# ODD MAN OUT:

## Bargaining Among Three Players

by

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*This paper considers three-player bargaining problems in which only the two-player coalitions can achieve any gains. The paper presents a cooperative model (the multilateral Nash solution of Bennett (1991a)) and a noncooperative model (the proposal-making model with discounting), and characterizes the cooperative and noncooperative solutions. These solutions are mutually reinforcing: For certain parameter values, the cooperative and noncooperative solutions coincide. For the remaining parameter values, the cooperative solutions form an interval; the noncooperative solution is one endpoint of this interval, and Binmore's (1985) market demand solution is the other endpoint of this interval.*

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## 1. INTRODUCTION

Much of the modern theory of bargaining has its origins in a series of papers by John Nash (1950, 1951, 1953). In these papers, Nash presents cooperative and noncooperative approaches to the two-person simple bargaining problem. His cooperative (axiomatic) approach lists desirable characteristics for a solution, and asks if any solution has these characteristics. His noncooperative approach sets out a particular bargaining procedure (a demand game), and asks what outcomes would result from rational, self-interested actions of individuals (i.e., as a "Nash" equilibrium). Nash showed that his axioms characterize a unique cooperative solution (the Nash bargaining solution), and that this is an equilibrium outcome of the demand game. Nash went on to argue, on the basis of equilibrium selection, that the cooperative Nash bargaining solution is the "correct" equilibrium outcome.

Nash viewed the cooperative and noncooperative approaches as complementary. It is typically difficult to assess the reasonableness of the intuition (axioms) underlying a cooperative solution, or the range of situations to which it applies, without having a specific bargaining procedure in mind. A cooperative solution is in serious doubt if it is not compatible with some sensible noncooperative bargaining procedure. Conversely, a noncooperative bargaining procedure is not likely to be sensible if its outcomes are not supported by the intuition of some cooperative solution. The search for such mutually reinforcing cooperative and noncooperative bargaining solutions has come to be called the "Nash program".

The Nash program was advanced by Rubinstein (1982), who presented a different noncooperative model for the two-person simple bargaining problem. In Rubinstein's model, the two players alternate making offers; the first accepted offer ends the game, and the players obtain the agreed-upon payoffs. Making an offer takes time, and players are impatient (and so discount future payoffs). Rubinstein showed that this game has a unique subgame perfect equilibrium outcome. Moreover, if players discount future payoffs at the same rate, then as the discount factor converges to 1, this outcome converges to the Nash bargaining solution.

This paper continues the Nash program by extending the Nash bargaining solution and the Rubinstein alternating offer model to a class of three-player bargaining problems that Binmore (1985) dubbed "three-player/three-cake" problems. In the simple bargaining problem studied by Nash, a pair of players bargain over the division of a "cake"; i.e., the choice of a utility vector from a set of attainable utility vectors. In the three-player problems we consider, each pair has a cake to divide, but at most one of these three cakes can be divided.[1] Individuals acting alone, and the coalition of all three players have no cake to divide, and no sidepayments of any kind are possible from any pair to the third player. Since only two of the three players will succeed in forming a coalition, we refer to these bargaining situations as "Odd Man Out". These are the simplest bargaining problems in which each player's "outside option" is to bargain in another coalition. Our interest in these problems is primarily as a first step in the analysis of the general bargaining problem with many players and many coalitions. However, there are certainly real situations of interest that can naturally be modeled as three-player/three-cake problems. For instance, after an election when none of three contending parties obtains a majority of votes, and the three parties together cannot agree on essential policies, the problem of forming a government (deciding on policies, allocating ministries etc.) is of the type we study here.

Our cooperative model views three-player/three-cake problems as a set of interrelated two-player bargaining problems. Each pair of players faces a bargaining problem: division of the cake they control. Within this bargaining problem, the role of the third player is indirect: either player could threaten to abandon the current negotiations and take up negotiation with the third player. We capture this threat by an endogenously determined outside option vector; given this vector, we assume that the division of the cake is that specified by the Nash bargaining solution.[2] A multilateral Nash solution specifies a "division" of the cake for each potential coalition that is consistent both with the bargaining within each coalition and with the evaluation of each player's outside option.

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[1] If, for instance, participation in a coalition is a full-time occupation then at most one coalition can form.

[2] We caution the reader that we use "outside option vector" in the sense of Binmore (1985) rather than the usual "disagreement point" or "threat point" in the sense of Nash (1950), and that we use the Nash bargaining solution corresponding to the disagreement point (0,0), constrained by the outside option vector. See Section 3 for details.

A solution specifies a division of *each* cake, but in general some of the specified divisions may not be feasible.[3] We interpret an infeasible division as a prediction that a particular coalition will *not* form. We use the term *outcome* to describe a pair [ij] and a *feasible* division $\bar{z}^{ij}$ of the cake V[ij] . An outcome is *consistent* with a multilateral Nash solution if $\bar{z}^{ij}$ is the prescribed division of V[ij]. Thus, given a multilateral Nash solution, each conditional division which is feasible corresponds to a particular outcome, while conditional divisions which are infeasible correspond to the prediction that the particular coalition will *not* form.

A solution therefore provides answers to a pair of questions: Which coalitions *might* form? *If* a given coalition forms, on what division will its members agree? In general, there seems to be no basis on which to resolve the question of which coalition will actually form.[4] Consider for example the simple majority game (with transferable utility): each pair of players can divide \$2 any way they choose, but no sidepayments are possible (so no other coalitions yield positive value). The complete symmetry of the situation suggests that, whatever cake is divided (i.e., whatever coalition forms), the division of the cake should be a dollar to each player; symmetry also suggests that any of the coalitions is as likely to form as any other. A solution reflecting these considerations would be the triple {(1,1,-), (1,-,1), (-,1,1)} where "-" indicates that the player does not belong to the coalition; i.e., if the coalition [12] forms the payoffs to its members will be \$1 each, and player 3 will obtain nothing. And this triple is the unique multilateral Nash solution for this problem.[5]

We show that multilateral Nash solutions exist, and characterize them. Three-player/three-cake bargaining problem fall into three classes, which we can intuitively think of as corresponding to the number of "strong" coalitions. If there are no strong coalitions, the tensions among them lead to a unique multilateral Nash solution. If there is only one strong coalition, its members solve their bargaining problem as if the other two coalitions did not exist; this again leads to a unique multilateral Nash solution. If there are two strong coalitions, the "pivotal" player who belongs to both of them can play off each of his potential partners against the

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3 We show that at least one of the specified divisions is feasible.

4 Although this will be possible in some cases.

5 See Binmore (1985) for similar discussion.

other to obtain a higher payoff; this leads to an interval of multilateral Nash solutions, the multiplicity reflecting the effectiveness of the pivotal player. (In the case of one seller and two *identical* buyers, for example, one extreme solution yields equal division between the seller and one of the buyers; the other extreme solution yields the seller *all* the gains from trade.) As part of our characterization, we show that every multilateral Nash solution is determined by a triple of prices; if $(p_1,p_2,p_3)$ is this triple of prices then the corresponding (conditional) agreements are $(p_1,p_2,-)$, $(p_1,-,p_3)$, $(-,p_2,p_3)$.

Our noncooperative model is an extension of the alternating offer model to the three-player/three-cake context. At the beginning of the bargaining process, a player (selected at random) is given the initiative. A player with the initiative has the right to make a proposal, specifying a (feasible) division of a specific cake. His potential partner may accept or reject the offer. If he accepts, the game is over and players divide the cake as agreed (the third player obtaining nothing); if he rejects, he obtains the initiative and play proceeds as above.[6] (Perpetual disagreement yields zero payoff to all players.) It takes time to make an offer, and players discount the utility of future agreements using a common discount factor. We are looking for the outcomes of rational play; the notions of rationality we use are pure strategy subgame perfect and stationary subgame perfect equilibria. For all discount rates, subgame perfect equilibria exist. Bargaining problems with one or two "strong" coalitions have essentially unique (up to an indeterminacy arising from indifference) subgame perfect equilibria; bargaining problems with no strong coalitions have multiple subgame perfect equilibria, but a unique stationary subgame perfect equilibrium. In each bargaining problem the "unique" equilibrium is characterized by a triple of reservation prices: each player accepts all offers at least equal to his reservation price, rejects offers below his reservation price, and proposes divisions consistent with his reservation price and the reservation prices of other players.[7] In general, in

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[6] This proposal-making model incorporates discounting into the proposal-making model of Bennett (1991a, 1991b), which in turn is an extension of the proposal-making model of Selten (1981).

[7] More precisely, each player accepts offers yielding at least the present value of his reservation price obtained one period later (rejects anything less) and offers his partner the present value of her reservation price obtained one period later.

such a subgame perfect equilibrium, all initial offers are accepted;[8] in particular, the player with the first initiative obtains his reservation price.

Note that the outcome of play depends on the strategy of each player and also on the random choice of the player with the first initiative. In general, each subgame perfect equilibrium is consistent with three outcomes, each corresponding to the choice of a particular player to have the first initiative.

As in Binmore, Rubinstein and Wolinsky (1986) we are primarily interested in limit results. We show that the limit (as the discount factor tends to 1) of reservation prices is the price vector of a multilateral Nash solution, and that the limit of the set of associated outcomes is the set of outcomes of this multilateral Nash solution. For bargaining problems with no strong coalition or only one strong coalition the limit outcomes coincide with the outcomes of the unique multilateral Nash solution. For bargaining problems with two strong coalitions, the limit outcomes coincide with the outcomes of the multilateral Nash solution which is one endpoint of the interval of multilateral Nash solutions.

Our analysis is parallel to that of Binmore (1985), who also presented mutually reinforcing cooperative and noncooperative models. In Binmore's noncooperative ("market demand") model, players announce demands (rather than making proposals); the order of play is fixed. For each order of play and each discount factor sufficiently close to 1, there is a unique subgame perfect equilibrium outcome. As the discount factor tends to 1, the set of outcomes converges to the outcomes of a multilateral Nash solution. For bargaining problems with no strong coalition or only one strong coalition the limit outcomes coincide with the outcomes of the unique multilateral Nash solution. For bargaining problems with two strong coalitions, the limit outcomes coincide with the outcomes of the multilateral Nash solution which is the "opposite" endpoint of the interval of multilateral Nash solutions.

To summarize these results (in the limit as the discount factor tends to 1): for bargaining problems with no strong coalition or only one strong coalition the outcomes of Binmore's noncooperative model, our cooperative model and our noncooperative model coincide; for bargaining problems with two strong coalitions, the

[8] The exceptional case occurs when there is one strong coalition and the player *not* in this coalition is selected to make the first proposal.

outcomes of Binmore's noncooperative model coincide with those of one endpoint of the interval of cooperative solutions while the outcomes of our noncooperative model coincides with those of the opposite endpoint of the interval of cooperative solutions.

After this Introduction, Section 2 gives a formal description and a classification of three-player/three-cake bargaining problems. Section 3 discusses the cooperative model and its multilateral Nash solution, and Section 4 discusses our noncooperative model and its solutions. Section 5 discusses Binmore's noncooperative solution. Section 6 presents some examples, and Section 7 presents a discussion of related literature. Finally, Section 8 presents conclusions and suggestions for further research.

## 2. THREE-PLAYER/THREE CAKE BARGAINING PROBLEMS

In the situations we consider, there are three players, $I = \{1,2,3\}$ . Each pair [ij] of players owns a "cake" that they can divide - but only one cake can be divided. Players individually, and the coalition of all three players, have no cake. No sidepayments of any kind are possible. For each pair [ij] of players, we describe their cake by the set V[ij] of utility pairs resulting from each possible division of their cake. We assume that each V[ij] is a subset of $\mathbf{R}^2_+$ that is compact, convex and strongly comprehensive. (By *strongly comprehensive* we mean that if $x \in V[ij]$ , $y \leq x$ and $y \neq x$ then $y$ belongs to the interior of V[ij] . Strong comprehensiveness guarantees that the weak and strong Pareto boundaries coincide, so that, along the Pareto boundary of V[ij] , any increase in the utility of one player comes at the expense of the utility of the other player.) We also assume that the problem is non-trivial, in the sense that $V[ij] \neq \{(0,0)\}$ for at least one pair [ij] and that utility of dividing no cake is 0. If $x \in V[ij]$ , $x_i$ is its i-th component and $x_j$ is its j-th component. In order to avoid a profusion of super- and sub-scripts, when $\bar{x}^{ij}$ is a vector in V[ij] then $x^{ij}$ is its i-th component and $x^{ji}$ is its j-th component.

Since V[ij] is strongly comprehensive, we may describe its boundary by a continuous function $g^{ij} : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ which specifies player i's maximum utility (in the pair [ij] ) as a function of player j's payoff; i.e.,

$$g^{ij}(t) = \begin{cases} \max \ \{x \mid (x,t) \in V[ij]\} & \text{if } (0,t) \in V[ij] \\ 0 & \text{otherwise} \end{cases}$$

Following the convention above, $g^{ji}$ specifies player j's maximum utility (in [ij] ) as a function of player i's payoff. Over the range where both $g^{ij}$ and $g^{ji}$ are strictly positive, they are inverses, and are strictly decreasing; convexity of V[ij] guarantees that $g^{ij}$ and $g^{ji}$ are concave (on this range).

The core plays a role in our classification and a central role in the classification of the solutions of Binmore's market demand model (Section 5). We say that $\bar{z}^{ij}$

(alternatively $(\bar{z}^{ij}, 0)$) is in the *core*[9] if $\bar{z}^{ij} \in V[ij]$ and no pair can (feasibly) improve upon its components of $\bar{z} = (\bar{z}^{ij}, 0)$. Using the notation above, it is easy to see that the coalition [ij] cannot improve upon[10] $\bar{z}$ if $z^{ij} = g^{ij}(z^{ji})$; [ik] cannot improve upon $\bar{z}$ if $g^{ki}(z^{ij}) = 0$ and [jk] cannot improve upon $\bar{z}$ if $g^{kj}(z^{ji}) = 0$. When $\bar{z}^{ij}$ is in the core, we refer to [ij] as the *core coalition.*

The Nash bargaining solution for two-player/one-cake bargaining problems plays a central role through out the paper. For the pair of players [ij] and the cake V[ij] , we write $\bar{N}^{ij}$ for the Nash bargaining solution to the simple two-person bargaining problem in which V[ij] is the set of feasible utilities and (0,0) is the disagreement vector. Following our earlier convention, $N^{ij}$ is the i-th component of $\bar{N}^{ij}$ . Thus, $\bar{N}^{ij} = (N^{ij},N^{ji})$ is the vector that maximizes the Nash product $x_i x_j$ over x in V[ij].

Our classification of bargaining problems is based on the notion of "Nash stability" given below. Although we use this notion only to classify bargaining problems, and not to justify any solution, it may be helpful to provide some intuition. Consider a world in which the "standard of behavior" is for each coalition to agree on its Nash payoffs. By this we mean that $\bar{N}^{ij}$ is the focal point for bargaining in each coalition [ij] and hence is the anticipated agreement for each coalition *in absence of compelling reasons* to change it. In which circumstances will this standard of behavior be stable?

Consider first bargaining problems such that $\bar{N}^{ij}$ is in the core. Formation of [ij] with $\bar{N}^{ij}$ would be stable because no alternative coalition could improve upon these payoffs. In this case we call [ij] a Nash dominant coalition. It is easy to see that if there is a Nash dominant coalition, it is unique.

Consider next bargaining problems such that $N^{ij} \geq N^{ik}$ and $N^{ji} \geq N^{jk}$. Formation of [ij] with $\bar{N}^{ij}$ would be stable because each player prefers his Nash

[9] More formally, the vector $\bar{z} = (z_i, z_j, z_k)$ is in the *core* for the coalition structure {[ij], [k]} if $(z_i, z_j) \in V[ij]$, $z_k \in V[k]$ and no pair can (feasibly) improve upon their components of z. Since $\{0\} = V[k]$, the definition given above is equivalent.

[10] The strong comprehensiveness assumption is important here.

payoff in [ij], to his (anticipated) Nash payoff with player k; this is the first alternative for Nash stability listed below. Clearly Nash dominant coalitions are also Nash stable.

