KERNELS OF REPLICATED MARKET GAMES

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1 Introduction

Our subject is replicated TU economies and the kernels of the cooperative games that they generate. The kernel is a multi-bilateral bargaining equilibrium; it contains the nucleolus and is contained in most bargaining sets. Two players are considered to be in equilibrium if at the given outcome they split their combined payoff equally between what they could claim on the basis of marginal worth to coalitions containing one but not the other. To qualify for the kernel, an outcome must have *all* pairs of players in equilibrium simultaneously. It is somewhat remarkable that the kernel is never empty.

The kernel has a special relationship to the core. Any core point is at the center of a (possibly degenerate) "asterisk" of line segments in the core, radiating in all directions of possible two-player transfers. A core point is a member of the kernel if and only if it *bisects* all the line segments that make up its "asterisk." Despite the fact that the core provides at most $n-1$ dimensions in which to satisfy up to $\binom{n}{2}$ conditions, there is always at least one point that does this, assuming of course that the core is not empty.[1] Indeed, the nucleolus has this property. Unlike the nucleolus, however, kernel points can also appear outside the core, even when the core is not empty.

Although the kernel often turns out to contain just one point, it is generally a formidable task to verify that there are no other points.[2] It is therefore pleasant to be able to report here on a class of games that have played a historical role in economic theory—the replicated market games—in which the kernels are easy to work with and yet exhibit very interesting behavior.

[0]This paper grew out of joint work with Martin Shubik, begun in the 1970's and supported then by both the National Science Foundation and the Office of Naval Research, but not heretofore published. It was presented at an International Conference on Game Theory at SUNY Stony Brook in July 1990.

[1]This bisection property also holds for any nonempty *strong ε-core*; see Maschler, Peleg and Shapley (1979).

[2]Even in the case of *convex* games, where most cooperative solutions shed their complexities under the benign influence of an all-powerful core, the kernel remains a "hard nut" to crack. Though it's easily shown that the kernel of a convex game is in the core, the additional fact that it is a single point requires a most elaborate argument (Maschler, Peleg, Shapley (1972)).

Just as the nucleolus was first presented as a way of explaining the nonemptiness of the kernel, the kernel began as a way of proving the nonemptiness of a certain bargaining set.[3] Our example in §6 of a nonconvergent kernel in a replication sequence of TU markets therefore serves as a counterexample to the convergence of certain bargaining sets as well. The counterexample will be seen to depend on the piecewise-linearity of the underlying utility functions and may be compared with the more positive results that have been obtained for smooth utility functions.[4]

Recognizing that many readers will be unfamiliar with the kernel concept, we have taken this opportunity to review many of its basic properties along the way.

2 Definitions and Notation

We shall follow, with a few exceptions, the terminology and notation of Maschler, Peleg and Shapley (1979).

Let N be a finite set of size $n = |N|$, and let $\mathcal{N}$ stand for its power set 2^N. It will be convenient to define[5]

$$\mathcal{N}_i =: \{S \in \mathcal{N} : i \in S\}, \qquad \text{all } i \in N, \text{ and}$$

$$\mathcal{N}_{i\setminus j} =: \{S \in \mathcal{N} : i \in S,\, j \notin S\}, \qquad \text{all } i,j \in N,\ i \neq j.$$

By a *game*[6] Γ we shall mean an ordered pair (N, v), where v is a function from $\mathcal{N}$ to the real numbers $\mathbf{R}$ such that $v(\emptyset) = 0$. We call v (or Γ) *monotonic* if

$$S \supseteq T \implies v(S) \geq v(T), \qquad \text{all } S,T \in \mathcal{N},$$

and *0-monotonic* if v_0 is monotonic, where v_0 is the "0-normalization" of v :

$$v_0(S) =: v(S) - \textstyle\sum_{i\in S} v(\{i\}), \qquad \text{all } S \in \mathcal{N}.$$

We note that in the standard interpretation of the characteristic function v is automatically superadditive, and hence 0-monotonic. As customary in this context, $\mathbf{R}^N$ will denote the n-dimensional product space $\mathbf{R}^n$ with coordinates indexed by $i \in N$. If $x \in \mathbf{R}^N$, then "x" will be used both for the real vector $(x_i : i \in N)$ and the additive set function $x(S) =: \sum_{i\in S} x_i$.

Given $\Gamma = (N, v)$, certain basic subsets of $\mathbf{R}^N$ are

$${}_{\mathrm{p}}\mathrm{X}(\Gamma) =: \{x \in \mathbf{R}^N : x(N) = v(N)\}, \qquad \text{(the \textit{pre-imputations} of } \Gamma)$$

$$\mathrm{X}(\Gamma) =: \{x \in {}_{\mathrm{p}}\mathrm{X}(\Gamma) : x_i \geq v(\{i\}),\ \text{all } i \in N\}, \qquad \text{(the \textit{imputations} of } \Gamma)$$

$$\mathrm{C}(\Gamma) =: \{x \in \mathrm{X}(\Gamma) : x(S) \geq v(S),\ \text{all } S \in \mathcal{N}\}. \qquad \text{(the \textit{core} of } \Gamma)$$

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[3]See Schmeidler (1969) for the nucleolus; Maschler (1966), Maschler and Peleg (1966), and Davis and Maschler (1965) for the kernel.

[4]Shapley and Shubik, in Shubik (1984; App. B); see also Mas-Colell (1989).

[5]The symbol "=:" indicates a definition.

[6]More fully, a "cooperative TU game in characteristic-function form."

(The argument "Γ" will be omitted when there is no danger of confusion.)

