

Efficiency and the Role of Default When Security Markets are Incomplete

William R. Zame*

UCLA
and
The Johns Hopkins University

August 2, 1992

WORKING PAPER NO. 673

*I am grateful to John Geanakoplos for many stimulating conversations and for access to his unpublished work with Pradeep Dubey and Martin Shubik. I have received useful comments from seminar audiences at the California Institute of Technology, the University of California at Berkeley, the University of California at Los Angeles, the University of California at San Diego, the University of California at Santa Barbara, the University of Illinois, Indiana University, the Johns Hopkins University, the University of Minnesota, the University of Southern California, Stanford University, SUNY at Albany, and Tulane University. Bryan Ellickson, two referees, and a co-editor provided much helpful advice on earlier drafts. Financial support was provided by grants from the National Science Foundation and the UCLA Academic Senate Committee on Research.

1. INTRODUCTION

Default, and the anticipation of default, appear to play an important role in the economy.¹ The possibility of default may deter lenders, the fear of penalties may deter borrowers, and the cost of enforcement represents a deadweight loss to society. In view of such consequences, it seems natural to view default in a negative light, and to ascribe the default which is observed to the impossibility of perfectly foreseeing all possible contingencies.

The purpose of this paper is to argue — precisely to the contrary — that default may also play an important *positive* role. In a world of uncertainty and incomplete financial markets, default may promote — indeed, even be necessary for — efficiency. We find this to be so because the possibility of default improves the efficiency of markets, and does so in a way that simply opening new markets does not.

The positive role which we find for default rests on three essential ideas. The first of these comes from Dubey, Geanakoplos and Shubik (1988). When markets are incomplete — that is, when only certain contingent contracts are available — opportunities for mutual risk-sharing are missing, and there may be a gap between equilibrium and efficiency. Suppose for instance that efficiency requires some trader to shift wealth from some particular future state (call it ω , say) into the present.² This could be accomplished by selling today a contract that promises delivery (of units of account or consumption goods) in the future, contingent on the occurrence of the state ω . However, if none of the contracts that are available today promise delivery in state ω , such wealth transfers will be impossible, and all feasible allocations will necessarily be inefficient. Market incompleteness may also lead to inefficiency in another way, however. Suppose that contracts are available today that promise delivery in the future contingent on the occurrence of state ω , but that these contracts *also* promise delivery in some other states, which have low probability of occurrence. To sell such contracts is to promise deliveries

¹The U.S. Bankruptcy Court reported 3.9 million filings for personal bankruptcy during the 10-year period ending June 30, 1989.

²Or between two future states.

in *all* of these future states — and not merely in state ω . In the absence of sufficient endowments in these other states — or of sufficiently many other contracts — such promises cannot be kept. Hence, if traders are only able to enter into agreements that they will be able to honor *regardless of the future state*, then opportunities for mutual risk-sharing may be severely limited. (In extreme cases, only spot market trades might be possible.) In such a circumstance, a dramatic improvement in efficiency may be obtained by allowing traders to enter into contracts that they will be able to execute with high probability, but not with certainty.^{3,4}

Allowing for default may shrink the gap between equilibrium and efficiency — but so might opening new markets, and it might be simpler (and less costly) to open new markets than to create the institutions necessary to regulate default. The second essential idea is that opening new markets does *not* necessarily shrink the gap between equilibrium and efficiency — even as the set of markets expands to an approximately complete set. (This idea comes from Zame (1988).) Although perhaps puzzling at first, this has a simple explanation, which builds on the previous discussion. As new markets are opened, the collection of conceivable portfolios of contracts is enlarged. However, a typical portfolio is comprised of both longs and shorts (purchases and sales); the net payoff of such a portfolio will be positive in some future states and negative in others. Negative net payoffs represent liabilities to be satisfied; if these liabilities are sufficiently great, they will exceed the value of endowments, and cannot be satisfied. If traders are only able to enter into agreements that they will be able to execute regardless of future states, portfolios which create unsatisfiable liabilities cannot be traded. Opening new markets may therefore expand the set of *conceivable* portfolios without (significantly) expanding the set of portfolios that can *actually be traded*; in such case, opening new markets will not lead to efficiency.

The third idea (which seems entirely new to this work) is closely related to the first two. As new markets are opened, the collection of conceivable

³We defer for the moment any discussion of the institutions which might accomplish this.

⁴Just as there may be an optimal amount of pollution, so there may be an optimal amount of default; see Dubey, Geanakoplos and Shubik for some discussion of this point.

portfolios is enlarged. Some of these new portfolios may create liabilities so large that they cannot be satisfied. In general, however, there will be many portfolios with the property that the occurrence of states in which liabilities are unsatisfiable is a very low probability event. In such a setting, default may dramatically improve efficiency by allowing people to enter into agreements that they will be able to execute with high probability — but not with certainty. Indeed, we find that this is precisely what occurs, and in the limit (as the set of available contracts expands to one that spans all the uncertainty), equilibrium allocations become (arbitrarily close to) efficient.⁵

Although the mathematical model we use to formalize these ideas is a bit complicated, the three essential points may be understood quite clearly in a simple example, which is presented in a very informal way in Section 2. We consider a two-date world with uncertainty about the state of nature at date 1; a single consumption good is available at date 0 and in each of the infinite number of states of nature at date 1. At date 0, there is trade in the (date 0) consumption good and in N given contingent contracts for date 1 consumption; since there are infinitely many states of nature, but only a finite number of contracts, contingent contracting is necessarily incomplete. In such a setting, equilibrium allocations need not be efficient (Pareto optimal). In the example, in fact, no feasible allocation is efficient, and we are able to calculate explicitly the (utility) distance from the set of feasible allocations to the set of efficient allocations. We find that this distance remains bounded away from 0 as $N \rightarrow \infty$. In particular, equilibrium allocations do *not* become nearly efficient when new markets are opened. To put it in language that we have used before: there is a gap between equilibrium and efficiency, and this gap does not shrink when new markets are opened. On the other hand, we show that the possibility of default shrinks this gap, and that in the limit as $N \rightarrow \infty$ it disappears entirely. In a very real sense, the remainder of the paper is devoted to building a rigorous formal framework around this example, and to showing that this example represents the typical case, not the pathological.

Our formalization of these ideas builds on what is by now a standard

⁵Provided that default penalties are sufficiently large.

model of a security market (as in Arrow (1953, 1964) or Radner (1972)), adapted to allow for an infinite set of states of nature. To examine the effect of expanding the set of available securities (= contingent contracts), we fix an infinite sequence of securities $\{A_n\}$, and consider the equilibrium allocations of the security market in which only the first N of these securities are available for trade.⁶ If these allocations converge, as $N \rightarrow \infty$, to a Pareto optimal allocation, we say the sequence of securities $\{A_n\}$ is *asymptotically efficient*; otherwise, it is *asymptotically inefficient*. The example discussed above demonstrates that asymptotic inefficiency is possible; in Section 4 we show that, in a sense we make precise, asymptotic inefficiency is typical.

To explore the role played by default, we adapt a model introduced by Dubey, Geanakoplos and Shubik (1988). In the present context, a security is simply a promise to pay; default means that (some) agents can not — or do not — keep (some of) these promises. In the real world, such default might entail many and varied consequences: creditors might be able to seize assets and be awarded judgments against future earnings, defaulters might be barred from future credit markets, etc. Rather than attempt to model such institutional details, we follow Dubey, Geanakoplos and Shubik in assuming that the only consequences of default are penalties assessed against the defaulters, and that these penalties are assessed directly in terms of utility. For the sake of simplicity, we assume that the default penalty is independent of the security and of the state of nature, is the same for all consumers, and is proportional to the amount of default; we write λ for the constant of proportionality.⁷ We work in a perfect foresight, general equilibrium framework.⁸ Thus, we assume that all default is perfectly anticipated — but anonymous — and that default is spread equally among all creditors.

For each default penalty λ (with $0 \leq \lambda \leq \infty$), a default equilibrium exists.

⁶To avoid trivialities, we assume that the sequence $\{A_n\}$ spans all the uncertainty, in an appropriate approximate sense.

⁷But there would be no difficulty in allowing for non-linear penalties which depend on the security, the state of nature, and the consumer. What is crucial is that penalties become arbitrarily large as the magnitude of default tends to infinity.

⁸Since we may assume that traders have common priors over states of nature, our framework is even consistent with rational expectations.

At one extreme, $\lambda = 0$ and default goes unpunished; in such a situation, no optimizing agent will ever keep promises to pay, and there will be no trade at the default equilibrium. At the other extreme, $\lambda = \infty$ and no optimizing agent will ever default; in such a situation, default equilibrium coincides with security market equilibrium in the sense discussed previously. (So the security market model might be viewed as a special case of the default model.) However, for all intermediate values of λ , there will generally be a positive amount of default at equilibrium (although the probability of default and the expected magnitude of default will both be small if λ is large).

When default is not possible, the requirement that liabilities be satisfied may severely restrict the portfolios that can be held. When default is possible, traders may plan not to keep all their promises (that is, to leave some liabilities unsatisfied); this will enlarge the set of portfolios that can be held. Moreover, since both the choice of portfolios and the default decision are endogenous, this enlargement might be in precisely the “efficient directions.” We show that this is indeed the case: if the sequence of securities $\{A_n\}$ spans all the uncertainty and the default penalty is sufficiently large, default equilibrium allocations will be close to Pareto optimal allocations — indeed, to Walrasian (complete markets) equilibrium allocations..

We emphasize that this positive role of default depends crucially on incompleteness of security markets. If security markets are complete (that is, if a complete set of contingent contracts is available), equilibrium allocations will already be Pareto optimal, and default — whenever it occurs — will necessarily have a Pareto worsening effect.

We have nothing to say here about *why* security markets are incomplete. Our point of view is simply to take market incompleteness as given, and to explore the consequences. Similarly, we offer no mechanism by which new markets may be opened. Rather, we take as given a specific infinite sequence of securities, which we view as a proxy for some unmodelled process of opening new markets. Market incompleteness and the origins of new markets are clearly important — and related — issues, and worthy of further study.

Another limitation of our framework is that we formulate default penalties entirely in terms of utility punishments. Of course this is unrealistic: debtor's prisons and flogging are no longer in common use (although they are certainly of historical importance). However, we do not intend that utility penalties should be taken literally. Rather, we intend that utility penalties should be viewed as proxies for real — but unmodelled — economic penalties: seizure of assets, loss of access to future credit markets, etc. It will surely be desirable to have a model which incorporates such real penalties, but such a model will necessarily be substantially more complicated.⁹

Our formalization also requires that the potential magnitude of default penalties be unlimited. In reality, society might find such penalties difficult or undesirable to enforce — for moral as well as economic reasons. In such circumstances we might expect to find alternative institutional arrangements. We might find, as well, that the particular institutional arrangements have an important effect on outcomes. We have chosen here to abstract away from institutional arrangements, but there is no doubt that they too are an important topic for future work.