Notice that if no coalition satisfies this first type of Nash-stability then there is a (re)numbering of the players such that $N^{12} > N^{13}$ , $N^{23} > N^{21}$ , $N^{31} > N^{32}$. At the Nash payoffs player 1 prefers the coalition [12] , player 2 prefers the coalition [23] , and player 3 prefers the coalition [13]. The second type of Nash stability for the coalition [ij] occurs when i prefers his Nash payoff in [ij] to his Nash payoff in [ik] while j prefers her Nash payoff in [jk] but cannot obtain k's cooperation to form [jk]. For this to be the case it is necessary that k's Nash payoff with i, be larger than k's Nash payoff with j, (so $N^{ki} > N^{kj}$), but this is not sufficient since it leaves open the possibility that j could "bribe" k into forming [jk] by offering more than $N^{kj}$ (by accepting less than $N^{jk}$) and still obtain more than $N^{ji}$. Player j will be satisfied with forming [ij] if she can foresee that she can't form a coalition with k because *no* feasible division of V[jk] which yields j as much as her $N^{ji}$ would still yield k as much as forming [ik] and giving player i his $N^{ij}$.

The third type of Nash stability is identical to the second except the roles of i and j are reversed.

Formally, we say that [ij] is *Nash dominant* (or simply *dominant*) if $N^{ij} \geq g^{ik}(0)$ and $N^{ji} \geq g^{jk}(0)$. We say that [ij] is *Nash stable* if

1. $N^{ij} \geq N^{ik}$ and $N^{ji} \geq N^{jk}$ *or*

2. $N^{ij} \geq N^{ik}$ and $N^{ji} < N^{jk}$ but $g^{ki}(N^{ij}) \geq g^{kj}(N^{ji})$ *or*

3. $N^{ji} \geq N^{jk}$ and $N^{ij} < N^{ik}$ but $g^{kj}(N^{ji}) \geq g^{ki}(N^{ij})$

There can be multiple Nash stable coalitions: there may one, two or three Nash stable pairs satisfying the first condition, or one Nash stable pair satisfying the first condition and one satisfying either of the other two, but there cannot be two

Nash stable pairs failing to satisfy the first condition.[11] We show (Theorem 2.1) that three-player/three-cake problems with a nonempty core always have a Nash stable coalition. The converse is not true; some three-player/three-cake problems with an empty core can have a Nash stable coalition.

For some bargaining problems there are *no* Nash stable coalitions and the Nash "standard of behavior" leads to instability. In this case not only is there a (re)numbering of the players such that $N^{12} > N^{13}$, $N^{23} > N^{21}$, $N^{31} > N^{32}$, but also in each pair there is a player who could improve upon his payoff (with the willing cooperation of the third player) by defecting. When this is the case the cooperative and noncooperative solutions presented here prescribe a solution payoff for each player in each coalition which is a compromise between his higher and his lower Nash payoff.[12] Following Binmore (1985) will call such a set of payoffs a "von Neumann-Morgenstern tuple".

Formally we call a set of payoff vectors $z = \{\bar{z}^{ij}\}$ (one payoff vector for each pair) a *von Neumann Morgenstern tuple* if

1. each $\bar{z}^{ij}$ belongs to the Pareto boundary of $V[ij]$ and

2. $z^{ij} = z^{ik}$ for each $i$

The first condition requires payoff vectors to be feasible and efficient for their coalitions and the second requires each player to obtain the *same* payoff in the two coalitions to which he belongs. It is easily verified that von Neumann Morgenstern tuples, if they exist, are unique. In view of the second condition above, a von Neumann Morgenstern tuple can always be represented by a single payoff for each player, i.e., $p_i = z^{ij} = z^{ik}$ ; we call such a summary vector $(p_1, p_2, p_3)$ a *von Neumann Morgenstern vector.*

Binmore (1985) shows that a von Neumann Morgenstern vector exists whenever the core is empty. For our purpose the crucial point is that a von Neumann

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[11] See the proof of Proposition 4.1.4 for the argument.

[12] One can intuitively think of these payoffs as resulting from an "equilibrium" set of bribes.

Morgenstern vector exists whenever there is no Nash stable coalition. In this event (and only then) the components of the von Neumann Morgenstern vector represent compromises between each player's "higher" and "lower" Nash payoffs, i.e., if $(p_1, p_2, p_3)$ is a von Neumann Morgenstern vector then (renumbering as above) $N^{12} > p_1 > N^{13}$, $N^{23} > p_2 > N^{21}$, $N^{31} > p_3 > N^{32}$.[13]

In the Introduction we spoke in an intuitive way about a classification of bargaining problems according to the number of "strong" coalitions; the following formalizes this classification using the notions of Nash stability and the core. Bargaining problems with "one strong coalition" are those with a Nash dominant coalition (Class I). Bargaining problems with no strong coalitions are those with no Nash stable coalitions (Class III). In the next section we shall see that when there is a Nash stable coalition but no dominant coalition (Class II), then there is a pivotal player who may be able to play off his partners in "two strong" coalitions.

**THEOREM 2.1:** For each three player/three cake bargaining problem, exactly one of the following holds:

I. Some coalition is Nash dominant and the core is non-empty.

II. Some coalition is Nash stable, no coalition is Nash dominant, and
IIA: the core is non-empty.
IIB: the core is empty and there is a von Neumann Morgenstern vector.

III. No coalition is Nash stable, the core is empty, and there is a von Neumann Morgenstern vector.

**PROOF:** If the coalition [ij] is Nash dominant, it is easily seen that the vector $(N^{ij}, N^{ji}, 0)$ is in the core. Binmore (1985) shows that if the core is empty then there is a von Neumann Morgenstern vector. In view of the definitions and our previous remarks, therefore, all that remains to be proved is that non-emptiness of the core implies the existence of a Nash stable coalition. To this end, suppose that [1,2] is

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[13] For the proof of the this statement see the arguments for Propositions 4.1.6 and 4.1.8.

the core coalition and $\bar{z}^{12}$ is in the core. If $g^{31}(N^{12})$ and $g^{32}(N^{21})$ are both strictly positive, one of the coalitions [13] , [23] can block $(\bar{z}^{12}, 0)$ because no movement from $\bar{N}^{12}$ along the boundary of $V(12)$ can decrease 3's payoff to 0 in *both* [13] and [23]. Without loss, therefore, we may assume that $g^{32}(N^{21}) = 0$. Note that $g^{32}(N^{23}) = N^{32} \geq 0 = g^{32}(N^{21})$, so that $N^{21} \geq g^{23}(0) \geq N^{23}$ . If $g^{31}(N^{12}) = 0$ , the coalition [12] is Nash dominant, hence Nash stable. For the remaining cases we have $N^{21} \geq N^{23}$ , that $g^{31}(N^{12}) > 0$ and $g^{32}(N^{21}) = 0$ . If $N^{12} \geq N^{13}$ then [12] is Nash stable. If $N^{13} > N^{12}$ and $N^{31} \geq N^{32}$ then [13] is Nash stable. Finally, if $N^{13} \geq N^{12}$ and $N^{31} < N^{32}$ , we assert that $g^{21}(N^{13}) \geq g^{23}(N^{31})$ (and hence [13] is Nash stable. To see this, let $t_1$ be the smallest value of $t \in \mathbf{R}$ such that $g^{31}(N^{12} + t) = 0$ , and let $t_2$ be the largest value of $t \in \mathbf{R}$ such that $g^{32}(g^{21}(N^{12} + t)) = 0$ ; note that $t_1 \leq t_2$ . Since $(N^{13},N^{31}) \in V[13]$ , it follows that $N^{13} \leq N^{12} + t_1 \leq N^{12} + t_2$ , and hence that $g^{21}(N^{13}) \geq g^{21}(N^{12} + t_2)$. Since, by definition, $g^{32}(g^{21}(N^{12} + t_2)) = 0$, $g^{21}(N^{12} + t_2) = g^{23}(0)$ and hence $g^{21}(N^{13}) \geq g^{23}(0)$. Since $N^{32} \geq 0$, $g^{23}(N^{32}) \leq g^{23}(0)$ and hence $g^{21}(N^{13}) \geq g^{23}(0) \geq g^{23}(N^{31})$. Since $g^{21}(N^{13}) \geq g^{23}(N^{31})$, we conclude that [13] is Nash stable. This completes the proof. §

It is worth noting that, although non-emptiness of the core entails the existence of a Nash stable coalition, it does not entail that the core coalition itself is Nash stable. Although this may seem strange, a little reflection reveals the intuition: the Nash bargaining solution, and hence Nash stability, depends only on the payoffs at the "middle" of the utility frontiers; the core, on the other hand, depends on how the utility frontiers extend.

In Figures 1 - 4 we illustrate this classification of bargaining problems.

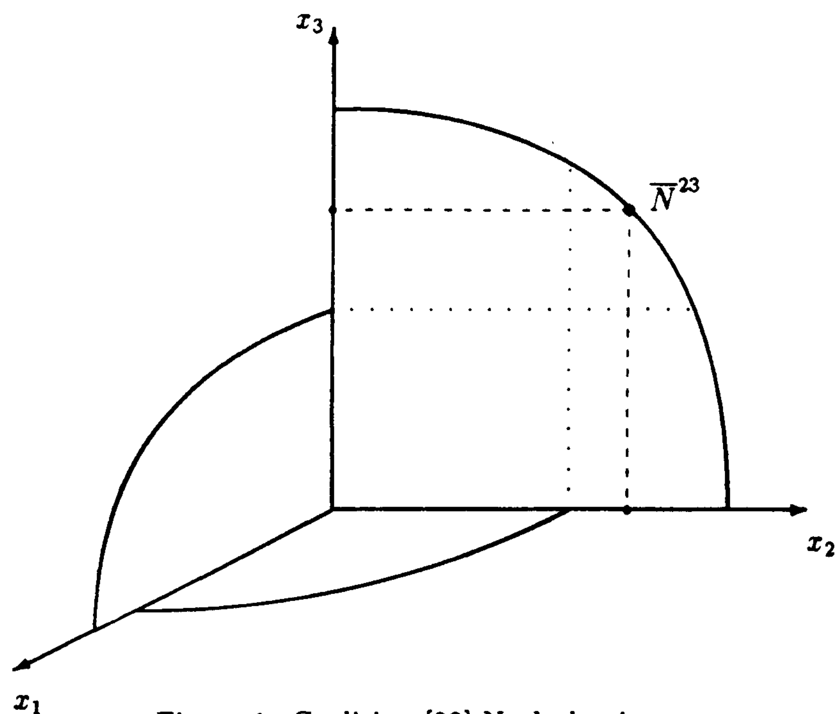


Figure 1: Coalition [23] Nash-dominant.

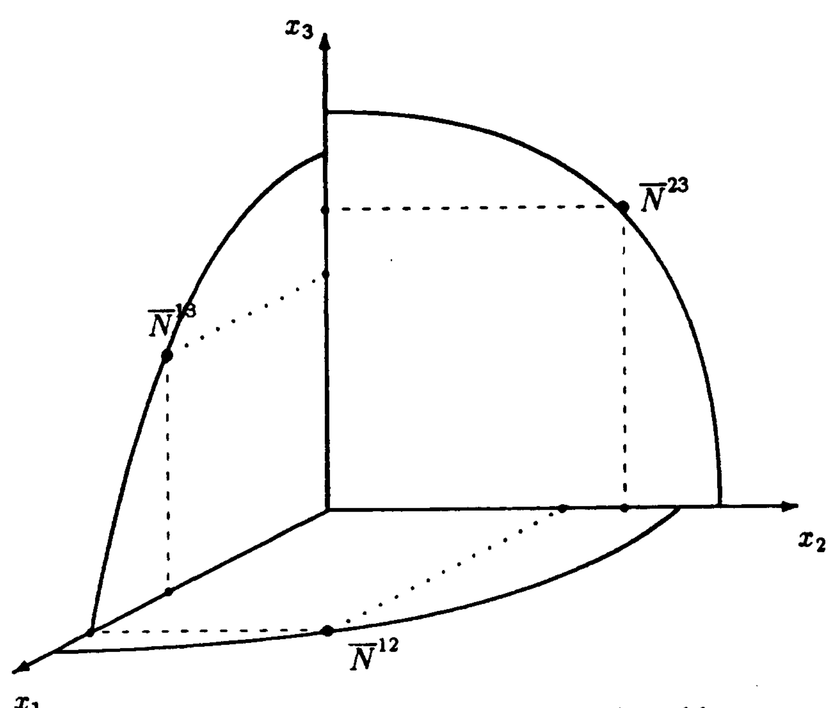


Figure 2: Coalition [23] is type 1 Nash-stable.

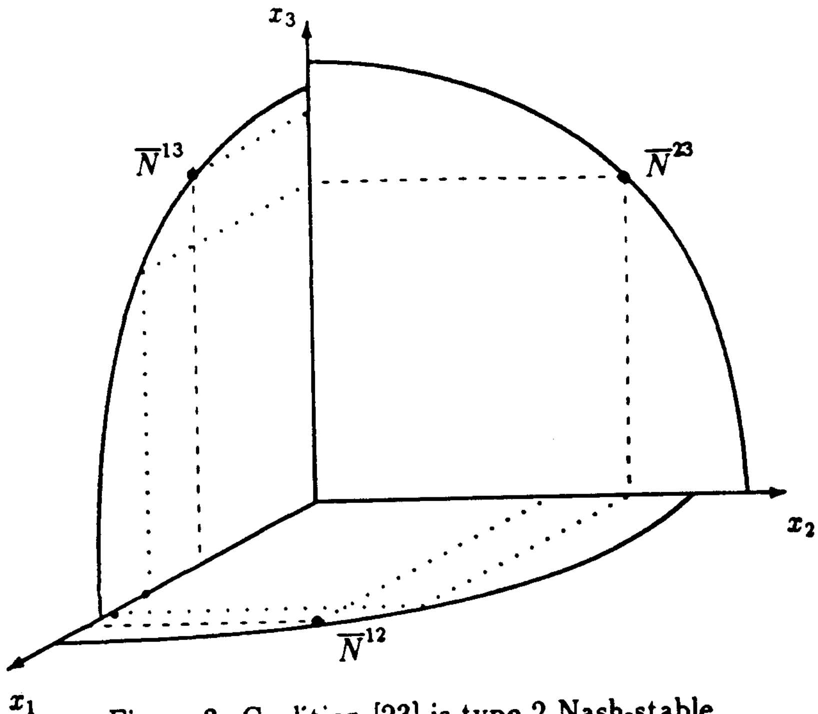

Figure 3: Coalition [23] is type 2 Nash-stable.

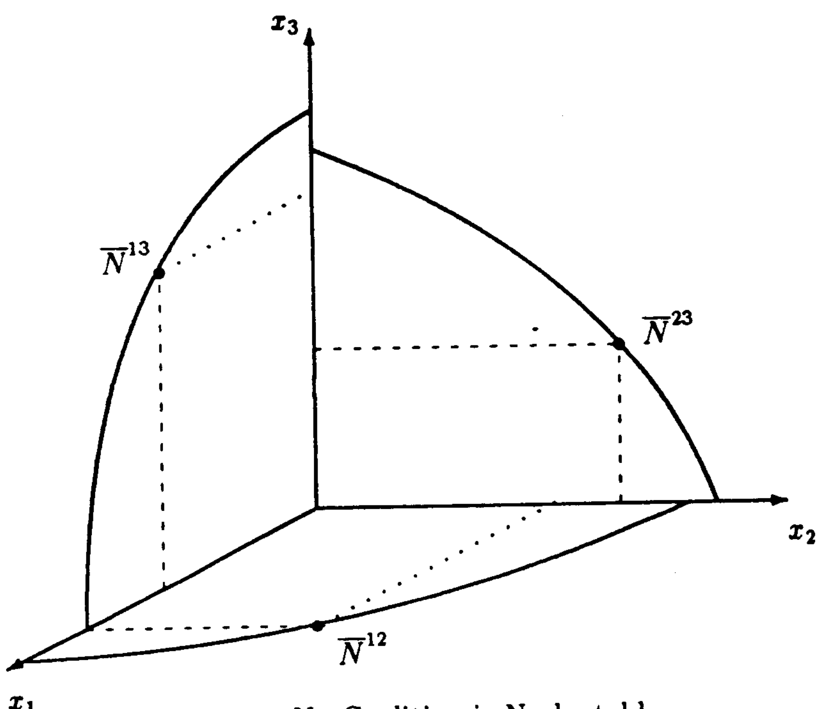

Figure 4: No Coalition is Nash-stable.

## 3. MULTILATERAL NASH SOLUTIONS

We view each three-player/three-cake bargaining problem as a set of interrelated two-player bargaining problems. Each pair of players faces a bargaining problem: division of the cake they control. Within this bargaining problem, the role of the third player is indirect: either player could abandon the current negotiations and take up negotiation with the third player. We capture the opportunity cost of this alternative for each player in an endogenously determined outside option utility. Given its players' outside options we assume that the pair bargains to an agreement and that the resulting division of the cake is that specified by the Nash bargaining solution constrained by its players' outside options. A multilateral Nash solution specifies a division of each cake that is consistent both with the bargaining within each coalition and with the evaluation of each player's outside option.