We now introduce the sequence of concepts leading to the "multi-bilateral" equilibrium. First the *excess* of S at x:

(2.1) $$e(S,x) =: v(S) - x(S), \qquad \text{all } S \in \mathcal{N},\ x \in {}_{\mathbf{p}}\mathbf{X};$$

then the *surplus of i against j* at x:

(2.2) $$\sigma_{ij}(x) =: \max_{S \in \mathcal{N}_{i \setminus j}} e(S,x), \qquad \text{all } i,j \in N,\ i \neq j;$$

then the *pre-kernel*:

(2.3) $${}_{\mathbf{p}}\mathbf{K}(\Gamma) =: \{ x \in {}_{\mathbf{p}}\mathbf{X} : \sigma_{ij}(x) = \sigma_{ji}(x),\ \text{all } i,j \in N,\ i \neq j \};$$

and finally, the *kernel* itself:

(2.4) $$\begin{aligned} \mathbf{K}(\Gamma) =: \{ x \in \mathbf{X} :\ & \text{for each } i,j \in N,\ i \neq j, \ldots \\ & \underline{\text{either}}\ \sigma_{ij}(x) = \sigma_{ji}(x), \\ & \underline{\text{or}}\ \sigma_{ij}(x) < \sigma_{ji}(x) \text{ and } x_i = v(\{i\}), \\ & \underline{\text{or}}\ \sigma_{ij}(x) > \sigma_{ji}(x) \text{ and } x_j i = v(\{j\}) \}. \end{aligned}$$

The last definition—though historically correct—is needlessly complicated in the standard interpretation, in view of Proposition 2 below. Indeed, (2.3) is increasingly being used now as the definition of $\mathbf{K}$, since for non 0-monotonic games the consideration of "individual rationality" that motivates (2.4) has no particular appeal.

The following results are basic to the theory of the kernel:[7]

Proposition 1. If $\mathbf{X} \neq \emptyset$ then $\mathbf{K} \neq \emptyset$ and $\mathbf{K} \subseteq \mathbf{X}$.

Proposition 2. If Γ is 0-monotonic, then $\mathbf{K}(\Gamma) = {}_{\mathbf{p}}\mathbf{K}(\Gamma)$.

Proposition 3. If $\mathbf{C} \neq \emptyset$ then $\mathbf{C} \cap \mathbf{K} \neq \emptyset$.

The next proposition sets forth the "bisection" property, which is helpful in both visualizing and calculating the kernel. Let $x \in \mathbf{C}$, and for any $i \neq j$ let $\mathbf{L}_{ij}(x)$ be the line that passes through x in the "i-j direction," i.e., the line in ${}_{\mathbf{p}}\mathbf{X}$ along which only x_i and x_j are allowed to vary.

Proposition 4. If $x \in \mathbf{C}$, then $x \in \mathbf{K}$ if and only if x bisects the segment $\mathbf{C} \cap \mathbf{L}_{ij}(x)$ for each $i \neq j$. (If $\mathbf{C} \cap \mathbf{L}_{ij}(x)$ is a single point, then x is that point.)

Figure 1 shows a kernel point bisecting the "asterisk" of line segments that pass through it in the six possible directions. No other core point in Figure 1 enjoys this

[7]See e.g. Maschler, Peleg and Shapley (1979).

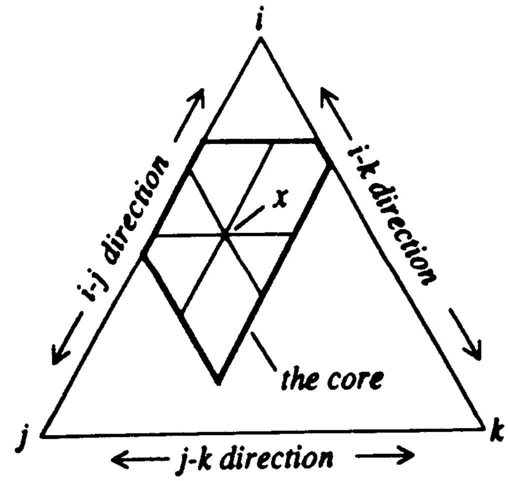


Figure 1. Illustrating the bisection property.

property, and indeed, as in all 3-person games, the kernel is a singleton. For larger n, however, there can be many core points with the bisection property. Indeed, we shall soon be dealing with a class of games in which *all* points in the core are multi-bilateral bisectors, and there may be kernel points outside the core as well.

For those who may be puzzled by the bisection property's success in imposing up to $(n^2-n)/2$ constraints on the kernel when the core itself is at most $(n-1)$-dimensional, the following remarks may provide some clues: (1) the constraints are actually not linear but piecewise linear; (2) the boundary faces of the core are severely restricted in the directions of their normals, since the coefficients of the defining inequalities are all 0 or 1 (note the parallel faces in Figure 1); (3) if $n \geq 4$ and the dimension of the core is less than $n-2$, it is quite possible for the core to be skewed in such a way that none of the transfer lines L_{ij} intersect it in more than a point. Then all the asterisks are trivial, and we have $\mathbf{K} \supseteq \mathbf{C}$. We shall soon see that this is a common occurrence in replicated market games.[8]

3 Symmetries, Types and Profiles

By a *symmetry* of a game, we shall mean any permutation of the players that leaves the characteristic function unchanged. Thus, if $\Gamma = (N, v)$ and $\pi : N \leftrightarrow N$ is a permutation of the players, then the symmetries of Γ comprise the set

$$\Pi = \Pi(\Gamma) =: \{\pi : \pi v = v\},$$

where πv is defined by $(\pi v)(S) =: v(\pi S) =: v(\{\pi(i) : i \in S\})$. Π is obviously a group under composition: $(\pi \cdot \rho)v =: \pi(\rho v)$.