Following this Introduction, Section 2 presents the example which lies at the heart of paper. Section 3 presents the formal security market model, and Section 4 addresses asymptotic inefficiency in the absence of default. Section 5 presents the formal default model, Section 6 addresses the efficiency-promoting role of default, and Section 7 concludes. Proofs are collected in the Appendix.

⁹Kehoe and Levine (1989) have constructed an infinite horizon model in which the penalty for default is loss of access to future credit markets. However, they work in a complete markets framework, and there is no equilibrium default. Geanakoplos and Zame (in progress) describe a model in which the penalty for default is the seizure of collateral, and there may be equilibrium default.

2. EXAMPLE

In this Section we present — in a very informal way — an example which illustrates the crucial ideas. In a very real sense, the function of the remainder of the paper is merely to formalize the insights of this example. Although the example we present has some special features, they serve only to simplify various calculations. As we shall show in the following sections, this example is entirely typical, and its properties are quite robust.

We consider a world in which there are two trading dates, 0 and 1, and uncertainty about the state of nature at date 1. We summarize uncertainty by a countably infinite set $\Omega = \{1, 2, \dots\}$ of possible states of nature. A single consumption good is available at date 0 and at each state in date 1. The economy is comprised of two traders who maximize the sum of utility for consumption at date 0 and expected utility for consumption at date 1. Each trader's consumption is constrained to be non-negative at date 0 and at each state of the world at date 1. Traders share a common probability assessment μ over the set of states of nature at date 1, where $\mu(1) = \mu(2) = 1/4$, $\mu(\omega) = 3^{-\omega+2}$ for $\omega > 2$. Traders also share a common time and state independent utility function $u(t) = \sqrt{1+t}$. Thus each trader's utility for a consumption plan x , representing consumption $x(0)$ at date 0 and state-dependent consumption $x(\omega)$ in state ω is:

$$U(x) = x(0) + \frac{1}{4}\sqrt{1+x(1)} + \frac{1}{4}\sqrt{1+x(2)} + \sum_{\omega=3}^{\infty} 3^{-\omega+2}\sqrt{1+x(\omega)}$$

Finally, endowments w^1, w^2 are given by:

$$\begin{aligned} w^1(0) &= 1, & w^2(0) &= 1 \\ w^1(1) &= 1, & w^2(1) &= 7 \\ w^1(2) &= 7, & w^2(2) &= 1 \\ w^1(\omega) &= 1, & w^2(\omega) &= 1 \quad \text{for } \omega > 2 \end{aligned}$$

Write $\bar{w} = w^1 + w^2$. A simple calculation with marginal rates of substitution shows that the Pareto optimal allocations for this economy are all of

the form:

$$x^1 = t\bar{w}, \quad x^2 = (1 - t)\bar{w}$$

for some t with $0 \leq t \leq 1$. In particular, at any Pareto optimal allocation each trader consumes equal amounts in the two states $\omega = 1$ and $\omega = 2$. Similarly, it is easy to see that the unique Walrasian equilibrium allocation is the symmetric one:

$$z^1 = \bar{w}/2, \quad z^2 = \bar{w}/2$$

In the Walrasian world, all contingent contracts are available for trade. Suppose however, that only date 0 consumption and a *finite* number of contingent contracts (*securities*) for date 1 consumption are available. To be precise, suppose that contracts A_1, A_2, \dots, A_N are available, where each A_n promises consumption at date 1, contingent on the state of the world. The only allocations (y^1, y^2) that can be obtained by trading date 0 consumption together with such contracts have the property that, for each state ω at date 1

$$y^1(\omega) = w^1(\omega) + \sum_{n=1}^N \theta_n A_n(\omega)$$

$$y^2(\omega) = w^2(\omega) - \sum_{n=1}^N \theta_n A_n(\omega)$$

where $\theta = (\theta_1, \dots, \theta_N)$ is a *portfolio*.

Now suppose that A_n promises delivery of one unit of the consumption good in the state $\omega = n$, two units of the consumption good in the state $\omega = n + 1$, and nothing in other states. If consumption is constrained to be non-negative, so that $y^1 \geq 0$ and $y^2 \geq 0$, expanding the above expressions for $y^1(\omega), y^2(\omega)$ and evaluating successively in states $\omega = N + 1, N, \dots, 1$ yields two systems of simultaneous inequalities:

$$\begin{aligned}
1 + 2\theta_N &\geq 0 \\
1 + 2\theta_{N-1} + \theta_N &\geq 0 \\
&\vdots \\
1 + 2\theta_2 + \theta_3 &\geq 0 \\
1 + 7\theta_1 + \theta_2 &\geq 0 \\
1 + \theta_1 &\geq 0
\end{aligned}$$

and

$$\begin{aligned}
1 - 2\theta_N &\geq 0 \\
1 - 2\theta_{N-1} - \theta_N &\geq 0 \\
&\vdots \\
1 - 2\theta_2 - \theta_3 &\geq 0 \\
1 - 2\theta_1 - \theta_2 &\geq 0 \\
7 - \theta_1 &\geq 0
\end{aligned}$$

Solving these systems of inequalities yields

$$-1 \leq \theta_1 \leq +1 \text{ and } -1 \leq \theta_2 \leq +1$$

This entails $y^1(1) \leq 2$ and $y^1(2) \geq 4$, whence $y^2(1) \geq 6$ and $y^2(2) \leq 4$.

As we have already noted, at every every Pareto optimal allocation, each trader is allocated equal amounts of the consumption good in states $\omega = 1$ and $\omega = 2$. Hence, no Pareto optimal allocation can be obtained by trading only the contingent contracts A_1, \dots, A_N . Indeed, if (y^1, y^2) can be obtained by trading these contracts, and (x^1, x^2) is a Pareto optimal allocation, then at least one trader i finds that $U(x^i)$ exceeds $U(y^i)$ by at least $[\sqrt{7} - 2]/4$. Thus there is a gap between the utilities that can be attained when complete contingent contracts are available, and the utilities that can be attained when only the contracts A_1, \dots, A_N are available. Most importantly from our point of view, *this gap does not depend on N , and hence it does not disappear, no matter how large is the number of contracts available for trade.*

To make the same point in another way, suppose we try to obtain the Walrasian equilibrium allocation $(\bar{w}/2, \bar{w}/2)$ by trading a portfolio θ of the contracts A_1, \dots, A_N . Suppose for the moment that N is even. Trader 1's state ω allocation will be

$$y^1(\omega) = w^1(\omega) + \sum_{n=1}^N \theta_n A_n(\omega)$$

Setting $y^1 = \bar{w}/2$ and evaluating successively in states $\omega = 1, 2, \dots$ yields

$$\theta_1 = 3 \tag{1}$$

$$\theta_n = (-1)^{n-1} 2^{n-1} 9 \quad \text{for } 2 \leq n \leq N \tag{2}$$

Since N is even, it follows that $y^1(N+1) = 1 - 2^{N-1} 9$ which of course is (very much) less than 0. In other words, the only portfolio θ which yields the desired consumption in states $\omega = 1, 2, \dots, N$ also creates a liability in state $\omega = N+1$ that cannot be satisfied. If N is odd we obtain the same conclusion, but with trader 2 playing the role of trader 1.

Thus, the requirement that all liabilities be met puts limitations on the portfolios that can be traded. The effect of these limitations is to constrain feasible allocations away from Pareto optimal allocations — even when the number of contracts available is very large.

Requiring that liabilities be met is the same as forbidding default; let's see what happens if default is permitted. For convenience, assume that $N \geq 2$ is even. In that case, trader 1 might purchase (and trader 2 might sell) the portfolio θ defined by equations (1) and (2) above, and trader 1 might simply default (make no payment at all) whenever his liability exceeds his endowment. This will yield both trader 1 and trader 2 the allocation $\bar{w}/2$. In each state $\omega \neq N+1$, each trader meets his obligations; in state $\omega = N+1$, trader 1 has a unsatisfied liability of $2^{N-1} 9$ units of the consumption good. However, the probability that this state occurs is only 3^{-N-1} , so the expected magnitude of trader 1's unsatisfied liability is only $2^N 3^{-N+1}$, which is very small if N is large. If the institutional structure is such that default which is of small expected magnitude incurs penalties which are small in expectation,

both traders will benefit from this arrangement.¹⁰

In the sections to follow, we shall try to formalize the intuitions of this example. In Section 3 we formalize a model in which markets are incomplete; that is, only some contingent contracts are traded. In Section 4 we show that opening new markets — expanding the set of contingent contracts that are available — will generally *not* lead to efficient allocations. In Section 5 we formalize a model in which default is contemplated, and in Section 6 we show that default promotes efficiency in precisely the manner suggested by this example.

An important part of our formal development will be to show that this example represents behavior that is quite typical, and not at all pathological. In particular, both asymptotic inefficiency in the absence of default and asymptotic efficiency in the presence of default are typical conclusions for sequences of assets.

¹⁰It might appear that the particular probability distribution μ and particular preference structure play important roles here, but these appearances are misleading. If the probability distribution μ and/or the preference structure were different, it might be necessary to choose the portfolio θ differently, and it might not be possible to achieve an allocation that agrees exactly with the Walrasian equilibrium allocation in the first N states and differs in other states by a total which is small in expectation. But it will *always* be possible to achieve an allocation that is *close* to the Walrasian equilibrium allocation in the first N states and differs from the Walrasian allocation in other states by a total which is small in expectation.

3. THE SECURITY MARKET MODEL

In this Section we describe our basic model of a security market. We use what seems to be the simplest possible model because it is adequate for our purposes and avoids the technical difficulties that would arise in more general models.

Our model has two dates, 0 and 1, with uncertainty about the state of nature at date 1. A single good is available for consumption at date 0 and in each state at date 1. At date 0, trade takes place in the single consumption good and in each of a finite number of securities, whose date 1 payoffs depend on the state of nature. At date 1, the state of nature is revealed, securities pay their returns, and consumption takes place. Since we consider only a single consumption good, there will of course be no trade at date 1.

More formally, we describe uncertainty by the set $\Omega = \{1, 2, \dots\}$ of states of nature. It is convenient to write $\Omega^* = \Omega \cup \{0\} = \{0, 1, 2, \dots\}$ for the set of spots. A consumption plan $x : \Omega^* \rightarrow \mathfrak{R}$ specifies consumption at date 0 and in each state at date 1; $x(s)$ is consumption at the spot s . It is sometimes convenient to write x_1 for the restriction of x to Ω , so that x_1 is a plan of consumption at date 1. Conversely, each date 1 consumption plan x_1 has a canonical extension to a plan $\Omega^* \rightarrow \mathfrak{R}$ which calls for 0 consumption at date 0; it is convenient to abuse notation and write x_1 for this latter plan also. For simplicity, we assume throughout that all conceivable consumption plans are bounded. Write l^∞ for the space of date 1 consumption plans and $\mathfrak{R} \times l^\infty$ for the space of all consumption plans. Given consumption plans $x, y \in \mathfrak{R} \times l^\infty$, we write: $x \geq y$ to mean $x(\omega) \geq y(\omega)$ for all $\omega \in \Omega^*$; $x > y$ to mean $x(\omega) > y(\omega)$ for all $\omega \in \Omega^*$, and $x \gg y$ to mean that there is a positive number $\epsilon > 0$ such that $x(\omega) > y(\omega) + \epsilon$ for all $\omega \in \Omega$. In particular, $x \gg 0$ means that x is bounded away from 0.