How are outside option utilities determined? If the players in the coalition [ij] fail to reach an agreement, one player or the other may enter into a coalition with the third player k; the opportunity cost for player i is the utility of the (foregone) partnership with k . This suggests that we use as i's outside option utility (i.e., his component of the pair's outside option vector) the utility he would receive if *he* entered into an agreement with k.[14] Similarly, player j's outside option is the utility she would receive if *she* entered into an agreement with k.

What do such outside options mean? Although either i or j may enter into a coalition with player k, it is not possible for *both* of them to do so. Thus, the outside option vector determined in this way cannot be interpreted as the *outcome* if

[14] One might think that the "correct" outside option utility should allocate player i only as much as his expected utility in his partnership with player k: player i's utility in the partnership times the probability that [ik] forms plus 0 times the probability that [jk] forms instead. This point of view is consistent with thinking of first forming the coalition [ij] and then negotiating the payoff -- player i might then accept as less than his agreement payoff with player k because there is a possibility that k will not accept. However, as a model of endogenous coalition formation the viewpoint is problematic: if player i anticipates receiving less with j than with k, player i would prefer [ik] to [ij] and assuming j wants i's cooperation, this would lead to an instability in the payoffs in [ij]. Intuitively speaking, for a coalition to remain competitive, i.e., have both players find the coalition desirable, the coalition must offer at least as much as its players can obtain elsewhere.

bargaining breaks down in the coalition [ij]. (Since this is the interpretation Nash gave his "disagreement point"[15] it is clear that such an outside option should not be interpreted as a disagreement point.) The interpretation we have in mind, rather, is that i's outside option represents what *he* could obtain if *he* broke off bargaining in the coalition [ij] and formed the coalition [ik] (in which case j would obtain 0). Similarly, j's outside option represents what *she* could obtain if *she* broke off bargaining in the coalition [ij] and formed the coalition [jk] (in which case i would obtain 0 ).

What utility would player i receive if he broke off negotiations with player j and entered into an agreement with k ? If it were *known* that i and k would come to a specific feasible agreement $\bar{z}^{ik}$ , the answer would clearly be $z^{ik}$ (or, equivalently, $g^{ik}(z^{ki})$). However, we envision that bargaining proceeds simultaneously in all coalitions, so the agreement in [ik] will *not* be known (in advance) either to i or to j . On the other hand, the agreement in [ik] will be *conjectured* (at every stage) by both i and j. In what follows, we assume that i and j make identical conjectures about the agreement in [ik].[16]

When the outside option vector is feasible for a pair, we assume that each player's outside option acts as a lower bound on his agreement payoff and that each pair of players bargains to an agreement which is the Nash bargaining solution over the set of feasible utility vectors that give each players at least his outside option utility.

If the outside option vector for the coalition [ij] is *not* feasible, players i and j will not agree to any feasible division in [ij] (because for any feasible division, at least one of them could do better by making an agreement with k ); in this case, we simply assume that the cake [ij] will not be divided.[17] It is convenient however to adopt the convention that when the outside option vector is *not*

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[15] "Disagreement point" is the term in current usage; Nash used the term "threat point".

[16] We could allow for different conjectures without affecting the final solution, since at a solution conjectures will be correct, and therefore identical.

[17] As we shall see, such situations are unavoidable.

feasible then the "agreement" is the outside option vector -- with the understanding that *infeasible* agreements are agreements *not* to divide the cake.

The possibility of infeasible agreements raises one final question. We intend to use as i's outside option (in bargaining with j ) the utility he would receive if he entered into an agreement with k , but we allow for the possibility that i and k would not come to a feasible agreement; what utility should we impute to player i in this case? We impute to player i the *maximum* utility he could obtain in any *feasible* agreement which yields k at least the utility she could otherwise obtain. Since, by the convention given above, the "infeasible agreement" $\bar{z}^{ik}$ in [ik] is equal to its members outside options, the utility player k can obtain "elsewhere" is $z^{ki} = d^{ki}$. We impute to player i what remains for player i in [ik] i.e., $g^{ik}(z^{ki})$. Hence, $g^{ik}(z^{ki})$ is player i's outside option in [ij].[18]

Informally, *a multilateral Nash solution* for a three-player/three-cake problem is a set of agreements (one for each coalition) which is consistent with the bargaining within each coalition and with the formation of outside options. Since agreements depend on outside options and outside options depend on (beliefs about) agreements, a solution represents a consistent set of beliefs about the outcomes of bargaining in each coalition with the property that, if players hold these beliefs about the agreements in other coalitions, then the agreement they reach will be precisely the agreement mandated for their coalition.

To make all of this precise we first formally define the constrained Nash bargaining solution $\bar{N}^{ij}(\bar{d}^{ij})$ for each outside option vector $\bar{d}^{ij} \in \mathbf{R}^2_+$. If the outside option vector $\bar{d}^{ij}$ is a feasible vector i.e., $\bar{d}^{ij} \in V(ij)$, then the value of the *constrained Nash bargaining solution* $\bar{N}^{ij}(\bar{d}^{ij})$ is the unique vector $x \in V(ij)$ that maximizes the Nash product $x_i x_j$ over all vectors in $V(ij)$ such that $x \geq \bar{d}^{ij}$. If the outside option vector $\bar{d}^{ij}$ is *not* a feasible vector, i.e., $\bar{d}^{ij} \notin V(ij)$, then $\bar{N}^{ij}(\bar{d}^{ij}) = \bar{d}^{ij}$. (Notice that only infeasible outside options lead to infeasible agreements, i.e., $\bar{N}^{ij}(\bar{d}^{ij})$ is feasible whenever $\bar{d}^{ij}$ is feasible.)

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[18] For a more detailed justification for our outside option conventions see Bennett (1986 revised 1992).

To formalize the notion of outside options, we take as given a set $\bar{z} = \{\bar{z}^{kl}\}$ of agreements (one for each coalition). For each pair [ij], we define the outside option vector $\bar{d}^{ij}$ by

$$d^{ij}(\bar{z}) = g^{ik}(z^{ki}) \qquad \text{and} \qquad d^{ji}(\bar{z}) = g^{jk}(z^{kj}).$$

Although the outside option vector $\bar{d}^{ij}$ depends only on the agreements in the coalitions [ik] and [jk] but not on the agreement in [ij], it is notationally convenient to view $\bar{d}^{ij}$ as depending on the entire set of agreements.

Finally, we define a ***multilateral Nash solution*** to be a set $\bar{z} = \{\bar{z}^{kl}\}$ of agreements (one for each coalition) with the property that $\bar{N}^{ij}(\bar{d}^{ij}(\bar{z})) = \bar{z}^{ij}$ for each pair [ij]. A multilateral Nash solution is thus a triple of conditional agreements. We stress that some of these agreements may not be feasible (but show below that at least one of them is always feasible). "Outcomes" correspond to the feasible agreements and are the ultimate objects of interest. An *outcome* consists of a pair [ij] and a feasible agreement $\bar{x}^{ij} \in V(ij)$. The outcome ([ij], $\bar{x}^{ij}$) is an *outcome of the multilateral Nash solution* $\{\bar{z}^{kl}\}$ if $\bar{x}^{ij} = \bar{z}^{ij}$. Notice that if the agreement $\bar{z}^{ij}$ *not* feasible for [ij] then ([ij], $\bar{z}^{ij}$) is *not* an outcome of the multilateral Nash solution.

A multilateral Nash solution and its associated outcomes provides answers to a pair of questions: Which coalitions might form? and *If* one of these coalitions forms, on what division will its members agree?

Three simple examples may serve to illustrate multilateral Nash solutions and outcomes. (1) For the simple majority game discussed in the Introduction, in which any pair may divide \$2, the unique multilateral Nash solution is {(1,1,-), (1,-,1), (-,1,1)}.[19] There are three outcomes of this solution: any of the three cakes might be divided, but whatever cake is divided will be divided equally. (2) For the game in which either [12] or [13] can divide \$2 but [23] can divide \$6, the unique multilateral Nash solution is {(0,3,-), (0,-,3), (-,3,3)}. Since the first two "divisions" are infeasible, there is a unique outcome of this solution: the coalition [23] forms and divides equally \$6. (3) For the game in which either [13] or [23] can divide \$6 but [12] can divide \$2, there is an interval of multilateral Nash solu-

[19] Recall our convention that (1,1,-) represents a division of the cake V[12] between the players 1 and 2, and that the "-" denotes the omission of player 3.

tions: for each $t \in [0,2]$, the corresponding solution is $\{(1+t,1+t,-), (1+t,-,5-t), (-,1+t,5-t)\}$ . For $t \neq 0$, the division of $V[12]$ is not feasible. In this case, there are two outcomes of each solution in the interval: either [13] or [23] forms, player 3 obtains $5-t$ while his partner obtains $1+t$.

Before going further, we make several observations. The first is that in a multilateral Nash solution the two agreement payoffs for each player are equal. To see this, let $(\bar{z}^{ij})$ be a multilateral Nash solution, and suppose that $z^{12} > z^{13}$ . Because $d^{12} \leq z^{13} < z^{12}$ , we conclude that $d^{12} \neq z^{12}$ and hence that $\bar{d}^{12}$ and $\bar{z}^{12}$ are both feasible. In that case, $d^{13} \geq z^{12}$ ; since $z^{13} \geq d^{13}$ we conclude that $z^{13} \geq z^{12}$ , a contradiction. We conclude that $z^{ij} = z^{ik}$ for every $i$, as asserted. Hence we can summarize the set $\{\bar{z}^{ij}\}$ of agreements by a vector $p = (p_1,p_2,p_3)$ of real numbers, where $p_i = z^{ij} = z^{ik}$ . We refer to $p$ as the *price vector* of the multilateral Nash solution $\{\bar{z}^{ij}\}$ , and say that $\{\bar{z}^{ij}\}$ is *generated* by the price vector $p$ .

The second observation is that no multilateral Nash agreement is Pareto dominated and at least one of the agreements must be feasible and non-zero. To see this let $p = (p_1,p_2,p_3)$ be the price vector of a multilateral Nash solution $\bar{z} = \{\bar{z}^{ij}\}$. By definition of $\bar{N}^{ij}(\bar{d}^{ij})$, if $\bar{d}^{ij} \in V(ij)$ then $\bar{N}^{ij}(\bar{d}^{ij})$ is on the Pareto boundary of $V(ij)$ and if $\bar{d}^{ij} \notin V(ij)$ then $\bar{N}^{ij}(\bar{d}^{ij}) = \bar{d}^{ij} \notin V(ij)$. Hence $\bar{N}^{ij}(\bar{d}^{ij}) \notin \text{Int } V(ij)$ so no pair can improve upon its agreement and hence the agreement is not Pareto dominated. To show that at least one of the agreements must be feasible and non-zero, recall that by assumption there is at least one nontrivial cake and hence $p_i > 0$ for some player $i$. If $p_i > 0$ then either $(p_i,p_j)$ is feasible for [ij] or $(p_i,p_k)$ is feasible for [jk]. To see this, suppose that $p_1 > 0$, $(p_1,p_2) \notin V[12]$ and $(p_1,p_3) \notin V[13]$ . Because feasible outside options lead to feasible agreements, it follows that $\bar{d}^{12} = \bar{z}^{12} = (p_1,p_2)$ and $\bar{d}^{13} = \bar{z}^{13} = (p_1,p_3)$ . Because of the way in which outside options are determined, this entails either $p_1 = d^{12} = 0$ or $d^{12} < z^{13} = d^{13} \leq z^{12} = d^{12}$ . However, the first set of equalities violates our assumption that $p_1 > 0$ and the second set of inequalities is inconsistent, so we have reached a contradiction. We conclude that either $(p_1,p_2) \in V[12]$ or $(p_1,p_3) \in V[13]$ . It follows therefore that each multilateral Nash solution prescribes at least one feasible division and has at least one consistent outcome.

Finally, every three-player/three-cake problem has a multilateral Nash solution. The proof is a simple fixed point argument; for the details (in a much more general setting), we refer to Bennett (1992). Summarizing these observations, we have the following result.

**THEOREM 3.1:** For every three-player/three-cake bargaining problem:

(i) a multilateral Nash solution exists;

(ii) for every multilateral Nash solution, no agreement is Pareto dominated and at least one agreement is feasible and non-zero;

(iii) every multilateral Nash solution is generated by a vector of prices.

It is worth noting explicitly that a von Neumann Morgenstern vector (whenever it exists) is the price vector of a multilateral Nash solution; we leave the easy verification to the reader.

We now turn to the characterization of multilateral Nash solutions. Theorem 2.1 provided a three-way classification of three player/three-cake bargaining problems. In Class I and Class III there is a unique multilateral Nash solution; in Class II we find a (possibly degenerate) curve of multilateral Nash solutions.

**THEOREM 3.2:** For every three-player/three-cake bargaining problem, exactly one of the following holds:

I. Some pair, say [12] , is Nash dominant. In this case there is a unique multilateral Nash solution, and its price vector is $(N^{12},\ N^{21},\ 0)$.

II. Some pair is Nash stable but not Nash dominant; we may, without loss, assume that $N^{12} \geq N^{13}$ , $g^{31}(N^{12}) > 0$ , and $g^{31}(N^{12}) \geq g^{32}(N^{21})$. In this case the set of multilateral Nash solutions is a curve, and its price vectors are given by $q(t) = (N^{12}+t,\ g^{21}(N^{12}+t),\ g^{31}(N^{12}+t)))$ for $t$ in an appropriate interval $[0,T]$.

IIA. The core is nonempty. In this case $q(T)$ is in the core.

IIB. The core is empty. In this case $q(T)$ is the von Neumann Morgenstern vector.

III. No pair is Nash stable. In this case the von Neumann Morgenstern vector is the price vector of the unique multilateral Nash solution.

## 4. THE PROPOSAL-MAKING MODEL

In this and the next section we discuss noncooperative bargaining models for three player/three cake problems. Both models are extensions of the Rubinstein alternating offer model to the three player/ three cake context. In this section we analyze the proposal-making model; in the next section we discuss Binmore's market demand model.

In the proposal-making model, players bargain by making, accepting, and rejecting proposals. A *proposal* by player $i$ specifies a coalition [ij] to which $i$ belongs (so that $j \in I$ and $j \neq i$) and a vector $x \in V[ij]$, representing a feasible division of the cake $V[ij]$. Bargaining begins when a player (selected at random) is given the initiative. A player with the initiative has the right to make a proposal. His designated partner may then accept or reject the proposal. If he accepts, the game ends and the players divide the cake as agreed (the third player obtaining nothing); if he rejects, he obtains the initiative and play proceeds as above. (Infinite plays - corresponding to perpetual disagreement - yield zero payoff to all players.)

We assume that it takes time to make proposals (but not to accept or reject them), and that players are impatient, and so discount the utility of future payoffs. Thus, in each time period $t = 0, 1, 2, \ldots$, there are two actions: a proposal will be made and then accepted or rejected. We assume that all players discount the utility of future agreements using the common discount factor $\delta < 1$.[20] Since payoffs are already denominated in utility, the utility of player $i$ for the agreement $x$ reached at time period $t$ is simply $\delta^t x_i$, if $i$ is a party to the agreement, and 0 otherwise.

For $k \geq 0$, a *k-history* $h$ is a sequence of $k$ actions that follow the above rules of play. Player $i$ is *active* following the k-history $h$ if according to the rules of play, player $i$ will have the next move (either to accept or reject the current proposal, or to make a new proposal). Player $i$'s *strategy* specifies his action following every history for which he is active.[21]

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[20] Different discount factors would lead to asymmetric Nash solutions.

[21] We consider only pure strategies. It appears that allowing for mixed strategies, while cumbersome, would not materially affect the conclusions.