While two players i, j will be called *symmetric* if $\pi(i) = j$ for some $\pi \in \Pi$, the stronger relationship of "perfect substitutes" is also important; it requires that the simple transposition of i and j—i.e., leaving the other players fixed—is itself an

[8]See Proposition 7.

element of Π. Both symmetry and substitutability are equivalence relations[9] and therefore induce partitions of N; we shall call them *symmetry classes* and *substitution classes* respectively, the latter being in general a refinement of the former.

For an example, let $N = \{1,2,3,4\}$ and let $v(S)$ be 1 if S is $\{1,2\}$, $\{2,3\}$, $\{3,4\}$ or $\{1,4\}$, 2 if $|S| = 3$ or 4, and 0 otherwise. Then $\Pi(v)$ is the familiar group of the square (with 1 and 3 sitting at opposite corners). All six pairs of players are symmetric, but players 1 and 2, for example, are not substitutes. The unique symmetry class is N, while the substitution classes are $\{1,3\}$ and $\{2,4\}$.

Given Γ, we may distinguish two special classes of pre-imputations:

$$_{\mathbf{p}}\mathbf{X}_{\text{sym}} =: \{x \in {}_{\mathbf{p}}\mathbf{X} : x_i = x_j \text{ whenever } i \text{ and } j \text{ are symmetric in } \Gamma\},$$

$$_{\mathbf{p}}\mathbf{X}_{\text{subs}} =: \{x \in {}_{\mathbf{p}}\mathbf{X} : x_i = x_j \text{ whenever } i \text{ and } j \text{ are substitutes in } \Gamma\}.$$

We observe that always $_{\mathbf{p}}\mathbf{X}_{\text{sym}} \subseteq {}_{\mathbf{p}}\mathbf{X}_{\text{subs}}$. (In the example above, $(1,0,1,0)$ is in $_{\mathbf{p}}\mathbf{X}_{\text{subs}}$ but not $_{\mathbf{p}}\mathbf{X}_{\text{sym}}$.) Of course if Γ has no symmetries except the identity permutation, then $_{\mathbf{p}}\mathbf{X}_{\text{sym}} = {}_{\mathbf{p}}\mathbf{X}_{\text{subs}} = {}_{\mathbf{p}}\mathbf{X}$. We shall write $\mathbf{X}_{\text{sym}}$ for $\mathbf{X} \cap {}_{\mathbf{p}}\mathbf{X}_{\text{sym}}$ and $\mathbf{X}_{\text{subs}}$ for $\mathbf{X} \cap {}_{\mathbf{p}}\mathbf{X}_{\text{subs}}$.

The following is an easily proved but important property of the kernel:

Proposition 5. $\mathbf{K}(\Gamma) \subseteq \mathbf{X}_{\text{subs}}(\Gamma)$.

In other words, every kernel point gives "equal treatment" to all members of each substitution class. This is not true in general for core points, nor is it generally true that kernel points give equal treatment to players who are symmetric but not substitutes.[10]

We shall be dealing in the sequel with games whose players can be classified into *types*, a type being any nonempty subset of a substitution class. We define a *regular type-partition of order k* to be any refinement of the substitution partition with the property that all types have size k. Thus, even though a game may have many substitution classes of different sizes, it may not have any regular type-partitions beyond the trivial $k = 1$. In fact, it is easily seen that the order of any regular type-partition must be a common divisor of all the substitution-class sizes.

It would appear, then, that games with nontrivial regular type-partitions are rare in practice. Nevertheless they are of much interest in economics because of their connection with the procedure of "replication," which traditionally has served as an important point of entry into the study of large economies. The present note continues that tradition.

[9]To verify the transitivity of substitution, let $(ijk...m)$ denote the *cyclic permutation* in which i takes j's seat, j takes k's seat ..., and m takes i's seat. Then $(ij)=(ji)$ is just the transposition of i and j. We must show that if (ij) and (jk) are in Π then so is (ik). But $(ij)\cdot(jk)$ (read "(jk), then (ij)") is (ikj), which is a symmetry but not a transposition. Another application of (jk) does the trick, however, since $(jk)\cdot(ikj) = (ik)$.

[10]But see Proposition 6 below. Of course, both the kernel and the core—*as sets*—enjoy the full symmetry of the game. It follows that in the case of a one-point kernel or a one-point core, all members of each *symmetry* class get equal treatment.

Let $\{N_t : t = 1, \ldots, m\}$ be a partition of N into m types of size k. To each $S \in \mathcal{N}$ we associate an integer-valued m-vector $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_m)$, called the *profile* of S, where $\mathbf{s}_t = |S \cap N_t|$. Because the members of a type are substitutes, the characteristic function on coalitions can be replaced by a simpler, "characterizing" function on profiles, defined by the identity

$$v(S) \;=\; \phi(\mathbf{s}). \tag{3.1}$$

The precise domain of ϕ is the cubic lattice of integer points $\mathbf{s} \in \mathbf{R}^m_+$ with $\mathbf{0} \le \mathbf{s} \le \mathbf{k}$, where $\mathbf{0}$ denotes $(0, 0, \ldots 0)$ and $\mathbf{k}$ denotes $(k, k, \ldots k)$.