Securities are claims to date 1 consumption plans, and thus are elements of l^∞ . The *dividend* of security A in state ω is $A(\omega)$; we assume for simplicity that $A(\omega) \geq 0$ for each ω . If there are N securities A_1, \dots, A_N , a *portfolio* is a vector $\theta = (\theta_1, \dots, \theta_N) \in \mathfrak{R}^N$; θ_n is the holding of the n -th security. The

dividends on the portfolio $\theta = (\theta_1, \dots, \theta_N) \in \mathfrak{R}^N$ are:

$$\text{div}(\theta) = \sum_n \theta_n A_n \in l^\infty$$

Security prices are vectors $q \in (\mathfrak{R}^N)^+$; if $\theta \in \mathfrak{R}^N$ is a portfolio, then $q \cdot \theta = \sum q_n \theta_n$ is the value of the portfolio θ at the prices q . We take date 0 consumption as numeraire, so that security prices are denominated in date 0 consumption.

Traders $h \in \{1, \dots, H\}$ are defined by consumption sets X^h , endowments $w^h \in X^h$, and utility functions $U^h : X^h \rightarrow \mathfrak{R}^+$. For convenience, we assume that all consumption sets are the positive cone $\mathfrak{R}^+ \times (l^\infty)^+$, and that each trader maximizes the sum of utility for consumption at date 0 and expected utility for consumption at date 1 (expectations taken according to some common prior probability distribution μ on Ω).¹¹ Our assumptions on utility functions mean that there are functions u^h, v^h such that:

$$\begin{aligned} U^h(x) &= v^h(x(0)) + \int u^h(x(\omega)) d\mu(\omega) \\ &= v^h(x(0)) + \sum u^h(x(\omega)) \mu(\omega) \end{aligned}$$

We assume that the functions $u^h, v^h : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ are continuous, strictly concave, and strictly monotone, and that the (right hand) derivatives $u^{h'}(0), v^{h'}(0)$ are finite. Finally, we assume that endowments are bounded away from 0; that is, $w^h \gg 0$ for all h .

A security market \mathcal{E} consists of a finite set of traders $\{(w^h, U^h)\}$ and a finite set of securities $\{A_n\}$; the assumptions above are understood to be in force at all times.

Because there is a single physical commodity, there will be no trading in commodities after the state of nature is revealed. Hence, given security prices

¹¹The assumption of common priors is made only for notational simplicity. Our arguments would remain unchanged if we allowed for traders to have different priors μ^h , provided that priors are consistent, in the weak sense that consumers find each of the possible states of nature to have positive probability: $\mu^h(\omega) > 0$ for each h, ω . Indeed, there would be no real difficulty in extending our results to allow for general utility functions.

q , we define the *budget set* $B^h(q, w^h)$ for a consumer with endowment w^h as the set of set of triples (x^h, ϕ^h, ψ^h) , where x^h is a non-negative consumption plan and ϕ^h, ψ^h are non-negative portfolios — of *security purchases* and *security sales*, respectively — such that:

- $q \cdot (\phi^h - \psi^h) \leq w^h(0) - x^h(0)$
- $x^h(\omega) = w^h(0) + \text{div}(\phi^h)(\omega) - \text{div}(\psi^h)(\omega)$

The first of these equalities says each trader finances purchases and sales of securities from date 0 consumption, while the second says that each trader's consumption in state ω is the sum of his endowment and the dividends on his purchases, less his payments on liabilities.¹²

An *equilibrium* for the security market \mathcal{E} is a 4-tuple $(q, (x^h), (\phi^h), (\psi^h))$, of security prices q , consumption plans x^h , purchases (ϕ^h) and sales (ψ^h) such that:

1. $\sum(x^h - w^h) = 0$ (commodity markets clear)
2. $\sum(\phi^h - \psi^h) = 0$ (securities are in zero net supply)
3. $(x^h, \phi^h, \psi^h) \in B^h(q, w^h)$ for each h (plans are budget feasible)
4. if $(y^h, \alpha^h, \beta^h) \in B^h(q, w^h)$ then $U^h(x^h) \geq U^h(y^h)$
(traders optimize in their budget sets)

The basic fact about security markets is that equilibria exist. We defer the simple proof to the Appendix.

¹²In the usual formulation, purchases are not separated from sales, but there is certainly no harm in doing so, and it will prove necessary in the default model.

Theorem 1 *Every security market has an equilibrium.*

Underlying the security market \mathcal{E} is a complete markets (Arrow-Debreu) economy \mathcal{E}^{CM} with the same traders, in which all contingent consumption patterns are available for sale. A price in \mathcal{E}^{CM} is a positive linear functional $\pi : \mathfrak{R} \times l^\infty \rightarrow \mathfrak{R}^+$ which is continuous with respect to the topology of convergence in expectation (with respect to the common prior probability distribution μ).¹³ Continuity implies that we may interpret π as a price list: there is a (unique) function $\pi^* : \Omega^* \rightarrow \mathfrak{R}^+$ with the property that, for each $x \in \mathfrak{R} \times l^\infty$, the value of the consumption plan x at the price π is

$$\pi \cdot x = \sum_{s=0}^{\infty} \pi^*(s)x(s)$$

Thus $\pi^*(s)$ is the value (at the price π) of the consumption plan promising 1 unit of consumption at the spot s and 0 elsewhere. Since $\pi \cdot x$ is finite for all consumption plans x , it follows that $\sum \pi^*(s) < \infty$.

As usual, a (Walrasian) equilibrium for \mathcal{E}^{CM} is a pair $(\pi, (x^h))$ consisting of prices π for contingent claims, and feasible consumption plans x^h , satisfying

1. $\sum (x^h - w^h) \leq 0$
2. $\pi \cdot (x^h - w^h) \leq 0$ for each h
3. if $U^h(y^h) > U^h(x^h)$ then $\pi \cdot (y^h - w^h) > 0$

¹³The sequence $\{x^n\}$ of consumption plans converges in expectation to the consumption plan x if $x^n(0) \rightarrow x(0)$ and

$$d_E(x_1^n - x_1) = \text{Exp}(|x_1^n - x_1|) = \int |x_1^n - x_1| d\mu \rightarrow 0$$

(Recall that x_1^n is the restriction of x^n to Ω , which is a date 1 consumption plan.)

4. ASYMPTOTIC INEFFICIENCY OF SECURITY MARKET EQUILIBRIUM

If security markets are incomplete, opportunities for insurance and risk-sharing are missing. As a consequence, there may be a gap between (security market) equilibrium and (Pareto) efficiency. As the example of Section 2 shows, this gap between equilibrium and efficiency may persist, even as the set of available securities expands.

In general, the limit behavior as the set of securities expands will depend (in a rather complicated way) on the particular securities. What moral (if any) we might draw should depend — to some extent at least — on our model of the process which gives rise to securities. Unfortunately, no convincing model seems available, and we have none to offer. Instead we take what seems to be a reasonable shortcut. We fix an infinite sequence $\{A_n\}$ of securities, consider the equilibria of the security market in which only the first N securities are available for trade, and then consider what happens as $N \rightarrow \infty$. To capture the idea that our results represent the “typical” situation, we parametrize the set of sequences of securities as a compact metric space, and appeal (as is frequently done) to a topological notion of size to represent the “typical” situation. In this framework, we show below that the example of Section 2 is typical: asymptotic inefficiency is the rule.

Of course, the efficiency or inefficiency of security market allocations depends on the particular securities available for trade, but it also depends on endowments. Equilibrium allocations in security markets will always be Pareto optimal if endowments are themselves Pareto optimal — even if no securities are available for trade. It seems natural therefore to focus on “typical” endowments as well as “typical” sequences of securities. Here too we shall parametrize the set of endowments as a compact metric space and appeal to a topological notion of size to represent the “typical” situation.

To parametrize endowments, recall that we have, throughout, required that endowments be bounded above and bounded away from 0. For the present purpose it is convenient to restrict attention to endowments bounded

above by 1 and below by $1/10H$; write

$$W = \{(w^1, \dots, w^H) : \frac{1}{10H} \leq w^h(s) \leq 1 \text{ for each } h, s\}$$

If we equip consumption sets $\mathfrak{R} \times l^\infty$ with the topology of simple convergence (so that $x^n \rightarrow x$ exactly when $x^n(s) \rightarrow x(s)$ for each state s), and equip W with the product topology, then W is a compact metric space.¹⁴ To parametrize sequences of securities, recall that we have required that security returns be positive and bounded; there is no loss of generality in requiring that they be bounded by 1. Write S for the set of such securities

$$S = \{A : 0 \leq A(\omega) \leq 1, \text{ for each } \omega\}$$

and let \mathcal{A} be the set of (infinite) sequences of securities (in S); we write \mathbf{A} for a sequence in \mathcal{A} . Equipped with the topology of simple convergence, S is a compact metric space; if we equip \mathcal{A} with the (infinite) product topology, it too becomes a compact metric space.

Recall that a subset of a topological space is *residual* if it is the intersection of a countable family of dense open sets. It is customary to view residual subsets of a compact metric space as large, and to view their complements as small; in particular, the Baire category theorem asserts that residual sets are dense. We say that a property holds for *almost all* values of some parameter if the set of parameter values for which it is valid contains a residual set.

We find it convenient to view utility functions U^1, \dots, U^H as fixed, and to view the endowments $w = (w^h) \in W$ and the sequence of securities $\mathbf{A} = \{A_n\} \in \mathcal{A}$ as parameters. For each integer N , denote by \mathcal{E}_N the security market populated by these traders and in which only the first N securities A_1, \dots, A_N are available for trade. We are interested in whether or not equilibrium allocations of \mathcal{E}_N are close to Pareto optimal allocations, when N is large. Anticipating the negative answer obtained below, we shall say that the sequence of securities $\{A_n\}$ is *asymptotically inefficient* (from

¹⁴Alternatively, we might endow W with the topology of convergence in expectation (defined in Section 3). However, the product topology and the topology of convergence in expectation coincide on bounded sets.

w) if the utilities of all feasible security market allocations of \mathcal{E}_N are bounded away from the utilities of Pareto optimal allocations.

