Debt Constraints and Equilibrium in Infinite Horizon Economies with Incomplete Markets

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July 10, 1992

WORKING PAPER NO. 666

*We are grateful to Michael Magill and Martine Quinzii for conversations which changed our thinking on this problem, and for access to their unpublished work; to Darrell Duffie and Wayne Shafer for patient answers to our questions; to Roger Farmer for helpful comments; and to the National Science Foundation for financial support.

1. INTRODUCTION

The primary purpose of this paper is to suggest a notion of equilibrium and pseudo-equilibrium for infinite horizon economies with incomplete asset markets, and to establish the existence of such a pseudo-equilibrium when assets are short-lived and denominated in general commodity bundles. When assets are denominated solely in a single numeraire commodity, or in units of account, we establish the existence of a true equilibrium.

The crucial issue that divides the infinite horizon setting from the finite horizon setting is the nature of debt constraints. In the finite horizon setting, the constraint that there be no debt following the terminal date, together with the budget constraint, imply limits on debt at earlier dates. In the infinite horizon setting, this terminal debt constraint — and the implied debt constraints at earlier dates — are absent. If no additional debt constaints were imposed, no equilibrium could possibly exist: all traders would attempt to finance unbounded levels of consumption by unbounded levels of borrowing without repayment. When markets are complete, such Ponzi schemes may be ruled out by the simple requirement that debt never grow so large that it cannot be repaid. Completeness of markets guarantees that this is an unambiguous requirement, and it is sufficient to guarantee that an equilibrium exists.[1] [2]

The most straightforward way to repay present debt is to convert all future endowments into present wealth; when markets are complete, it is possible to accomplish this directly. When markets are incomplete, however, future endowments cannot be exchanged directly for present wealth; the optimal strategy for converting future endowments into wealth today may involve borrowing at many future date events. Thus there is no unambiguous way to require that present debt can be repaid without simultaneously specifying debt constraints at all subsequent date events.

This suggests the point of view we take here: We should view debt

[1] Provided, of course, that we make suitable assumptions about preferences and endowments; see Bewley (1972) for instance.

[2] Note that, even in the complete markets setting, debt may not be entirely repaid in finite time, but the present value of debt will tend to zero.

constraints as an entire *system*, and specify debt constraints *simultantaneously* at all date events, rather than *individually* at each date event. Moreover, these debt constraints should be incorporated into the *definition* of equilibrium.[3] Thus an equilibrium consists of a list of asset prices, commodity prices, consumption plans, portfolio plans, *and* a system of debt constraints, such that the plans satisfy the usual market clearing conditions and budget constraints and the given debt constraints, and are utility optimal among all such plans.

In addition to debt constraints, there is an additional difficulty that we must face because we treat real assets: the dividend matrix may fail to have constant rank.[4] In this paper, we content ourselves to follow Duffie and Shafer (1985, 1986) and establish the existence of a pseudo-equilibrium. We conjecture that, as in the finite horizon setting, pseudo-equilibria will generically be equilibria, but the precise notion of genericity required here seems to be a subtle one.

We are primarily interested in systems of debt constraints that satisfy two conditions. Roughly speaking, a system of debt constraints is *loose* if liabilities which satisfy tomorrow's debt constraints can be acquired today. A system of debt constraints is *consistent* if liabilities that do not exceed today's debt constraint can be satisfied (paid off) without exceeding tomorrow's debt constraints. To say that a system of debt constraints is both loose and consistent is to say that the debt constraint at each date event reflects an accurate summary of relevant information about future debt constraints. In the finite horizon setting, the implicit debt constraints are loose and consistent, and are the only such debt constraints. Thus, the notion of (pseudo-)equilibrium we propose reduces to the usual one in the finite horizon setting.

Because our main purpose here is to emphasize the role of debt constraints, we restrict ourselves to the case of short-lived assets.[5] Given our

[3] We formulate the debt constraint at each node as a value of the portfolios that a trader may acquire at the preceding node.

[4] In this regard, the infinite horizon setting is no different from the finite horizon setting. See Hart (1975).

[5] There would be only notational difficulties in allowing for long-lived assets, provided that they pay off in finite time; infinitely-lived assets — including consols — present more

notion of debt constraints, establishing the existence of pseudo-equilibrium is rather straightforward (following Levine (1989)). Every suitable finite truncation of the economy has a pseudo-equilibrium (with no debt constraints other than those implied by the constraint that there be no liabilities following the terminal date). The limit of these finite horizon pseudo-equilibria provides a pseudo-equilibrium for the infinite horizon economy, in which the debt constraints are taken to be the limit of the implicit debt constraints for the finite horizon truncations.

For short-lived numeraire assets (that is, assets denominated in a single commodity), pseudo-equilibria are necessarily equilibria, so in this case we obtain the existence of an equilibrium.[6] Since the case of short-lived financial assets (that is, assets denominated in units of account) can be reduced to the case of numeraire assets, we obtain an equilibrium in this case as well.[7]

Our approach to debt constraints is certainly not the only one possible, and two recent papers dealing with infinite horizon economies with incomplete asset markets treat debt constraints in quite a different way. Hernandez and Santos (1991) require that the present value of debt never exceed the present value of future endowments. Magill and Quinzii (forthcoming) require that it be possible to pay off debt in a (given) finite amount of time.

Our attention here is on infinite horizon economies populated by (a finite number of) infinitely lived traders. In an infinite horizon economy populated by finitely-lived traders — for example, an overlapping generations economy — the issue of debt constraints can be resolved exactly as in the finite horizon setting: Each individual faces the constraint that he cannot have liabilities after the terminal period of his life; debt at other dates is constrained implicitly by this requirement and by the budget constraints. For the existence of equilibrium in an overlapping generations economy (with purely financial assets), see Schmachtenburg (1989).

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serious complications.