4 TU Markets and Market Games

Following Shapley and Shubik (1969, 1975), a *TU market* $\mathrm{M} = (T, G, a, u)$ is composed of (1) a finite set T of *traders*, (2) a finite set G of *commodities* with the associated *consumption set* $\mathbf{R}^G_+$, (3) a family $a = \{a^t\}_{t \in T} \subset \mathbf{R}^G_+$ of *initial bundles*, and (4) a family $u = \{u^t\}_{t \in T}$ of *utility functions*, $u^t : \mathbf{R}^G_+ \to \mathbf{R}(t)$, assumed to be continuous, concave, and nondecreasing. (Here the $\mathbf{R}(t)$ are copies of $\mathbf{R}$ representing the individual *utility scales* of the respective traders t.)

The "TU" designation means that utility is treated as an additional, money-like good, valuable in itself and freely transferable, but serving also as a common unit of measurement for the $\mathbf{R}(t)$. To obtain the final utility levels or "payoffs" of the traders we take their utilities of consumption $u^t(z^t)$, where $z^t \in \mathbf{R}^G_+$ is t's final bundle, and add their net balances of transfered utility.[11]

The market $\mathrm{M} = (T, G, a, u)$ generates a cooperative TU game $\Gamma = (T, v)$ in a very natural way. First, define an *S-allocation* to be any $x^S = \{x^i \in \mathbf{R}^G_+ : i \in S\}$, and call it *feasible* if

$$\sum_{i \in S} x^i \;=\; \sum_{i \in S} a^i. \tag{4.1}$$

Then, for each nonempty $S \subseteq T$ define[12]

$$v(S) \;=:\; \max\left\{\sum_{i \in S} u^i(x^i) : x^S \text{ is a feasible } S\text{-allocation}\right\}. \tag{4.2}$$

Finally, set $v(\emptyset) = 0$. It is easy to see that v is automatically superadditive.

By a *TU market game* we shall mean any $\Gamma = (T, v)$ that can be derived from a TU market $\mathrm{M} = (T, G, a, u)$ in this way. These games have been widely studied, a central result being that the TU market games are precisely those that are "totally balanced," a property which, among other things, assures a non-empty core.[13]

[11] An early statement of the role of transferable utility in economic theory, applicable to the present context, will be found in Shapley and Shubik (1966; pp. 807-808).

[12] Note that transfered utility does not enter into the definition. No help for S can be expected from outside, and utility transfers among members of S do not affect the total.

[13] Shapley and Shubik (1969, 1975); see also Shapley (1967), Scarf (1967), Billera (1970), Billera and Bixby (1973) (who apply Rader's (1972) "Principle of Equivalence" between production economies and pure exchange economies), Mas-Colell (1975), Hart (1982), Qin (1991).

A *replicated* TU market has a special structure. We may imagine constructing one by starting with any TU market M, making k exact copies, arranging them side by side on a map, then finally erasing all boundaries and barriers to create a "common market," denoted by ^{k}M. If we start with $|T| = n$ traders in M then we end up with $|{}^kT| = kn$ traders in ^{k}M; moreover ${}^kT$ will admit a regular type-partition $\{T_1, \ldots, T_n\}$ of order k. Of course, this type-partition applies equally to the *market* ^{k}M and the associated TU market *game*, which we denote by ${}^k\Gamma$.[14]

It is not difficult to see that ${}^k\Gamma = ({}^kT, {}^kv)$ is positively homogeneous in the following, discrete sense: let $S \in {}^kT$, and let ${}^hS$ be any coalition in ${}^kT$ having exactly h times as many traders of each type as S; then ${}^kv({}^hS) = h\,{}^kv(S)$.

This suggests that in dealing with games of the form ${}^k\Gamma$ we should speak of *profiles* rather than coalitions, and introduce an alternative characterizing function ϕ, as in (3.1), defined on the lattice of integer n-vectors between $\mathbf{0}$ and $\mathbf{k}$, inclusive. The homogeneity property then has a more familiar appearance:

$$\phi(h\mathbf{s}) = h\phi(\mathbf{s}), \tag{4.3}$$

for all positive integers h such that $\mathbf{0} \leq h\mathbf{s} \leq \mathbf{k}$.

Referring to (4.2), we see that ϕ is given by

$$\phi(\mathbf{s}) = \max_x \sum_{t \in T} \mathbf{s}_t u^t(x^t), \tag{4.4}$$

where x runs over n-dimensional space of the feasible *type-symmetric* allocations—i.e., sets of bundles $\{x^t\}_{t \in T}$ such that $\sum_{t\in T} s_t x^t = \sum_{t \in T} s_t a^t$, since the concavity of the u^t ensures that the maximum over all allocations will be achieved at an equal-treatment allocation. To see this, observe that if two members of S are of the same type but receive different bundles, then the total utility to S will not be decreased if they both are given the average of the two bundles.

We see also that the natural superadditivity of ${}^kv$ implies that ϕ is (discretely) *concave*, in the sense that if $p\mathbf{r} + (1-p)\mathbf{s}$ is an integer vector for any $\mathbf{r} \leq \mathbf{k}$, $\mathbf{s} \leq \mathbf{k}$ and $0 \leq p \leq 1$, then $\phi(p\mathbf{r} + (1-p)\mathbf{s}) \geq p\phi(\mathbf{r}) + (1-p)\phi(\mathbf{s})$.

We now state the well-known "equal treatment" property of core points in k-replicated economies, $k \geq 2$. With its aid we shall establish an important fact about the kernel in the TU case.

> **Proposition 6.** Let $x \in \mathrm{C}({}^k\Gamma)$, $k \geq 2$. Then if i and j are members of the same type, we have $x_i = x_j$.

The proof consists in forming an n-player coalition S with profile $\mathbf{s} = (1, \ldots, 1)$ by selecting a worst-treated player of each type. Then in the absence of equal treatment, S can improve.