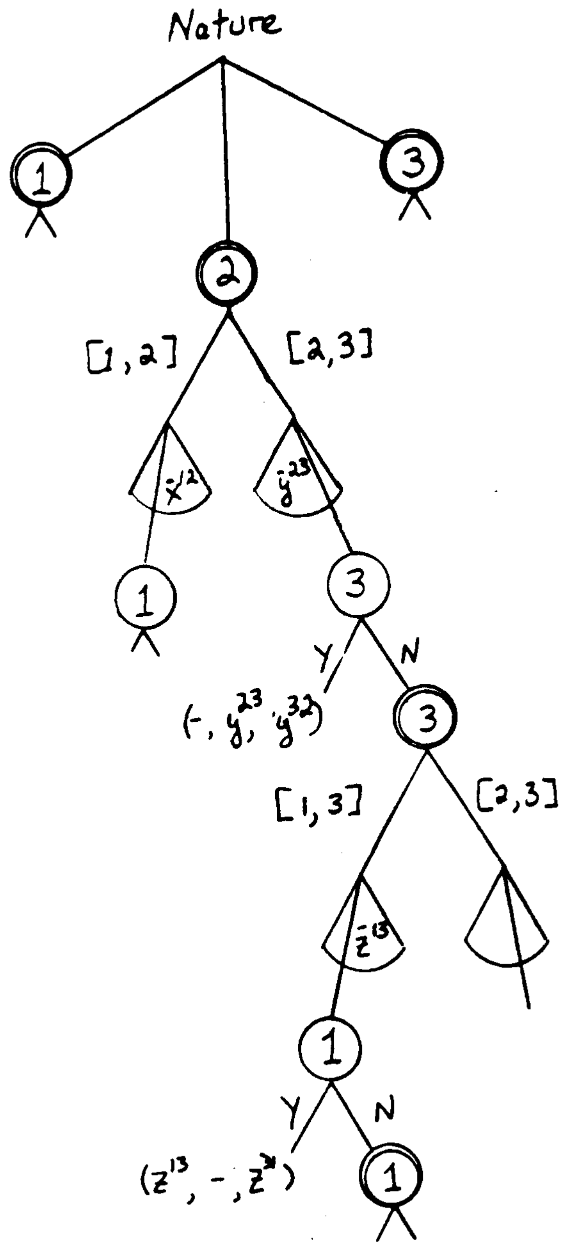


Figure 5: The Proposal-Making Model

The above specifications formalize the bargaining procedure as an extensive form game of perfect information (with a random move of nature at the beginning, selecting the first player to have the initiative); a sketch of the game tree is shown in Figure 5. The solution concepts we employ are subgame perfect equilibrium strategies (SPE) and stationary subgame perfect equilibrium strategies (SSP). (Subgame perfection of a strategy profile requires it to be a Nash equilibrium in every subgame while stationarity of a player's strategy requires the player to make the same decision in similar situations: as initiator the player must make the same proposal every time he has the initiative and accepts/rejects every time the same proposal is made to him.

Our analysis has two goals. We first characterize the subgame perfect and the stationary subgame perfect equilibria (depending on the case) of the proposal-making model for each $\delta < 1$ : we show that equilibria exist and are essentially unique. We also show that at these equilibria, players behave as if they had reservation prices. We then relate the proposal-making model to the cooperative model of Section 3 by showing that, in the limit as $\delta$ tends to 1 , the vector of reservation prices converges to the price vector of a multilateral Nash solution and, moreover, the sets of equilibrium converge to the outcomes of this multilateral Nash solution. (We say that ([ij], $(x_i,x_j)$) is an *equilibrium outcome* if there is an equilibrium strategy profile $\sigma$ such that the outcome ([ij], $(x_i,x_j)$) occurs with positive probability[22] when players follow $\sigma$. In order to make these statements precise, we need some additional notions.

Central to our analysis are solutions to the following system of equations:

$$\clubsuit \qquad \begin{aligned} p_1 &= \max \{ g^{12}(\delta p_2) , g^{13}(\delta p_3) \} \\ p_2 &= \max \{ g^{21}(\delta p_1) , g^{23}(\delta p_3) \} \\ p_3 &= \max \{ g^{31}(\delta p_1) , g^{32}(\delta p_2) \} \end{aligned}$$

For reasons that will become clear shortly, we refer to any solution $p(\delta) = (p_1,p_2,p_3)$ of this system as a *reservation price vector*. It is easy to see that there are two kinds

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22 Although players are following pure strategies, nature randomly selects the first player to have the initiative; and hence up to three outcomes can occur with positive probability.

of solutions to the system ♣ : either there is a pair [ij] of players so that the maximum for player i occurs with player j and the maximum for player j occurs with player i (we refer to such reservation price vectors as *reflexive*, and to [ij] as the *reflexive pair*) or, for some renumbering, the maximum for player i occurs with player j and not with player k , the maximum for player j occurs with player k and not with player i , and the maximum for player k occurs with player i and not with player j (we refer to such reservation price vectors as *triangular*).

Given a vector $p = (p_1,p_2,p_3) \in \mathbf{R}^3_+$, we say that a strategy $\sigma_i$ for player i is a *price strategy* corresponding to p (and p is the *price* corresponding to the strategy $\sigma_i$ ) if:

(1) player i always accepts any proposal ([ij], x) such that $x_i \geq \delta p_i$ , and rejects any proposal ([ij], x) such that $x_i < \delta p_i$;

(2a) if $p_i > 0$ and $(p_i,\delta p_j) \in V[ij]$ but $(p_i,\delta p_k) \notin V[ik]$ , then player i always proposes the partnership [ij] and the agreement $(p_i,\delta p_j)$;

(2b) if $p_i > 0$ and $(p_i,\delta p_j) \in V[ij]$ and $(p_i,\delta p_k) \in V[ik]$ then player i proposes either the partnership [ij] and the agreement $(p_i,\delta p_j)$ or the partnership [ik] and the agreement $(p_i,\delta p_k)$.

(If $p_i = 0$ or if $(p_i,\delta p_j) \notin V[ij]$ and $(p_i,\delta p_k) \notin V[ik]$ , we make no restrictions as to player i's behavior because it is irrelevant to his own payoff; the reasoning is explained below.) A strategy profile $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ is a *price strategy profile* if each strategy $\sigma_i$ is a price strategy. Informally, $\sigma$ is a price strategy profile if each player proposes divisions which yield him his component of p and his partner the present value of his partner's component of p (obtained one period later), and each player accepts proposals which yield him at least the present value of his component of p (obtained one period later) and rejects all others. In particular, player i behaves as if $p_i$ were his reservation price (hence the terminology). Notice also that the outcomes of the price strategy $\sigma$ with price vector p are necessarily in the set $\{ ([ij], (p_i,p_j)) \mid (p_i,p_j) \in V(ij) \}$ and that every outcome in this set is the outcome of some price strategy with price vector p.

Every solution $p(\delta)$ to the system ♣ corresponds to at least one price strategy profile; some solutions may correspond to multiple price strategy profiles.

This multiplicity may occur in either of two ways. If $(p_i,\delta p_j) \in V[ij]$ and $(p_i,\delta p_k) \in V[ik]$ then player i is indifferent between his two partners, and our requirements allow him to choose either (or to alternate between them, or ... ). For $\delta < 1$, price strategies which differ in this way may correspond to different sets of outcomes. However, as $\delta \to 1$, these sets of outcomes have the same limit. If $(p_i,\delta p_j) \notin V[ij]$ and $(p_i,\delta p_k) \notin V[ik]$, we do not restrict player i's proposals: If all players follow the price strategy profile $\sigma$, player i cannot make any acceptable proposals. Hence the only outcomes of $\sigma$ involve formation of the coalition [jk], and yield player i a payoff of 0. When nature chooses player i to have the first move, his behavior determines only the subsequent order of play, and hence whether the division in [jk] will be $(p_j, \delta p_k)$ or $(\delta p_j, p_k)$. As $\delta \to 1$, these divisions converge to the same limit. If $p_i = 0$ we also do not restrict player i's proposals, if $(0, \delta p_j) \in V(ij)$, player j can make an acceptable proposal, it yields player i only a payoff of 0. Moreover, if $(0, \delta p_j) \in V(ij)$, then $(0, \delta' p_j) \notin V(ij)$ for $\delta' > \delta$, so in the limit as $\delta \to 1$, this situation reduces to the one above.

In view of these definitions and remarks, the following result asserts the existence and essential uniqueness of equilibria.

**THEOREM 4.1:** For each $\delta < 1$,

(1) the system ♣ has a unique solution $p(\delta)$ (which is either reflexive or triangular).

(2) every price strategy corresponding to $p(\delta)$ is subgame perfect.

(3) if $p(\delta)$ is reflexive then every subgame perfect equilibrium is a price strategy corresponding to $p(\delta)$.

(4) if $p(\delta)$ is triangular then every stationary subgame perfect equilibrium is a price strategy corresponding to $p(\delta)$.

The proof of this result is quite long, and is deferred to the Appendix.

When $p(\delta)$ is triangular, there is a unique stationary subgame perfect equilibrium (and it is a price strategy) but there may be many non-stationary subgame perfect equilibria. In the Appendix (following the proof of Theorem 4.1) we discusses the range of outcomes associated with non-stationary subgame perfect equilibria.

We now ask about the behavior in the limit, as $\delta \to 1$, of these equilibrium outcomes. In view of Theorem 4.1, the question revolves around the limit behavior of the reservation price vector $p(\delta)$. In the following result, we refer to the classification of three-player/three-cake problems given in Theorem 2.1 and the description of multilateral Nash solutions given in Theorem 3.2

**THEOREM 4.2:** For every three-player/three-cake problem,

(i) The limit, $\lim_{\delta \to 1} p(\delta)$, exists and is the price vector of a multilateral Nash solution.

(ii) If there is a dominant coalition or if there is no Nash stable coalition (i.e., for problems of Class I or Class III), then $\lim_{\delta \to 1} p(\delta)$ is the price vector of the unique multilateral Nash solution.

(iii) If there is a Nash stable coalition but no dominant coalition (i.e., for Class II), then $\lim_{\delta \to 1} p(\delta)$ is the price vector of the endpoint of the set of multilateral Nash solutions corresponding to $t = 0$.

The proof of Theorem 4.2 is also given in the Appendix to this Section.

As we have pointed out there may be several price strategies corresponding to the same price vector $p(\delta)$ and different strategies may result in different outcomes. However, as $\delta$ approaches 1, the different payoff vectors for each coalition converge; in the limit, there is at most one outcome corresponding to the formation of each coalition.

We can recast the results of Theorem 4.2 into the language of outcomes; in view of our earlier discussion, no proof is required. We refer to the set of limits of outcomes of price strategies (as $\delta \to 1$) as the set of *limit outcomes* of the proposal-making model.

**THEOREM 4.2'**: For each three-player/three-cake bargaining problem, exactly one of the following holds:

I. Some pair is Nash dominant. In this case the set of limit outcomes of the proposal-making model coincides with the set of outcomes of the unique multilateral Nash solution.

II. Some pair is Nash stable but not Nash dominant. Without loss renumber players as in II of Theorem 3.2. In this case the set of limit outcomes of the proposal-making model coincides with the set of outcomes of the extreme multilateral Nash solution whose price vector is $q(0)$.[23]

III. No pair is Nash stable. In this case the set of limit outcomes of the proposal-making model coincides with the set of outcomes of the unique multilateral Nash solution and the set of payoff vectors of these outcomes is a von Neumann Morgenstern tuple.

[23] See Theorem 3.2 for the definition of $q(0)$.

## 5. BINMORE'S MARKET DEMAND MODEL

As in the present paper, Binmore (1985) provides mutually reinforcing cooperative and non-cooperative models of bargaining. In this section we give a brief description of his non-cooperative model -- the market demand model -- and then use our classification scheme to relate its solution to our those of our cooperative and non-cooperative models. We refer the reader to Binmore (1985) for a detailed description of his cooperative model.

In Binmore's market demand model, players bargain by making *demands*; a demand of player i represents the willingness to form *either* coalition to which player i belongs, provided that player i obtains at least that demand. Binmore fixes the order of play (each choice leads to a distinct outcome --- see below); for convenience, we assume the order of play is 1, 2, 3. Player 1 begins by making a demand. Player 2 may accept that demand, in which case the game ends, the coalition [12] forms, player 1 obtains his demand, and player 2 obtains the remainder. Alternatively, player 2 may reject the demand of player 1 and make a demand of his own, in which case player 3 has the next move. Player 3 considers the demands of players 1 and 2 in turn. If he accepts the demand of player 1, the game ends, the coalition [13] forms, player 1 obtains his demand, and player 3 obtains the remainder. If player 3 rejects the demand of player 1, he may accept the demand of player 2, in which case the game ends, the coalition [23] forms, player 2 obtains his demand, and player 3 obtains the remainder. If player 3 rejects the demand of player 2, then player 3 makes a demand of his own, and player 1 has the next move. From this point on, each player considers in turn the demands of the previous two players and, if he rejects both of them, makes a new demand of his own. Players discount the utility of future agreements, using the same discount factor $\delta < 1$. The utility of player i for the agreement x reached at time period[24] t is $\delta^t x_i$, if i is a party to the agreement, and 0 otherwise. Infinite plays - corresponding to perpetual disagreement - yield zero payoff to all players.

[24] From Binmore's paper it is not clear when a new period is to begin. He says simply, "provided suitable assumptions are made about discounting and the timing of offers" (p. 291). The following timing does work. For bargaining problems with empty core: a new period begins after each rejection; for bargaining problems with a nonempty core: a new period begins *only* after rejections by players in the core coalition.

The above specifications formalize the bargaining procedure as an extensive form game of perfect information. The solution concept Binmore employs is subgame perfect equilibrium in pure strategies. For each choice of the order of play and each $\delta$ sufficiently close to 1, Binmore establishes the existence of a unique subgame perfect equilibrium outcome. Since there are 6 orders of play, this yields, 6 (potentially) different subgame perfect equilibrium outcomes -- two different payoff vectors for each potential coalition. However, as $\delta$ approaches 1, the different payoff vectors for each coalition converge; in the limit, there is at most one outcome corresponding to the formation of each coalition.[25] Since Binmore refers to his non-cooperative model as a *market model*, we refer to this set of limiting outcomes as the *market solution*.

To characterize the market solution, it is convenient to consider two cases depending on whether or not the core is empty. Binmore showed that when the core is empty, the market solution contains exactly three outcomes, and the set of payoff vectors of these outcomes is a von Neumann Morgenstern tuple. For the case that the core is *not* empty, number the players so that [12] is the core coalition. In this case all core payoffs yield player 3 a payoff of 0. For $i = 1$ or 2 let $v_i$ for the largest payoff player $i$ can obtain with player 3, i.e., $v_i = g^{i3}(0)$ and set

$$b^{12} = (N^{12}(v_1,v_2), N^{21}(v_1,v_2))$$

$$b^{13} = (N^{12}(v_1,v_2), 0)$$

$$b^{23} = (N^{21}(v_1,v_2), 0)$$

When [12] is a core coalition, ([12], $b^{12}$) is always an outcome of the market solution. For $i = 1$ or 2, if $b^{i3} \in V[i3]$ then ([i3], $b^{i3}$) is also an outcome of the market solution. (Note that the outcomes ([i3], $b^{i3}$) are in a sense degenerate, in that player 3 obtains 0 as his payoff.) Binmore showed that these and only these are outcomes of the market solution.

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25 This is not too surprising. Rubinstein's alternating offer model has two distinct payoff vectors (depending on which of the two player makes the first offer); and these payoff vectors converge as $\delta \rightarrow 1$.

With this discussion in hand, we can relate the market solution to our cooperative and non-cooperative solutions. For problems of Class I and Class III, there is a unique multilateral Nash solution and its outcomes coincide with the market solution and the outcomes of the proposal-making model. For problems of Class II there is an interval of multilateral Nash solutions and the outcomes of the market solution coincide with those of one extreme multilateral Nash solution while those of the proposal-making model coincide with those of the other extreme multilateral Nash solution. The following Theorem proves the results for the market solution.

**THEOREM 5.1:** For each three-player/three-cake bargaining problem, exactly one of the following holds:

I. Some pair is Nash dominant. In this case the market solution coincides with the set of outcomes of the unique multilateral Nash solution.

II. Some pair is Nash stable but not Nash dominant. Without loss renumber players as in II of Theorem 3.2. In this case the market solution coincides with the set of outcomes of the extreme multilateral Nash solution whose price vector is $q(T)$.[26]

III. No pair is Nash stable. In this case the market solution coincides with the set of outcomes of the unique multilateral Nash solution and the set of payoff vectors of these outcomes is a von Neumann Morgenstern tuple.

**PROOF:** In the first two cases we consider (**I** and **IIA**) the core of the bargaining problem is not empty. In each case it is sufficient to show that $\{b^{ij}\}$ the are agreements of the unique multilateral Nash solution, because the $([ij], b^{ij})$ is in the market solution precisely when it is an outcome of the multilateral Nash solution, i.e., when $b^{ij} \in V(ij)$.

26 Recall that the limit outcomes of the proposal-making model coincides with the set of outcomes of the multilateral Nash solution whose price vector is $q(0)$. See Theorem 3.2 for the definition of $q(t)$.