[6]The restriction to short-lived assets is important here; the rank difficulty identified by Hart can occur even for numeraire assets that are long-lived.

[7]We do not consider economies with long-lived financial assets, but the existence of equilibrium for such economies should be not problematical.

2. INFINITE HORIZON ECONOMIES

Time and uncertainty are represented by a (countably) infinite tree S. A node $s \in S$ represents a finite history of exogenous events; we denote by $t(s)$ the length of that history. The root of the tree is denoted by $s = 0$; thus $t(0) = 0$. The node immediately preceding s is denoted by $s - 1$, and the set of nodes immediately following s is denoted by s^+.

There are L commodities $1, \ldots, L$ available at each node. Write $p_s \in \Re^L_+$ for the vector of commodity spot prices at the node s, p_{sl} for the price of commodity l at s, and $p : S \to \Re^L_+$ for the function which assigns commodity spot prices at each node. It is convenient to normalize so that p_s lies on the unit simplex in $\Re^L_+$. [8] A *consumption plan* is a bounded function $x : S \to \Re^L_+$; so the consumption set (for each trader) is $X = (l^\infty_+)^L$.[9] Write x_s for the vector of consumption at node s, and x_{sl} for consumption of commodity l.

There are I traders $1, \ldots, I$ characterized by endowments $w^i \in X$ and utility functions $U^i : X \to \Re$. We assume that endowments and utility functions satisfy the following assumptions.

Assumption 1 *Utility functions U^i are concave, monotonically increasing, and continuous in the product topology.*[10]

Assumption 2 *Endowments are strictly positive and commensurable, in the sense that there is a constant $\rho > 0$ such that $w^j_s \geq \rho w^i_s$ for each node s and each pair of traders i, j.*

[8]We emphasize that p_s is a vector of *spot* prices, *not* present value prices.

[9]The restriction to bounded consumption plans is innocuous; after a re-scaling, we may always assume that the social endowment is bounded, whence all feasible consumption plans are bounded. Of course, traders do not take social feasibility into account when they choose optimal plans. However, under extremely mild conditions, if a trader finds that a given bounded consumption plan is dominated by an unbounded consumption plan (satisfying appropriate constraints), it will also be dominated by a bounded consumption plan (satisfying the same constraints). See Bewley (1972) for a similar discussion.

[10]It would suffice to assume that utility functions are continuous on the set of feasible consumption plans. Without the restriction to bounded consumption plans, such an assumption would be more natural.

Monotonicity and concavity are standard assumptions. Continuity in the product topology is an assumption about time preference: additional consumption today is more desirable than additional consumption in the distant future.[11] The assumption that endowments are strictly positive and commensurable is strong. As we shall see, it serves three functions: It guarantees that some short selling is always possible (independent of prices), that income is strictly positive, and that debt constraints for different traders are commensurable (in the same sense that endowments are commensurable).

Intertemporal transactions and insurance are carried out through the trade of short-lived (one period) assets. For convenience, we assume that the number of assets available at each node is a constant M.[12] We write $q_s \in \Re^M$ for the vector of *asset prices* at node s, q_{sm} for the price of asset m at s, and $q : S \to \Re^M$ for the function which assigns asset prices to nodes. The portfolio of assets held by trader i at node s is denoted by y^i_s. A *portfolio plan* $y : S \to \Re^M$ assigns a portfolio choice at each node s.

We treat *real* assets, so that each asset purchased at node s returns a vector of commodities at each node $\sigma \in s^+$. We write R_σ for the *returns operator* at node σ; thus, if y_s is the portfolio held at the end of the node s preceding σ, then $R_\sigma y_s$ is the commodity bundle promised by the portfolio y_s at the node σ. For convenience, we assume that asset returns are non-negative, so the returns operator R_σ is non-negative. However, since portfolios may have negative components, the yield of a portfolio may have any combination of signs.

We make two assumptions about asset returns.

Assumption 3 (Positive Returns) *For each node s there is a portfolio $y_s \geq 0$ such that $R_\sigma y_s \geq 0$ and $R_\sigma y_s \neq 0$ for each node $\sigma \in s^+$.*

Assumption 4 (No Redundant Assets) *For each node s and each portfolio $y_s \neq 0$, there is a node $\sigma \in s^+$ such that $R_\sigma y_s \neq 0$.*

[11]See Brown and Lewis (1981) for a detailed discussion.

[12]There would be no difficulty in allowing for a different number of assets at each node.

Given commodity spot prices p_σ, the portfolio y_s yields a *dividend* of $p_\sigma \cdot R_\sigma y_s$ (units of account) at the node σ. It is convenient to write $V_s(p)$ for the *dividend operator* which maps portfolios at the node s to the vector of dividends at nodes in s^+;

$$(V_s(p)y_s)(\sigma) = p_\sigma \cdot R_\sigma y_s$$

Since there are M assets, the dividend operator has rank at most M, but it may have lower rank for some prices. However, since there are no redundant assets, there is a closed subset $E \subset (\Re^M)^{s^+}$ of measure 0 such that the dividend operator has rank precisely M for spot prices $\{p_\sigma\} \notin E$.

No production or intertemporal storage is possible, and assets are in zero net supply, so the social feasibility conditions for the economy are

$$\sum_i x_s^i \leq \sum_i w_s^i$$

$$\sum_i y_s^i = 0$$

Initial holdings of securities are zero. When $s = 0$, it is convenient to write $y_{s-1}^i = 0$. Thus the budget constraint facing trader i at the node s may be written

$$p_s \cdot (x_s^i - w_s^i) + q_s \cdot y_s^i \leq p_s \cdot R_s y_{s-1}^i$$

(Note that this inequality is homogeneous in (p_s, q_s), so that we are indeed free to normalize so that p_s lies in the unit simplex.)