> **Proposition 7.** If $k \geq 2$, then $\mathrm{K}({}^k\Gamma) \supseteq \mathrm{C}({}^k\Gamma)$.

[14] We do not define the replication of games *per se*. Indeed, two TU markets that generate the same game can easily have k-replications that generate different games.

Proof. In defining the asterisk at any $x \in C(^k\Gamma)$, only two players can depart from their x coordinates at the same time—one up and one down. Thus, the equal treatment that exists at x by Proposition 6 is necessarily spoiled by any movement away from x on a transfer line $L_{ij}(x)$. So all core points have trivial asterisks, and the result follows at once by Proposition 4.

It will be convenient now to extend the domain of the "characterizing" function ϕ to a continuous, nondecreasing, concave, positively homogeneous function $\bar{\phi}$ on the full positive orthant $\mathbf{R}^n_+$. We begin by releasing the h of (4.3) from its integer constraint, thereby extending ϕ to all the rays in $\mathbf{R}^n_+$ that hit lattice points in the k-cube. If k is allowed to increase indefinitely these "rational rays" become dense in $\mathbf{R}^n_+$, and an appeal to continuity completes the extension.

Proposition 8. Given a TU market $M = (T, G, a, u)$ together with its k-replications, $k = 2, 3, \ldots$, there is a unique continuous function $\bar{\phi} : \mathbf{R}^n_+ \to \mathbf{R}$ such that the characteristic function ^kv of $^k\Gamma(M)$ is given by

$$^kv(S) = \bar{\phi}(\mathbf{s}), \qquad \text{all } S \in {}^kT;$$

moreover, $\bar{\phi}$ is positively homogeneous, nondecreasing, and concave.

The converse is also of interest:

Proposition 9. Every continuous, concave, nondecreasing, positively homogeneous function $\bar{\phi} : \mathbf{R}^n_+ \to \mathbf{R}$ is represented by some TU market $M(T, G, a, u)$ together with its k-replications.

Proof. An easy way to set up an $M(T, G, a, u)$ with the desired properties is to take $|T| = |G| = n$ and match trader types to commodities.[15] Initially endow each member of type t with one unit of good t, and nothing else, so that a^i for i in type t is just the t-th unit vector of $\mathbf{R}^G_+$. The utility functions u^t are then defined to be equal to each other and to the given function $\bar{\phi}$:

$$u^t(x_1, \ldots x_n) \equiv u^o(x_1, \ldots x_n) = \bar{\phi}(x_1, \ldots x_n), \qquad \text{all } t \in T. \tag{4.5}$$

Under this set-up, the total bundle available to a coalition to distribute is exactly its profile. Concavity of u^o implies that total utility is maximized for any S is attained by an equal split, each player turning in his endowment for the bundle $\mathbf{s}/|S|$. This of course yields $v(S)$ or $^kv(S)$ equal to $\phi(\mathbf{s})$, as in (4.4).

Of course with the positive homogeneity of (4.5), the maximum is attained by by other, nonsymmetric distributions—for example, by giving the whole of $\mathbf{s}$ to one player. But homogeneity of u^o is not at all essential to the construction. It suffices that u^o be concave, continuous and nondecreasing, and agree with $\bar{\phi} + u^o(0)$ on the simplex formed by the convex hull of the unit vectors of $\mathbf{R}^G_+$. These remarks are easily verified.

[15]Cf. the *direct market of a game*, utilized by Shapley and Shubik (1969, 1975).

5 The Kernel for Two Types.

We now restrict ourselves to the case $n = 2$ and consider the sequence of $2k$-person games ${}^k\Gamma$ that derive from a fixed continuous, concave, positively homogeneous function $\bar{\phi}$ on $\mathbf{R}^2_+$, restricted for each ${}^k\Gamma$ to the profiles $\mathbf{s}$ in the kxk-square $\mathbf{0} \leq \mathbf{s} \leq \mathbf{k}$.

Consider an equal-treatment imputation of ${}^k\Gamma$:

$$(5.1) \qquad (a, \ldots, a, b, \ldots, b) \quad \text{or, for short,} \quad (a; b).$$

where $a \geq \phi(1,0)$, $b \geq \phi(0,1)$ and $a + b = \phi(1,1)$. The excess at $(a;b)$ of any coalition $S \subseteq {}^kT$ with profile $\mathbf{s}$ is given by[16]

$$(5.2) \qquad e(S,(a;b)) \; = \; \varepsilon(\mathbf{s},(a,b)) \; =: \; \phi(\mathbf{s}_1 + \mathbf{s}_2) - (a\mathbf{s}_1 + b\mathbf{s}_2)$$

(cf. (2.1)), and the surplus of any $i \in T_1$ against any $j \in T_2$ is given by

$$(5.3) \qquad \sigma_{12}((a,b)) \; =: \; \max_{\mathbf{s} \in \mathrm{D}_{12}} \{\varepsilon(\mathbf{s},(a,b))\}$$

where

$$(5.4) \qquad \mathrm{D}_{12} \; =: \; \{\mathbf{s} : 1 \leq \mathbf{s}_1 \leq k \text{ and } 0 \leq \mathbf{s}_2 \leq k - 1\}$$

(cf. (2.2)), with corresponding expressions defining $\sigma_{21}((a,b))$ and D_{21}. Of course, if the two players are of the same type their surpluses against each other are automatically equal, since the maximization problems they face are the same.

So in order for $(a;b)$ to be in the kernel, it is necessary and sufficient that

$$(5.5) \qquad \sigma_{12}((a,b)) \; = \; \sigma_{21}((a,b)),$$

by (5.3), (2.3) and Proposition 2. Our problem therefore reduces to maximizing ε over the two finite domains D_{12} and D_{21} and comparing the results (Figure 2).

