re)\}, \\ r_x &= \sup R(x), \text{ and} \\ u^*(x) &= u(r_x e) \end{aligned}$$

where e is the unit vector in $\mathbf{R}^\ell$ (see Figure 5). For any $x \in \mathbf{R}^\ell_+$, let $A^*(x)$ denote the at-least-as-good-as x set for preferences u^*; and let Δ^ℓ be the ℓ-dimensional unit simplex.

[5]Note that 'u' was used in a different sense in Section 5.

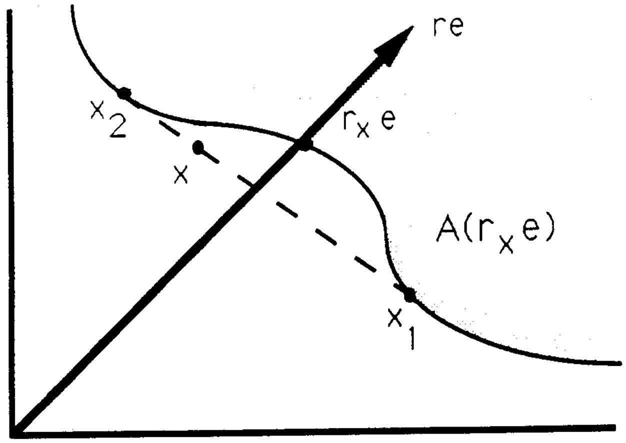

Figure 5: The convexified preferences. Notice that x is a convex combination of x_1 and x_2.

6.10 Representation Theorem for convexified preferences *$u^* : \mathbf{R}^\ell_+ \to \mathbf{R}$ is a well-defined (i.e., sup $R(x) < \infty$ for all x), quasi-concave, weakly monotone and continuous function, with $u^*(0) = u(0)$. Specifically, for any $x \in \mathbf{R}^\ell_+$:*

$$A^*(x) = \text{conv } A(r_x e)$$

and

$$u^*(x) = \max_{(x_k)_{k=1,\ldots,\ell+1}} \{u(x_1) : u(x_1) = \ldots = u(x_{\ell+1}),\ x = \sum_{k=1}^{\ell+1} \alpha_k x_k \ \&\ \alpha \in \Delta^\ell\}.$$

Proof: See Appendix B. □

The result says that u^* inherits the continuity and monotonicity properties of u. Further, it suggests an interpretation for the representation: If there is a unit mass of type u individuals consuming x each, $u^*(x)$ is the largest (equal-utility) utility level they can achieve by re-trading $x \cdot \mathbf{1}$ among themselves: each fraction α_k of the group could obtain the allocation x_k and hence utility $u(x_k)$. We remark that u^* is a convexified version of the numerical representation of preferences used by Kannai [1970] (without his strong monotonicity assumption).

To translate into trade space, for any type v let $S_v(z)$ represent the at-least-as-good-as set for type v with respect to *changes* from z, i.e., $S_v(z) = \{y : v(z+y) \geq v(z)\}$. Consequently, $S_v(z) + \{z\}$ represents the set of trades that are at-least-as-good-as z for v.

Corollary 6.11 *Assume $v : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ satisfies (A.1-3). Then there exists a function $v^* : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ satisfying (A.1-4), with $Z_{v^*} = Z_v$. The function v^* is* *a numerical representation of the convexification of the preferences in v. Specifically, for any $z \in Z_{v^*}$, there are $(z_k)_{k=1,\ldots,\ell+1} \subset Z_v$ such that*

(i) $z = \sum_{k=1}^{\ell+1} \alpha_k z_k$ *for some* $(\alpha_k) \in \Delta^\ell$,

(ii) $v(z_1) = \ldots = v(z_{\ell+1}) \geq v^*(z)$, *and*

(iii) $S_{v^*}(z) + \{z\} = \text{conv } S_v(z_k) + \{z_k\}$ *(for any k).*

Proof: See Appendix B. □

Thus v^*, the convexified version of v, satisfies (A.1-4) and has the same effective domain as v. Specifically, (iii) gives the relationship between the at-least-as-good-as sets for v^* and v. With the aid of the Corollary, we now provide a full extension of the finite types existence theorem.

6.12 Extended Existence Theorem *Assume (F′) and (C). Then for any economy* $\mathbf{v}$ *in* $\mathcal{V}$*, there exists a perfectly competitive equilibrium.*

Proof: As in the proof of Theorem 6.9, assume first that supp $\mu_{\mathbf{v}}$ is finite and $\mu_{\mathbf{v}}(v)$ is a rational number for all $v \in$ supp $\mu_{\mathbf{v}}$. Define $\mathbf{E} = \{v_i \cdot \mathbf{1}\}_{i=1,\ldots,m}$ as in 6.9, and let $\mathbf{E}^* = \{v_i^* \cdot \mathbf{1}\}_{i=1,\ldots,m}$, where v_i^* is the convexification of v_i. To verify that $\mathbf{E}^*$ has a perfectly competitive equilibrium, it suffices to show that $\mathbf{E}^*$ satisfies (F). (Note that some v_i^* may not be in V, hence this does not follow immediately from (F′).)

Accordingly, let $\mathbf{z}^*$ be in $\mathbf{Z}_{\text{NA}} \cap \mathbf{Z}_{\text{IR}}$ for the economy $\mathbf{E}^*$. By Lemmas 3.2 and 4.3, there exist prices p ($p \neq 0$) such that

$$0 \leq pS_{\mathbf{z}^*}$$

where $S_{\mathbf{z}^*}$ is the aggregate at-least-as-good-as $\mathbf{z}^*$ set for $\mathbf{E}^*$, i.e., $S_{\mathbf{z}^*} = \sum S_{v_i^*}(z_i^*) \cdot \mathbf{1}$. Since 0 is in each $S_{v_i^*}(z_i^*)$ set, this may be strengthened to

$$0 \leq pS_{v_i^*}(z_i^*) \text{ for each } i.$$

Or, using (iii) of Corollary 6.11, for each i and any i_k

$$p(z_i - z_{i_k}) \leq pS_{v_i}(z_{i_k})$$

where $z_i^* = \sum_{k=1}^{\ell+1} \alpha_{i_k} z_{i_k}$, $\alpha_{i_k} \geq 0$, $\sum_k \alpha_{i_k} = 1$, and $v_i^*(z_i^*) \leq v_i(z_{i_k})$.

Let $\mathbf{z} \equiv \mathbf{z}(\mathbf{z}^*)$ be such that $\mu_{\mathbf{z}}(v, z) = \frac{1}{m} \sum_{i \text{ s.t. } v_i^* = v^*} \sum_{k \text{ s.t. } z_{i_k} = z} \alpha_{i_k}$ for each subtype (v, z), where v^* is the convexification of v. By construction, $\mathbf{z}$ is a feasible allocation for v. Further, since $\mathbf{z}^* \in \mathbf{Z}_{\text{IR}}$ for $\mathbf{E}^*$, $\mathbf{z} \in \mathbf{Z}_{\text{IR}}$ for $\mathbf{v}$ (recall for each i and k, $v_{i_k}(z_{i_k}) \geq v_i(z_i^*) \geq 0$). Now observe that $S_{\mathbf{z}} = \frac{1}{m} \sum_i \sum_k \alpha_{i_k} \text{conv } S_{v_i}(z_{i_k})$. Hence,

$p(z_i^* - z_{i_k}) \le pS_{v_i}(z_{i_k})$ for each i_k implies $0 \le pS_{\mathbf{z}}$. So, $\mathbf{z} \in \mathbf{Z}_{\mathrm{NA}}$ for the economy $\mathbf{v}$ and, furthermore,

$$0 \le pS_{\mathbf{z}^*} \text{ implies } 0 \le pS_{\mathbf{z}}.$$

We conclude from (F$'$) and (ii) of Corollary 6.2 that dim $N(S_{\mathbf{z}}) = 1$; hence dim $N(S_{\mathbf{z}^*}) = 1$. That is, $\mathbf{E}^*$ satisfies (F); so it possesses a perfectly competitive equilibrium.