A system of *debt constraints* for trader i is a function $D^i : S \to (-\infty, 0]$. Given commodity prices p, the portfolio $y_s \in \Re^M$ *satisfies the debt constraint* at $\sigma \in s^+$ if

$$V_\sigma(p)y_s = p_\sigma \cdot R_\sigma y_s \geq D_\sigma^i$$

Write $Y_s \subset \Re^M$ for the set of portfolios y_s that satisfy the debt constraint at each node $\sigma \in s^+$.[13]

[13]Note that the debt constraint is automatically satisfied at the initial node, since there are no initial portfolios.

The role of debt constraints is to rule out Ponzi schemes in the infinite horizon economy. Notice that the debt limits are non-positive — that is, traders cannot be forced to save.

To motivate these constraints, it is useful to see how they are connected to the usual finite horizon incomplete markets model. In that model, debt cannot be held at the end of the terminal period. Implicitly, this gives rise to debt constraints at earlier nodes as well. The budget constraint forces repayment by the terminal date, so the debt limit at any node s is the greatest amount of debt that the trader could hold, entering node s, and still be able to repay by the terminal date. Our approach is to make these implicit constraints explicit, because in the infinite horizon model there is no terminal constraint.

Notice that debt constraints are defined in terms of the value of the portfolio held at the beginning of the period, rather than at the end of the period. To understand why, consider again the implicit debt constraints in the finite horizon model. With incomplete markets, the amount of debt that can be held at the end of the period depends on the form in which it is held. If a trader is short in securities which promise repayment in future states in which his endowment is large, then a substantial debt can be repaid; if he is short in securities which promise repayment in future states in which his endowment is small, then he can repay very little. If debt were defined in terms of end-of-period holdings, it would be necessary to distinguish various portfolios of debt. Our definition in terms of beginning-of-period holdings is therefore convenient because it enables us to work entirely in terms of value.

Given endowments w^i, prices p, q, and debt constraints D^i, the consumption/portfolio plan (x^i, y^i) belongs to the *budget set* $B^i_s(w^i, p, q, D^i)$ for trader i at the node s if:

- the budget constraint is satisfied at s; i.e.,

$$p_s \cdot (x^i_s - w^i_s) + q_s \cdot y^i_s \leq p_s \cdot R_s y^i_{s-1}$$

- the debt constraint is satisfied at s; i.e.,

$$V_s(p) y_{s-1} = p_s \cdot R_s y_{s-1} \geq D^i_s$$

In this circumstance, we frequently say that the portfolio plan y^i *finances* the consumption plan x^i.

An *equilibrium* consists of prices p, q, consumption plans (x^i), portfolio plans (y^i), and systems of debt constraints (D^i) such that

- consumption and portfolio plans are socially feasible
- for each trader i, x^i maximizes trader i's utility over all plans belonging to the budget set $B^i_s(w^i, p, q, D^i)$ at every node s

Unfortunately, the assumptions we have made will not in general suffice to guarantee the existence of a sensible equilibrium. As noted in the Introduction, the difficulty is that, for some prices p, the dividend operator $V_s(p)$ may have rank less than M. To deal with this difficulty, we shall follow Duffie and Shafer (1985, 1986) and introduce the notion of a pseudo-equilibrium. We find it convenient to formulate this notion in a different — albeit equivalent — way.

For each node s, we consider an M-dimensional subspace $K_s \subset \Re^{s^+}$ of *income transfers*, and a *pricing functional* $Q_s : K_s \to \Re$. An *income transfer plan* is a family of vectors $k_s \in K_s$. For $\sigma \in s^+$, write $k_s(\sigma)$ for the σ-component of k_s. Given commodity prices p, the consumption/income transfer plan (x^i, k^i) *satisfies the budget constraint* at s if

$$p_s \cdot (x^i_s - w^i_s) + Q_s \cdot k_s \leq k_{s-1}(s)$$

Similarly, the consumption/income transfer plan (x^i, k^i) *satisfies the debt constraint* at s if

$$k_{s-1}(s) \geq D^i_s$$

Finally, (x^i, k^i) belongs to the *budget set* $B^i_s(w^i, p, K, Q, D^i)$ for trader i at the node s if it satisfies the budget and debt constraints. Again, in this circumstance we frequently say that the income transfer plan k^i *finances* the consumption plan x^i.

A *pseudo-equilibrium* consists of prices p, a family K of subspaces of income transfers, pricing functionals Q, consumption plans (x^i), income transfer plans (k^i), and systems of debt constraints (D^i) such that

- consumption plans are socially feasible
- income transfer plans are socially feasible (i.e., $\sum_i k^i_s = 0$ for each s)
- for each trader i, the plan (x^i, k^i) maximizes trader i's utility over all plans belonging to the budget set $B^i_s(w^i, p, K_s, Q, D^i)$ at every node s
- for each s, the range of the dividend operator $V_s(p)$ is a subspace of K_s

A pseudo-equilibrium is *proper* if, for each s, the range of the dividend operator $V_s(p)$ is equal to K_s.

If trader i acquires the portfolio y_s at the node s, he will effect the income transfers $V_\sigma(p)y_s$ at nodes $\sigma \in s^+$. Since the definition of pseudo-equilibrium requires that the range of the dividend operator $V_s(p)$ be a subspace of the space K_s of income transfers, allowing income transfers to lie in K_s expands the possibilites for each trader. Thus, the notion of pseudo-equilibrium is more general than the notion of equilibrium. Moreover, proper pseudo-equilibria are actually equilibria. More precisely, if

$$\left\langle p, K, Q, (x^i), (k^i), (D^i) \right\rangle$$

is a proper pseudo-equilibrium, then there are asset prices q and portfolio plans (y^i) such that

$$\left\langle p, q, (x^i), (y^i), (D^i) \right\rangle$$

is an equilibrium. To see this, we need only note that the pricing functional Q_s defines prices q_s for asset portfolios by the rule

$$q_s \cdot y_s = Q_s \cdot V_s(p)$$

and that the income transfer plans k^i define portfolio plans y^i by the rule

$$V_s(p)y^i_s = k^i_s$$

It is straightforward to verify that the equilibrium conditions are satisfied.