(I) Let [12] be the Nash dominant coalition. As we noted in Theorem 2.1, a Nash dominant coalition is necessarily also a core coalition so, in particular, the core is nonempty. In this case the unique multilateral Nash solution has price vector $(N^{12}, N^{21}, 0)$ and hence the set of agreements are: $(N^{12}, N^{21})$ for the coalition [12], $(N^{12}, 0)$ for the coalition [13], and $(N^{21}, 0)$ for the coalition [23]. Nash dominance means that $N^{12} \geq g^{12}(0) = v_1$ and $N^{21} \geq g^{21}(0) = v_2$, so $b^{12} = (N^{12}, N^{21})$, $b^{13} = (N^{12}, 0)$ and $b^{23} = (N^{21}, 0)$. This completes the proof of (I).

(II) There are two cases of bargaining problems with a Nash stable but no Nash dominant coalition to consider depending on whether or not the bargaining problem has a nonempty core.

**Case IIA:** The bargaining problem has a nonempty core. Again it is sufficient to show that $\{b^{ij}\}$ the are agreements of the unique multilateral Nash solution. After renumbering players as in II of Theorem 3.2, one can show that the core coalition must be either [12] or [13].

Suppose [12] is the core coalition. In this case the corresponding multilateral Nash solution is the extreme multilateral Nash solution with price vector $q(T) = (N^{12} + T,\ g^{21}(N^{12} + T),\ g^{31}(N^{12} + T))$. The binding constraint that determines $T$ is $g^{31}(N^{12} + T) = 0$. The argument of Theorem 3.2 shows that $(N^{12}+T, 0) \in V[13]$, and hence $N^{12}+T = v_1$. Since $v_1 \geq N^{12}$ and $\bar{N}^{12}$ is feasible, we have $N^{12}(v_1,v_2) = v_1$ so $N^{12}(v_1,v_2) = N^{12} + T$. Clearly $N^{21}(v_1,v_2) = g^{21}(N^{12}(v_1,v_2))$ and hence $N^{21}(v_1,v_2) = g^{21}(v_1) = g^{21}(N^{12} + T)$. We may therefore write $q(T) = (N^{12}(v_1,v_2),\ N^{21}(v_1, v_2), 0)$. From this it follows directly that $\{b^{ij}\}$ are the agreements of the extreme Nash solution.

Suppose instead [13] is the core coalition. In this case we must redefine the $v_i$ and the $b^{ij}$. For $i = 1$ or $3$ let $\bar{v}_i$ for the largest payoff player $i$ can obtain with player 2, i.e., $\bar{v}_i = g^{i2}(0)$ and set $\bar{b}^{12} = (N^{13}(\bar{v}_1,\bar{v}_3), 0)$, $\bar{b}^{13} = (N^{13}(\bar{v}_1,\bar{v}_3), N^{31}(\bar{v}_1,\bar{v}_3))$, and $\bar{b}^{23} = (0, N^{31}(\bar{v}_1,\bar{v}_3))$. Clearly, in this case the market solution consists of the $([ij], \bar{b}^{ij})$ for which $\bar{b}^{ij} \in V(ij)$. In this case we again show that $\{\bar{b}^{ij}\}$ are the agreements of the multilateral Nash solution by showing that its price vector is $(N^{13}(\bar{v}_1,\bar{v}_3), 0, N^{31}(\bar{v}_1,\bar{v}_3))$. For this case the corresponding multilateral Nash solution has the price vector $q(T) = (N^{12} + T, g^{21}(N^{12} + T),\ g^{31}(N^{12} + T))$, where $T$ is determined by the binding constraint $g^{21}(N^{12} + T) = 0$. The argument of Theorem 3.2 shows that $(N^{12}+T, 0) \in V[12]$, so that $N^{12}+T = \bar{v}_1$. Since

$N^{12} \geq N^{13}$, $N^{13}(\bar{v}_1,\bar{v}_3) = \bar{v}_1$. Hence $N^{12} + T = N^{13}(\bar{v}_1,\bar{v}_3)$. Clearly $N^{31}(\bar{v}_1,\bar{v}_3) = g^{31}(N^{13}(\bar{v}_1,\bar{v}_3))$ and hence $N^{31}(\bar{v}_1,\bar{v}_3) = g^{31}(\bar{v}_1) = g^{31}(N^{12} + T)$. We may therefore write $q(T) = (N^{13}(\bar{v}_1,\bar{v}_3), 0, N^{31}(\bar{v}^1,\bar{v}_3))$. From this it follows directly that $\{\bar{b}^{ij}\}$ are the agreements of this multilateral Nash solution. This completes the proof for Case **IIA**.

In the remaining two cases (**IIB** and **III**) the core is empty. Binmore showed that in this case there are three outcomes in the market solution and its payoff vectors form a von Neumann Morgenstern tuple. The von Neumann Morgenstern tuple, when it exists, is unique and the von Neumann Morgenstern payoff vectors are feasible for their coalitions. Hence to show that the market solution coincides with the outcomes of the multilateral Nash solution, we need only show in each case that the agreements of the multilateral Nash solution form a von Neumann Morgenstern tuple.

**Case IIB:** The bargaining problem has an empty core. In this case the corresponding multilateral Nash solution is the extreme multilateral solution whose price vector is $q(T) = = (N12 + T,\ g^{21}(N^{12} + T),\ g^{31}(N^{12} + T))$, and the binding constraint is $g^{31}(N^{12} + T) = g^{32}(g^{21}(N^{12} + T))$; the argument of Theorem 3.2 shows that $g^{31}(N^{12} + T) > 0$ (otherwise the core would not be empty) and hence $(q_1, q_2) \in V(12)$, $(q_2, q_3) \in V(13)$ and $(q_1, q_3) \in V(13)$. Since each $(q_i, q_j)$ is feasible, by Theorem 3.1 $(q_i, q_j)$ is on Pareto boundary of $V(ij)$ and hence $\{(q_i, q_j)\}$ is a von Neumann Morgenstern tuple. This completes the proof of Case **IIB** and with it the proof of **(II)**.

**(III)** This is immediate since III of Theorem 3.2 proved that the agreements of the unique multilateral Nash solution form a von Neumann Morgenstern tuple. This establishes **(III)** and completes the proof of Theorem 5.1.§

## 6. EXAMPLES

In this section we present a series of examples to illustrate the relationship between the cooperative and noncooperative solutions for various classes of bargaining problems. The simplest class of bargaining problems are those with transferable utility: those problems in which each pair has a number of utils to divide. If $r$ is the total number of utils for the pair [ij] then the Nash agreements for the pair are simply $N^{ij} = r/2$ and $N^{ji} = r/2$. Moreover transferable utility bargaining problems always have a Nash stable coalition: the coalition with the maximum number of utils is Nash stable of the first type. To provide an example of Nash stability of the second type and an example of a bargaining problem with no Nash stable coalition we will have to consider bargaining problems with nontransferable utility.

**EXAMPLE 6.1:** For $r$ in the interval $[0,30]$ and $s$ in the interval $[0,20]$, consider the transferable utility bargaining problems in which the feasible sets for the coalitions are:

$$V[12] = \{ (x_1,x_2) \in \mathbf{R}^2_+ \mid x_1 + x_2 \leq 40 \}$$

$$V[23] = \{ (x_2,x_3) \in \mathbf{R}^2_+ \mid x_2 + x_3 \leq r \}$$

$$V[13] = \{ (x_1,x_3) \in \mathbf{R}^2_+ \mid x_1 + x_3 \leq s \}$$

$$V(S) = \{0\} \quad \text{for all other coalitions } S$$

For $r = s = 0$ we may think of the bargaining problem as representing bargaining between one seller and one buyer. For $r > 0$ , $s = 0$ we may think of the bargaining problem as representing the bargaining when there is one seller and two buyers (with different reservation values). For $r > 0$, $s > 0$ we may think of the bargaining problem as representing the negotiations over the formation of a government in a parliamentary system (when no party holds a majority of seats). We distinguish three classes of bargaining problems (I), (IIA) and (IIB), according to the values of the parameters $r$ and $s$.

(I) If $0 \leq r \leq 20$ and $s = 0$ , then the coalition [12] is Nash dominant (Class I), so there is a unique multilateral Nash solution; its price vector is (20, 20, 0). For each $r < 20$, ( [12], (20, 20, -)) is the unique outcome of each of

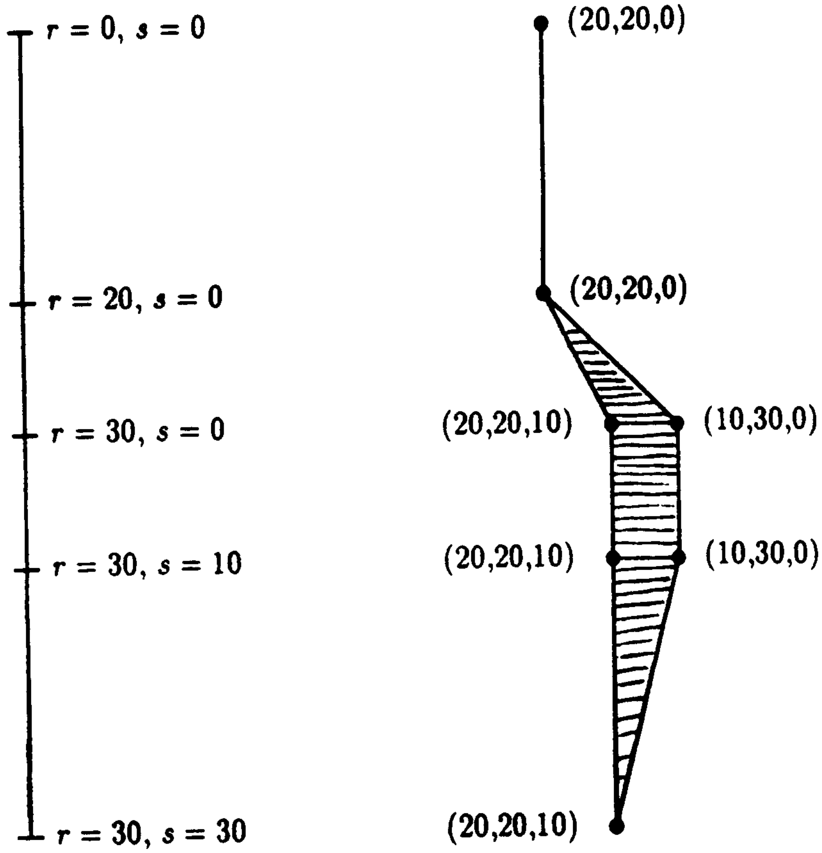


Figure 6: The set of multilateral Nash solutions of example 6.1.

the three models[27]; for $r = 20$, $([23], (-, 20, 0))$ is a second outcome of the three models.

(IIA) If $20 < r \leq 30$ and $s = 0$, the coalition [12] is Nash stable but not Nash dominant, and the core is not empty. The price vectors of multilateral Nash solutions form the interval $\{(20-t, 20+t, r-t) : 0 \leq t \leq r-20\}$. The endpoint $(20, 20, r-20)$ corresponds to the limit solution of the proposal-making model while the endpoint $(40-r, r, 0)$ corresponds to the market solution. For all three models the outcomes correspond to the formation of either [12] or [23].

Notice that for the proposal-making model the division in [12] is unaffected by the possibility of forming [23]; player 2 is unable to use his "outside option" to obtain more favorable terms from player 1. For the market solution the division in [12] is very sensitive to [23]; player 2 obtains the full value of [23] from player 1. Notice too that $(40-r, r, 0)$ is in the core.

For $0 < s \leq 10$, the multilateral solutions and the outcomes of all three models remain unchanged: the coalition [13] is not sufficiently profitable to affect the bargaining.

(IIB) If $r = 30$ and $10 < s \leq 30$, coalition [12] remains Nash stable but not Nash dominant, the core becomes empty, and a von Neumann Morgenstern vector appears. The price vectors of the multilateral Nash solutions form the interval $\{(10+t, 30-t, t) : -5 + s/2 \leq t \leq 10\}$. The endpoint $(20,20,10)$ corresponds to the limit solution of the proposal-making model while the endpoint $(5+s/2, 35-s/2, -5+s/2)$, corresponds to the market solution and is the von Neumann Morgenstern vector. For $s < 30$ the outcomes of the proposal-making model correspond to the formation of [12] and [23]; for $s = 30$ the coalition [13] is also possible. For $10 < s \leq 30$ formation of any of the three pairs is possible in the market solution.

[27] When we refer to "all three models" we mean the cooperative (multilateral Nash solution), our noncooperative model (the proposal-making model) and Binmore's model (the market demand model).

Notice that for the pair [12] the proposal-making model again ignores [23] and [13] while the market solution does not. In the market model the possibility that [13] will form places an upper limit on the maximum concession player 2 can obtain from player 1 based on player 2's outside option.

Figure 6 gives a pictorial representation of the situation for the parameter values discussed above.

As we have earlier pointed out, if there is no Nash stable coalition of type 1 then necessarily, (after renumbering) $N^{12} > N^{13}$, $N^{23} > N^{21}$, and $N^{31} > N^{32}$, i.e., at the Nash payoffs player 1 wants to form a coalition with 2, player 2 wants to form a coalition with 3 and player 3 with 1. It is easy to see that such a situation cannot be created within a transferable utility bargaining problem. The simplest bargaining problems with this structure are hyperplane games. We first present an example of a hyperplane bargaining problem with a Nash stable coalition of type 2 and follow it with a "similar" bargaining problem that has *no* Nash stable coalition.

**EXAMPLE 6.2:** Consider the hyperplane bargaining problem in which the feasible sets for the coalitions are:

$$V[12] = \{ (y_1,y_2) \in \mathbf{R}^2_+ \mid y_1 + 2y_2 \leq 40 \}$$

$$V[23] = \{ (y_2,y_3) \in \mathbf{R}^2_+ \mid y_2 + 2y_3 \leq 30 \}$$

$$V[13] = \{ (y_1,y_3) \in \mathbf{R}^2_+ \mid 2y_1 + y_3 \leq 16 \}$$

$$V(S) = \{0\} \quad \text{for all other coalitions } S$$

From the intuition developed in transferable utility bargaining problems one might expect that [12], the most "profitable" coalition, to be Nash stable, but as we will see, this is not the case. The Nash bargaining solution for $x_i + 2x_j = r$ is simply $N^{ij} = r/2$ and $N^{ji} = r/4$. Hence $\bar{N}^{12} = (20, 10)$, $\bar{N}^{23} = (15, 7.5)$ and $\bar{N}^{13} = (4, 8)$ and $N^{12} > N^{13}$, $N^{23} > N^{21}$, and $N^{31} > N^{32}$.

The coalition [12] is not Nash stable because player 2 can obtain more than $N^{21} = 10$ by offering player 3 even more than her higher Nash payoff $N^{31} = 8$

(proposing ([23], (12, 9)), for instance). The coalition [23] is Nash stable, because player 3 (who would prefer $N^{31} = 8$ to $N^{32} = 7.5$) cannot obtain player 1's cooperation because not only is $N^{12} > N^{13}$, but also $g^{12}(N^{23}) = 10 > g^{13}(N^{32}) = 4.25$.

This bargaining problem is of Class IIB: There is a Nash stable coalition but an empty core. The price vectors of multilateral Nash solutions form the interval $\{(10\text{-}2t, 15+t, 7.5+.5t) \mid 0 \leq t \leq 23/9\}$. The endpoint (10, 15, 7.5) corresponds to the limit solution of the proposal-making model while endpoint $(4.\bar{8}, 17.\bar{5}, 6.\bar{2})$ -- the von Neumann Morgenstern vector-- corresponds to the market solution. Formation of the coalitions [12] and [23] is consistent with limit outcomes of the proposal-making model; while formation of any of the three pairs is consistent with the market solution.

Notice that the two "strong" coalitions are [12] and [23] so that player 2 is the pivotal player. The proposal-making model gives player 2 his higher Nash payoff $N^{23}$ while (by playing off player 1 against 3) the market solution gives player 2 an even higher payoff -- so high indeed that [13] can form without him. Notice also that the components of the von Neumann Morgenstern vector do not represent a compromise between a players higher and lower Nash payoffs; in particular $17.\bar{5} > N^{23} > N^{21}$.

We next consider a Class III bargaining problem: a bargaining problem with no Nash stable coalition.