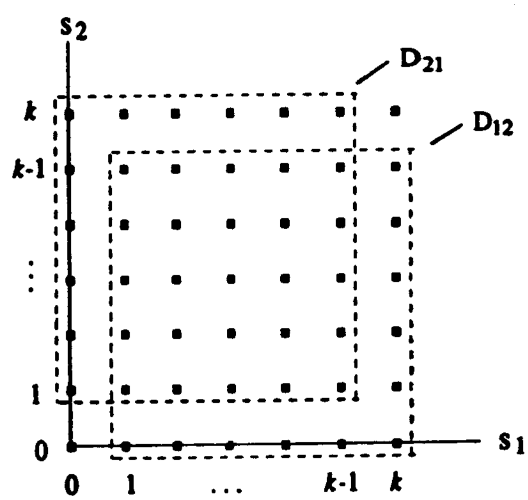


Figure 2. Illustrating the squares D_{12} and D_{21}.

[16]Note that "$(a;b)$" denotes a point in $\mathbf{R}^{2k}$, "(a,b)" a point in $\mathbf{R}^2$.

If we extend ϕ and ε to functions $\hat{\phi}$, $\hat{\varepsilon}$ on the full $k \times k$-square by replacing $\mathbf{s}_1$ and $\mathbf{s}_2$ by real variables s_1 and s_2 in $[0, k]$, we see that $\hat{\varepsilon}$ is just another continuous, concave, positively homogeneous function for each (a, b), resembling ϕ but with the added property that it is 0 along the diagonal $s_1 = s_2$.

Figure 3 below shows the positive and negative regions of a typical $\hat{\varepsilon}$ for $k = 6$. Because of the concavity there can be at most one "+" sector; moreover, there can be no "0" sector of positive area if a "+" sector exists. If there is no "+" sector, the maximum of $\varepsilon(\mathbf{s}, (a, b))$ is 0 in both D_{12} and D_{21} and so $(a; b)$ is in the core *and* the kernel.[17]

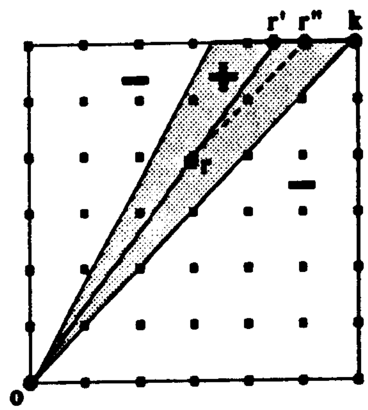


Figure 3. An excess function in the $k \times k$-square.

Attempting to construct kernel points $(a; b)$ outside the core, we choose (a, b) so that the "+" sector of $\hat{\varepsilon}(\mathbf{s}, (a, b))$ contains at least one lattice point, like $\langle k-1, k\rangle$ in Figure 3, say. Then $\max \varepsilon(\mathbf{s})$[18] is positive in $D_{21} \backslash D_{12}$ but negative in $D_{12} \backslash D_{21}$. So the only way that (5.5) can hold is if $\max \varepsilon(\mathbf{s})$ over $\mathbf{s} \in D_{12} \cup D_{21}$ is attained at some $\mathbf{r} \in D_{12} \cap D_{21}$.

Let $\mu > 0$ be that maximum, and let it be attained at a lattice point $\mathbf{r}$ with $r_1 < r_2 \le k-1$, like $\langle 3, 4\rangle$ in Figure 3. Extend the ray $0\mathbf{r}$ to the boundary of the square at $\mathbf{r}' = \langle r_1 k/r_2, k\rangle$. By homogeneity, $\hat{\varepsilon}(\mathbf{r}') = k\mu/r_2 > \mu$. So $r_1 k/r_2$ cannot be an integer. Indeed, there must be no lattice points on the segment $\mathbf{r}\mathbf{r}'$ other than $\mathbf{r}$ itself. But note that since $\mathbf{r}_1 < \mathbf{r}_2$, there is at least one boundary lattice point strictly between $\mathbf{r}'$ and the corner $\mathbf{k} = \langle k, k\rangle$, namely $\mathbf{r}'' =: \langle k - (r_2 - r_1), k\rangle$. So concavity of $\hat{\varepsilon}$ entails that

$$\hat{\varepsilon}(\mathbf{r}'') \ \ge\ \frac{k - \mathbf{r}_1''}{k - \mathbf{r}_1'}\,\hat{\varepsilon}(\mathbf{r}') + \frac{\mathbf{r}_1'' - \mathbf{r}_1'}{k - \mathbf{r}_1'}\,\hat{\varepsilon}(\mathbf{k}').$$

Observing that the triangles $\mathbf{r}\mathbf{r}'\mathbf{r}''$ and $0\mathbf{r}'\mathbf{k}$ are similar, we have at once

$$(5.6) \qquad \hat{\varepsilon}(\mathbf{r}'') \ \ge\ \frac{\mathbf{r}_2}{k}\frac{k\mu}{\mathbf{r}} + \frac{k - \mathbf{r}_2}{k}0 \ =\ \mu.$$

[17]Indeed, $(a; b)$ is a CE payoff of the underlying market. A sharper condition for core membership is that the "+" sector contain no lattice points. Thus, as k increases we get a glimpse of how and why the core is generally larger than the CE set but closes in on it as $k \to \infty$. (The "shrinkage" takes place in $\mathbf{R}^2$, however, not $\mathbf{R}^{2k}$.)

[18]We shall suppress the argument (a, b) until it is needed again.