To this point, we have placed no restrictions on debt constraints, but it should be clear that some debt constraints are less interesting than others. We shall restrict attention to debt constraints that satisfy two conditions.

The first is a consistency condition: if debt can be acquired (that is, satisfies the current debt limit), then it can be repaid (while satisfying future debt limits). The second is a condition that debt is not overly restricted: if debt can be repaid, then it can be acquired.

The internal consistency condition we use is that, if the current debt limit is satisfied, then there is a plan that meets today's liabilities and satisfies tomorrow's debt constraints. Formally, the debt constraint D^i is *consistent* at node s if for every income transfer plan $k^i_{s-1} \in K_{s-1}$ that meets the debt constraint at s — that is, $k^i_{s-1}(s) \geq D^i_s$ — there is an income transfer plan $k^i_s \in K_s$ such that

$$k^i_{s-1}(s) + p_s \cdot w^i_s - Q_s \cdot k^i_s \geq 0$$

and $k^i_s(\sigma) \geq D^i_\sigma$ for each $\sigma \in s^+$. Since the only requirement on the income transfer plan k^i_{s-1} is that it meet the debt constraint at the node s, and it is always possible to find such a k^i_{s-1} such that $k^i_{s-1}(s) = D^i_s$, an alternative formulation of consistency is: there is a plan $k^i_s \in K_s$ such that

$$D^i_s + p_s \cdot w^i_s - Q_s \cdot k^i_s \geq 0$$

and $k^i_s(\sigma) \geq D^i_\sigma$ for each $\sigma \in s^+$. That is, it is possible to meet a liability equal to or greater than today's debt constraint by consuming nothing today and acquiring an income transfer (that is, borrowing) that meets tomorrow's debt constraints. The system D^i of debt constraints is *consistent* if it is consistent at each node.

Note that consistency of the entire system expresses the idea desired: If debt satisfies the limit at a particular node, and debt constraints are consistent at that node, then debt can be rolled over to satisfy the constraints next period. If the entire system of debt constraints is consistent, this process can be repeated, so constraints can be satisfied at every future node. In other words, the current constraint correctly summarizes future constraints.

The requirement that the system of debt constraints be consistent is important, but not limiting in itself: any given system of debt constraints — consistent or not — can be modified to a system that is consistent and yields exactly the same budget sets. (Recall that the definition of budget sets involves both budget constraints and debt constraints.)

Merely to establish the existence of equilibrium with some system of debt constraints — even some consistent system of debt constraints — does not seem very satisfactory. There is, for example, an equilibrium in which $D^i_s = 0$ for every i, s, in which there is no intertemporal trade or insurance. (Zero debt constraints are clearly consistent.) In the finite horizon model, the usual (implicit) assumption is that, if debt can be repaid then it can be acquired. We wish to make a similar requirement in the infinite horizon model as well. To formalize this requirement, consider an income transfer plan $k^i_{s-1} \in K_{s-1}$, and the liability $k^i_{s-1}(s)$ it creates at the node s. This liability can be repaid satisfying next period's debt limits if there is a plan $k^i_s \in K_s$ such that

$$k^i_{s-1}(s) + p_s \cdot w^i_s - Q_s \cdot k^i_s \geq 0$$

and $k^i_s(\sigma) \geq D^i_\sigma$ for every $\sigma \in s^+$. If debt that can be repaid can also be acquired, it must be the case that $k^i_{s-1}(s) \geq D^i_s$. Since this must hold for all such choices, it reduces to saying that if $k^i_s \in K_s$ and $k^i_s(\sigma) \geq D^i_\sigma$ for each node $\sigma \in s^+$, then

$$D^i_s + p_s \cdot w^i_s - Q_s \cdot k^i_s \leq 0$$

and we take this as the formal definition of *loose* at the node s. The system D^i is *loose* if it is loose at every node.

In the finite horizon setting, it is easy to see that the implicit debt constraints at each node (that is, debt at each node is constrained to the level that can be repaid by the terminal period) are both loose and consistent. Moreover, the implicit debt constraints are the only debt constraints that are both loose and consistent. In the infinite horizon setting, the debt constraint of 0 is consistent, and the debt constraint of $-\infty$ is loose — independently of prices.[14] However, some price systems do not support any finite debt constraints that are both loose and consistent. Indeed, it is not at all apparent that there exist *any* price systems supporting finite debt constraints that are both loose and consistent. In particular, the debt constraint of 0 is not loose if endowments are positive; it will always be possible to sustain some debt at the beginning of each node.

It may be useful to turn this last remark around: if debt constraints are loose, it will always be possible to hold today a level of debt that can be

[14] Of course, the debt constraint of $-\infty$ is not consistent with any equilibrium.

repaid from tomorrow's endowments. More generally, for any given τ, it will always be possible to hold today a level of debt that can be repaid from endowments in the next τ periods. This notion of repayment in finite time is one used by Magill and Quinzii (forthcoming). Note however, that we allow for debt that cannot be repaid in finite time. This should not be a surprise. If markets are complete, the usual infinite horizon budget constraint allows for such debt, and it is indeed quite easy to construct examples of complete markets, infinite horizon equilbria in which some trader never exactly repays a debt, but rather makes interest payments forever.

Similarly, note that, because we work entirely with spot prices, we do *not* require that debt constraints be uniformly bounded below — nor is there any reason we should do so. Even in the complete markets setting, for which we might want to insist that the date 0 present value of debt tends to 0, the spot price of debt might well be unbounded below. We shall return to this point in Section 3.