**EXAMPLE 6.3:** Consider the hyperplane bargaining problem in which the feasible sets for the coalitions are:

$$V[12] = \{ (y_1,y_2) \in \mathbf{R}^2_+ \mid y_1 + 2y_2 \leq 30 \}$$

$$V[23] = \{ (y_2,y_3) \in \mathbf{R}^2_+ \mid y_2 + 2y_3 \leq 30 \}$$

$$V[13] = \{ (y_1,y_3) \in \mathbf{R}^2_+ \mid 2y_1 + y_3 \leq 30 \}$$

$$V(S) = \{0\} \quad \text{for all other coalitions } S$$

For this bargaining problem $\bar{N}^{12} = (15, 7.5)$, $\bar{N}^{23} = (15, 7.5)$ and $\bar{N}^{31} = (15, 7.5)$ so $N^{12} > N^{13}$, $N^{23} > N^{21}$, and $N^{31} > N^{32}$. The coalition [12] is not Nash stable because player 2 can obtain more than $N^{21} = 7.5$ by making player 3 the proposal ([23], (10,10)), for example. Player 3 accepts (10,10) because, although lower than her higher Nash payoff, it is more than she could obtain if she offered player 1 his $N^{12}$ i.e., $g^{31}(N^{12}) = 0 < 10 < g^{32}(N^{21})$. Similarly, no other pair is Nash stable.

This bargaining problem is of Class III; it has an empty core and no Nash stable coalition. The unique multilateral Nash solution has as its price vector (10, 10, 10) which is the von Neumann Morgenstern vector. Formation for any of the three pairs with payoff division of (10,10) are the outcomes for all three models. Notice that each component of the von Neumann Morgenstern vector is a compromise between the player's higher and lower Nash payoff, eg., $N^{12} > 10 > N^{13}$.

## 7. RELATED LITERATURE

The cooperative model presented here is a special case of the multilateral bargaining model introduced in Bennett (1986, revised 1992). For a brief overview of this approach see Bennett (1991a). The cooperative model introduced in Binmore (1985) takes a similar approach to three-player/three-cake problems but uses a selection criterion to guarantee a unique solution for each bargaining problem.

The proposal-making model was invented by Selten (1981). Selten used the framework of recursive games (rather than extensive form games) to model bargaining in multi-player games with transferable utility. The recursive framework requires that players use stationary strategies and allows no discounting. This model has many equilibrium outcomes; Selten provides arguments based on equilibrium selection to choose among them. Bennett (1991a) recasts the proposal-making model as an extensive form game, and uses it to model bargaining in multi-player games without transferable utility. Bennett (1991b) examines the role of stationarity and discounting in such models by showing that, in their absence, anything can happen: every individually rational outcome can be supported by a subgame perfect equilibrium. Chatterjee et al.(1987) introduces discounting into the model, however, the analysis is restricted to stationary strategies and transferable utility.

Binmore's noncooperative model differs from proposal-making models in a fundamental way. In Binmore's model each player in turn states a demand for his coalitional participation and this demand "stays on the table" even if the next player rejects the demand and states his own demand. In setting his demand each player must therefore take into account the other demands "on the table"; this introduces a note of competition into the demand-setting process. In proposal-making models the proposal disappears when it is rejected; alternative proposals therefore only compete implicitly. As a result, Binmore's model is "more competitive": in the one buyer two seller case, for instance, the buyer can extract more of the gains from trade. So far, however, this model has not been extended to more general bargaining situations. Bennett and van Damme (1991) present a related model in which each player in turn states a demand, however, once set, demands cannot be changed. Bennett and van Damme use their model to analyze a class of multi-player games with transferable utility (apex games).

## 8. CONCLUDING REMARKS

In this paper, we have analyzed a class of three-player bargaining problems, which we call "three-player/three-cake" problems. We have addressed the questions of which coalitions might form, and what the agreements within such coalitions might be, supposing that they do form. In the tradition of the Nash program, we have presented cooperative and non-cooperative models, and show that they reinforce each other. Our cooperative analysis is based on an extension of the Nash bargaining solution; our non-cooperative model is based on extensions of the Rubinstein alternating-offer model with discounting. For some problems, our cooperative analysis yields a unique solution (i.e., a unique set of conditional agreements, one for each coalition); for the remainder, our cooperative analysis yields an interval of solutions. Our non-cooperative model yields, for each discount factor less than 1 , a unique solution; as the discount factor tends to 1, these conditional agreements converge to the cooperative solution when it is unique, or to one endpoint of the interval of cooperative solutions.

We have related our cooperative model to a noncooperative model of Binmore (1985). When the cooperative solution is unique, Binmore's solution coincides with our cooperative and non-cooperative solutions; when the cooperative solution is an interval, so that our non-cooperative solution is one endpoint of this interval, Binmore's solution is the other endpoint.

We regard this work as a step in the analysis of general multi-player bargaining problems with many coalitions. We expect the insights obtained here to be of value in more general situations. However, we have gotten a lot of mileage in our noncooperative model from the fact that all relevant coalitions involve only two players, so that we may apply the very sharp predictions of the Rubinstein alternating-offer model. The general case will surely be much more difficult.

## APPENDIX

**PROOF OF THEOREM 3.2:** It is convenient to isolate one simple bit of reasoning, which will be used repeatedly.

- ♦ Let $z = \{\bar{z}^{ij}\}$ be a multilateral Nash solution and let $\{\bar{d}^{ij}\}$ be the corresponding set of disagreements. The definition of $N^{ij}(d^{ij})$ implies that for each $i$ and $j$ , if $\bar{d}^{ij} \neq \bar{z}^{ij}$ then both $\bar{d}^{ij}$ and $\bar{z}^{ij}$ are feasible for [ij]. Moreover, if $z^{ij} > N^{ij}$ then $z^{ij} = d^{ij}$ and this entails feasibility of $\bar{z}^{ik}$ for the pair [ik] (to see why notice that i's outside option $d^{ij}$ is computed from the pair [ik] and i's agreement payoff must be the same both [ij] and [ik]).

**(I)** As we noted in Section 2, a Nash dominant coalition must be unique. Write $p = (N^{12}, N^{21}, 0)$ and $z = \{ (N^{12}, N^{21}, 0), (N^{12}, 0, 0), (0, N^{21}, 0) \}$

To see that $z$ is indeed a solution, notice that dominance means that $g^{31}(N^{12}) = g^{32}(N^{21}) = 0$ . Hence (0,0) is the outside option vector for [12] , so $\bar{N}^{12}$ is its agreement vector. $(N^{12}, 0)$ is the outside option vector for [13] , but means that this outside option vector is not in the interior of V(13) , whence the agreement is also $(N^{12}, 0)$. Similarly, $(N^{21},0)$ is the agreement for [2,3]. This solution is unique because no feasible agreement in any other coalition could cause [12] to change its agreement from $\bar{N}^{12}$. This completes the proof of **(I)**.

**(II)** It is easily verified that if some coalition is Nash-stable then there is a renumbering so that $N^{12} \geq N^{13}$ and $g^{31}(N^{12}) \geq g^{32}(N^{21})$; for [12] *not* to be Nash dominant, necessarily $g^{31}(N^{12}) > 0$. We next show that $q(0) = (N^{12},N^{21}, g^{31}(N^{12}))$ is the price vector of a multilateral Nash solution. If $g^{31}(N^{12}) = g^{32}(N^{21})$ this is evident because then q(0) is a von Neumann Morgenstern vector. To see this when $g^{31}(N^{12}) > g^{32}(N^{21})$ , we analyze the outside options and agreements in each pair in turn. Note first that, in the pair [12] , the outside option value for player 1 is just $N^{12}$ (because $(N^{12}, g^{31}(N^{12}))$ is a feasible agreement for [13] ) and that the outside option value for player 2 is strictly less than $N^{21}$ (since $(N^{21}, g^{32}(N^{(21})})$ is on the boundary of V(23) and $g^{31}(N^{12}) > g^{32}(N^{21})$. Hence the agreement in [12] is $(N^{12}, N^{21})$. Similarly, in [23] the outside option vector is $(N^{21},g^{31}(N^{12}))$ , which is infeasible for [23] and hence the "agreement" is the outside option vector $(N^{21}, g^{31}(N^{12}))$. For the coalition [13] the outside option vector is $(N^{12},g^{32}(N^{21}))$. Since $g^{31}(N^{12}) > g^{32}(N^{21})$, this outside option vector is in the interior of V[13], so the agreement will be the constrained Nash solution from this outside option. Since $N^{12} \geq N^{13}$, player 1's outside option $N^{12}$ is a binding constraint, and $(N^{12}, g^{31}(N^{12}))$ is the agreement. Hence q(0) is indeed the price vector of a multilateral Nash solution, as asserted.

Now for $t \geq 0$, we consider price vectors of the form

$$q(t) = (N^{12}+t,\ g^{21}(N^{12}+t),\ g^{31}(N^{12}+t)).$$

The same reasoning as above shows that q(t) is the price vector of a multilateral Nash solution provided that $g^{21}(N^{12}+t) > 0$, $g^{31}(N^{12}+t) > 0$ and $g^{31}(N^{12}+t) > g^{32}(g^{21}(N^{12}+t))$. These inequalities are satisfied for values of the parameter t in some open interval [0,T) ; continuity implies that q(T) is also the price vector of a multilateral Nash solution, so this yields the desired curve parameter by t in the closed interval [0,T]. Notice that the interval [0,T] might be degenerate, and indeed will be degenerate if $g^{31}(N^{12}) = g^{32}(N^{21})$. Moreover one can easily show that: if $g^{21}(N^{12} + T) = 0$ then q(T) is in the core with [13] as the core coalition; if $g^{31}(N^{21} + T) = 0$ then q(T) is in the core with [12] as the core coalition; if $g^{31}(N^{12}+T) = g^{32}(g^{21}(N^{12}+T))$ is the only binding constraint (and hence $(g^{21}(N^{12}+T), g^{31}(N^{12}+T)) \in V(23)$) then q(T) is a von Neumann Morgenstern vector.

We claim that these are the only price vectors of multilateral Nash solutions. To this end, let $q = (q_1,q_2,q_3)$ be the price vector of a multilateral Nash solution; we must show that $q = q(t)$ for some t. The proof proceeds in a series of steps.

**Step 1:** We show first that $q_1 \geq N^{12}$. Suppose to the contrary that $q_1 < N^{12}$. Since $(q_1,q_3)$ is the agreement in [13], it follows that $q_3 \geq g^{31}(q_1) > g^{31}(N^{12})$. By Case II assumption $g^{31}(N^{12}) \geq g^{32}(N^{21})$. Since $N^{12}$ is feasible for the pair [12], and $q_1 < N^{12}$, we must have $q_2 > N^{21}$ and hence $g^{32}(q_2) \leq g^{32}(N^{21})$. Combining these expressions we have $q_3 > g^{31}(N^{12}) \geq g^{32}(N^{21}) \geq$ g32(q2) and hence $(q_2,q_3)$ is not feasible for the pair [23]. Since $q_2 > N^{21}$ we know $q_2 > 0$, it follows from Theorem 3.1 that $(q_1,q_2)$ must be feasible for [12]. Since $q_1 < N^{12}$ and $q_2 > N^{21}$, reasoning as in ♦ yields that $d^{21} = q_2$. However, since $(q_2,q_3)$ is not feasible for [23], reasoning as in ♦ also yields $d^{21} = g^{23}(q_3) < q_2$.

a contradiction. We conclude that $q_1 \geq N^{12}$, as asserted. Since [12] is Nash stable, $V[12] \neq \{(0,0)\}$ ; in particular, $q_1 > 0$ .

**Step 2:** We show next that $(q_1,q_2)$ is feasible for [12]. If not, Theorem 3.1 and the fact that $q_1 > 0$ imply that $(q_1,q_3)$ is feasible for [13]. Since $(q_1,q_2)$ is not feasible for [12] $d^{13} \leq q_1$ and hence $q_1 \leq N^{13}$ -- and 1 obtain as much as $N^{13}$ only if $q_3 \leq N^{31}$. By case assumption we have $N^{12} \geq N^{13}$ and in Step 1 we proved $q_1 \geq N^{12}$. Hence $q_1 \leq N^{13} \leq N^{12} \leq q_1$ and hence $q_1 = N^{12} = N^{13}$. Since $(q_1,q_2)$ is not feasible for [12] , we must have $q_2 > N^{21}$ and in particular that $q_2 > 0$ . Hence Theorem 3.1 implies that $(q_2,q_3)$ is feasible for [23]. We conclude that $g^{32}(q_2) = q_3 = g^{31}(q_1) = g^{31}(N^{12}) \geq g^{32}(N^{21})$. Since $g^{32}$ is decreasing in player 2's payoff, we conclude that $q_2 \leq N^{21}$ , which is a contradiction. We conclude that $(q_1,q_2)$ is feasible for [12] , as asserted.

**Step 3:** We now show that $(q_1,q_3)$ is feasible for [13]. Assume not, then it is not the case that $g^{31}(q_1) \geq q_3$ ; so $g^{31}(q_1) < q_3$. Reasoning as in ♦, since $(q_1,q_3)$ is not feasible for [13], player 1 can obtain no more than $N^{12}$ in [12]; this and Step 1 imply $q_1 = N^{12}$ ; since $(q_1,q_2)$ is feasible for [12] , we conclude that $q_2 = N^{21}$ . Since [12] is not Nash dominant, we must have $q_3 > 0$, and hence $(q_2,q_3)$ must be feasible for [23]. Hence $q_3 = g^{32}(q_2) = g^{32}(N^{21}) \leq g^{31}(N^{12}) = g^{31}(q_1) < q_3$ . Since this chain of inequalities is inconsistent, we conclude that $(q_1,q_3)$ is feasible for [13] , as asserted.

**Step 4:** We set $t = q_1 - N^{12}$ , so that $q_1 = N^{12} + t$ ; feasibility of $(q_1,q_2)$ for [12] and $(q_1,q_3)$ guarantees that $q_2 = g^{21}(N^{12}+t)$ and $q_3 = g^{31}(N^{12}+t)$ , so $q = q(t)$ , as desired.

This completes the proof of **(II)**.

**(III)** If no coalition is Nash stable then, after a suitable renumbering,

$$(*) \qquad N^{12} > N^{13} \quad , \quad N^{23} > N^{21} \quad , \quad N^{31} > N^{32}$$

It is convenient to first establish two claims.

**CLAIM 1:** If $\{\bar{z}^{ij}\}$ is a multilateral solution then $\bar{z}^{ij} \neq \bar{N}^{ij}$ for every pair [ij] . To see this suppose that $\bar{z}^{12} = \bar{N}^{12}$ (and hence that $\bar{z}^{12}$ is feasible for [12] ). Let

$q$ be the price vector of $\{\bar{z}^{ij}\}$ . Then $(q_1,q_2) = \bar{N}^{12}$ ; since $N^{12} > N^{13}$ and $N^{23} > N^{13}$ and agreements lie on the Pareto boundary, it follows that $q_3 > N^{32}$ , whence (by ♦ ) $q_3 = d^{32} = z^{31}$ and $\bar{z}^{13}$ is feasible for the pair [13]. Hence $q_3 = g^{31}(q_1) = g^{31}(N^{12})$. The definition of $d^{32}$ yields that $q_3 \geq g^{32}(q_2)$ and hence $q_3 \geq g^{32}(N^{21})$. Combining inequalities yields $g^{31}(N^{12}) \geq g^{32}(N^{21})$. This, together with (*), implies that [12] is Nash stable, a contradiction; this establishes CLAIM 1.

**CLAIM 2:** No player obtains more than his higher Nash payoff in any agreement of a multilateral Nash solution. Assume not then there is a player, say player 1, who obtains more than $\max\{N^{12},N^{13}\}$ at the multilateral solution $\{\bar{z}^{ij}\}$. Let $q = (q_1,q_2,q_3)$ be the price vector of $\{\bar{z}^{ij}\}$. Clearly $q_1 > \max\{N^{12},N^{13}\}$ (i.e., player 1 must obtain more than his higher Nash in his agreement in *both* coalitions so $z^{12} > N^{12}$ and $z^{13} > N^{13}$ ). Reasoning as in ♦ we know that $\bar{z}^{12}$ and $\bar{z}^{13}$ must be feasible for their coalitions. Let $Z$ be the set of multilateral solutions $\{\bar{z}^{ij}\}$ with the property that $\bar{z}^{12}$ and $\bar{z}^{13}$ are feasible and $z^{12} \geq N^{12}$ and $z^{13} \geq N^{13}$. It is evident that $Z$ is a compact set, so there is a multilateral solution $\{\bar{y}^{ij}\} \in Z$ for which $y^{12}$ is minimized. In view of CLAIM 1, $y^{12} > N^{12}$ and $y^{13} > N^{13}$. Hence, if $\varepsilon > 0$ is small enough, $y^{12} - \varepsilon > N^{12}$ and $y^{13} - \varepsilon > N^{13}$. One can then easily verify that the vector $(y^{12} - \varepsilon, g^{21}(y^{12} - \varepsilon), g^{31}(y^{12} - \varepsilon))$ is the price vector of a multilateral Nash solution belonging to $Z$ , which is a contradiction. This proves CLAIM 2.