Now let $\mathbf{z}^*$ be a perfectly competitive equilibrium for $\mathbf{E}^*$ supported by p (i.e., $p \in K^\circ_{\mathbf{z}^*}$ for $\mathbf{E}^*$). We will show that $\mathbf{z} = \mathbf{z}(\mathbf{z}^*)$ is a perfectly competitive equilibrium for $\mathbf{v}$ with prices p. Since $p(z_i^* - z_{i_k}) \le pS_{v_i}(z_{i_k})$ for each i_k, it suffices to show that $pz_{i_k} = 0$ for each i_k. Observe that $pz_{i_k} \ge pz_i^* = 0$ for all i_k since $(\mathbf{z}^*, p)$ is Walrasian for $\mathbf{E}^*$ and $z_{i_k} \in S_{v_i^*}(z_i^*) + \{z_i^*\}$. Hence $\sum p\alpha_{i_k} z_{i_k} \ge pz_i^* = 0$. Since the left hand side just equals pz_i^*, we conclude $pz_{i_k} = pz_i^* = 0$ for each i_k, as required.

To extend the existence result to economies $\mathbf{v}$ for which supp $\mu_{\mathbf{v}}$ has a continuum of types, now proceed as in the proof of Theorem 6.9. □

To translate the Existence Theorem into a statement only involving conditions on individual characteristics, we give sufficient conditions for (F$'$) and (C).

From Lemma 6.5, if every utility function v in V is C^1 on the interior of Z_v and satisfies the boundary condition

(B) $v(y') > v(y)$ whenever $y' \in \text{int } Z_v$ and $y \in \text{bdry } Z_v$

then (F$'$) will hold.[6] To also ensure (C), assume preferences v are *strictly monotone* on the interior of Z_v, i.e., for any $z \in \text{int } Z_v$, $z' > z$ implies $v(z') > v(z)$.

Lemma 6.13 *If every $v \in V$ is strictly monotone on the interior of Z_v and satisfies the boundary condition (B), then (C) holds.*

(For a proof see Hildenbrand [1974], Proposition 4, Chapter 2.2.[7]) Combining these observations yields:

Theorem 6.14 (Extended sufficiency condition for existence) *Assume every utility function $v \in V$ is C^1, strictly monotone on the interior of Z_v, and satisfies the boundary condition (B). Then (F$'$) and (C) hold; hence, for any economy $\mathbf{v} \in \mathcal{V}$, there exists a perfectly competitive equilibrium.*

[6] It should again be mentioned that while (B) is convenient, it is also restrictive. In applications, other boundary conditions can be substituted that assure the flatness of $K_{\mathbf{z}}$.

[7] Hildenbrand assumes preferences are strictly monotone everywhere, even on the boundary, which would be inconsistent with the convenient indispensability condition (B). But this difference is of no consequence since, given (B), any Walrasian equilibrium will be interior.

7. Historical and Concluding Remarks

The purpose of this section is to call attention to an historical precedent for the arbitrage approach to competitive equilibrium in the work of Jevons [1879]; to emphasize the Jevonian connection to Edgeworth [1881]; and, to contrast Jevons' formulation with Edgeworth's (the core) and with the arbitrage approach adopted here. It will be argued that our version is a blend of the others and that this blend provides another perspective on the meaning of 'marginalism' and its relation to competitive equilibrium.

Jevons employed a simple and clever argument to derive equilibrium from arbitrage, although his treatment was somewhat casual. A key limitation of Jevons' presentation is the (still) standard one that arbitrage prices could be established independent of utility considerations. Edgeworth responded with a deeper and more ambitious formulation.[8] It was deeper in its examination of the process by which bargains are struck: Edgeworth's was a utility-based form of arbitrage. It also was more ambitious in describing a process that could operates in both competitive and non-competitive environments.[9]

As a utility-based argument for getting at the Law of One Price, our derivation resembles Edgeworth's; but in avoiding any possible implications for imperfect competition and in emphasizing individualistic behavior, it resembles Jevons'. Of course, a single theory applicable to all environments would be superior to one that applies only under perfect competition. But, the implications of the core—notably that a core allocation is always Pareto optimal—call into question its domain of applicability for imperfectly competitive environments. If, however, the core is regarded as an equilibrium condition more appropriate to a competitive setting, then the version of arbitrage adopted in this paper may be interpreted as simply a different tack from the one Edgeworth took for elaborating an arbitrage foundation for competitive equilibrium.

Jevons' Formulation of Equilibrium in Exchange

An important construction in Jevons' theory, as well as a principal source of confusion, is his construction of a 'trading body'.

> By a *trading body* I mean, in the most general manner, any body of buyers

[8] For the writing of *Mathematical Psychics*, Edgeworth's personal contact with Jevons (they were neighbors) as well as Jevons' then recently published second edition [1879] were evidently very influential. See Creedy [1986]. On Jevons, see Schabas [1990].

[9] "The advantage of this general method is that it is applicable to the particular cases of imperfect competition; where the conception of *demand and supply at a price* are no longer appropriate." (p. 31, original italics)

or sellers. The trading body may be a single individual in one case; it may be the whole inhabitants of a continent in another; it may be the individuals of a trade diffused through a country in a third. (p. 88, italics in original)

Commentators such as Stigler [1941, p.17, fn.4], Blaug [1985, p.310], and an anonymous contemporary of Jevons who reviewed his work[10] interpret the two trading bodies in Jevons' theory of exchange as two isolated individuals, but there is considerable evidence that this is not what he intended. (See, below.) Edgeworth [1881] devotes an extensive appendix to Jevons' theory and he pointedly objects to this reading of Jevons as unwarranted.[11]

A brief outline of Jevons' theory is: In a perfect market there are two trading bodies A and B and two commodities 1 and 2. The trading body A has a stock a of the first commodity and B has a stock b of the second commodity. Let x be the quantity of the first commodity given up by A and y the quantity of the second commodity given up by B. Ambiguity arises from interpretating x and y (and a and b) as *total* or as *per capita* quantities—is x supplied by A or by a member of A? Of course, if each trading body were to consist of one individual, there would be no distinction. In our reading, Jevons makes them stand both for total and per capita quantities as needed and, as a result, there is the inevitable confusion of 'the many in the one'.

The first step in Jevons' description of equilibrium in exchange is to demonstrate that arbitrage leads to uniform market prices. He expressed this *Law of Indifference* as the equality

$$(LoI) \qquad \frac{dy}{dx} = \frac{y}{x},$$

where dy/dx is the rate of exchange between infinitesimal units of the commodities. The Law of Indifference ("there cannot be two prices for the same kind of article", p. 92) was well-accepted in Classical economics and was certainly not the point of departure for his new theory; in particular, although he tries to give it a utility underpinning by referring to 'Indifference', his argument for the Law does not rely on more than a trivial application of utility.[12]

[10] *Saturday Review*, 1871.