Our debt constraints are based on what a trader could repay, not on his wealth (that is, the present value of future endowment). A simple example, adapted from Hernandez and Santos (1991), may illustrate why this is an important distinction. Consider a tree S that has two branches at each node, so that each node s has two successors; write $s^+ = \{s_0, s_1\}$. Assume that there is a single commodity available for consumption at each node, and a single one-period asset, which promises delivery of one unit of consumption at each successor node. Consider a trader i whose endowment w^i is given by $w^i(0) = 0$ and $w^i(s_0) = 0, w^i(s_1) = 1$ for each node s^+. If prices are strictly positive, this trader's wealth (that is, the present value of his future endowment) is strictly positive at each node. However, the debt constraints that are identically 0 at each node are loose and consistent; with such debt constraints, no borrowing is possible at any node. Zero debt constraints are perfectly sensible here: in any finite horizon truncation of this tree, it will be impossible for this trader to borrow at any node, since it might be impossible for him to repay his debt by the terminal node. The implicit debt constraints in these finite horizon truncations are therefore identically zero at each node. Zero debt constraints in the infinite horizon setting therefore correctly capture the finite horizon limit.[15]

[15] We do not assert that a sensible theory of debt constraints cannot be based on present values, only that our theory is not, and that the distinction is a real one.

3. EXISTENCE OF PSEUDO-EQUILIBRIUM

Our basic result is the following.

Theorem 1 *The infinite horizon economy admits a pseudo-equilibrium*

$$\left\langle p, K, Q, (x^i), (k^i), (D^i) \right\rangle$$

for which each D^i is a loose, consistent system of debt constraints.

Before beginning the proof proper, it will be useful to isolate a technical point that will be used repeatedly. We first introduce some notation. Let x and y be consumption plans, c a real number, and s a node. By the *splice* $\langle x, c, y|s\rangle$ we mean the consumption plan defined by

$$\langle x, c, y|s\rangle_\tau = \begin{cases} c\mathbf{1} = (1, \ldots, 1) & \text{if } \tau = s \\ y_\tau & \text{if } \tau \text{ follows } s \\ x_\tau & \text{otherwise} \end{cases}$$

Lemma 2 *For each trader i, feasible consumption plan x^i, and node s, there are real numbers c, δ with $c > 0$ and $0 < \delta < 1$, such that the consumption plan $(1-\delta)x^i + \delta\langle x^i, c, w^i|s\rangle$ is preferred to x^i.*

Proof: Concavity implies that U^i has right-hand directional derivatives at x^i in every direction. We claim that, for c sufficiently large, the right-hand derivative (call it β_c) of U^i at x^i in the direction $\langle x^i, c, w^i|s\rangle - x^i$ is strictly positive. Assuming this claim, the remainder of the argument is simple. For $\delta > 0$, the definition of the right-hand derivative yields:

$$\begin{aligned} U^i((1-\delta)x^i + \delta\langle x^i, c, w^i|s\rangle) &= U^i(x^i + \delta[\langle x^i, c, w^i|s\rangle - x^i]) \\ &= U^i(x^i) + \beta_c\delta + o(\delta) \end{aligned}$$

where $o(\delta)/\delta \to 0$ as $\delta \to 0$. Since $\beta_c > 0$, we conclude that

$$U^i((1-\delta)x^i + \delta\langle x^i, c, w^i|s\rangle) > U^i(x^i)$$

provided that δ is sufficiently small, as asserted.

It remains to establish the claim. Write $x^* = \langle x^i, 0, w^i|s\rangle - x^i$ and $z = \langle 0,1,0|s\rangle$, so that $\langle x^i, c, w^i|s\rangle - x^i = x^* + cz$. Concavity implies that the right-hand derivatives are at least as large as the difference quotients:

$$\beta_c \geq \frac{U^i(x^i + \epsilon[x^* + cz]) - U^i(x^i)}{\epsilon}$$

for each $\epsilon > 0$. Setting $\epsilon = 1/c$ and expanding yields

$$\begin{aligned} \beta_c &\geq \frac{U^i(x^i + (1/c)[x^* + cz]) - U^i(x^i)}{1/c} \\ &= c[U^i(x^i + (1/c)x^* + z) - U^i(x^i)] \end{aligned}$$

Since U^i is monotonically increasing and is continuous at x^i, it follows that $U^i(x^i + (1/c)x^* + z) - U^i(x^i) > 0$ if c is sufficiently large, so that

$$\beta_c \geq c[U^i(x^i + (1/c)x^* + z) - U^i(x^i)] \to \infty$$

as $c \to \infty$. In particular, $\beta_c > 0$ for c sufficiently large. □

Proof of Theorem 1: We construct a pseudo-equilibrium for our infinite-horizon economy as a limit of pseudo-equilibria for appropriate finite-horizon truncations. To this end, fix a time horizon T and consider the finite-horizon economy $\mathcal{E}(T)$ obtained in the following way:

- time and uncertainty are described by the tree $S(T)$ consisting of all nodes $s \in S$ for which $t(s) \leq T$
- the commodities and assets available for trade at each node of $S(T)$ are the same as at the corresponding node of S, except that no assets are available at terminal nodes of $S(T)$
- there are I traders; endowments at each node of $S(T)$ are the same as at the corresponding node of S
- trader i's utility $\overline{U}^i_T(x^i)$ for the consumption plan $x : S(T) \to \Re^L_+$ is set equal to his utility for the plan x^* which coincides with x at each node $s \in S(T)$ and with w^i_s at each node $s \notin S(T)$

According to Geanakoplos and Shafer (1990), the finite-horizon economy $\mathcal{E}(T)$ has a pseudo-equilibrium

$$E(T) = \left\langle p(T), K(T), Q(T), (x^i(T)), (k^i(T)) \right\rangle$$

with no debt constraints (other than the terminal ones).[16]

We would like to let $T \to \infty$ and pass to a convergent subsequence. In order to do this, we must first verify that the various components of the pseudo-equilibrium $E(T)$ all lie in compact sets. For some of these components, this is a triviality:

- Commodity prices $p_s(T)$ lie in the unit simplex
- Subspaces $K_s(T)$ of income transfers lie in the compact Grassman manifold of M-dimensional subspaces of $\Re^{s^+}$
- Consumption vectors $x^i_s(T)$ are non-negative and bounded by aggregate endowments

Passing to a subsequence if necessary, write p_s for the limit commodity spot prices, K_s for the limit subspaces of income transfers, and x^i_s for the limit consumption vectors.