There is another point to be addressed here. Inefficiency may arise because securities do not span all the uncertainty, but it is the combination of inefficiency and spanning that is of most interest to us. In the case of a finite state space, spanning (completeness) has an unambiguous meaning: every date 1 consumption pattern can be obtained as the dividends of a finite portfolio. When the state space is infinite, it seems natural to require only that every date 1 consumption pattern be approximable by the dividends of a finite portfolio. Since we have assumed that traders share a common probability distribution μ over the set of states Ω , it seems natural to require approximation in the topology of convergence in expectation with respect to this common probability distribution.¹⁵ Formally, say that the sequence $\{A_n\}$ spans all the uncertainty if for each date 1 consumption pattern x and each $\epsilon > 0$ there is a finite portfolio θ (of securities in the sequence $\{A_n\}$) such that

$$\text{Exp}(|x - \text{div}(\theta)|) = \int |x - \text{div}(\theta)| d\mu < \epsilon$$

As the following result shows, spanning and inefficiency are the rule. Looking ahead to the default model and the discussion in Sections 5 and 6, we take this opportunity to record the additional fact that almost all sequences of securities are linearly independent.

¹⁵As noted earlier, convergence in expectation (defined in Section 3) coincides with convergence in the product topology for bounded sequences, but convergence in expectation is a stronger requirement for unbounded sequences. Our assumptions imply that feasible consumption plans are uniformly bounded, and that dividends on feasible portfolios are bounded, but dividends on feasible portfolios need not be *uniformly* bounded.

Theorem 2 *Fix utility functions U^h . Then*

- *Almost all sequences of securities are linearly independent.*
- *Almost all sequences of securities span all the uncertainty.*
- *For almost all endowments, almost all sequences of securities are asymptotically inefficient.*

5. THE DEFAULT MODEL

In this section, we describe an adaptation of the security market model which allows for the possibility of endogenous default. The model we use is a variant of a model due to Dubey, Geanakoplos and Shubik (1988). Our discussion follows rather closely that paper and related work of Dubey and Geanakoplos (1989); we refer the reader to these papers for further discussion.

The basic data of the model are essentially the same as the data of the security market model described in Section 2. However, we modify the definitions of the budget set and of optimizing behavior (and consequently of equilibrium) to allow for the possibility that agents do not meet their liabilities. To be more precise, we allow each trader h to choose, in addition to a consumption plan and portfolios of purchases and sales, the amount $D^h(n, \omega)$ that he actually delivers on security A_n in the state ω .¹⁶ Of course, if trader h chooses not to sell the security A_n , he will have no obligation to meet; in this case we require that his delivery $D^h(n, \omega) = 0$; in every case, we require that $D^h(n, \omega) \geq 0$.¹⁷ Since additional consumption is always desirable, consumer h will never choose to deliver more than the full amount of his promise, which is $\psi^h(n)A_n(\omega)$; hence we will always have

$$0 \leq D^h(n, \omega) \leq \psi^h(n)A_n(\omega)$$

Several things are important to note. First of all, the decision not to deliver on a security is voluntary; in particular, there is no requirement that traders meet their obligations whenever they are able. Default may occur either from necessity or for strategic reasons. Second, by separating purchases from sales, we have allowed for the possibility that agents go long and short in (i.e., buy and sell) the same security. We have implicitly contemplated this possibility in the security market model, but when default is not possible,

¹⁶Recall that sale of a security entails a promise to pay in the future. Since we require securities to have non-negative returns, we avoid the need to interpret failure to deliver on promises of negative quantities.

¹⁷To allow $D^h(n, \omega)$ to be negative would amount to allowing additional borrowing at date 1.

such an action is irrelevant. However, when default is possible, such an action may not be irrelevant; it may benefit a trader to go long and short in the same security *if he does not intend to meet all his obligations*. In particular, there is nothing to prevent a trader from buying one share of a security, selling one share of the same security, collecting the returns from his purchase, and defaulting on his obligations.

With the possibility that others will default on their obligations, a rational agent will make conjectures about this default, and act accordingly. We view purchases and sales of securities as implemented through some central bank, and assume that shortfalls on promised deliveries are spread uniformly among all creditors. In this sense default is anonymous, and each creditor is affected only by the aggregate default. (Mortgage-backed securities provide a reasonable real world approximation to such securities.) Moreover, although we consider explicitly only a model with a finite number of consumers, we implicitly take the view that there are actually a continuum of consumers, but only a finite number of types. Hence, each trader correctly views the effect of his own actions on the aggregate level of default as negligible. Write $K^h(n, \omega)$ for trader h 's conjecture about the aggregate fraction of promised deliveries on security A_n that will actually be made in state ω . Of course, $0 \leq K^h(n, \omega) \leq 1$.

If trader h 's conjectures are correct, then 1 share of security A_n will yield the actual return $K^h(n, \omega)A_n(\omega)$ in state ω , rather than the promised nominal return $A_n(\omega)$. Trader h 's budget set $B^h(q, w^h, K^h)$, given security prices q , endowment w^h and conjectures K^h , will therefore consist of 4-tuples $(x^h, \varphi^h, \psi^h, D^h)$, where x^h is a consumption plan, φ^h is a portfolio of security purchases, ψ^h is a portfolio of security sales, and D^h is a plan of deliveries, such that:

- $q \cdot (\varphi^h - \psi^h) \leq w^h(0) - x^h(0)$
- $x^h(\omega) \leq w^h(\omega) + \sum \{K^h(n, \omega)\varphi^h(n)A_n(\omega)\} - \sum D^h(n, \omega)$

As before, the first of these inequalities says that traders finance purchases and sales of securities from date 0 consumption, and the second says that

trader h 's consumption in state ω is no bigger than the sum of his endowment and the dividends on his purchases (taking into account the default against him), less his own deliveries on liabilities.

To this point we have not spoken of the consequences of default. In reality, creditors might be able to seize assets and be awarded judgments against future earnings, defaulters might be barred from future credit markets, etc. To simplify matters as much as possible, we assume here that the only consequences of default are penalties assessed against the defaulters, and that these penalties are assessed directly in terms of utility. We shall also assume that default penalties are independent of the state of nature and of the security, are the same for all agents, and are proportional to the amount of default; we write λ for this constant of proportionality.¹⁸

Given a default penalty of λ , the utility consumer h will achieve by following the plan $(x^h, \varphi^h, \psi^h, D^h)$ is:

$$\tilde{U}^h(x^h, \varphi^h, \psi^h, D^h) = U^h(x^h) - \int \lambda \sum [\psi^h(n)A_n(\omega) - D^n(n, \omega)] d\mu$$

That is, trader h enjoys the utility of his consumption, less his expected penalties. (Note that $\psi^h(n)A_n(\omega) - D^n(n, \omega)$ is the amount of trader h 's default on the security A_n .)

A *default equilibrium* is a list q of security prices, a collection of conjectures K^h , and a collection of plans $(x^h, \varphi^h, \psi^h, D^h)$ such that

1. $\sum (x^h - w^h) \leq 0$

¹⁸As we have discussed in the Introduction, we view utility penalties as proxies for real economic penalties; it would surely be desirable — although much more complicated — to formulate these explicitly. However, the assumptions about the precise nature of the utility penalty are made solely for notational convenience. There would be no difficulty in allowing for default penalties that depend on the state and the security, are agent-specific, and are not proportional to the amount of default. All that is really required is that default penalties be concave and be sufficiently severe when default is large to ensure that no agent will seek to acquire liabilities that exceed the aggregate resources of society.

2. $\sum(\varphi^h - \psi^h) = 0$
3. $(x^h, \varphi^h, \psi^h, D^h) \in B^h(q, w^h, K^h)$ for each h
4. if $(y^h, \alpha^h, \beta^h, \bar{D}^h) \in B^h(q, w^h, K^h)$ then

$$\hat{U}^h(y^h, \alpha^h, \beta^h, \bar{D}^h) \leq \hat{U}^h(x^h, \varphi^h, \psi^h, D^h)$$

5. conjectures K^h are correct

In other words, commodity markets clear, security markets clear, agents optimize (among budget feasible plans) given security prices and their own conjectures, and agents have correct (and therefore identical) conjectures.

It remains to formalize the last requirement, that conjectures K^h be correct at equilibrium. This is straightforward for conjectures about securities that are traded and pay positive dividends in a given state ω , and is irrelevant for securities that do not yield dividends in the state ω ; the requirements are:

- if $A_n(\omega) > 0$ and $\sum_h \varphi^h > 0$ then

$$K^h(n, \omega) = \frac{\sum D^h(n, \omega)}{\sum \varphi^h(n) A_n(\omega)}$$

- if $A_n(\omega) = 0$ then $K^h(n, \omega)$ may be arbitrary

However, if the security A_n yields positive returns in state ω and is not traded at equilibrium, there is some question as to the proper requirement. Were we to allow for arbitrary conjectures in this case, there would always be trivial equilibria in which no assets are traded because all agents conjecture total default — even if default penalties were so high that purchasers of securities would not actually be willing to default. A similar issue is familiar from the theory of extensive form games. As Selten (1975) points out, the Nash equilibrium notion imposes no restrictions off the equilibrium path. Requiring that agents optimize at all decision nodes — even those

off the equilibrium path — leads to the stronger notion of subgame perfect equilibrium. Of course the bite of subgame perfection is in the restrictions it imposes on beliefs. In our framework, insisting that traders conjecture that others always choose optimal actions — even out of equilibrium — rules out the trivial equilibria described above. There are several ways to formalize the requirement that consumers always conjecture optimal behavior by others; such formalizations have been given by Dubey, Geanakoplos and Shubik (1988) and by Dubey and Geanakoplos (1989). Our formalization is weaker than either of these, so that we allow more equilibria (as the reader will see, this is in keeping with our aims).

To motivate our requirement, we show that, if the default penalty λ is sufficiently high, and traders are optimizing, then the probability of default will be low, and there will be no voluntary default at all. To make these assertions precise, write w^0 for the aggregate endowment. Fix a trader k , and consider an optimal budget-feasible plan $(x^k, \varphi^k, \psi^k, D^k)$. Let Ω_k be the set of states in which trader k defaults on some security (i.e., $\psi^k(n)A_n(\omega) - D^k(n, \omega) \neq 0$ for some n) and let $\hat{\Omega}_k$ be the set of states in which consumer k 's default is voluntary (i.e., $\psi^k(n)A_n(\omega) - D^k(n, \omega) \neq 0$ for some n and $x^k(\omega) \neq 0$). Write $w^0 = \sum w^h$. We assert that

- (i) if $\lambda > u^{k'}(0)$ then $\hat{\Omega}_k = \emptyset$
- (ii) $\mu(\Omega_k) \leq (1/\lambda)(u^{k'}(0) + v^{k'}(0))(\sup\{w^0(\omega) : \omega \in \Omega^*\})$

The first of these assertions means that trader k will not default voluntarily if the penalty is sufficiently high; the second provides bounds the probability that trader k will default at all.