**CLAIM 3:** *Every* multilateral Nash solution is a von Neumann Morgenstern tuple. To this end, let $\{z^{ij}\}$ be a multilateral Nash solution, and let $q = (q_1,q_2,q_3)$ be its price vector. At least one of the agreements, say $\bar{z}^{12}$ , is feasible. In view of CLAIM 1, there are two possibilities to consider: either $q_1 < N^{12}$ and $q_2 > N^{21}$, or $q_1 > N^{12}$ and $q_1 < N^{21}$.

If $q_1 < N^{12}$ and $q_2 > N^{21}$ , it follows from ♦ that $\bar{z}^{23}$ is feasible. From CLAIM 1 it follows that $\bar{z}^{23} \neq N^{23}$ and from CLAIM 2 it follows that $q_2 < N^{23}$ and hence that $q_3 > N^{23}$ . Another application of ♦ yields that $\bar{z}^{13}$ is feasible. If $q_1 > N^{12}$ and $q_2 < N^{12}$ , the argument is exactly the same, except that the roles of players 1 and 2 are reversed.

In either case all three agreements are feasible (and hence, by Theorem 3.1, on the Pareto boundaries for their coalitions, with $z^{ij} = z^{ik}$ for all i, j, k). We conclude that $\{\bar{z}^{ij}\}$ is a von Neumann Morgenstern tuple. This proves CLAIM 3.

With these claims in hand, the remainder of the proof of **(III)** is simple. Theorem 3.1 guarantees the existence of a multilateral Nash solution. Claim III showed that every multilateral solution is a von Neumann Morgenstern tuple. As we have already noted, if a von Neumann Morgenstern tuple exists it is unique. Hence in this case there is a unique multilateral Nash solution whose price vector is a von Neumann Morgenstern vector.

This completes the proof of **(III)**, and with it the proof of Theorem 3.2. §

**PROOF OF THEOREM 4.1:** It is convenient to break the proof of Theorem 4.1 into a series of lemmas and propositions. Before beginning, we define, for each $i$, the auxiliary function $g^{i*} : \mathbf{R}_+^2 \to \mathbf{R}_+$ by setting $g^{i*}(x_j,x_k) = \max\{ g^{ij}(x_j) , g^{ik}(x_k) \}$. Note that $g^{i*}$ is continuous and (weakly) decreasing. The first proposition guarantees that the system ♣ has at least one solution.

**PROPOSITION 4.1.1:** For every $\delta < 1$, the system ♣ has at least one solution.

**PROOF:** Choose $M$ so large that, for each $i, j$, $V[ij]$ is contained in the square $[0,M]^2$. Define $G : [0,M]^3 \to [0,M]^3$ by

$$G(p_1,p_2,p_3) = (g^{1*}(\delta p_2,\delta p_3), g^{2*}(\delta p_1,\delta p_3), g^{3*}(\delta p_1,\delta p_2))$$

It is easily seen that $G$ is a continuous mapping of the cube into itself, and hence has a fixed point; such a fixed point solves the system ♣. §

The argument that price vectors are unique makes use of a notion similar to that of Nash stability (see Section 2), but deriving from the payoffs of the alternating offer model. Write $A^{ij}(\delta)$ for the payoff to player $i$ in the alternating offer model in which players $i$ and $j$ bargain over $V[ij]$, with the common discount factor $\delta$, using the disagreement point $(0,0)$, and player $i$ makes the first offer; note that

$p_i = A^{ij}(\delta)$ and $p_j = A^{ji}(\delta)$ is the unique solution to the pair of equations $p_i = g^{ij}(\delta p_j)$ and $p_j = g^{ji}(\delta p_i)$. When the discount factor $\delta$ is fixed (which will be the case for the remainder of the proof of Theorem 4.1), we will avoid distraction by writing $A^{ij}$ instead of $A^{ij}(\delta)$. We say that the pair [ij] is *A-stable* if either

1. $A^{ij} \geq A^{ik}$ and $A^{ji} \geq A^{jk}$, or

2. $A^{ij} \geq A^{ik}$, $A^{ji} < A^{jk}$, and $\delta g^{ki}(\delta A^{ij}) \geq g^{kj}(A^{ji})$, or

3. $A^{ij} < A^{ik}$, $A^{ji} \geq A^{jk}$, and $\delta g^{kj}(\delta A^{ji}) \geq g^{ki}(A^{ij})$

In the first case (type 1 stability ), players $i$, $j$ prefer their alternating offer payoffs in the partnership [ij] to their payoffs in the partnership with player $k$. In the second case and third cases (type 2 stability ), one of $i$, $j$ would prefer to be matched with player $k$, but cannot obtain $k$'s cooperation, because $k$ would prefer to wait a period and form a partnership with the other.

The crucial information we require about the alternating offer model is embodied in the following lemma. The reader familiar with the alternating offer model will recognize the argument.

**LEMMA 4.1.2:** Let $p$ be a reservation price vector such that $(\delta p_i, p_j)$ is on the boundary of V[ij]. If $p_i < A^{ij}$, then $(p_i, \delta p_j)$ is in the interior of V[ij]; if $p_i = A^{ij}$ then $(p_i, \delta p_j)$ is on the boundary of V[ij]; if $p_i > A^{ij}$ then $(p_i, \delta p_j)$ is outside V[ij].

**PROOF:** Write $f(x) = g^{ji}(x) - \delta g^{ji}(\delta x)$; we assert that $f$ is strictly decreasing over the range $0 \leq x \leq x^*$ (where $x^*$ is the largest value of $x$ such that $(x,0) \in$ V[ij]). To see this, consider first the case in which $g^{ji}$ (and hence $f$) is continuously differentiable. Since $g^{ji}$ is decreasing and concave, its derivative $(g^{ji})'$ is negative and decreasing; thus $f'(x) = (g^{ji})'(x) - \delta^2(g^{ji})'(\delta x) < 0$, as desired. If $g^{ji}$ is not continuously differentiable, neither is $f$. However, $g^{ji}$ is decreasing and concave, so its subdifferential is negative (i.e., consists entirely of negative numbers) and decreasing (in the sense that, if $x > x'$ then every element of the subdifferential at $x$ is smaller than every element of the subdifferential at $x'$). Arguing as above allows us to conclude that the subdifferential of $f$ is negative, so that $f$ is decreasing, as desired. We also note that $f(A^{ij}) = 0$ .

If $p_i < A^{ij}$ , then $f(p_i) > f(A^{ij}) = 0$ and $g^{ji}(p_i) > \delta g^{ji}(\delta p_i)$ . Since $(\delta p_i, p_j)$ is on the boundary of $V[ij]$ , it follows that $p_j = g^{ji}(\delta p_i)$ . Hence $g^{ji}(p_i) > \delta p_j$ , so $(p_i, \delta p_j)$ is in the interior of $V[ij]$ .

If $p_i > A^{ij}$ , similar reasoning shows that $g^{ji}(p_i) < \delta p_j$ , so that $(p_i, \delta p_j)$ is not in $V[ij]$ .

Finally, if $p_i = A^{ij}$ then $g^{ji}(\delta A^{ij}) = A^{ji}$ so $p_j = A^{ji}$ , and $(\delta p_i, p_j)$ is on the boundary of $V[ij]$ . §

Our analysis of reservation price vectors proceeds by analyzing reflexive and triangular reservation price vectors in turn. Our first task is to relate reflexive reservation price vectors to A-stability.

**PROPOSITION 4.1.3:** The vector $p = (A^{12}, A^{21}, g^{3*}(A^{12}, A^{21}))$ is a reservation price vector if and only if $[12]$ is an A-stable coalition.

**PROOF:** Suppose that $p$ is a reservation price vector and that $[12]$ is not A-stable. Renumbering if necessary, we must have $A^{12} < A^{13}$ and $\delta g^{32}(\delta A^{21}) < g^{31}(A^{12})$ , so that $p_1 < A^{13}$ and $\delta g^{32}(\delta p_2) < g^{31}(p_1)$ . Since $p_3 = g^{3*}(A^{12}, A^{21})$ , there are two cases to consider. If $p_3 = g^{31}(\delta p_1)$ then $(\delta p_1, p_3)$ is on the boundary of $V[13]$ . Since $p_1 < A^{13}$ , Lemma 4.1.2 implies that $(p_1, \delta p_3)$ is in the interior of $V[13]$ , contradicting the fact that $p$ solves the system ♣ . On the other hand, if $p_3 = g^{32}(\delta p_2)$ then $\delta g^{32}(\delta A^{21}) < g^{31}(A^{12})$ , so that $g^{31}(p_1) > \delta p_3$ . Hence, $p_1 < g^{13}(\delta p_3)$ , again contradicting the fact that $p$ solves the system ♣ . We conclude that $[12]$ is A-stable, as desired.

Conversely, suppose that $[12]$ is A-stable; we show that $p$ is a reservation price vector. The definitions of the alternating offer payoffs and of the function $g^{3*}$ mean that $p_1 = g^{12}(\delta p_2)$ , $p_2 = g^{21}(\delta p_1)$ , and $p_3 = \max\{g^{31}(\delta p_1), g^{32}(\delta p_2)\}$ ; what remains is to show that $(p_1, \delta p_3)$ is not in the interior of $V[13]$ and that $(p_2, \delta p_3)$ is not in the interior of $V[23]$ . We give the argument for $(p_1, \delta p_3)$ ; the argument for

$(p_2,\delta p_3)$ involves only reversing the roles of 1 and 2. There are two cases to consider.

**Case 1:** $p_1 \geq A^{13}$. If $g^{31}(\delta p_1) \geq g^{32}(\delta p_2)$ then $p_3 = g^{31}(\delta p_1)$. Hence $(\delta p_1, p_3)$ is on the boundary of V[13] and Lemma 4.1.2 shows that $(p_1,\delta p_3)$ is not interior to V[13]. On the other hand, if $g^{31}(\delta p_1) < g^{32}(\delta p_2)$ then $p_3 = g^{32}(\delta p_2) > g^{31}(\delta p_1)$ and $(p_1,\delta p_3)$ does not belong to V[13] at all.

**Case 2:** $p_1 < A^{13}$. Since [12] is A-stable, $\delta g^{32}(\delta A^{21}) \geq g^{31}(A^{12})$ so that $\delta g^{32}(\delta p_2) \geq g^{31}(p_1)$. Since $p_1 < A^{13}$, it follows (from the proof of 4.1.2) that $g^{31}(p_1) > \delta g^{31}(\delta p_1)$ so that $g^{32}(\delta p_2) > g^{31}(\delta p_1)$, and hence $p_3 = \max\{ g^{31}(\delta p_1), \delta g^{31}(\delta p_1)\} = g^{32}(\delta p_2)$. Since $p_3 > g^{31}(\delta p_1)$ we again conclude that $(p_1,\delta p_3)$ does not belong to V[13] at all.

We conclude that p solves ♣, as desired. §

After one small lemma, we can now show that there is at most one reflexive reservation price vector.

**LEMMA 4.1.4:** If [ij] is A-stable with $A^{ij} \geq A^{ik}$ and $A^{ji} < A^{jk}$, then $A^{ki} > A^{kj}$.

**PROOF:** Since $A^{ij} \geq A^{ik}$, it follows that $\delta g^{ki}(\delta A^{ij}) \leq \delta g^{ki}(\delta A^{ik}) = \delta A^{ki}$. Similarly, from the fact that $A^{ji} < A^{jk}$, we obtain $g^{kj}(A^{ji}) > g^{kj}(A^{jk}) = \delta A^{jk}$. Stability requires $g^{kj}(A^{ji}) \leq \delta g^{ki}(\delta A^{ij})$, and combining all three expressions yields the desired result. §

**PROPOSITION 4.1.5:** The system ♣ admits at most one reflexive reservation price vector.

**PROOF:** Let p and q be reflexive reservation price vectors. If they are distinct, we may renumber so that $p = (A^{12},A^{21},p_3)$ and $q = (q_1,A^{23},A^{32})$. Hence [12] and [23] are A-stable; we distinguish three cases, according to the type of stability.

**Case 1:** [12] and [23] are type 1 stable. Necessarily $A^{12} \geq A^{13}$, $A^{21} = A^{23}$, $A^{32} \geq A^{31}$, so that $g^{31}(\delta A^{12}) \leq g^{31}(\delta A^{13}) = A^{31}$ and $g^{32}(\delta A^{21}) = g^{32}(\delta A^{23}) = A^{32}$. Combining these yields $g^{32}(\delta A^{21}) \geq g^{31}(\delta A^{12})$, so that $p_3 = g^{32}(\delta A^{21}) = A^{32} = q_3$. Similar reasoning shows that $g^{13}(\delta A^{32}) \leq A^{13} \leq A^{12} = g^{12}(\delta A^{23})$ and thus $q_1 = A^{12} = p_1$. Hence $p = q$.

**Case 2:** one coalition is type 1 stable and the other is type 2 stable. In light of Lemma 4.1.4, [12] must be the type 1 stable coalition and [23] must be the type 2 stable coalition and moreover $A^{31} > A^{32}$, $A^{12} > A^{13}$ and $A^{23} = A^{21}$. Stability of [23] requires $\delta g^{12}(\delta A^{23}) \geq g^{13}(A^{32})$ or equivalently (since $A^{23} = A^{21}$) $\delta A^{12} \geq g^{13}(A^{32})$ or equivalently, $g^{31}(\delta A^{12}) \leq A^{32}$. Now $A^{32} = g^{32}(\delta A^{23})$ and hence $g^{32}(\delta A^{23}) \geq g^{31}(\delta A^{12})$ or equivalently (since $A^{23} = A^{21}$) $g^{32}(\delta A^{21}) \geq g^{31}(\delta A^{12})$ so that $p_3 = \max\{g^{31}(\delta p_1), g^{32}(\delta p_2)\} = g^{32}(\delta A^{21}) = g^{32}(\delta A^{23}) = A^{32} = q_3$. Similarly, $g^{12}(\delta A^{23}) \geq g^{13}(\delta A^{32})$ so that $q_1 = A^{12} = p_1$. Again, this forces $p = q$.

**Case 3:** both of [12] and [23] are type 2 stable. We show this leads to a contradiction; i.e., it is impossible for two coalitions to be A-stable of type 2. We may assume that $A^{12} \geq A^{13}$ and $A^{23} > A^{21}$; Lemma 4.1.4 implies that $A^{31} > A^{32}$. Stability of [12] implies that $\delta g^{31}(\delta A^{12}) \geq g^{32}(A^{21})$; since $A^{21} < A^{23}$ and $g^{32}(A^{23}) = \delta A^{32}$, this yields $\delta g^{31}(\delta A^{12}) > \delta A^{32}$, which in turn yields $\delta A^{12} = g^{13}(g^{31}(\delta A^{12})) < g^{13}(A^{32})$. On the other hand, we know that $A^{23} > A^{21}$, so that $\delta g^{12}(\delta A^{23}) < \delta g^{12}(\delta A^{21}) = \delta A^{12}$. Combining these inequalities yields $\delta g^{12}(\delta A^{23}) < g^{13}(A^{32})$, which contradicts stability of [23].

This completes the proof of Case 3, and with it, the proof of Proposition 4.1.5. §

The analysis now turns to triangular reservation price vectors. It is convenient to say that the triangular vector $p$ is in *standard form* if $(p_1,\delta p_2) \in V[12]$, $(p_2,\delta p_3) \in V[23]$ and $(p_3,\delta p_1) \in V[31]$; $p$ is in *reverse form* if $(p_1,\delta p_3) \in V[13]$, $(p_2,\delta p_1) \in V[21]$ and $(p_3,\delta p_2) \in V[32]$. Every triangular vector is in one of these two forms; note that permuting the numbering of players 1, 2 converts standard form to reverse form. In either case, since $p$ solves the system ♣, it follows that the respective vectors are on the boundaries of the respective sets. Our first task is to obtain bounds for the components of any triangular reservation price vector.

**PROPOSITION 4.1.6:** If $p$ is a triangular reservation price vector in standard form, then $A^{13} \leq p_1 \leq A^{12}$, $A^{21} \leq p_2 \leq A^{23}$ and $A^{32} \leq p_3 \leq A^{31}$.

**PROOF:** Suppose, to the contrary, that $p_1 < A^{13} \leq A^{12}$. Standard form implies that $(\delta p_1, p_3)$ is on the boundary of $V[13]$. Since $p_1 < A^{13}$, Lemma 4.1.2 implies that $(p_1, \delta p_3)$ is in the interior of $V[13]$, contradicting the fact that $p$ solves the system ♣ ; we conclude that $p_1 \geq A^{13}$, a contradiction. The other inequalities follow by similar reasoning. §

We can now show that triangular reservation price vectors are unique.