- Income transfers $k^i_s(T)$ are bounded above. For, if not, we could find a node $\sigma \in s^+$ for which $k^i_s(T)(\sigma)$ is unbounded above. For real numbers c, δ, consider the consumption plans $z^i, z^i(T)$ defined by

$$z^i = (1 - \delta)x^i + \delta\langle x^i, c, w^i | s\rangle$$

[16]Geanakoplos and Shafer formulate pseudo-equilibrium in terms of present value prices, rather than spot prices, but the notions are equivalent for finite horizon economies. They also assume that the indifference surface through any interior consumption plan is a closed subset of the strictly positive orthant, an assumption that we have not made. However, this assumption is unnecessary. To see this, let $\overline{U}$ be any quasi-concave utility function having the desired indifference surfaces; for each $\epsilon > 0$, consider utility functions $U^i_\epsilon = U^i + \epsilon\overline{U}$. Evidently, the utility functions U^i_ϵ also have the desired indifference surfaces. Write $\mathcal{E}_\epsilon(T)$ for the economy obtained by substituting these utility functions. Applying the result of Geanakoplos and Shafer, we conclude that $\mathcal{E}_\epsilon(T)$ has a pseudo-equilibrium. Letting $\epsilon \to 0$, and passing to the limit (of a subsequence, if necessary) we obtain a pseudo-equilibrium for the economy $\mathcal{E}(T)$.

$$z^i(T) = (1-\delta)x^i(T) + \delta\langle x^i(T), c, w^i|s\rangle$$

According to Lemma 2, we can choose c, δ so that z^i is preferred to x^i. Continuity of utility functions in the product topology entails that

$$U^i(z^i(T)) \to U^i(z^i)$$

Hence, $z^i(T)$ is preferred to $x^i(T)$ for T sufficiently large. Since $k^i_s(T)(\sigma)$ is unbounded above, the consumption plan $\langle x^i(T), c, w^i|s\rangle$ is budget feasible if T is sufficiently large.[17] Hence $z^i(T)$ is a convex combination of budget feasible plans, and therefore is itself budget feasible for T sufficiently large. This is a contradiction, so we conclude that income transfers are indeed bounded above.

- Income transfers $k^i_s(T)$ are bounded below, since they are bounded above, and the sum of income transfers of all traders is identically 0.

- Prices $Q_s(T)$ are non-negative and bounded above. Non-negativity is clear, since preferences are increasing. If the prices $Q_s(T)$ are not bounded above, we may choose, for each T, a trader $i(T)$ such that $k^{i(T)}_{s-1}(T)(s) \geq 0$; for notational convenience, we henceforward suppress the dependence of i on T. As before, we can use Lemma 2 to choose real numbers c, δ with $c > 0, 0 < \delta < 1$ and define a consumption plan

  $$z^i = (1-\delta)x^i + \delta\langle x^i, c, w^i|s\rangle$$

  so that z^i is preferred to x^i. For $0 < r < 1$, set

  $$Z^i = (1-\delta)x^i + \delta\langle x^i, c, rw^i|s\rangle$$

  Continuity of preferences guarantees that Z^i is preferred to x^i, provided that r is sufficiently close to 1. Set:

  $$Z^i(T) = (1-\delta)x^i(T) + \delta\langle x^i(T), c, rw^i|s\rangle$$

  Continuity again guarantees that $Z^i(T)$ is preferred to $x^i(T)$, provided that T is sufficiently large. However, if T is sufficiently large, the

[17]That is, this is the consumption part of a consumption/income transfer plan in the budget set at each node.

consumption plan $\langle x^i(T), c, rw^i|s\rangle$ is budget feasible. (To see this, note first that, since only a finite number of assets A are available at s, their payoffs at nodes in s^+ are bounded by some multiple of the consumption vector $\mathbf{1} = (1, \ldots, 1)$; say $A(\sigma) \leq \alpha\mathbf{1}$, for each asset A. Endowments are strictly positive, so $w^i > \beta\mathbf{1}$ for some $\beta > 0$. Choose a real number ϵ with $0 < \epsilon < (1-r)\beta/\alpha$. We have supposed that prices $Q_s(T)$ are unbounded above, so, for T sufficiently large, there is an asset A^* whose price is at least cL/ϵ. The consumption plan $\langle x^i(T), c, rw^i|s\rangle$ can then be financed by the following plan of income transfers: at the node s, sell ϵ units of the asset A^* at the node s (this yields income sufficient to purchase c$\mathbf{1}$); do nothing at nodes following s (liabilities at $\sigma \in s^+$ arising from the sale of A^* at s can be covered by the fraction of endowment $(1-r)w^i_\sigma$), and follow the income transfer plan $k^i_\tau(T)$ at every other node τ. Thus, the consumption plan $Z^i(T)$ is the convex combination of budget feasible plans, and therefore is itself budget feasible if T is sufficiently large. This is a contradiction, so we conclude that prices $Q_s(T)$ are bounded above, as asserted.