To verify (i), suppose to the contrary that $\lambda > u^{k'}(0)$ and that there is a state ω in which trader k defaults voluntarily. If ϵ is less than trader k 's consumption and default in state ω , then trader k can alter his plan by making a larger payment on his state ω debt, thereby decreasing both his consumption and default in the state ω by ϵ . Reducing default by ϵ reduces the utility penalty by $\epsilon\lambda\mu(\omega)$; reducing consumption by ϵ reduces the

utility of consumption by at most $\epsilon u^{k'}(0)\mu(\omega)$. If $\lambda > u^{k'}(0)$, this contradicts the optimality of the plan $(x^k, \varphi^k, \psi^k, D^k)$; this contradiction establishes the assertion.

To verify (ii), let t be a real number with $0 < t < 1$, and consider the alternate plan $(w^k + t(x^k - w^k), t\varphi^k, t\psi^k, D^k)$. Because $w^k \gg 0$, this plan will be budget-feasible if t is sufficiently close to 1. However, this alternate plan would entail a utility cost of foregone consumption not exceeding

$$(1 - t)(u^{k'}(0) + v^{k'}(0))(\sup\{w^0(\omega) : \omega \in \Omega^*\})$$

while avoiding a utility penalty of $(1 - t)\lambda\mu(\Omega_k)$. The plan $(x^k, \varphi^k, \psi^k, D^k)$ is optimal, so this alternate plan cannot be an improvement; this yields (ii).

The rationality requirement we wish to impose when securities are not traded is that conjectures should be consistent with the above observations. To express this requirement most simply, write

$$M_0 = \sup\{u^{k'}(0) + v^{k'}(0) : 1 \leq k \leq H\}$$

and

$$\Omega_h = \{\omega : K^h(n, \omega) \neq 0 \text{ for some } n\}$$

Applying (ii) for each trader leads to the requirement:

- $\mu(\Omega_h) \leq H(M_0/\lambda)(\sup\{w^0(\omega) : \omega \in \Omega^*\})$ for each h

As we have said, our rationality requirements are weaker than those imposed in Dubey, Geanakoplos and Shubik (1988) and Dubey and Geanakoplos (1989). In particular, our requirement is vacuously satisfied when

$$H(M_0/\lambda)(\sup\{w^0(\omega) : \omega \in \Omega^*\}) > 1$$

that is, when the default penalty is sufficiently small. Hence, we allow here for a potentially larger set of default equilibria.¹⁹

¹⁹This is in keeping with our goal, which is to show that all default equilibria are nearly optimal; see Section 6.

For $\epsilon > 0$, we define an ϵ -equilibrium to be a list q of security prices, a collection of conjectures K^h , and a collection of plans $(x^h, \varphi^h, \psi^h, D^h)$ satisfying 1., 2., 3., 5., and

6. if $(y^h, \alpha^h, \beta^h, \bar{D}^h) \in B^h(q, w^h, K^h)$ then

$$\hat{U}^h(y^h, \alpha^h, \beta^h, \bar{D}^h) \leq \hat{U}^h(x^h, \varphi^h, \psi^h, D^h) + \epsilon$$

This completes the description of the default model. The next step is to show that the model is consistent; i.e., that equilibria exist. Again, the proof is deferred to the Appendix.

Theorem 3 *If securities are linearly independent, then for each λ a default equilibrium exists.*²⁰

It is instructive to consider the two extreme cases $\lambda = 0$ and $\lambda = \infty$. If $\lambda = 0$, there is no penalty for default. In such case, optimizing agents will never honor any of their commitments, and agents with correct expectations will never lend (i.e., sell securities). Thus, at equilibrium, there will be no trade in securities, and hence (given that there is a single commodity), no trade at all. If $\lambda = \infty$, optimizing agents will never default (else they would incur infinite penalties). Hence such default equilibria coincide precisely with the security market equilibria of Section 3.

Thus, in extreme cases, we see no equilibrium default. However, for all intermediate penalties, there will generally be some (probability of) default at equilibrium. We have already noted that, if the default penalty is high, the probability that default will occur will be small. As we shall see, if the default penalty is high, the expected amount of default will also be small.

²⁰In the security market model, there is no need to assume that securities are linearly independent, since redundant securities can be priced by arbitrage. However, when default is possible, this is no longer the case; see Dubey, Geanakoplos and Shubik (1988) for further discussion.

6. THE ROLE OF DEFAULT

Default creates inefficiencies: the direct losses of imposing the default penalties, and the indirect losses due to the reluctance of lenders to lend and the inability and reluctance of borrowers to borrow. In this section we show that, despite the inefficiencies it creates, default may promote overall efficiency. The example of Section 2 and Theorem 2 in Section 4 provide the intuition necessary for understanding this apparent contradiction. When default is not possible, the requirement that consumers keep all their promises (i.e., that terminal wealth constraints be met) may severely restrict the portfolios that can actually be traded, and hence the effective span of securities. Indeed, as we have argued in Section 4, this is typically the case. Default gives consumers some ability to tailor security payoffs to their own requirements. In this way, default endogenously increases the effective span of securities, and this may (more than) compensate for the direct and indirect losses associated with default.

We formalize this intuition in two ways. If the default penalty is large enough to discourage voluntary default and the number of securities is sufficiently large, then near-efficiency is *possible*: some default ϵ -equilibrium allocation is close (in utility) to a Walrasian equilibrium.²¹ If both the default penalty and the number of securities are sufficiently large, then near-efficiency is *necessary*: every default equilibrium allocation is close (in utility) to a Walrasian equilibrium.

In what follows, we fix a set of H traders, specified by endowments w^h and utility functions U^h , and an infinite sequence $\{A_n\}$ of securities. We assume that this sequence is linearly independent and spans all the uncertainty (in the sense discussed in Section 4). (In view of Theorem 2, almost all sequences of securities satisfy these requirements.) Given an integer N and a default penalty λ , write $\mathcal{E}^{N,\lambda}$ for the security market in which the first N securities $\{A_1, \dots, A_N\}$ are available for trade and the default penalty is λ . Recall that, given a consumption plan x^h , purchases φ^h , sales ψ^h , delivery plans D^h

²¹In fact, every Walrasian equilibrium is approximated by a default ϵ -equilibrium.

and conjectures K^h , trader h 's utility is

$$\hat{U}^h(x^h, \varphi^h, \psi^h, D^h) = U^h(x^h) - \int \lambda \left[\sum \psi^h(n) A_n(\omega) - D^h(n, \omega) \right] d\mu$$

Theorem 4 For every $\epsilon > 0$ there is a default penalty $\lambda(\epsilon)$ and a function $N(\epsilon, \cdot) : (\lambda(\epsilon), \infty) \rightarrow (0, \infty)$ such that:

If $\lambda > \lambda(\epsilon)$ and $(q, (K^h, x^h, \varphi^h, \psi^h, D^h))$ is a default equilibrium of the security market $\mathcal{E}^{N, \lambda}$ then there is a Walrasian equilibrium allocation (y^h) for which

- $\text{Exp}(|x^h - y^h|) < \epsilon$
- $|\hat{U}^h(x^h, \varphi^h, \psi^h, D^h) - U^h(y^h)| < \epsilon$

for each trader h .

Theorem 5 For every $\epsilon > 0$ and every $\lambda > \lambda_0 = \max \{u^{h'}(0) : 1 \leq h \leq H\}$ there is an integer N_0 such that:

If y is a Walrasian equilibrium allocation of the underlying complete markets economy, and $N > N_0$ then there is a default ϵ -equilibrium $(q, (K^h, x^h, \varphi^h, \psi^h, D^h))$ of the security market $\mathcal{E}^{N, \lambda}$ for which

- $\text{Exp}(|x^h - y^h|) < \epsilon$
- $|\hat{U}^h(x^h, \varphi^h, \psi^h, D^h) - U^h(y^h)| < \epsilon$

for each trader h .

It is natural to think of an economy with complete markets as an idealized limit of economies with incomplete markets. From this point of view, we might interpret Theorems 4 and 5 as statements about the continuity of equilibrium allocations. Theorem 4 — which provides conditions under which all default equilibria are close to Walrasian equilibria — asserts a kind of upper hemi-continuity. Conversely, Theorem 5 — which provides conditions under which Walrasian equilibria can be approximated by default ϵ -equilibria — represents a kind of lower hemi-continuity.²²

It should be noted that the conclusions of Theorems 4 and 5 are not quite parallel. To obtain the conclusion that *all* default equilibrium allocations are close to Walrasian equilibrium allocations, the default penalty might need to be extremely large. However, in order to obtain the weaker conclusion that *some* default ϵ -equilibrium allocation is close to a Walrasian equilibrium allocation, the default penalty will only need to be sufficiently large to discourage voluntary default.

A final point. In the proofs we will show that, if the number of securities and the default penalty are sufficiently large, then the probability of default and the expected magnitude of default are both small; in particular, there is no default in most states of the world. However, we have nothing to say about the magnitude (or fraction) of default, conditional on the event that default actually occurs. In particular, we do not rule out the possibility that when default occurs it is total: no deliveries at all are made.

²²Theorem 2 shows that, in the absence of default, both upper and lower hemi-continuity fail.

7. CONCLUSION

In this paper, we have argued that default plays an important, positive role in the economy. When markets are incomplete, and traders are only able to enter into contracts that they will be able to execute regardless of future events, contingent contracting may be severely restricted. Moreover, opening new markets may not relieve these restrictions. Default makes it possible for traders to enter into contracts that they will be able to execute with high probability, but not with certainty — and that possibility may greatly expand opportunities for contingent contracting.

The only consequences of default that we have considered here are direct utility penalties imposed on defaulters. This has made it possible for us to explore the role of default without exploring particular mechanisms or institutions. There is no doubt however, that institutions matter, and that this represents an important direction for future research.

The institutions currently used in this country for dealing with default involve primarily the withdrawal of credit and the seizure of collateral and other assets. Most of the previous analysis of these institutions has been in the context of game-theoretic and/or partial equilibrium models (Hart and Moore (1989) is a recent example), but there has been some work in the general equilibrium spirit of the present model. Kehoe and Levine (1989) have formulated a model in which the threat of withdrawal of credit plays a central role. In their model, however, markets are complete and there is no equilibrium default. This is entirely in keeping with their goals, which are to see how the threats of penalties can force traders to honor their commitments. Expanding their model to allow for incomplete markets and equilibrium default does not seem at all straightforward. In work in progress, Geanakoplos and the author construct a general equilibrium model in which short sales of securities (that is, borrowing) is collateralized.

APPENDIX

Here we collect proofs of the Theorems. In our one-commodity world, establishing the existence of security market equilibrium is easy.

Proof of Theorem 1: There is no loss of generality in assuming that the securities A_n have linearly independent returns (since redundant securities can be priced by arbitrage). We reduce the existence of a security market equilibrium to the existence of a Walrasian (Arrow-Debreu) equilibrium for a complete markets shadow economy in which the commodity bundles represent date 0 consumption and portfolios of securities.