Since μ is the maximum on lattice points, equality holds in (5.6), and it follows that the triangle $\mathbf{0kr'}$ represents a "flat" sector of $\hat{\varepsilon}$ (i.e., linear, not level). Hence $\hat{\varepsilon}$ is identically μ along the line $\mathbf{rr''}$ which runs parallel to the diagonal.

In short, a lattice point in $D_{12} \cap D_{21}$ that is maximal in $D_{12} \cup D_{21}$ *can* be constructed, but only just barely, by making a considerable portion of $\hat{\varepsilon}$ linear.

What more is demanded of $\hat{\varepsilon}$ (and hence $\bar{\phi}$) to ensure that $(a; b)$ has the kernel property? Not much. To the right of the "flat" sector, concavity keeps $\hat{\varepsilon}$ safely negative. To the left, however, $\hat{\varepsilon}$ must drop off rather quickly, so as not to exceed μ at any lattice point. But apart from this condition, which is easily fulfilled, there is nothing more to assume. Indeed, other than the need for a flat region adjacent to the diagonal $\mathbf{0k}$, the existence of kernel points outside the core is a robust phenomenon. Let us make two further observations:

(1) The kernel is readily seen to be convex. If the $(a; b)$ of Figure 3 is thought of as a plane in $\mathbf{R}^3$ that contains the line $\mathbf{0k}$ in $\mathbf{R}^2$, then consider decreasing a and increasing b by the same amount.[19] This will tilt it to the right, and so by (5.2) cause the graph of $\hat{\varepsilon}$ to tilt to the left, turning on the $\mathbf{0k}$ hinge, until the "flat" becomes level and the "+" sector disappears. This means that $(a; b)$ has entered the core. But throughout the tilting process the point $\mathbf{r}$ continues to maximize $\varepsilon(\mathbf{s})$ on both D_{12} and D_{21}, even as $\mu \to 0$. So the kernel extends all the way from the original $(a; b)$ to the core.

(2) Due to the absence of requirements on $\hat{\varepsilon}$ on the negative side of $\mathbf{0k}$, there is nothing to prevent a similar construction in that region as well, resulting in another segment of the kernel protruding from the core in the opposite direction. If the two "flats" are given the same slope then the core will be a single point in the interior of $\mathbf{K}$. But if they are given different slopes there will be a "crease" along $\mathbf{0k}$, and the core will be a closed line segment interior to $\mathbf{K}$, representing the family of planes that support $\phi(\mathbf{s})$ along the crease.

We expect that the situation described for $n = 2$ carries over in most respects to larger n. The core, at least, is always convex and generically $(n-1)$-dimensional, and it appears that the kernel too will be convex.

6 A Nonconvergent Example

We now apply the analysis of §5 to a special case. The characterizing function, $\bar{\phi} : \mathbf{R}^2_+ \to \mathbf{R}$ is deceptively simple:

$$(6.1) \qquad \bar{\phi}(s) = \min\{s_1, \alpha s_2\},$$

where α is a parameter > 1; $\bar{\phi}$ is of course also the utility function for all traders in the two-commodity "direct market" that represents ϕ.[20] This piecewise linear function has two flat sectors joined along a ray R whose slope is $1/\alpha$, as depicted

[19] Recall that $a + b$ is a constant.

[20] See the proof of Proposition 9.

in Figure 4(a) for $\alpha = 2$. (The small numbers are the values of $\bar{\phi}$ on the L-shaped contours.) By (5.2), we have

$$\begin{aligned}\bar{\varepsilon}(s, (a,b)) &= \min\{s_1 - as_1 - bs_2,\ \ \alpha s_2 - as_1 - bs_2\} \\ &= bs_1 - bs_2 + \min\{0,\ -s_1 + \alpha s_2\},\end{aligned}$$

(eliminating a by the identity $a + b = \bar{\varepsilon}(1,1) = 1$). Figure 4(b) indicates the signed sectors of $\bar{\varepsilon}$ for $(a,b) = (.2,.8)$ with $\alpha = 2$; the shading is parallel to the contours.

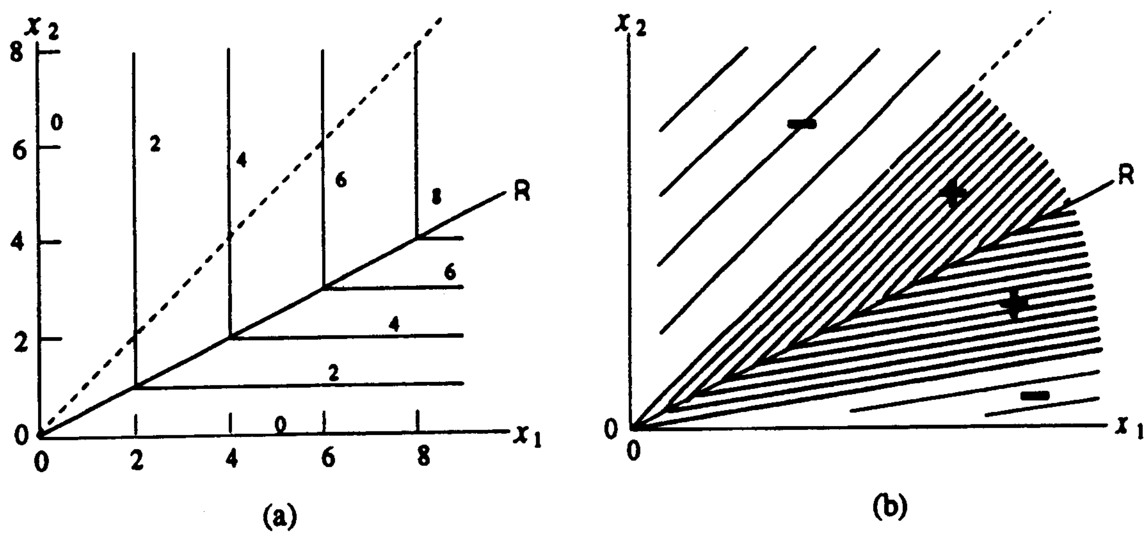


Figure 4. The extended functions $\bar{\phi}$ and $\bar{\varepsilon}$.