Having established that the components of the equilibria $E(T)$ lie in compact sets, we may extract a subsequence converging to

$$E = \left\langle p, K, Q, (x^i), (k^i)\right\rangle$$

The next step is to construct suitable debt constraints, as limits of implicit debt constraints for each of the economies $\mathcal{E}(T)$. To this end, fix a trader i, a node s and an index $T > t(s)$. Define the implicit debt constraint $D^i_s(T)$ for the economy $\mathcal{E}(T)$ as:

$$D^i_s(T) = inf\{-p_s \cdot (\overline{x}^i_s - w^i_s) - Q_s \cdot \overline{k}^i_s\}$$

where the infimum is taken over all consumption and income transfer plans $(\overline{x}^i, \overline{k}^i)$ which meet the budget constraints (relative to commodity prices $p(T)$ and pricing functionals $Q(T)$) at s and at every node τ following s.[18]

- The implicit debt constraints $D^i_s(T)$ are bounded below (at each node). If not, suppose that trader i's implicit debt constraints are not bounded

[18] We make no restrictions on $(\overline{x}^i, \overline{k}^i)$ at other nodes.

below at the node s. For each T, choose a trader j such that $k^j_{s-1}(T)(s) \geq 0$. (We suppress the dependence of j on T.) According to Assumption 2, there is a real number $\rho > 0$ such that $w^j_s \geq \rho w^i_s$ for each trader j. Arguing as before, we may find real numbers c, δ, r with $c > 0$, $0 < \delta < 1$, $0 < r < 1$ so that the consumption plan

$$Z^j = (1-\delta)x^j + \delta\langle x^j, c, rw^j|s\rangle$$

is preferred to x^j. Continuity implies that

$$Z^j(T) = (1-\delta)x^j(T) + \delta\langle x^j(T), c, rw^j|s\rangle$$

is preferred to $x^j(T)$ if T is sufficiently large. We assert that $Z^j(T)$ is budget feasible if T is sufficiently large. To establish this, it is sufficient to show that $\langle x^j(T), c, rw^j|s\rangle$ is budget feasible if T is sufficiently large (since $Z^j(T)$ is a convex combination of $\langle x^j(T), c, rw^j|s\rangle$ and the equilibrium consumption $x^j(T)$). By definition of the implicit debt constraint $D^i_s(T)$, there is an income transfer plan $h^i(T)$ for trader i that, beginning at the node s, repays the debt $D^i_s(T)$ (provided that trader i consumes nothing at subsequent nodes). In other words, the plan $h^i(T)$ yields the income $-D^i_s(T)$ at the node s, and involves no liabilities at the nodes at time T. By Assumption 2, the endowments of trader i and trader j are commensurable: $w^j \geq \rho w^i$. Hence, by following the plan $\rho h^i(T)$, beginning at the node s, trader j can obtain the income $-\rho D^i_s(T)$ at the node s, and still meet all his liabilities at subsequent nodes (provided he consumes nothing). And if trader j follows the plan $h^j = (1-r)\rho h^i(T)$, beginning at the node s, he can obtain an income of $-(1-r)\rho D^i_s(T)$ at the node s, consume the portion rw^j of his endowment at all subsequent nodes, and still meet all his liabilities. Define the income transfer plan H^j by $H^j_\tau = h^j_\tau$ if $\tau = s$ or τ follows s, and $H^j_\tau = k^j_\tau(T)$ for all other nodes τ. This income transfer plan finances the consumption plan $\langle x^j(T), c, rw^j|s\rangle$, provided that the income it generates at node s is sufficient to purchase the consumption bundle $c\mathbf{1}$. The income generated by H^j at s is equal to $-(1-r)\rho D^i_s(T)$. Since we have assumed that $D^i_s(T)$ is is unbounded below, we conclude that the consumption plan $\langle x^j(T), c, rw^j|s\rangle$ is budget feasible, provided that T is sufficiently large. But then $Z^j(T)$ is budget feasible and preferred to $x^j(T)$, a contradiction. We conclude that implicit debt constraints are bounded below.

Having established that the implicit debt constraints are bounded below, we may, passing to a subsequence if necessary, assume that

$$D_s^i(T) \to D_s^i$$

for each trader i and node s. This provides us with a tuple

$$E^* = \left\langle p, K, Q, (x^i), (k^i), (D^i) \right\rangle$$

which we claim to be a pseudo-equilibrium for the infinite horizon economy.

It is trivial to verify that that individual consumption plans and transfer plans belong to the individual budget sets at each node, that consumption plans and income transfer plans are socially feasible, and that the range of each dividend operator lies in the appropriate income transfer subspace. It remains only to verify that individual plans are optimal. To this end, suppose that there is a trader i and consumption/income transfer plan (a^i, h^i) for trader i which belongs to the budget set at each node and has the property that $U^i(a^i) > U^i(x^i) + \delta$, for some $\delta > 0$. For each horizon T^*, consider the consumption plan $a^i|T^*$ which coincides with a^i at each node s with $t(s) < T^*$, and is 0 at every node s with $t(s) \geq T^*$. Continuity of preferences and the definition of the utility functions $\overline{U}_T^i$ guarantees that

$$\overline{U}_T^i(a^i|T^*) \geq U^i(a^i|T^*) > \overline{U}_T^i(x^i(T)) + \delta/2$$

for all $T > T^*$, provided that T^* is sufficiently large. Set $\underline{a}^i = (1-\epsilon)a^i|T^*$; continuity of preferences also guarantees that $U^i(\underline{a}^i) > U^i(x^i) + \delta/3$ for $\epsilon > 0$ sufficiently small. Set $\underline{h}^i = (1-\epsilon)h^i$. Because endowments are bounded away from 0, the consumption/income transfer plan $(\underline{a}^i, \underline{h}^i)$ has the property that the budget and debt constraints are satisfied (for prices p, pricing functionals Q) with *strict* inequalities at every node.