This Walrasian shadow economy is defined in the following way. The commodity space for the shadow market is $\mathfrak{R} \times \mathfrak{R}^N$ (where N is the number of securities). For $(t, \theta) \in \mathfrak{R} \times \mathfrak{R}^N$, we interpret t as date 0 consumption and θ as a portfolio (not necessarily non-negative) of securities. The consumption set for consumer h is the set X^h of pairs $(t^h, \theta^h) \in \mathfrak{R} \times \mathfrak{R}^N$ such that $w^h + (t^h, \text{div}(\theta^h)) \geq 0$; it is easily seen that X^h is a closed convex subset of $\mathfrak{R} \times \mathfrak{R}^N$, and is bounded below (because security returns are linearly independent.) The utility function $V^h : X^h \rightarrow \mathfrak{R}$ of consumer h is defined by

$$V^h(t^h, \theta^h) = U^h(w^h + (t^h, \text{div}(\theta^h)))$$

It is easily seen that V^h is continuous, quasi-concave, and strictly monotone (because security returns are non-negative). Finally, consumer h 's endowment is $e^h = (w^h(0), 0) \in \mathfrak{R} \times \mathfrak{R}^N$. (Keep in mind that securities are in zero net supply.) Our assumption that security returns are bounded and that endowments are bounded away from 0 guarantee that e^h belongs to the interior of X^h .

It follows from standard existence theorems that this Walrasian economy has an equilibrium $(q, (\bar{t}^h, \bar{\theta}^h))$. Write φ^h, ψ^h for the positive and negative parts (respectively) of $\bar{\theta}^h$, and set $x^h = w^h + (\bar{t}^h, \text{div}(\bar{\theta}^h))$. It is easily checked that the tuple $(q, (x^h), (\varphi^h), (\psi^h))$ is an equilibrium for the security market \mathcal{E} , as desired. \square

We note that this construction provides an equivalence between the equilibria of the security market \mathcal{E} and the equilibria of the Walrasian shadow economy. It follows that equilibria of the security market \mathcal{E} are constrained optimal, in the sense of being Pareto undominated by any allocation attainable by trading date 0 consumption and available securities. This is an observation first made by Diamond (1967), in the context of a finite state space.

Before embarking on the proof of Theorem 2, it is convenient to collect some notation and isolate two portions of the argument as lemmas. If $w \in W$ is an endowment vector, write $PO(w)$ for the set of Pareto optimal allocations (from initial allocations w). For each state ω , set

$$W_\omega = \{ w \in W : \text{for every } (x^h) \in PO(w), \text{ there is a trader } h \text{ such that} \\ |x^h(\omega) - w^h(\omega)| > 1/3 \text{ or } |x^h(\omega + 1) - w^h(\omega + 1)| > 1/3 \}$$

Write $W_0 = \bigcup W_\omega$. The first lemma establishes that, for almost all endowments, every Pareto optimal allocation entails at least one large net trade.

Lemma 1 W_0 is a dense open subset of W .

Proof: That each W_ω — and hence W_0 itself — is open follows directly from the definition and the fact that the Pareto correspondence is compact-valued and upper hemi-continuous. To see that W_0 is dense, fix an endowment vector $w \in W$ and a state ω ; define new endowment vectors \bar{w}, \underline{w} by

$$\begin{aligned} \bar{w}^h(\tau) &= \underline{w}^h(\tau) = w^h(\tau) \quad \text{for } \tau < \omega \\ \bar{w}^1(\omega) &= \bar{w}^2(\omega + 1) = 1 \\ \underline{w}^2(\omega) &= \underline{w}^1(\omega + 1) = 1 \\ \bar{w}^h(\tau) &= \underline{w}^h(\tau) = \frac{1}{8H} \quad \text{otherwise} \end{aligned}$$

The endowment vectors \bar{w} and \underline{w} represent different distributions of the same aggregate, so $PO(\bar{w}) = PO(\underline{w})$. Suppose that neither \bar{w} nor \underline{w} belong to W_ω .

Then we could find Pareto optimal allocations $(\bar{x}^h), (\underline{x}^h) \in PO(\bar{w}) = PO(\underline{w})$ satisfying all the following inequalities:

$$\begin{aligned}
\bar{x}^1(\omega) &> 2/3 \\
\bar{x}^1(\omega + 1) &< 1/2 \\
\bar{x}^2(\omega) &< 1/2 \\
\bar{x}^2(\omega) &> 2/3 \\
\underline{x}^1(\omega) &< 1/2 \\
\underline{x}^1(\omega + 1) &> 2/3 \\
\underline{x}^2(\omega) &> 2/3 \\
\underline{x}^2(\omega) &< 1/2
\end{aligned}$$

We now use separability of preferences to compare marginal rates of intertemporal substitution at (\bar{x}^h) to those at (\underline{x}^h) ; we see that consumer 1's increases while consumer 2's decreases. Since marginal rates of substitution are equal at a Pareto optimal allocation, this means that (\bar{x}^h) and (\underline{x}^h) cannot both be Pareto optimal allocations. We conclude that at least one of \bar{w} or \underline{w} belongs to W_ω . By choosing ω sufficiently large, we can make \bar{w} and \underline{w} as close to w as we like, so it follows that W_0 is dense in W , as asserted. \square

For $x, y \in l^\infty$, write $\|x\|_\infty = \sup \{x(\omega) : \omega \in \Omega\}$ and $d_\infty(x, y) = \|x - y\|_\infty$. If $I \subset \Omega$ is any set of states, write $x[I]$ for the vector that agrees with x at all states in I , and is 0 elsewhere. It is convenient to abbreviate $x[\{1, \dots, n\}]$ by $x[n]$ and $x[I \cap \{1, \dots, n\}]$ by $x[I[n]]$. If $\mathbf{A} = \{A_n\}$ is a sequence of securities, we write $\mathbf{A}[I] = \{A_n[I]\}$. Write $\text{span}\mathbf{A}$ for the linear subspace of l^∞ spanned by \mathbf{A} ; i.e., the set of finite linear combinations of the securities A_n , or equivalently, the set of dividends on finite portfolios of elements of \mathbf{A} . Write \mathcal{F} for the set of sequences in l^∞ that are 0 from some point on. For $v \in \mathcal{F}$ and $I \subset \Omega$, write $Q_I(v)$ for the set of sequences \mathbf{A} such that

$$d_\infty(v[I], \text{span}\mathbf{A}[I]) \geq \frac{1}{2}\|v[I]\|_\infty$$

Set $Q_I = \bigcap Q_I(v)$, the intersection extending over all $v \in \mathcal{F}$. The following lemma is closely related to a result in Zame (1988).

Lemma 2 *If $I \subset \Omega$ is an infinite set, then Q_I is a residual subset of \mathcal{A} .*

Proof: We show first that $Q_I[v]$ is residual for each v . To this end, fix $v \in \mathcal{F}$; if $v[I] = 0$ there is nothing to prove, so assume $v[I] \neq 0$. Fix integers m, r such that $v(\omega) = 0$ for $\omega > r$, and let k be the first integer such that $I[k]$ has precisely r elements. (Such a k exists because I is infinite.) Write $\rho = (1/2)(1 - 2^{-m})$, and let $Q_I(v, r, m)$ be the set of security sequences \mathbf{A} such that:

1. $A_i[I[k]], \dots, A_r[I[k]]$ are linearly independent
2. $d_\infty(v[I], \text{span}\{A_i[I], \dots, A_r[I]\}) > \rho \|v[I]\|_\infty$

It is evident that $Q_I[v] = \bigcap Q_I(v, r, m)$, so to show that $Q_I[v]$ is residual, it suffices to show that each $Q_I(v, r, m)$ is a dense open set.

Note first that 1. is equivalent to the non-vanishing of a $k \times k$ determinant, and so remains valid if we replace \mathbf{A} by any perturbation that is small in the first k states and for the first k terms of the sequence. Hence the set of sequences satisfying 1. is dense and open. Note next that 2. is satisfied if and only if there is a state ω such that

3. $d_\infty(v[I[\omega]], \text{span}\{A_1[I[\omega]], \dots, A_r[I[\omega]]\}) > \rho \|v[I]\|_\infty$

Because the vectors $v[I[\omega]], A_n[I[\omega]]$ all lie in a finite dimensional space, a simple continuity argument shows that 3. remains valid if we replace \mathbf{A} by any perturbation that is sufficiently small in the states $\{1, \dots, \omega\}$ and for the first r terms of the sequence. Hence the set of security sequences satisfying condition 2. is open.

To see that the set of sequences satisfying condition 2. is dense, fix a sequence \mathbf{A} ; we must find arbitrarily small perturbations of \mathbf{A} satisfying 2. Because the set of sequences satisfying 1. is dense, there is no loss of generality in assuming that \mathbf{A} already satisfies 1. We may also assume without

loss that there is a state $\zeta > r$ such that $A_i(\tau) = 0$ for $\tau \geq \zeta$. Consider the dividend operator $\Delta : \mathfrak{R}^r \rightarrow l^\infty$ defined by $\Delta(\theta) = \sum \theta_n A_n[I]$. Since $A_1[I], \dots, A_r[I]$ is a linearly independent set, this transformation is an isomorphism of \mathfrak{R}^r with a finite dimensional subspace of l^∞ . Let Θ be the set of portfolios $\theta \in \mathfrak{R}^r$ such that

$$d_\infty(\sum \theta_n A_n[I], v[I]) \leq \rho \|v[I]\|_\infty$$

Because the dividend operator Δ is an isomorphism, Θ is a compact set.

Fix $\theta \in \Theta$, and a state $\tau > \zeta$; define a perturbation \mathbf{A}' of \mathbf{A} by $A'_n(\omega) = A_n(\omega)$ for $\omega \neq \tau$ and $A'_n(\tau) = 1$. The dividends of each security A_i are non-negative and bounded by 1, and $\rho < 1/2$, so $|\sum \theta_i| > \rho \|v\|_\infty$. Because $v(\omega) = 0$ for $\omega > r$, it follows that

$$(*) \quad d_\infty(\sum \theta_n A'_n[I], v[I]) > \rho \|v\|_\infty$$

Continuity implies that $(*)$ remains valid if we replace θ by any portfolio θ' in some neighborhood B_θ of θ ; moreover, this neighborhood B_θ may be chosen independently of the choice of the state τ . Since Θ is compact, we can cover it with a finite number of these neighborhoods, and make perturbations in different states to achieve a security sequence \mathbf{A}'' such that

$$(**) \quad d_\infty(\sum \theta_n A''_n[I], v[I]) > \rho \|v\|_\infty$$

for all $\theta \in \Theta$. Because we have made perturbations only in states where v and each A_i vanish, we conclude that $(**)$ holds for all $\theta \in \mathfrak{R}^r$. Because we can make these perturbations in states with ζ arbitrarily large, these perturbations can be made arbitrarily small. Hence $Q_I(v, r, m)$ is a dense set; as we have noted, this implies that $Q_I(v)$ is residual.