We note that the type-symmetric imputations form a line segment in $\mathbf{R}^{2n}$:

$$\mathbf{X}_{sym} = [(1;0),(0;1)] = \{(a;b) : 0 \le a = 1 - b \le 1\},$$

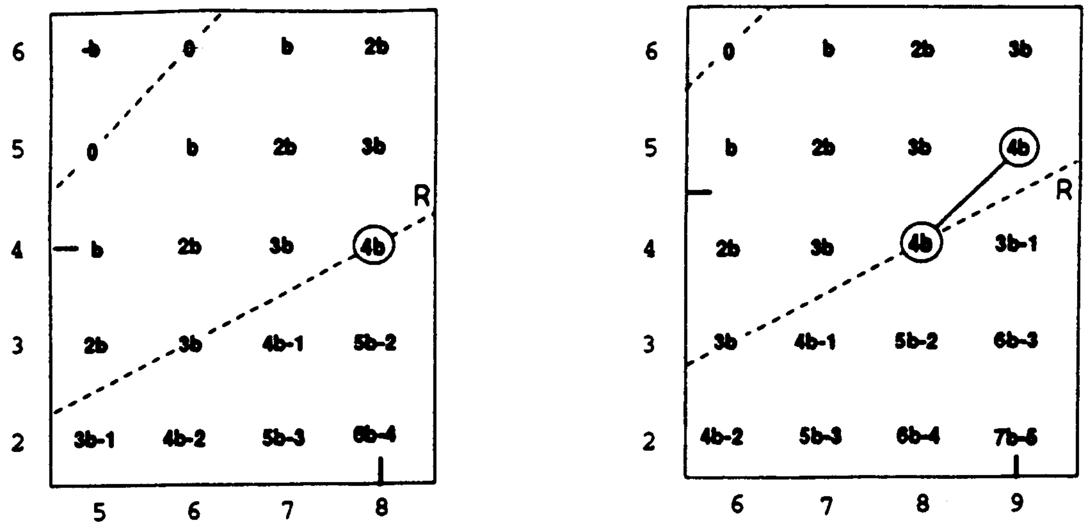
while the core $\mathbf{C}$ consists of the single point $(1;\ 0)$, which is situated at one end of $\mathbf{X}_{sym}$. As we saw in §5, $\mathbf{K}$ is a closed line segment (or point) containing $\mathbf{C}$ and contained in $\mathbf{X}_{sym}$.[21]

We now fix $\alpha = 2$ and determine $\mathbf{K}$ for all values of k. Figure 5 (next page) graphically tabulates the $\varepsilon(\mathbf{s})$ values at lattice points in the critical region around $(k, k/2)$. The parity of k now becomes critical.

If k happens to be even, then $\varepsilon(\mathbf{s})$ takes its maximum in D_{12} at $\langle k, k/2\rangle$ and—if $b > 0$—nowhere else in D_{12}. This means that we can never equate the two surpluses: $\sigma_{12} = \sigma_{21}$ unless $(a;b)$ is the core point $(1;0)$. So, by Proposition 7, $\mathbf{K} = \mathbf{C}$.

If k is odd, however, the situation is quite different. From Figure 5(b) we see that $\varepsilon(\mathbf{s})$ is maximized at both $\langle k, (k+1)/2\rangle$ and $\langle k-1, (k-1)/2\rangle$. So $\sigma_{12} = \sigma_{21}$ throughout the entire range, and we have $\mathbf{K} = \mathbf{X}_{sym}$.

[21]Note that in the present example, the type partition *is* the symmetry partition.



(a) $k = 8$, unique maximum. (b) $k = 9$, two maxima.

Figure 5. Excesses in the neighborhood of $\langle k, k/2 \rangle$.

To summarize: The kernel flips in and out of the core according to whether the replication number is even or odd. Viewed in the "type" space (i.e., $\mathbf{R}^2$), the oscillation is exact:

$$[(1,0),\ (0,1)] \text{ for } k = 1,3,5,\ldots, \qquad \{(1,0)\} \text{ for } k = 2,4,6,\ldots.$$

Viewed in the payoff space, or rather, the increasing sequence of payoff spaces of dimension $kn-1$, the even-odd oscillation is at least as bad. Indeed, depending on the metric or sequence of metrics employed, the diameter of the "odd" kernels may even grow without bound—e.g., like $\sqrt{k}$ if the Euclidean norm is used. Certainly in this case the kernel cannot be said to "shrink" to the core.[22]

It is worth remarking that $\alpha = 2$ is not a special case. In fact, any real number $\alpha > 1$ will yield double maxima as in Figure 5(b) for infinitely many k. If α is rational, the ray R will pass through a lattice point at regular intervals, and a periodic pattern of $\mathbf{C}$ kernels and $\mathbf{X}_{sym}$ kernels will result. If α is irrational, the ray will not hit any lattice points and the oscillation will be irregular. Nevertheless, infinitely many kernels of each kind will occur.

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[22]Similar metric considerations arise in connection with the "shrinking" of both the classical core (see e.g. Shapley (1975)) and the bargaining set B treated by Shapley and Shubik (see Shubik (1984)).

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