[11] "It must be carefully remembered that Prof. Jevons' Formulæ of Exchange apply not to bare individuals, an isolated couple, but (as he himself sufficiently indicates, p. 98), to individuals clothed with the properties of a market, a typical couple (see Appendix V.)." Edgeworth [1881, p. 31, fn. 1]. See also Edgeworth, p. 109 and p. 115.

[12] "The principle above expressed is a general law of the utmost importance in Economics, and I propose to call it the Law of Indifference, meaning that, when two objects or commodities are subject to no important difference as the regards the purpose in view, they will either of them by

Jevons put the Law in calculus terminology to prepare the way for what determined the ratio y/x. An individual A will trade until[13]

$$\frac{\mathrm{MU}_1^A(a-x)}{\mathrm{MU}_2^A(y)} = \frac{dy}{dx},$$

while B will trade until

$$\frac{\mathrm{MU}_1^B(x)}{\mathrm{MU}_2^B(b-y)} = \frac{dy}{dx}.$$

Invoking the Law of Indifference, Jevons obtains his two "equations of exchange",

$$(*) \qquad \frac{y}{x} = \frac{\mathrm{MU}_1^A(a-x)}{\mathrm{MU}_2^A(y)} = \frac{\mathrm{MU}_1^B(x)}{\mathrm{MU}_2^B(b-y)}$$

to solve for the two unknowns x and y.

We now reformulate Jevons' equations in terms of our framework. Using a notation adopted above to emphasize the distinction between total and per capita quantities, write $\mathbf{z} = (z_A \cdot \mathbf{1}, z_B \cdot \mathbf{1})$, where $z_A = (-x, y) = -z_B$ and $\mathbf{1}$ is the mass of each (collective) trading body. Then by Lemma 3.1, the no arbitrage cone is $K_{\mathbf{z}} = -[\text{cone } S_A(z_A) + \text{cone } S_B(z_B)]$ with boundary $H_p = \{(d\mathbf{x}, d\mathbf{y}) : p_1 d\mathbf{x} + p_2 d\mathbf{y} = 0\}$. Hence, $\frac{d\mathbf{y}}{d\mathbf{x}}$ represents the terms of trade facing any individual. In place of Jevons' (LoI), we rephrase the no-arbitrage condition in Jevons' setting as,

$$(\mathrm{NA}) \qquad \frac{d\mathbf{y}}{d\mathbf{x}} = \frac{\mathrm{MU}_1^A(a-x)}{\mathrm{MU}_2^A(y)} \cdot \mathbf{1} = \frac{\mathrm{MU}_1^B(x)}{\mathrm{MU}_2^B(b-y)} \cdot \mathbf{1}$$

The MRS's on the right hand side are multiplied by $\mathbf{1}$ since each MRS equals $\frac{dy}{dx}$, an infinitesimal quantity relative to any individual's total trade; while $d\mathbf{x} = dx \cdot \mathbf{1}$ and $d\mathbf{y} = dy \cdot \mathbf{1}$ are of the same order of magnitude as x and y (although infinitesimal relative to economy-wide trading). The formula (NA) thus reflects the message of Figure 2: the boundary of the arbitrage cone, $\frac{d\mathbf{y}}{d\mathbf{x}}$, is determined by the arbitrager making tiny trades with *many* indviduals at terms reflecting each one's MRS; that is, trades leaving each of the arbitrager's trading partners *indifferent* between making or not making the trade.

Now by the ability to *individually recontract*, everyone has the option of withdrawing from the economy and starting over. An individual will take advantage of this option whenever he is losing money at the available terms of trade, i.e., whenever

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taken instead of the other with perfect indifference by a purchaser. Every such act of indifferent choice gives rise to an equation of degrees of utility, so that in this principle of indifference we have one of the central pivots of the theory." (p. 92–93.)

[13]Marginal utilities are functions of only one variable because Jevons used a separable utility function of the form $u(x,y) = f(x) + g(y)$. Throughout this section we follow Jevons' notation, *except* that we use MU_1 and MU_2 in place of his ϕ_1 and ψ_2.

$y/x \neq \mathbf{dy}/\mathbf{dx}$ a member of trading body A or B is losing money. Therefore besides (NA), the remaining condition for equilibrium is that trade be a value-preserving activity,

$$\frac{y}{x} = \frac{\mathbf{dy}}{\mathbf{dx}}.$$

Putting these equations together, our formulae for exchange equilibrium are:

$$(**) \qquad \frac{y}{x} = \frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathrm{MU}_1^A(a-x)}{\mathrm{MU}_2^A(y)} \cdot \mathbf{1} = \frac{\mathrm{MU}_1^B(x)}{\mathrm{MU}_2^B(b-y)} \cdot \mathbf{1}.$$

The similarities between $(*)$ and $(**)$ are evident, but there are also the following important differences:

- The term 'Law of Indifference' is a more accurate description of (NA) than of Jevons (LoI) since (NA) relies on the marginal utility concept of indifference whereas (LoI) ignores such matters.
- Jevons uses only one margin of analysis in his equations (the infinitesimal margin for the individual, dy/dx) whereas the approach adopted here exploits two margins (the infinitesimal margin for the individual, dy/dx, *and* the infinitesimal margin for the economy as a whole, $\frac{\mathbf{dy}}{\mathbf{dx}}$, whose magnitude is of the same order as the total quantities traded by an individual).

To build an argument for arbitrage rather than *tâtonnement* as the equilibrating mechanism for competitive equilibrium, we shall argue below that it is essential to incorporate both of these departures from Jevons. Here, however, we want to point out that our attention to scale considerations—the second of the above two departures from Jevons, was not foreign to him. Rather our reformulation is a clarification to avoid the confusion of the 'many in the one'. Think of $\mathbf{dy}/\mathbf{dx}$ as the terms of trade available to an individual about to enter the economy, i.e., the individual C in the excerpt from Jevons below. Keep in mind our emphasis that the entering individual would be able to trade y for x at the rate $\mathbf{dy}/\mathbf{dx}$ because the members of the economy would be *indifferent* between making or not making such a trade.

> We may, firstly, express the conditions of a great market where vast quantities of some stock are available, so that any one small trader will not appreciably affect the ratio of exchange. This ratio is, then approximately a fixed number, and each trader exchanges just so much as suits him. These circumstances may be represented by supposing A to be a trading body possessing two very large stocks of commodities, a and b. Let C be a person who possesses a comparatively small quantity c of the second commodity, and gives a portion of it, y, which is very small compared

with b. Then, after exchange, we find A in possession of the quantities $a - x$ and $b + y$, and C in possession of x and $c - y$. These equations become

$$\frac{\mathrm{MU}_1^A(a-x)}{\mathrm{MU}_2^A(b+y)} = \frac{y}{x} = \frac{\mathrm{MU}_1^C(x)}{\mathrm{MU}_2^C(c-y)}.$$

Suppose $a - x$ and $b + y$, by supposition, do not appreciably differ from a and b, we may substitute the latter quantities, and we have, for the first equation, approximately,

$$\frac{\mathrm{MU}_1^A(a)}{\mathrm{MU}_2^A(b)} = \frac{y}{x} = m.$$

[p.112]

Except for the difference in boldface notation, the similarities between Jevons' description and our version of the equilibrium position of the individual arbitrager are evident.