To see that Q_I is residual, observe that, if $v, v' \in \mathcal{F}$ then the triangle inequality yields

$$d_\infty(v', \text{span } \mathbf{A}) > d_\infty(v, \text{span } \mathbf{A}) - d_\infty(v, v')$$

Since the subset of F consisting of vectors with rational entries forms a countable dense subset, it follows that, if $\mathbf{A} \in Q_I(v)$ for each $v \in \mathcal{F}$ having only rational entries, then in fact $\mathbf{A} \in Q_I(v)$ for every $v \in \mathcal{F}$. Hence Q_I can be written as the intersection of countably many residual sets, and is hence residual. \square

With these technical results in hand, we turn to the proof of Theorem 2.

Proof of Theorem 2: That almost all sequences are linearly independent was established in the proof of Lemma 2 above. To see that almost all sequences span all the uncertainty, we use a similar argument. For each state τ , write $\delta_\tau \in l^\infty$ for the consumption plan which is 1 in state τ and 0 in every other state. For each state τ and each positive integer k , write

$$A_{\tau,k} = \{\mathbf{A} \in \mathcal{A} : d_E(\delta_\tau, \text{span}\mathbf{A}) < 1/k\}$$

(Recall that $d_E(x, y) = \text{Exp}(|x - y|)$.) Evidently, every sequence in $\bigcap A_{\tau,k}$ spans all the uncertainty, so it suffices to show that each $A_{\tau,k}$ is a dense open set. To this end, observe first that every vector in $\text{span}\mathbf{A}$ is a linear combination of a finite number of securities. If $d_E(\delta_\tau, \sum \theta_i A_i) < 1/k$, then $d_E(\delta_\tau, \sum \theta_i A'_i) < 1/k$, provided that $d_E(A_i, A'_i)$ is small enough (for each i). Hence $A_{\tau,k}$ is an open set. To see that it is dense, fix any sequence of securities $\mathbf{A} = \{A_n\}$. For each index m , let \mathbf{A}^m be the sequence which is identical to \mathbf{A} except for the m -th security, with $A_m^m = \delta_\tau$. It is evident that $\mathbf{A}^m \in A_{\tau,k}$ and that $\mathbf{A}^m \rightarrow \mathbf{A}$ (as $m \rightarrow \infty$) in the product topology, so that $A_{\tau,k}$ is dense, as desired.

It remains to address asymptotic inefficiency. For each state ω , set

$$W^\omega = \{w \in W : \text{there is a } \tau > \omega \text{ such that } w^h(\tau) < 1/9H \text{ for each } h\}$$

and

$$W^0 = \bigcap W^\omega = \{w \in W : \text{for infinitely many } \tau, w^h(\tau) < 1/9H \text{ for each } h\}$$

Arguing just as above, we may see that each W^ω is a dense open set, so W^0 is a residual set. It follows that $W^* = W^0 \cap W_0$ is also a residual set.

Fix $w \in W^*$; say that $w \in W_\omega$. Set

$$I = \{\omega, \omega + 1\} \cup \{\tau : w^h(\tau) < 1/9H \text{ for each } h\}$$

Note that I is an infinite set of states. According to Lemma 2, the set Q_I of security sequences is residual, so it suffices to prove that every sequence $\mathbf{A} \in Q_I$ is asymptotically inefficient (for the initial endowment vector w). Since $w \in W_\omega$, every Pareto optimal allocation requires a net trade of at least $1/3$, either in state ω or in state $\omega + 1$. Hence every allocation which is close to a Pareto optimal allocation requires a net trade of at least $1/4$, either in state ω or in state $\omega + 1$. We claim that no such allocation (x^h) can be obtained by trading a finite number of the securities A_n .

To see this, write $z^h = x^h - w^h$ for the net trade of consumer h . For the sake of definiteness, assume that consumer 1's net trade in state ω is large:

$$|z^1(\omega)| = |x^1(\omega) - w^1(\omega)| \geq \frac{1}{4}$$

If (x^h) can be obtained by trading n securities, there is a profile of portfolios $\theta^h \in \mathfrak{R}^n$ such that

$$z^h = \text{div}(\theta^h) \in \text{span} \mathbf{A}$$

for each h . Set $v = z^1[\omega]$. Because $\mathbf{A} \in Q_I$, it follows that

$$d_\infty(v[I], \text{span} \mathbf{A}[I]) \geq (1/2) \|v[I]\|_\infty \geq |z^1(\omega)| \geq 1/4$$

Hence, $d_\infty(v[I], z^1[I]) \geq (1/8)$. Since $v(\omega) = z^1(\omega)$, there is a state $\tau \in I, \tau \neq \omega$ such that $|z^1(\tau)| \geq 1/8$. On the other hand, if $\tau \in I$ then $w^h(\tau) \leq 1/9H$. Since consumption vectors are constrained to be positive and the sum of net trades is 0, this entails $|z^1(\tau)| \leq 1/9H$. We have obtained a contradiction, so we conclude that (x^h) cannot be implemented by trading a finite number of the securities A^n ; this completes the proof. \square

Proof of Theorem 3: We have already noted that, if $\lambda = 0$, autarky is an equilibrium, and if $\lambda = \infty$ the default model reduces to the usual security market model, so it suffices to treat the case $0 < \lambda < \infty$. We construct a default equilibrium for the security market E as the limit of equilibria in security markets with a finite number of states.

For each index r , define a security market $\mathcal{E}[r]$ by truncating all the data of \mathcal{E} to the first r spots. (That is, endowments $w^h[r](s) = w^h(s)$ if $s \leq r$, $w^h[r](s) = 0$ if $s > r$, etc.) It follows from work of Dubey, Geanakoplos and Shubik (1988) and Dubey and Geanakoplos (1989) that $\mathcal{E}[r]$ has a default equilibrium

$$(q[r], (K^h[r], x^h[r], \varphi^h[r], \psi^h[r], D^h[r]))$$

We claim that some subsequence of these equilibria converge to an equilibrium for \mathcal{E} .

To demonstrate this, we need to show first that the components of equilibria lie in compact sets. Note that boundedness of the set of feasible date 0 consumptions implies that marginal utilities for date 0 consumption (evaluated at feasible consumption bundles) are bounded away from 0. By assumption, marginal utilities for date 1 consumption are bounded above. It follows that the security prices $q[r]$ are bounded (independently of r), and hence lie in a compact subset of \mathfrak{R}^N . To see that portfolios lie in a compact set, note first that, since the collection of securities is finite, there is an index r_0 with the property that each of the securities yields a strictly positive return in at least one state $\omega \leq r_0$, and that the truncations $A_1[r_0], \dots, A_N[r_0]$ are linearly independent. If the portfolios of sales $\psi^h[r]$ were not bounded (independently of r), linear independence of securities would guarantee that would be at least one state $\omega \leq r_0$ in which liabilities were unbounded. Since aggregate consumption is finite in each state, there would be at least one state $\omega \leq r_0$ in which default would be unbounded. Since default penalties become unbounded with unbounded defaults, such actions would be incompatible with individual rationality, and hence with equilibrium. It follows that portfolios of sales $\psi^h[r]$ are bounded; since securities are in 0 net supply, portfolios of purchases $\varphi^h[r]$ are also bounded. Hence, portfolios of purchases and sales lie in a compact subset of \mathfrak{R}^N . It follows that promises, and hence deliveries $D^h[r]$ lie in a compact subset of $\mathfrak{R} \times l^\infty$; we have already noted that consumption plans lie in a compact subset of $\mathfrak{R} \times l^\infty$. Of course conjectures also lie in a compact set. Passing to a subsequence if necessary, we see that equilibria of $\mathcal{E}[r]$ converge to some tuple $(q, (K^h, x^h, \varphi^h, \psi^h, D^h))$, which we assert to be an equilibrium of \mathcal{E} .

With the exception of individual optimization, verification of the equilibrium conditions is straightforward, and left to the reader. To verify individual optimization, suppose that $\chi = (x, \varphi, \psi, D)$ is an alternative plan for consumer h , which is feasible and superior to the equilibrium plan $\chi^h = (x^h, \varphi^h, \psi^h, D^h)$ (given endowment w^h , conjectures K^h , and security prices q). We are going to find an index s and a plan $\chi^* = (x^*, \varphi^*, \psi^*, D^*)$ in the economy $\mathcal{E}[s]$ which is superior to consumer h 's equilibrium plan $\chi[s]$. To this end, let $\alpha, \beta, \delta > 0$ be parameters (to be chosen later) and let s be an integer (also to be chosen later). Define the plan $\chi' = (x', \varphi', \psi', D')$ in the following way:

$$\begin{aligned} x'(t) &= \max\{x(t) - \alpha, 0\} && \text{for } 0 \leq t \leq s \\ &= 0 && \text{for } t > s \\ \varphi' &= (1 - \beta)\varphi \\ \psi' &= \psi \\ D'(n, t) &= \max\{D(n, t) - \delta, 0\} && \text{for } 1 \leq t \leq s \\ &= 0 && \text{for } t > s \end{aligned}$$

If we choose s sufficiently large and α, δ sufficiently small, the plan χ' achieves almost as much utility as χ ; in particular, χ' achieves more utility than χ^h . (Note that the utility achieved by a plan depends on consumption, on sales, and on deliveries, but not on purchases or on conjectures.) Since conjectures $K^h[s]$ converge to conjectures K^h , if we choose β sufficiently small and s sufficiently large, the plan χ' will be feasible (i.e., meet the non-negativity constraints) in the economy $\mathcal{E}[s]$. Since prices $q[s]$ converge to q , if we choose s sufficiently large, the plan χ' will be budget feasible in $\mathcal{E}[s]$. Finally, if we choose s sufficiently large, the plan $\chi^h[s]$ (the equilibrium plan for the economy $\mathcal{E}[s]$) and the plan χ^h (the equilibrium plan for the economy \mathcal{E}) achieve almost the same utility, so the plan χ' will achieve more utility than $\chi^h[s]$. Since this contradicts the equilibrium conditions for the economy $\mathcal{E}[s]$, we conclude that χ cannot be superior to χ^h . Hence $(q, (K^h, x^h, \varphi^h, \psi^h, D^h))$ is an equilibrium for \mathcal{E} , as desired. \square

Proof of Theorem 4: For each N, λ , consider a default equilibrium $\eta(N, \lambda)$ of

$\mathcal{E}^{N,\lambda}$; write $x(N, \lambda)$ for the vector of consumption plans. Passing to subsequences if necessary, we assume that $x(N, \lambda) \rightarrow x(\lambda)$ (as $N \rightarrow \infty$) and that $x(\lambda) \rightarrow x$ (as $\lambda \rightarrow \infty$). The desired result follows if we can show that (for all choices just made) x is a Walrasian equilibrium of the underlying complete markets economy, and that utilities converge: $W^h(\eta(N, \lambda)) \rightarrow U^h(x^h)$ for each h .