For $T > T^*$ define an income transfer plan $\underline{h}^i(T)$ by letting $\underline{h}_s^i(T)$ be the point of $K_s(T)$ closest to $\underline{h}_s^i$. Because $(\underline{a}^i, \underline{h}^i)$ satisfies the budget and debt constraints with strict inequalities at every node, convergence of income transfer subspaces $K_s(T) \to K_s$ and commodity spot prices $p_s(T) \to p_s$ implies that, for T sufficiently large, the consumption/income transfer plan $(\underline{a}^i, \underline{h}^i)$ strictly satisfies the budget and debt constraints (for prices $p(T)$, pricing functionals $Q(T)$) at all nodes s with $t(s) \leq T^*$. Moreover, if T is large enough, the plan $(\underline{a}^i, \underline{h}^i)$ also strictly satisfies the implicit debt constraints $D^i_s(T)$ (for prices $p(T)$, pricing functionals $Q(T)$) at every node s with $t(s) = T^*$.

The definition of the implicit debt constraints guarantees that it is therefore possible to find a consumption/income transfer plan (A^i, H^i) for the economy $\mathcal{E}(T)$ that agrees with $(\underline{a}^i, \underline{h}^i)$ for $t(s) < T^*$ and satisfies the budget constraints for the economy $\mathcal{E}(T)$ at every node. Since the consumption plan $\underline{a}^i$ is 0 at every node s with $t(s) \geq T^*$, monotonicity of preferences means that $U^i(A^i) > U(\underline{a}^i)$. Hence, for T sufficiently large, $\overline{U}^i_T(A^i) > \overline{U}^i(x^i(T)) + \delta/5$. Since (A^i, H^i) is feasible for the economy $\mathcal{E}(T)$, this is a contradiction. We conclude that the consumption/income transfer plans (x^i, h^i) are optimal, and hence that E^* is a pseudo-equilibrium, as desired.

It remains to see that the debt constraints D^i are loose and consistent. To this end, note first that our construction guarantees that the implicit debt constraints $D^i(T)$ are loose and consistent at each node (with respect to the prices $p(T), Q(T)$) at every node s with $t(s) < T$. To see that the debt constraint D^i is loose, fix a node s, an $\epsilon > 0$, and an income transfer plan $k^i_s \in K_s$ which satisfies the debt constraints at every node $\sigma \in s^+$; i.e., $k^i_s(\sigma) \geq D^i_\sigma$ for every $\sigma \in s^+$. Assumption 4 (Positive Returns), together with the fact that all commodity spot prices are strictly positive and the fact that the range of the dividend operator $V_s(p)$ lies in the income transfer subspace K_s, implies that we can find an income transfer plan $h^i_s \in K_s$ which strictly satisfies the debt constraints at every node $\sigma \in s^+$ (i.e., $h^i_s(\sigma) > D^i_\sigma$ for every $\sigma \in s^+$) and which differs from k^i_s by at most ϵ at every node $\sigma \in s^+$. As before, write $h^i_s(T)$ for the income transfer plan in the subspace $K_s(T)$ closest to h^i_s. Convergence of prices guarantees that, for $T > t(s)$ sufficiently large, the income transfer plan $h^i_s(T)$ satisfies the debt constraints at every node $\sigma \in s^+$; i.e., $h^i_s(T)(\sigma) \geq D^i_\sigma$ for every $\sigma \in s^+$. Because the debt constraints $D^i(T)$ are loose at s, it follows that

$$D^i_s(T) + p_s(T) \cdot w^i_s - Q_s(T) \cdot h^i_s(T) \leq 0$$

Because $D_s^i(T) \to D_s^i, p_s(T) \to p_s, Q_s(T) \to Q_s$, and $h_s^i(T) \to h_s^i$, it follows that

$$D_s^i + p_s \cdot w_s^i - Q_s \cdot h_s^i \leq 0$$

Since h_s^i differs from k_s^i by at most ϵ at every node $\sigma \in s^+$, and ϵ can be made as small as we like, we conclude that

$$D_s^i + p_s \cdot w_s^i - Q_s \cdot k_s^i \leq 0$$

That is, the debt constraint D^i is loose at s. The argument that debt constraints are consistent is essentially the reverse of this argument; details are left to the reader. This completes the proof. □

As we have noted earlier, debt constraints are denominated with respect to spot prices, not present value prices. Hence, we do not require that debt constraints be uniformly bounded below — and the debt constraints D_s^i we have constructed above may indeed *be* unbounded below. Obtaining uniform lower bounds on debt constraints would appear to require assumptions stronger than the ones we have made. (It would suffice, for instance, to know that the constant c of Lemma 2 could be chosen independently of the consumption plan x^i and the node s. This will be possible, for instance, if each trader maximizes discounted, state-independent, expected utility, and endowments are uniformly bounded above.)

In the finite-horizon setting, Duffie and Shafer (1985, 1986) have shown (with the additional assumption of smooth preferences) that, generically in endowments and asset structure, pseudo-equilibria are in fact equilibria. We conjecture that a similar result holds in our setting; however, giving a precise meaning to "generically in endowments and asset structure" does not seem an easy task in the infinite-horizon context. We can however, obtain the existence of an equilibrium in two cases: if all assets are denominated in a single commodity (*numeraire assets*), or if all assets are denominated in units of account (*financial assets*).

Corollary 3 (Numeraire Assets) *If all assets are denominated in a single commodity, then there is an an infinite horizon equilibrium* $p, q, (x^i), (y^i), (D^i)$ *for which each* D^i *is a loose, consistent system of debt constraints.*

Corollary 4 (Financial Assets) *If all assets are denominated in units of account, then there is an an infinite horizon equilibrium* $p, q, (x^i), (y^i), (D^i)$ *for which each* D^i *is a loose, consistent system of debt constraints.*

To obtain the first of these corollaries, observe that, for one-period numeraire assets, the returns operator necessarily has constant rank M at each node, so that the notions of equilibrium and pseudo-equilibrium coincide. To obtain the second of these corollaries, observe that purely financial assets can be reinterpreted as numeraire assets provided that we normalize so that the price of the numeraire commodity is always 1.

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