Edgeworth's Formulation of Equilibrium in Exchange

The point of departure for Edgeworth's contribution was a more explicit description of the trading body. By postulating that the economy consisted of a definite and *finite* number of individuals, he eliminated the confusion of the 'many in the one'. The finiteness assumption had the express purpose of allowing Edgeworth "to study how far contract is determinate in the case of imperfect competition".

Our notation $\mathbf{x}$ stands for a trading body with members of infinitesimal size. With n individuals in each trading body, we would set $[x_i]^n = x_i \cdot n^{-1}$ to distinguish the many from the one, where x_i is the quantity sold by a member of A and n^{-1} is the size of the individual relative to the trading body; so that for any n the size of the trading body is unity. In contrast, Edgeworth set $[x_i]^n = x_i$ so that the size of each trading body is n.

Edgeworth showed that what we now call core bargaining would lead to

$$\frac{y_i}{x_i} = \frac{y}{x} \qquad \text{and} \qquad \frac{\mathrm{MU}_1^A(a-x_i)}{\mathrm{MU}_2^A(y_i)} = \frac{\mathrm{MU}_1^B(x_i)}{\mathrm{MU}_2^B(b-y_i)}.$$

The first equality is the well-known equal treatment property (assuming an equal number of each type) and the second is the efficiency condition. With small numbers of each type, Edgeworth emphasized that there was nothing to tie the two sets of equalities together and he rejected $(*)$ saying that bargaining was *indeterminate*. He then argued that as the number of traders increased, contracting and recontracting among groups of individuals would cause the two sets of inequalities to merge, i.e., only then would Jevons' equations of exchange be established.

REMARK 4: Edgeworth used his well known Master-Servant Example [1881, p. 46] to illustrate an exception to his overall conclusion that Jevons' equations would be confirmed as the number of traders grows. The example is similar to Example 1 above (except that, for Edgeworth, the source of non- differentiability was indivisibility). It illustrates that even with a large number of individuals there still may remain room for *bargaining* over the terms of trade: "higgling dodges and designing obstinancy, and other incalculable and often disreputable accidents" (p. 46). Rephrased in our terms, even with a continuum of individuals, the arbitrage cone need not be flat.

The Master-Servant example exposes a certain conflict between Edgeworth's view of the relation between competition and large numbers and the modern core equivalence theorem. For the latter, flatness of the arbitrage cone is an unnecessary restriction. However, in the Master-Servant example and other models of the assignment type, the equivalence does not occur because the core shrinks, but because the core expands to fill out the set of Walrasian equilibria. (See Gretsky, Ostroy and Zame [1992].) Edgeworth objected to this form of core equivalence because it violated his position that perfect competition requires *determinacy.* Given this context, the 'emergence of a flat arbitrage cone', as opposed to merely the 'emergence of prices' (core equivalence), seems to capture more of what we mean by perfect competition.

COMPARISONS: THE PLACE OF MARGINAL UTILITY IN THE THEORY OF VALUE

The key contrast between Jevons' description of arbitrage (his LoI) and the one we have adopted (NA) is that arbitrage is utility-based. Since Jevons' usage is more or less the standard, call it 'arbitrage' in contrast to 'entrepreneurial arbitrage', the kind we rely on. Arbitrage has the appeal that whatever its implications, they are independent of the tastes and endowments of the economy. And it is exactly for this reason that arbitrage is an insufficient foundation for the theory of value since it is the details of tastes and endowments which determine competitive prices. Entrepreneurial arbitrage fills the gap by taking the arbitrage activity one step further: instead of limiting the search for profit opportunities to those based on existing market prices (and the knowledge that everyone prefers more to less), the entrepreneur-arbitrager also searches for opportunities by determining what individuals would be willing to pay for 'innovations', i.e., changes to the status quo. It is this extra step which leads, in a competitive environment, to the arbitrage cone of competitively determined prices.

We now consider the implications of entrepreneurial arbitrage for a marginalist derivation of competitive equilibrium. Marginal utility was, and is still regarded as, the key ingredient of the marginalist revolution: wherever marginal utility enters into the description of equilibrium, that it where attention will be focused. To elaborate on this claim, distinguish between (1) price-taking behavior and (2) entrepreneurial arbitrage behavior. The Walrasian description of equilibrium clearly points to (1) as the building block for equilibrium through the following: (a) Marginal utility is the key to the formation of (price-taking) individual demand and supply schedules which naturally leads to (b) the description of equilibrium as the equality of aggregate demand and supply and to (c) the *tâtonnement* view of the equilibration process. Parts (b) and (c) represent a unified construction driven by (a).

In contrast to this Walrasian scheme, Jevons' description of equilibrium is more ambiguous. For example, although he had a clear enough conception of the consequences of price-taking behavior—i.e., individual demand and supply schedules, Jevons does not employ them in his description of equilibrium, thereby suggesting that arbitrage *might* be an essential component of equilibration. But, because Jevons limits himself to a utility-free description of arbitrage and *only* highlights the role of marginal utility as it appears in price-taking behavior, he has been read as a precursor reaching only the foothills of the mountain scaled by Walras. Even if Jevons were trying to tell a non-*tâtonnement* story, there seems to be no way, other than the price-taking path that Walras took, to go from his utility-free conception of arbitrage to the equilibration of competitive markets.

From our point of view, Edgeworth's contribution was pioneering because he showed that arbitrage could incorporate preferences. This is the basic feature that entrepreneurial arbitrage has in common with the core. Not surprisingly, therefore, both the core and entrepreneurial arbitrage make no use of *tâtonnement*.

The connection that Walras established between marginal utility and competitive equilibrium is one way to proceed, but it is not the only way. Walras exploited only the marginal utility underpinnings of the consequences of perfect competition (i.e., price-taking behavior) rather than the marginal utility underpinnings of perfect competition itself. Although the two margins are not the same, 'marginal utility' underlies both the maximizing behavior of a price-taker and also the maximizing behavior of an entrepreneur arbitrager. From the individual's point of view, the marginal utility underlying price-taking behavior is his own (i.e., $\mathrm{MU}_1/\mathrm{MU}_2$, involving infinitesimal quantities dy/dx), whereas the marginal utility underlying arbitrage behavior is that of the rest of the market (i.e., the boundary of $\acute{K}_{\mathbf{z}} = \mathrm{MU}_1/\mathrm{MU}_2 \cdot \mathbf{1}$, involving quantities $dy/d\mathrm{x}$ that are infinitesimal with respect to the economy as a whole, but not to the individual arbitrager).

The standard interpretation of marginalism identifies it with price-taking behavior; i.e., in standard marginalism, price-taking is 'where the action is'. This reinforces the *tâtonnement* view of equilibration as apparently *the* logical path from marginalism to competitive equilibrium. The entrepreneurial arbitrage approach suggests a reappraisal of the *locus* of marginal utility which highlights another form of marginalism.