**PROPOSITION 4.1.7:** The system ♣ admits at most one triangular reservation price vector.

**PROOF:** If $p$, $q$ are triangular, renumber so that $p$ is in standard form; by definition, $p_1 = g^{12}(\delta g^{23}(\delta g^{31}(\delta p_1)))$. If $q$ is also in standard form, then $q_1 = g^{12}(\delta g^{23}(\delta g^{31}(\delta q_1)))$. However, since the functions $g^{ij}$ are decreasing, the equation $x = g^{12}(\delta g^{23}(\delta g^{31}(\delta x)))$ can have at most one fixed point, whence $p_1 = q_1$ ; it follows immediately that $p_2 = q_2$ and $p_3 = q_3$. If $q$ is in reverse form, Proposition 4.1.6 shows that $A^{13} \leq p_1 \leq A^{12}$ and $A^{12} \leq q_1 \leq A^{13}$. Hence $A^{13} = p_1 = q_1 = A^{12}$ ; it again follows immediately that $p_2 = q_2$ and $p_3 = q_3$, so the proof is complete. §

**PROPOSITION 4.1.8:** The system ♣ admits a unique reservation price vector.

**PROOF:** In view of our previous results, we need only show that the existence of a reflexive vector $p$ and a triangular vector $q$ which is not reflexive leads to a contradiction. Without loss, we may assume that $p = (A^{12}, A^{21}, g^{3*}(A^{12}, A^{21}))$ and (permuting the numbering of players 1 ,2 if necessary) that $q$ is in standard form. According to Proposition 4.1.6, $A^{13} \leq q_1 \leq A^{12}$, $A^{21} \leq q_2 \leq A^{23}$, and $A^{32} \leq q_3 \leq A^{31}$ ; because $q$ is not reflexive, all of these inequalities must be strict: $A^{13} < q_1 < A^{12}$, $A^{21} < q_2 < A^{23}$, and $A^{32} < q_3 < A^{31}$.

The vector p is reflexive, so Proposition 4.1.3 guarantees that [12] is A-stable. Since $A^{21} < A^{23}$, the pair [12] cannot be A-stable of type 1, so it must be A-stable of type 2 (with $i = 1$ and $j = 2$).

Since $q_3 = g^{31}(\delta q_1)$, the vector $(\delta q_1, q_3)$ is on the boundary of $V[13]$, whence $\delta q_1 = g^{13}(q_3)$ ; similarly, $\delta q_3 = g^{32}(q_2)$. Thus $q_1 = (1/\delta)g^{13}[(1/\delta)g^{32}(q_2)]$. Triangularity implies that $q_2 \geq A^{21}$ ; applying $g^{32}$ yields $g^{32}(q_2) \leq g^{32}(A^{21})$. Stability means that $\delta g^{31}(\delta A^{12}) \geq g^{32}(A^{21})$. Combining the last two inequalities, multiplying through by $1/\delta$ and reading the extremes yields $g^{31}(\delta A^{12}) \geq (1/\delta)g^{32}(q_2)$. Now applying $g^{13}$ yields $\delta A^{12} = g^{13}[g^{31}(\delta A^{12})] \leq g^{13}((1/\delta)g^{32}(q_2))$. Plugging into the expression for $q_1$ ,we obtain

$$q_1 = (1/\delta)g^{13}((1/\delta)g^{32}(q_2)) \geq (1/\delta)\delta A^{12} = A^{12}$$

However, the assumption that q is in standard form and is not reflexive means that $q_1 < A^{12}$ ; this is the contradiction we seek, and the proof is complete. §

The last Proposition establishes the uniqueness of reservation price vectors, and completes the proof of the first part of Theorem 4.1.

We now turn to the second part of Theorem 4.1, concerning subgame perfect equilibrium prices.

**PROPOSITION 4.1.9:** Every price strategy profile corresponding to $p(\delta)$ constitutes a subgame perfect equilibrium.

**PROOF:** Let $\sigma$ be a price strategy corresponding to $p(\delta)$. Consider a player i who is to make a proposal following the history h . Playing according to $\sigma$ will lead i to make a proposal which is accepted and yields $p_i(\delta)$. Since $p(\delta)$ solves ♣ , no higher proposal will be accepted, and any lower proposal will yield less than $p_i(\delta)$ (assuming other players follow $\sigma$ ). Moreover, any proposal at a later period, either made to i or made by i and accepted by another player, will yield less than $p_i(\delta)$ (in present terms). Hence no unilateral deviation by i improves upon $\sigma_i$ ; i.e., $\sigma$ is subgame perfect. §

The remaining two parts of Theorem 4.1 characterize the SPE and SSP equilibria. The next proposition provides an upper bound on subgame perfect equilibrium payoffs of each player. Consider the set of all subgame perfect equilibria, and, for each of these subgame perfect equilibria, the payoffs obtained by player $i$ in subgames in which he has the initiative; let $M_i$ be the supremum and $m_i$ the infimum of these payoffs. For simplicity let $p$ denote the (unique) reservation price vector $p(\delta)$ of the bargaining game. We first argue that no player can obtain a payoff higher than his "larger" alternating offer payoff.

**PROPOSITION 4.1.10:** For each $\delta < 1$ and each player $i$, if $x_i$ is the payoff to player $i$ in a subgame perfect outcome then $x_i \leq \max \{A^{ij},A^{ik}\}$, in particular $M_i \leq \max \{A^{ij},A^{ik}\}$.

**PROOF:** Subgame perfection requires that player $i$ accept any offer exceeding $\delta M_i$ ; thus, subgame perfection also requires that neither $j$ nor $k$ ever offer player $i$ more than $\delta M_i$; hence $x_i \leq M_i$. The only way player $i$ can obtain as much as $M_i$ is if he proposes it and his proposal is accepted. Suppose, contrary to hypothesis, that $M_i > \max \{A^{ij},A^{ik}\}$. To obtain this payoff suppose that $i$ proposes $(z_i,\delta z_j)$ to player $j$ with $z_i \geq M_i$. Since $M_i > A^{ij}$, Lemma 4.1.2 guarantees that $(\delta z_i, z_j)$ is in the interior of $V(ij)$. Hence there is a $\bar{z} \in V(ij)$ with $\bar{z} >> z$ such that player $j$ prefers $\bar{z}_j$ one period from now (after rejecting the proposal) to accepting $\delta z_j$ now and player $i$ prefers $\delta\bar{z}_i$ one period from now to $z_i$ two periods from now (after rejecting $\bar{z}_i$). Hence player $j$ will reject the offer. Similar reasoning proves that $M_i \leq A^{ik}$. Hence $i$ cannot obtain more than $\max \{A^{ij},A^{ik}\}$, as asserted. This completes the proof of Proposition 4.1.10. §

**PROPOSITION 4.1.11:** If $p(\delta)$ is reflexive and $\sigma$ is a subgame perfect equilibrium then $\sigma$ is a price strategy corresponding to the reservation price vector $p(\delta)$.

**PROOF:** We first show that $m_i = p_i = M_i$. Renumber players so that $p = p(\delta) = (A^{12}, A^{21}, g^{31}(A^{12}))$ . Proposition 4.1.3 shows that $[12]$ is an A-stable coalition; it is convenient to consider the two types of A-stability separately.

**Case 1:** $[12]$ is A-stable of type $1$ . Then $A^{12} \geq A^{13}$ and $A^{21} \geq A^{23}$ . Since player $1$ obtains $p_1 = A^{12}$ when all players follow a price strategy, $M_1 \geq A^{12}$ , so our previous inequality guarantees that $M_1 = A^{12}$. Similarly, $M_2 = A^{21}$. On the other hand, if player 1 has the initiative, he can offer player 2 slightly more than $\delta M_2 = \delta A^{21}$, and be sure that player 2 will accept. Hence $m_1 \geq g^{12}(\delta A^{21}) = A^{12}$, whence $m_1 = p_1 = A^{12} = M_1$. Similarly, $m_2 = p_2 = A^{21} = M_2$. When player 3 has the initiative, he can offer player 1 slightly more than $\delta M_1$, which will certainly be accepted; however, any offer less than $\delta m_1$ will certainly be rejected. Since $m_1 = M_1 = p_1$, this means that player 3 will obtain exactly $g^{31}(\delta A^{12}) = g^{31}(\delta p_1)$ in any subgame in which he is initiator and proposes to player 1. By assumption, $g^{31}(\delta A^{12}) \geq g^{32}(\delta A^{21})$, so player 3 would do no better proposing to player 2. Hence $m_3 = g^{31}(\delta A^{12}) = p_3 = M_3$, which completes the desired conclusion for this case.

**Case 2:** [12] is A-stable of type 2. Without loss suppose that $A^{23} > A^{21}$; then necessarily $A^{12} \geq A^{13}$ and $A^{31} > A^{32}$. Arguing as in Case 1, we see that $M_1 = A^{12}$ and $M_2 \leq A^{23}$. An offer by player 3 to player 1 will be accepted if it yields player 1 more than $\delta M_1 = \delta A^{12}$; hence $m_3 \geq g^{31}(\delta A^{12})$. Any proposal by player 2 to player 3 that yields player 3 less than $\delta m_3$ will be rejected, so the most player 2 can obtain from player 3 is $g^{23}(\delta m_3) \leq g^{23}(\delta g^{31}(\delta A^{12}))$. Stability implies that $\delta g^{31}(\delta A^{12}) \geq g^{32}(A^{21})$; substituting into the previous inequality yields $g^{23}(\delta m_3) \leq g^{23}(g^{32}(A^{21})) = A^{21}$. Hence player 2 could obtain no more than $A^{21}$ by proposing to player 3.

We need to determine what player 2 can obtain by proposing to player 1; to this end we first ask what offer player 1 would make to player 2. In the absence of player 3, Rubinstein's analysis tells us that player 1 would always offer $(A^{12},\delta A^{21})$, and that such an offer would be accepted. In the presence of player 3, such an offer would be rejected only if player 2 could do better with player 3; as we have shown, he cannot. So player 1 would offer $(A^{12},\delta A^{21})$, which player 2 would have to accept. Finally, what will player 2 propose to player 1? Demanding more than $A^{21}$ will be rejected, since player 1 will do better by rejecting and making the counterproposal $(A^{12},\delta A^{21})$ in the following period. Hence player 2 obtains no more than $A^{21}$ from player 1. Combining the analyses of this and the preceding paragraph, we conclude that $M_2 = A^{21}$. We may now argue as in Case 1 to conclude that $m_1 = p_1 = A^{12} = M_1$, $m_2 = p_2 = A^{21} = M_2$, and $m_3 = p_3 = g^{31}(\delta A^{12}) = M_3$, which is the desired result for this case, We next argue that every subgame perfect equilibrium strategy is a price strategy. Player $i$ will not accept any proposal that assigns her less than $\delta m_i = \delta p_i$ (since she can obtain at least $m_i$ one period later by the rejecting the proposal and taking the initiative). Player $i$ must accept any offer that assigns her at least $\delta M_i = \delta p_i$. If $p_i > 0$, as initiator player $i$ must propose $m_i = M_i = p_i$ for herself and (to make the offer acceptable) $\delta p_j$ to a player $j$ for which $(p_i, \delta p_j) \in V(ij)$. Hence $\sigma$ is a price strategy corresponding to $p = p(\delta)$.§

This completes the proof of the third part of Theorem 4.1. The fourth and final part of Theorem 4.1 asserts that if $p(\delta)$ is triangular and $\sigma$ is a stationary subgame perfect strategy then $\sigma$ is a price strategy corresponding to $p(\delta)$. In fact this is so whether or not $p(\delta)$ is triangular.

**PROPOSITION 4.1.12:** If $\sigma$ is stationary and subgame perfect then $\sigma$ is a price strategy corresponding to $p(\delta)$.

**PROOF:** Since $\sigma$ is stationary, each player $i$ obtains the same payoff in every subgame in which he has the initiative; call this common value $q_i$. Player $i$ must accept any proposal yielding more than $\delta q_i$ (since that is the most $i$ can obtain following rejection): hence other players need never offer $i$ more than $\delta q_i$. Similarly, $i$ will never offer $j$ more then $\delta q_j$ and $j$ will accept no less. Moreover, $i$ will never obtain $q_i$ unless he is the initiator. It follows that $q_i = \max \{g^{ij}(\delta q_j), g^{ik}(\delta q_k)\}$ for each $i$, so that $q = (q_1, q_2, q_3)$ satisfies ♣. Hence $q = p(\delta)$. Our reasoning above shows that $\sigma$ is a price strategy. §

This completes the proof of Theorem 4.1.

Before turning to the proof of Theorem 4.2, we record some remarks about the case of triangular reservation prices. As we have shown, if $p(\delta)$ is triangular, there is a unique stationary subgame perfect equilibrium; there may, however, be many non-stationary subgame perfect equilibria. To describe the range of subgame perfect outcomes in this case, fix a player $i$ and suppose for the sake of definiteness that $A^{ij} \leq A^{ik}$. There is a subgame perfect equilibrium outcome in which player $i$ belongs to the coalition that forms and obtains the payoff $x$ if and only if $\delta A^{ij} \leq x \leq A^{ik}$. For those bargaining problems for which $p(\delta)$ remains triangular as $\delta \to 1$, the limit outcomes for player $i$ span the range from $N^{ij}$ to $N^{ik}$. We note that such bargaining problems are exactly those of Class III (no Nash stable coalition). (Details are available on request.)

We turn next to Theorem 4.2. Since discount factors are no longer fixed, we resume writing $A^{ij}(\delta)$ rather than $A^{ij}$.

**PROOF OF THEOREM 4.2:** It is convenient to first isolate two bits of reasoning. Suppose first that $\{\delta_n\}$ is a sequence of discount factors tending to 1 and that the corresponding reservation price vectors are triangular and converge to some vector p. Continuity of the functions $g^{ij}$ implies that p is a von Neumann Morgenstern vector, and convergence of the alternating offer solutions to the Nash solution implies that each component of p lies between the corresponding components of the Nash solution; i.e. $N^{ij} \leq p_i \leq N^{ik}$ .

Suppose next that $\{\delta_n\}$ is a sequence of discount factors tending to 1 and that the corresponding reservation price vectors are reflexive and converge to some vector p ; renumbering and passing to a subsequence if necessary, assume that $p(\delta_n) = (A^{12}(\delta_n), A^{21}(\delta_n), g^{31}(\delta_n A^{12}(\delta_n)))$ , and that the coalition [12] is A-stable of type $\ell$ ($\ell = 1$ or $\ell = 2$). Rubinstein's results on the alternating offer model imply that $A^{12}(\delta_n) \to N^{12}$ and $A^{21}(\delta_n) \to N^{21}$ , and continuity implies that $g^{31}(\delta_n A^{12}(\delta_n)) \to g^{31}(N^{12})$ . It follows that the coalition [12] is Nash stable of type $\ell$ . It then follows from Theorem 3.1 that $p' = q(0) = (N^{12}, N^{21}, g^{31}(N^{12}))$ , which is the price vector of a multilateral Nash solution (and corresponds to one endpoint for problems of class II).

Now consider the entire family $\{p(\delta)\}$ of reservation price vectors. This is a bounded subset of $\mathbf{R}^3$ , so to prove that it converges it suffices to show that any two convergent subsequences must have the same limits. To this end, suppose that $\{\delta_n\}$ , $\{\delta_n'\}$ are sequences of discount factors tending to 1 , that $p(\delta_n) \to p$ and that $p(\delta_n') \to p'$ , with $p \neq p'$ . There are three cases to consider.

**Case 1:** If both sequences of reservation price vectors are triangular, the limits are von Neumann Morgenstern vectors. Since von Neumann Morgenstern vectors are unique (when they exist), we conclude in this case that $p = p'$ .

**Case 2:** If the first sequence is triangular and the second is reflexive, it follows that p is a von Neumann Morgenstern vector and that some pair is Nash stable, so that the problem is of class II. In view of Theorem 3.1, the von Neumann Morgenstern vector corresponds to the endpoint of the set of multilateral Nash solutions with $t = T$ . However, since each component of $p$ lies between the corresponding components of the Nash solution, this can only be the case when $T = 0$ , so that $p = q(0) = p'$ .

**Case 3:** If both sequences are reflexive, both $p$ and $p'$ are the price vectors of the same endpoint of the set of multilateral Nash solutions, and so coincide.

We conclude that $\lim_{\delta \to 1} p(\delta)$ exists. The analysis above provides the requisite information about the limit, so this completes the proof of Theorem 4.2. §

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