We show first that x is in the core of the complete markets economy; that is, no group of traders can improve on x using only their own resources. To see this, suppose not. Then there is a set of consumers (whom we may suppose to be $\{1, \dots, M\}$) and a vector y of consumption plans that is feasible for the group $\{1, \dots, M\}$ and strictly preferred (by each member of the group) to the plan x . Let $\alpha, \beta, \gamma, \delta > 0$ be positive real parameters and let r be an integer (all to be chosen later). Define \hat{w}, \hat{y} by

$$\begin{aligned} \hat{y}^h(t) &= \max\{x^h(t) - \alpha, \alpha\} && \text{for } 0 \leq t \leq r \\ &= 0 && \text{for } t > r \\ \hat{w}^h(t) &= w^h(t) && \text{for } 0 \leq t \leq r \\ &= 0 && \text{for } t > r \end{aligned}$$

If α is sufficiently small and r is sufficiently large, then \hat{y} is feasible for the group $\{1, \dots, M\}$ and strictly preferred to x . Write \hat{y}_1 for the restriction of y to Ω ; \hat{y}_1 is a vector of date 1 consumption plans.

Now let λ^* be any default penalty so large that, if $\lambda > \lambda^*$, and N is arbitrary, then in the security market $\mathcal{E}^{N,\lambda}$ there is no default in states $\omega \leq r$. For $1 \leq h \leq M - 1$, we may use the fact that the securities $\{A_n\}$ span all the uncertainty to choose a finite portfolio θ^h such that:

$$\begin{aligned} |w^h(\omega) + \text{div}(\theta^h)(\omega) - y^h(\omega)| &< \frac{\alpha}{M} && \text{for } 1 \leq \omega \leq r \\ \text{Exp}(|\hat{w}^h + \text{div}(\theta^h) - \hat{y}^h|) &< \frac{\beta}{M} \end{aligned}$$

Set

$$\theta^M = - \sum_{h=1}^{M-1} \theta^h$$

and let φ^h, ψ^h be the positive and negative parts of θ^h (respectively). Our construction guarantees that

$$\begin{aligned} |w^M(\omega) + \text{div}(\theta^M)(\omega) - y^M(\omega)| &< \alpha \quad \text{for } 1 \leq \omega \leq r \\ \text{Exp}(|\hat{w}^M + \text{div}(\theta^M) - \hat{y}^M|) &< \beta \\ \sum(\varphi^h - \psi^h) &= 0 \end{aligned}$$

We now define plans $(z^h, \varphi^h, \psi^h, D^h)$ (for the security markets $\mathcal{E}^{N,\lambda}$) as follows. Consumption plans z^h are given by

$$z^h(s) = \begin{cases} y^h(0) & \text{if } s = 0 \\ w^h(\omega) + \text{div}(\theta^h)(\omega) & \text{if } 1 \leq s \leq r \\ 0 & \text{if } s > r \end{cases}$$

Portfolios φ^h, ψ^h of purchases and sales are defined to be the positive and negative parts of θ^h (respectively). Plans of delivery D^h are defined by

$$D^h(n, \omega) = \begin{cases} \psi^h(n)A_n(\omega) & \text{if } 1 \leq \omega \leq r \\ 0 & \text{if } \omega > r \end{cases}$$

Our choice of λ and our rationality requirement guarantees that no trader conjectures default in states $\omega \leq r$. Hence the constructed plans are feasible (that is, satisfy the non-negativity constraints). If α is small enough then $U^h(z^h) > U^h(x^h)$ for $1 \leq h \leq M$. If β is enough, these plans all incur small default penalties; in particular, if β is small enough, $\hat{U}^h(z^h, \varphi^h, \psi^h, D^h) > U^h(x^h)$ for $1 \leq h \leq M$. On the other hand, continuity of utility functions implies that

$$U^h(x^h(N, \lambda)) \rightarrow U^h(x^h(\lambda)) \quad \text{as } N \rightarrow \infty$$

and

$$U^h(x^h(\lambda)) \rightarrow U^h(x^h) \quad \text{as } \lambda \rightarrow \infty$$

It follows that, if N, λ are sufficiently large then

$$\hat{U}^h(z^h, \varphi^h, \psi^h, D^h) > U^h(x^h(N, \lambda))$$

Since $U^h(x^h(N, \lambda)) \geq \hat{U}^h(\eta^h(N, \lambda))$, we conclude that

$$\hat{U}^h(z^h, \varphi^h, \psi^h, D^h) > \hat{U}^h(\eta^h(N, \lambda))$$

provided that N, λ are sufficiently large. This contradicts individual optimization at equilibrium, so we conclude that x is in the core of the complete markets economy, as asserted.

The same argument shows that (the replication of) x belongs to the core of every replication of the complete markets economy. By a result of Aliprantis, Brown and Burkinshaw (1987), which is the infinite dimensional version of the Debreu and Scarf (1963) core convergence theorem, it follows that x is a Walrasian equilibrium allocation of the complete markets economy. In particular, x is a Pareto optimal allocation.

It remains to show that $\hat{U}^h(\eta^h(N, \lambda)) \rightarrow U^h(x^h)$ for each trader h . If not we could (passing to subsequences if necessary) find a trader (say trader 1) and a $\zeta > 0$ such that $\hat{U}^1(\eta^1(N, \lambda)) \leq U^1(x^1) - \zeta/2$ for N, λ sufficiently large. We may then find a sufficiently large index r and a feasible profile of consumption plans (y^h) such that

$$y^h(\omega) = 0$$

for $\omega > r$ and all h ,

$$U^h(y^h) > U^h(x^h) \geq \hat{U}^h(\eta^h(N, \lambda))$$

for $h \neq 1$, all N, λ , and

$$U^1(y^1) > \hat{U}^1(\eta^1(N, \lambda))$$

for all N, λ . Let $\alpha, \beta > 0$ be positive real parameters. Using the same ideas as above, we may construct portfolios φ^h, ψ^h such that

$$|w^h(\omega) + \text{div}(\varphi^h)(\omega) - \text{div}(\psi^h)(\omega) - y^h(\omega)| < \alpha \quad \text{for } 1 \leq \omega \leq r$$

and

$$\text{Exp}(|\hat{w}^h + \text{div}(\varphi^h) - y^h|) < \beta$$

and $\sum(\varphi^h - \psi^h) = 0$. If λ is sufficiently large then there will be no default in states $\omega \leq r$. If β is sufficiently small the total default penalty will also be small. Hence, if we choose α sufficiently small we may construct, just as above, a collection of plans that are superior to the equilibrium plans in

$\mathcal{E}^{N,\lambda}$ (provided N is sufficiently large) and have the property that at least one of them is budget feasible. This is a contradiction, so we conclude that $\hat{U}^h(\eta^h(N, \lambda)) \rightarrow U^h(x^h)$, as desired. This completes the proof. \square

Proof of Theorem 5: The proof is quite similar to the proof of Theorem 4. Fix $\epsilon > 0$, the default penalty $\lambda > \lambda_0$, and a Walrasian equilibrium $(\pi, (x^h))$; there is no loss in normalizing so that the price of date 0 consumption is 1. Let $\alpha, \beta, \gamma > 0$ be positive real parameters and let r be an integer (all to be chosen later). Define consumption plans \hat{x}^h by

$$\hat{x}^h = \begin{cases} \max\{x^h(s) - 2\alpha\} & \text{if } 0 \leq s \leq r \\ 0 & \text{if } s > r \end{cases}$$

As in the proof of Theorem 4, we may choose a collection of finite portfolios φ^h, ψ^h of the securities $\{A_n\}$ such that

$$0 < w^h(\omega) + \text{div}(\varphi^h)(\omega) - \text{div}(\psi^h)(\omega) < \hat{x}^h(\omega) + \alpha$$

for $\omega \leq r$, and

$$\text{Exp}(|w^h(\omega) + \text{div}(\varphi^h) - \text{div}(\psi^h) - \hat{x}^h|) < \beta$$

and

$$\sum(\varphi^h - \psi^h) = 0$$

If we choose r sufficiently large and α sufficiently small then

$$\begin{aligned} \text{Exp}(|x^h - \hat{x}^h|) &< \epsilon/2 \\ |U^h(x^h) - U^h(\hat{x}^h)| &< \epsilon/4 \end{aligned}$$

The portfolios φ^h, ψ^h involve only finitely many securities; A_1, \dots, A_{N_0} , say. For $N \geq N_0$, let $\mathcal{E}^{N,\lambda}$ be the security market in which the securities A_1, \dots, A_N are available for trade and the default penalty is λ . From this data, we construct an ϵ -equilibrium $(q, (K^h, \hat{x}^h, \varphi^h, \psi^h, D^h))$ for $\mathcal{E}^{N,\lambda}$. Define security prices q by $q(n) = \pi \cdot A_n$, Conjectures K^h concerning the securities A_1, \dots, A_{N_0} predict no default in states $\omega \leq r$ and complete default in states $\omega > r$; conjectures concerning the securities A_n with $n > N_0$ are that default

is total in all states. For traders $1, \dots, H - 1$, portfolios φ^h, ψ^h are as above; we require that consumer H purchase (in addition to the portfolio φ^H above) and sell (in addition to the portfolio ψ^H above) a small quantity γ of each security A_n with $n > N_0$. Finally, we arrange deliveries consistent with no default on securities A_1, \dots, A_{N_0} in states $\omega \leq r$, total default on securities A_1, \dots, A_{N_0} in states $\omega > r$, and total default in all states on securities A_n with $n > N_0$. These plans are feasible. If β is sufficiently small, then default penalties for each consumer do not exceed $\epsilon/2$. Using this fact, and keeping in mind the conjectures and that $\lambda > \lambda_0$, it is easily checked that $(q, (K^h, \hat{x}^h, \varphi^h, \psi^h, D^h))$ is an ϵ -equilibrium. Moreover, the fact that default penalties for each consumer do not exceed $\epsilon/2$ implies that

$$|\hat{U}^h(\hat{x}^h, \varphi^h, \psi^h, D^h) - U^h(x^h)| < \epsilon/2$$

What we have accomplished is not quite what was called for, since we have chosen the index N_0 in a way that depends on the particular Walrasian equilibrium $(\pi, (x^h))$, while the statement of Theorem 5 calls on us to choose N_0 in a way that depends only on λ and ϵ , but the desired uniformity is easily obtained. The ϵ -equilibrium allocation we have constructed is close to the Walrasian equilibrium allocation (x^h) and hence close to every Walrasian equilibrium allocation (\bar{x}^h) that is close to (x^h) . Since the set of Walrasian equilibrium allocations is compact, uniformity follows by an obvious compactness argument. \square

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