The (relocated) significance of marginal utility for the theory of value is: it is *others'* MRS's that determine the boundary of anyone's arbitrage cone. This placement serves to make entrepreneurial arbitrage an *alternative* logical link between marginalism and competitive equilibrium.[14]

The reader will perhaps have already recognized that in our proposal to replace *tâtonnement* by arbitrage as the equilibration story behind competitive equilibrium, one is led to the following conclusion: while demand and supply functions (individual and aggregate) may be essentials of partial equilibrium theory, they are dispensable elements of competitive general equilibrium theory. It is interesting to observe that Walras' ideas on general equilibrium followed a more or less contrary path. Even before he saw how to make marginal utility the engine of his general equilibrium system, Walras had already formulated his conception of general equilibrium in terms of the equality of demand and supply schedules.[15] In this respect, therefore, we may say that the arbitrage approach represents a 'non-Walrasian' formulation of competitive equilibrium in which marginal utility figures even more prominently.

Finally, the fact remains that an arbitrage equilibrium is (modulo the flat cone condition) a Walrasian equilibrium that is explicitly situated in a thick markets environment, where it has historically been understood to belong. Does this coincidence mean that there are no practical distinctions to be drawn between the arbitrage and Walrasian versions of competitive equilibrium? In our view, there is an important difference which goes back to the first paragraph of this paper on the way we think about competitive behavior. The Walrasian tradition of price-taking reinforces the view that the perfect competitor responds passively to his environment whereas in the arbitrage approach the perfect competitor is actively opportunistic. The difference in 'psychology' between the competitor-as-price-taker vs. the competitor-as-arbitrager are alternative perspectives which significantly influence the way one interprets market behavior.

[14] At a deeper level, the importance of marginal utility for the competitive theory of value is illustrative rather than absolutely basic. For example, when commodities are indivisible (or, more generally, when choices are on the boundary of the feasible trading set so that derivatives are not so clearly defined), the principles of perfect competition continue to be represented by an arbitrage cone, but one that is not necessarily defined by marginal utilities. In modern versions of Walrasian equilibrium (e.g., Debreu [1959]), differentiability properties of individual utility/production functions are also eliminated, but for different reasons; they are not required for the existence of the price-taking definition of equilibrium.

[15] For this development of Walras' ideas, see Jaffe [1976] (reprinted in Walker [1983]), where he concludes: "Instead of climbing up from marginal utility to the level of his general equilibrium system, Walras actually climbed down from that level to marginal utility."

APPENDIX

A. PROOF OF THE EXTENDED ARBITRAGE LEMMA AND ITS COROLLARY

6.1 Extended Arbitrage Lemma

(i) $\hat{K}_{\mathbf{z}} = \text{cl (cone } K_{\mathbf{z}})$ = -cl $(\cup_{G \in \mathcal{G}} [\sum_{t \in G} \text{cone } S_t(z_t)])$, *a convex cone containing the origin.*

Further,

(ii) $\hat{K}_{\mathbf{z}} = \text{cl } K_{\mathbf{z}}$ *if all* $v \in V$ *are quasi-concave.*

Proof: The proof is divided into a series of steps. Steps (1)-(2) are preliminary; (3)-4) prove the first claim of the Lemma; (5) proves the second claim.

(1) $y \in K_{\mathbf{z}}$ implies $ny \in \text{cl } K_{\mathbf{z}}$ for any integer $n \geq 1$.

Proof: By assumption, $y = \sum_{t \in G} y_t$, where $-y_t \in S_t(z_t)$ and $(v_t, z_t) \in \text{supp } \mu_{\mathbf{z}}$. For any $t \in G$, monotonicity of preferences implies that for each $\delta > 0$ there is an $\epsilon > 0$ such that

$$v(-y_t + \delta e) \geq v(z) \quad \text{for all } (v, z) \in B_\epsilon(v_t, z_t),$$

where e denotes the unit vector and B_ϵ denotes an open ϵ-ball around the given point. Since $(v_t, z_t) \in \text{supp } \mu_{\mathbf{z}}$, $\mathbf{z}$ has a nonnull set of subtypes in $B_\epsilon(v_t, z_t)$. Hence $n(y_t - \delta e) \in K_{\mathbf{z}}$ for any $\delta > 0$ and each $t \in G$. Letting $\delta \to 0$, we conclude that $ny \in \text{cl } K_{\mathbf{z}}$. □

(2) $y \in \text{cone } K_{\mathbf{z}}$ implies that for any $\epsilon > 0$ there is a y' such that $\| y - y' \| < \epsilon$ and $ny' \in \text{cl } K_{\mathbf{z}}$ for some integer $n \geq 1$.

Proof: Since cone $K_{\mathbf{z}}$ = ray (conv $K_{\mathbf{z}}$), by Carathéodory's Theorem $y = \sum_{k=1}^{\ell+1} \alpha_k y_k$, where $\alpha \in \mathbf{R}_+^{\ell+1}$, $y_k \in K_{\mathbf{z}}$. If all the α_k are rational numbers then there is an integer $n > 0$ such that $ny = \sum n\alpha_k y_k$ with each $n\alpha_k$ an integer. Hence, by (2) above, $n\alpha_k y_k \in \text{cl } K_{\mathbf{z}}$ and, therefore, $ny = \sum n\alpha_k y_k \in \text{cl } K_{\mathbf{z}}$. If instead some of the α_k are irrational then for all $\epsilon > 0$ there is an α' close to α such that all the α'_k are rational and $\| y - y' \| < \epsilon$, where $y' = \sum \alpha'_k y_k$. Then, as above, $ny' \in \text{cl } K_{\mathbf{z}}$ for some integer $n > 0$. □

(3) $\hat{K}_{\mathbf{z}} = \text{cl (cone } K_{\mathbf{z}})$

Proof: By (2), $y \in \text{cone } K_{\mathbf{z}}$ implies $ny' \in \text{cl } K_{\mathbf{z}}$ for some y' arbitrarily close to y and $n \geq 1$. By construction of $\hat{K}_{\mathbf{z}}$, $ny' \in \hat{K}_{\mathbf{z}}$, and thus $y' = \frac{1}{n}(ny') \in \hat{K}_{\mathbf{z}}$ for y' arbitrarily close to y. Since $\hat{K}_{\mathbf{z}}$ is closed, we conclude that $y \in \hat{K}_{\mathbf{z}}$; i.e., cone $K_{\mathbf{z}} \subset \hat{K}_{\mathbf{z}}$. Or, since $\hat{K}_{\mathbf{z}}$ is closed, cl (cone $K_{\mathbf{z}}) \subset \hat{K}_{\mathbf{z}}$.

Conversely, $y \in \hat{K}_{\mathbf{z}}$ implies there is a sequence $y_m \to y$, where $y_m = \alpha_m y'_m$ for some $y'_m \in K_{\mathbf{z}}$ and $\alpha_m \in [0,1]$. Since each $y_m \in$ cone $K_{\mathbf{z}}$, $y \in$ cl (cone $K_{\mathbf{z}}$); i.e., $\hat{K}_{\mathbf{z}} \subset$ cl (cone $K_{\mathbf{z}}$). □

(4) cone $K_{\mathbf{z}} = - \cup_{G \in \mathcal{G}} \sum_{t \in G}$ cone $S_t(z_t)$

Proof: $y \in K_{\mathbf{z}}$ implies $-y = \sum_{t \in G} -y_t \in \sum_{t \in G}$ cone $S_t(z_t)$ for some $G \in \mathcal{G}$. That is,

$$K_{\mathbf{z}} \subset - \cup_{G \in \mathcal{G}} \sum_{t \in G} \text{cone } S_t(z_t) = - \cup_{G \in \mathcal{G}} C_G,$$

where $C_G \equiv \sum_{t \in G}$ cone $S_t(z_t)$. Since the sum of convex cones is a convex cone, each C_G is a convex cone. And since the union of cones is a cone, $\cup_{G \in \mathcal{G}} C_G$ is a cone. To verify it is convex, let it contain y_1 and y_2. Hence, $y_1 \in C_{G_1}$ and $y_2 \in C_{G_2}$ for some G_1, $G_2 \in \mathcal{G}$. Note $G_1 \cup G_2 \in \mathcal{G}$, so $C_3 \equiv \sum_{t \in G_1 \cup G_2}$ cone $S_t(z_t) \subset \cup_{G \in \mathcal{G}} C_G$. Further, C_3 contains both C_1 and C_2 (since $0 \in$ cone $S_t(z_t)$ for all $t \in G_1 \cup G_2$); in particular, y_1 and y_2 are in C_3. Hence, $\alpha y_1 + (1-\alpha) y_2 \in C_3$ for all $\alpha \in [0,1]$, since C_3 is convex. So, $\alpha y_1 + (1-\alpha) y_2 \in \cup_{G \in \mathcal{G}} C_G$. We conclude that cone $K_{\mathbf{z}} \subset - \cup_{G \in \mathcal{G}} C_G$ since the latter is a convex cone.

Conversely, observe that $-S_t(z_t) \subset K_{\mathbf{z}}$ for all t such that $(v_t, z_t) \in$ supp $\mu_{\mathbf{z}}$. Also, $-\sum_{t \in G} S_t(z_t) \subset K_{\mathbf{z}}$ for all $G \in \mathcal{G}$. Hence, -cone $\sum_{t \in G} S_t(z_t) = -\sum_{t \in G}$ cone $S_t(z_t) \equiv -C_G \subset$ cone $K_{\mathbf{z}}$ for all $G \in \mathcal{G}$. So, $- \cup_{G \in \mathcal{G}} C_G \subset$ cone $K_{\mathbf{z}}$. □

(5) $\hat{K}_{\mathbf{z}} =$ cl $K_{\mathbf{z}}$ if all $v \in V$ are quasi-concave

Proof: Clearly cl $K_{\mathbf{z}} \subset \hat{K}_{\mathbf{z}}$, so it will suffice to show that the latter is a subset of the former. If $y' \in K_{\mathbf{z}}$ then $y' = \sum_{t \in G} y_t$, $-y_t \in S_t(z_t)$. Hence, since $\alpha y' = \sum_{t \in G} \alpha y_t$ and each $-\alpha y_t \in S_t(z_t)$ by convexity, $\alpha y' \in K_{\mathbf{z}}$ for any $\alpha \in [0,1]$. Now $y \in \hat{K}_{\mathbf{z}}$ implies there is a sequence $y_n \to y$, where $y_n = \alpha_n y'_n$ for some $y'_n \in K_{\mathbf{z}}$ and $\alpha_n \in [0,1]$. By the above, each $y_n \in K_{\mathbf{z}}$; hence, $y \in$ cl $K_{\mathbf{z}}$. We conclude that $\hat{K}_{\mathbf{z}} \subset$ cl $K_{\mathbf{z}}$, as was to be shown. □

□

Corollary 6.2

(i) $K_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$ *if and only if* $\hat{K}_{\mathbf{z}} \cap \mathbf{R}^{\ell}_{++} = \emptyset$.

(ii) $N(K_{\mathbf{z}}) = \hat{K}^{\circ}_{\mathbf{z}} = -N(S_{\mathbf{z}})$.

(iii) $\hat{K}_{\mathbf{z}} = -$cl (cone $S_{\mathbf{z}}$).

Proof: (i): Since $K_{\mathbf{z}} \subset \hat{K}_{\mathbf{z}}$, the "if" direction is immediate. For the converse, suppose $y \in \hat{K}_{\mathbf{z}}$ and $y \gg 0$. Then there is a sequence $y_n \to y$, where $y_n = \alpha_n y'_n$ for some $y'_n \in K_{\mathbf{z}}$ and $\alpha_n \in [0,1]$. Since $y \gg 0$, $y'_n \gg 0$ for n sufficiently large.

(ii): First we show $N(K_{\mathbf{z}}) = \hat{K}^\circ_{\mathbf{z}}$. Since $K_{\mathbf{z}} \subset \hat{K}_{\mathbf{z}}$, clearly $p\hat{K}_{\mathbf{z}} \leq 0$ implies $pK_{\mathbf{z}} \leq 0$. Conversely, $pK_{\mathbf{z}} \leq 0$ implies $p \cdot \text{cl (cone } K_{\mathbf{z}}) \leq 0$ since cl (cone $K_{\mathbf{z}}$) $\subset \{y : py \leq 0\}$, another closed convex cone. Hence, by (i) of the Extended Arbitrage Lemma, $p\hat{K}_{\mathbf{z}} \leq 0$.

To show $N(K_{\mathbf{z}}) = -N(S_{\mathbf{z}})$, suppose first that $pS_{\mathbf{z}} \geq 0$. Then, by a standard argument, $pS_t(z_t) \geq 0$ for all $t \in G$ and all $G \in \mathcal{G}$; hence $p \cdot \text{cone } S_t(z_t) \geq 0$ for all $t \in G$ and all $G \in \mathcal{G}$. That is, cone $S_t(z_t) \subset \{y : py \geq 0\}$ for all $t \in G$ and all $G \in \mathcal{G}$; hence $\sum_{t \in G} \text{cone } S_t(z_t) \subset \{y : py \geq 0\}$ for all $G \in \mathcal{G}$ or $\cup_{G \in \mathcal{G}} \sum_{t \in G} \text{cone } S_t(z_t) \subset \{y : py \geq 0\}$. But by (i) of the Extended Arbitrage Lemma, the latter just equals -$K_{\mathbf{z}}$; hence $pK_{\mathbf{z}} \leq 0$.

Conversely, $pK_{\mathbf{z}} \leq 0$ implies $p[\cup_{G \in \mathcal{G}} \sum_{t \in G} \text{cone } S_t(z_t)] \geq 0$. Hence

$$p \sum_{t \in G} \text{cone } S_t(z_t) \geq 0$$

for all $G \in \mathcal{G}$ or $p \cdot \text{cone } S_t(z_t) \geq 0$ for all $t \in G$ and all $G \in \mathcal{G}$ since $0 \in \text{cone } S_t(z_t)$ for all t. We conclude that $pS_t(z_t) \geq 0$ for all representative t. Hence $p \int S_t(z_t) d\lambda(t) \geq 0$, i.e. $pS_{\mathbf{z}} \geq 0$.

(iii): Since for any convex cone S, $(S^\circ)^\circ = \text{cl } S$, (iii) now follows readily from (ii). □

B. Proof of the Representation Theorem and its Corollary

6.10 Representation Theorem for Convexified Preferences $u^* : \mathbf{R}^\ell_+ \to \mathbf{R}$ *is a well-defined (i.e., sup $R(x) < \infty$ for all x), quasi-concave, weakly monotone and continuous function, wtih $u^*(0) = u(0)$. Specifically, for any $x \in \mathbf{R}^\ell_+$:*

$$A^*(x) = \text{conv } A(r_x e)$$

and

$$u^*(x) = \max_{(x_k)_{k=1,\ldots,\ell+1}} \{u(x_1) : u(x_1) = \ldots = u(x_{\ell+1}),\ x = \sum_{k=1}^{\ell+1} \alpha_k x_k \ \&\ \alpha \in \Delta^\ell\}.$$

Proof: The proof is divided into a series of 6 steps. Step (1) establishes that u^* is well-defined; (2) is preliminary to proving (3), that u^* is quasi-concave, in particular it satisfies the next to last property of the theorem; (4) establishes weak monotonicity and that $u^*(0) = u(0)$; (5) proves continuity; finally, (6) proves the last property of u^* in the theorem.

(1) For all x, $\sup R(x) < \infty$.

Proof: Suppose the contrary. By monotonicity, the sets $A(er)$ are nested, i.e., $r' > r$ implies $A(r'e) \subset A(re)$. Hence, $\sup R(x) = \infty$ implies $x \in \text{conv } A(re)$ for all r. So, there is a sequence $r^n \to \infty$ with

$$x = \sum_{k=1}^{\ell+1} \alpha_k^n x_k^n \quad \text{for each } n,$$

where $u(x_k^n) \geq u(r^n e)$ and $\alpha^n \in \Delta^\ell$. Let $k(n) = \text{argmax } \{a_k^n : k = 1, \ldots \ \ell+1\}$; note $\alpha_{k(n)}^n \geq \frac{1}{\ell+1}$. Since $r^n \to \infty$, $\{x_{k(n)}^n\}$ is not bounded. Hence $\{\alpha_{k(n)}^n x_{k(n)}^n\}$ is not bounded. But for all n and all k, x_k^n is bounded from below by 0. This contradicts $\sum_{k=1}^{\ell+1} \alpha_k^n x_k^n = x$ for all n. □

(2) For all x, $x \in \text{bdry } (\text{conv } A(r_x e))$

Proof: If $x \in \text{int } (\text{conv } A(r_x e))$ then a contradiction of $r_x = \sup R(x)$ would result. To show $x \in \text{conv } A(r_x e)$, observe that, by definition of r_x, there is a sequence $r^n \to r_x$ from below with

$$x = \sum_{k=1}^{\ell+1} \alpha_k^n x_k^n \quad \text{for each } n,$$

where $u(x_k^n) \geq u(r^n e)$ and $\alpha^n \in \Delta^\ell$. Let $\alpha^n \to \alpha$ on a subsequence; and let k_1 (respectively, k_0) index the k for which $\alpha_k > 0$ ($\alpha_k = 0$). Then for each k_1, there is an x_{k_1} such that $x_{k_1}^n \to x_{k_1}$ on a subsequence (otherwise $x = \sum_k \alpha_k^n x_k^n$ for all n would be contradicted). Similarly, there is an $x_o \geq 0$ such that $\sum_{k_0} \alpha_{k_0}^n x_{k_0}^n \to x_o$ on a subsequence. So, $x = \sum_{k_1} \alpha_{k_1} x_{k_1} + x_0$. Since $x_0 \geq 0$, there is an $\overline{x}_{k_1} \geq x_{k_1}$ such that $x = \sum_{k_1} \alpha_{k_1} \overline{x}_{k_1}$. Now $r^n \to r_x$ implies that for each k_1 $u(\overline{x}_{k_1}) \geq u(r_x e)$, i.e., $\overline{x}_{k_1} \in A(r_x e)$. Thus, since $\sum_{k_1} \alpha_{k_1} = 1$, $x \in \text{conv } A(r_x e)$. □

(3) For all x, $A^*(x) = \text{conv } A(r_x e)$.

Proof: By (2), if $u^*(x') \geq u^*(x)$ then $x' \in \text{conv } A(r'e)$, $r' \geq r_x$. Hence $x' \in \text{conv } A(r_x e)$. Conversely, if $x' \in \text{conv } A(r_x e)$ then $u^*(x') \geq u(r_x e) = u^*(x)$ □

(4) u^* is weakly monotone, with $u^*(0) = u(0)$.

Proof: Let $x' > x$. By (2),

$$x = \sum_{k=1}^{n} \alpha_k x_k$$

with $u(x_k) \geq u(r_x e)$, $\alpha_k > 0$, $\sum_{k=1}^n \alpha_k = 1$. Let $y = x' - x > 0$, and let $x_k' = x_k + \alpha_k y$. By construction $x' = \sum \alpha_k x_k'$; and by monotonicity, for each k, $u(x_k') \geq u(r_x e)$, with strict inequality of $x' \gg x$. Hence $x' \in \text{conv } A(r_x e)$; and $x' \in \text{conv } A(r'e)$ for some $r' > r_x$ if $x' \gg x$. That is, $u^*(x') \geq u^*(x)$; and $u^*(x') > u^*(x)$ if $x' \gg x$.

To show $u^*(0) = u(0)$, observe that by monotonicity $R(0) = \{0\}$. Hence the conclusion follows. □

(5) u^* is continuous.

Proof: Assume $x^n \to x$ but $u^*(x^n) \not\to u^*(x)$. We show a contradiction.

Since for all n sufficiently large $x^n < x+e$, monotonicity implies for all n sufficiently large $\{u^*(x^n)\}$ is bounded. Hence there is a $\overline{u}$ such that $u^*(x^n) \to \overline{u}$ on a subsequence, say $s(n)$, with $\overline{u} \neq u^*(x)$.

Let $r^n = r_{x^n}$ and let $\overline{r} = \lim r^{s(n)}$ (hence $\overline{u} = u(\overline{r}e)$). By (2), for all n

$$x^n = \sum_{k=1}^{\ell+1} \alpha_k^n x_k^n$$

with $u(x_k^n) \geq u(r^n e)$ and $\alpha^n \in \Delta^\ell$. As in the proof of (2), there are numbers α, x_{k_1}, and x_0 such that

$$\begin{gathered} \alpha^n \to \alpha \in \Delta^\ell \text{ on a subsequence of } s(n), \\ x_{k_1}^n \to x_{k_1} \text{ on a subsequence of } s(n) \text{ (for each } k_1\text{), and} \\ \sum_{k_0} \alpha_{k_0}^n x_{k_0}^n \to x_0 \geq 0 \text{ on a subsequence of } s(n). \end{gathered}$$

Further, since $x = \lim x^n$,

$$x = \sum_{k_1} \alpha_{k_1} x_{k_1} + x_0$$

or

$$x = \sum_{k_1} \alpha_{k_1} \overline{x}_{k_1}$$

for some $(\overline{x}_{k_1}) \subset A(\overline{r}e)$. (Note each $x_{k_1} \in A(\overline{r}e)$ since $u(x_{k_1}^n) \geq u(r^n e)$ and $r^{s(n)} \to \overline{r}$.) Hence $u^*(x) \geq u(\overline{r}e) = \overline{u}$.

To complete the proof, it suffices to show that $u^*(x) \not> \overline{u}$. By (2),

$$x = \sum_{k=1}^{m} \beta_k y_k$$

with $u(y_k) \geq u^*(x)$, $\beta_k > 0$, $\sum_{k=1}^m \beta_k = 1$. Without loss of generality, we can suppose $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_m$. For each commodity h ($h = 1, \ldots, \ell$), let

$$x^h(\epsilon) = \begin{cases} x^h - \epsilon & \text{if } x^h \geq \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

Since $x^n \to x$, monotonicity implies that for all $\epsilon > 0$ there is an integer $N(\epsilon)$ such that

$$u^*(x(\epsilon)) \leq u^*(x^n) \quad \text{for all } n > N(\epsilon).$$

Adjust the y_k's downward so that the adjusted y_k's, say $y_k(\epsilon)$, satisfy

$$\begin{aligned} x(\epsilon) &= \sum_{k=1}^{m} \beta_k y_k(\epsilon), \\ y_k(\epsilon) &\geq 0 \text{ for all } k, \text{ and} \\ \| y_k(\epsilon) - y_k \| &\leq \| \frac{\epsilon e}{\beta_1} \| \quad \text{for all } k. \end{aligned}$$

(The worst case is that $x - x(\epsilon) = \epsilon e$ and that the k with the smallest β (i.e., $k = 1$) takes all the adjustment, which results in the above bound.)

As $\epsilon \to 0$, each $y_k(\epsilon) \to y_k$. Hence $u^*(x(\epsilon)) \to u^*(x)$. Since $u^*(x(\epsilon)) \leq u^*(x^n)$ for all n sufficiently large, we conclude that $\lim u^*(x^{s(n)}) = \overline{u} \geq u^*(x)$, as was to be shown. □

(6) For all x, $u^*(x) = \max_{(x_k)_{k=1,\ldots,\ell+1}}\{u(x_1) : u(x_1) = \ldots = u(x_{\ell+1}),\ x = \sum_{k=1}^{\ell+1} \alpha_k x_k \;\&\; \alpha \in \Delta^\ell\}$.

Proof: It will suffice to show that for all x, $x = \sum_{k=1}^{\ell+1} \alpha_k x_k$ where $u(x_k) = u(r_x e)$ for each k and $\alpha \in \Delta^\ell$. (If there exists an $(\overline{x}_k)$ such that $u(\overline{x}_1) > u^*(x)$ with $u(\overline{x}_1) = \ldots = u(\overline{x}_{\ell+1})$, $x = \sum \overline{\alpha}_k \overline{x}_k$, and $\overline{\alpha} \in \Delta^\ell$, then there would be a contradiction of $u^*(x) = u(r_x e)$ where $r_x = \sup R(x)$.)

By (2), $x = \sum_{k=1}^{n} \alpha_k x_k$ where $u(x_k) \geq u(r_x e)$, $\alpha_k > 0$, $\sum_{k=1}^{n} \alpha_k = 1$, $n \leq \ell + 1$. Let i (respectively, j') index those k for which $u(x_k) = u(r_x e)$ $(u(x_k) > u(r_x e))$; and let $\#i$ and $\#j'$ represent the number of i's and j''s, respectively. If $\#i = 0$ then $r_x = \sup R(x)$ would be contradicted; so we can take $\#i > 0$.

Suppose $\#j' > 0$. Pick a j', say j, and consider taking $(1-\beta)x_j$ from j and giving each i some additional goods along the ray βx_j $(\beta \geq 0)$. Let $\beta_j = \inf\{\beta \in [0,1] : u((1-\beta)x_j) = u(r_x e)\}$; hence $u((1-\beta_j)x_j) = u(r_x e)$. Let $\beta_i = \sup\{\beta \in [0,\infty) : u(x_i + \beta x_j) = u(r_x e)\}$.

There are two possibilities, either (i) $\sum \alpha_i \beta_i \geq \alpha_j \beta_j$ or (ii) $\sum \alpha_i \beta_i < \alpha_j \beta_j$. If (i), then there exists (x'_k) such that $x'_k = x_k$ for all $j' \neq j$; $x'_j = (1-\beta_j)x_j$; and $x'_i = x_i + \beta'_i x_j$ for each i, where $\sum \alpha_i \beta'_i = \alpha_j \beta_j$. Hence, by construction, $\sum \alpha_k x'_k = x$ and $\#k$ such that $u(x'_k) > u(r_x e)$ has been reduced to $\#j' - 1$. If instead (ii) holds, then there exist (x'_k) such that $x'_k = x_k$ for all $j' \neq j$; $x'_j = (1-\beta'_j)x_j$; and $x'_i = x_i + \beta'_i x_j$ for each i; where $\beta'_j < \beta_j$, $\beta'_i > \beta_i$, and $\sum \alpha_i \beta'_i = \alpha_j \beta_j$. By construction, $\sum \alpha_k x'_k = x$ and $u_k(x'_k) > u(r_x e)$ for all k, contradicting $r_x = \sup R(x)$. Hence (i) must hold. That is, we can construct an (x'_k) that reduced $\#j'$ by 1.

Repeating the construction starting from (x'_k) instead of from (x_k), will reduce $\#j'$ by 2. Hence, after a finite number of repetitions, we will arrive at an $(\overline{x}_k)$ such that $\sum \alpha_k \overline{x}_k = x$ and $u(\overline{x}_k) = u(r_x e)$ for all k, as required. □

□

Corollary 6.11 *Assume $v : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ satisfies (A.1-3). Then there exists a function $v^* : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ satisfying (A.1-4), with $Z_{v^*} = Z_v$. The function v^* is a numerical representation of the convexification of the preferences in v. Specifically, for any $z \in Z_{v^*}$, there are $(z_k)_{k=1,\ldots,\ell+1} \subset Z_v$ such that*

(i) $z = \sum_{k=1}^{\ell+1} \alpha_k z_k$ *for some* $(\alpha_k) \in \Delta^\ell$,

(ii) $v(z_1) = \ldots = v(z_{\ell+1}) \geq v^*(z)$, *and*

(iii) $S_{v^*}(z) + \{z\} = \text{conv } S_v(z_k) + \{z_k\}$ *(for any k)* .

Proof: Define $u : \mathbf{R}^\ell_+ \to \mathbf{R}$ by $u(x) = v(x - \omega_v)$. Hence, u is continuous and weakly monotone. Now define $v_0^* : \mathbf{R}^\ell \to \mathbf{R} \cup \{-\infty\}$ by $v_0^*(z) = u^*(z + \omega_v)$ for all $z \in Z_v$, $v_0^*(z) = -\infty$ for all $z \notin Z_v$. By construction $Z_{v_0^*} = Z_v$; and, by Theorem 6.10, v_0^* satisfies (A.1-4)—except v_0^* may not equal 0. To verify that v_0^* also satisfies (i)-(iii), observe that for any given trade $z \in Z_{v_0^*}$, the corresponding consumption is $x = z + \omega_v$. By Theorem 6.10, this $x = \sum_{k=1}^{\ell+1} \alpha_k x_k$, where $u^*(x) = u(x_1) = \ldots = u(x_{\ell+1})$, $\alpha \in \Delta^\ell$, and $A^*(x) = \text{conv } A(x_k)$ (for any k). Letting $z_k = x_k - \omega_v$, it immediately follows that $(z_k) \subset Z_v$ and satisfies (i)-(iii) ((ii) with equality).

Thus, if $v_0^*(0) = 0$ then v_0^* satisfies all the requirements of the Corollary. If not, re-normalize. That is, let $v^* = v_0^* - v_0^*(0)$. Since $v_0^*(0) = u^*(\omega_v) \geq u(\omega_v) = v(0) = 0$, v^* satisfies all the requirements, including (ii). □

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