A REPRESENTATION THEOREM FOR RIESZ SPACES AND ITS APPLICATIONS TO ECONOMICS

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A representation theorem for Riesz spaces and its applications to economics*

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Summary. We show that a Dedekind complete Riesz space which contains a weak unit e and admits a strictly positive order continuous linear functional can be represented as a subspace of the space L_1 of integrable functions on a probability measure space in such a way that the order ideal generated by e is carried onto L_∞. As a consequence, we obtain a characterization of abstract M-spaces that are isomorphic to concrete L_∞-spaces. Although these results are implicit in the literature on representation of Riesz spaces, they are not available in this form. This research is motivated by, and has applications in, general equilibrium theory in infinite dimensional spaces.

1. Introduction

Bewley's seminal work [12] on general equilibrium theory with infinitely many commodities is set in the space $L_\infty(\mu)$ of (equivalence classes of) essentially bounded measurable functions on a (finite) measure space (Ω, Σ, μ).[1] This work has been immensely influential, but subsequent work has made it clear that $L_\infty(\mu)$ is a very special environment, both economically and mathematically. As we shall see here, although the space L_∞ appears to be special, it turns out that many other spaces that are employed in

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[1] Note that the space ℓ_∞ of all bounded sequences can be viewed as L_∞ of a counting measure—and also as L_∞ of an appropriate probability measure.

the economic literature can be viewed as $L_\infty(\mu)$-spaces. Following the work of Aliprantis and Brown [3] and the subsequent work of Mas-Colell [22], Riesz spaces have appeared as the most natural general environments in which to set infinite dimensional general equilibrium theory.[2]

The objective of the present paper is to present a representation theorem that connects the general Riesz space environment with the special environment $L_\infty(\mu)$. We believe that this representation theorem is a useful tool for economic analysis in Riesz spaces—see Zame [31] and Anderson and Zame [9].

A *Riesz space* (or *vector lattice*) is an ordered real vector space E which is also a lattice in the sense that for every pair $x, y \in E$ the supremum (least upper bound) and the infimum (greatest lower bound) of the set $\{x, y\}$ exist in E, where as usual we denote the supremum and infimum by the symbols

$$x \vee y = \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}\,.$$

The vector $|x| = x \vee (-x)$ is known as the absolute value of x. A vector subspace J of a Riesz space E is said to be an *(order) ideal* if $|x| \le |y|$ and $y \in J$ imply $x \in J$. The *principal* ideal generated by a positive vector e is the ideal

$$\begin{aligned} E_e &= \{x \in E\colon\ \exists\ \lambda > 0 \text{ such that } |x| \le \lambda e\} \\ &= \{x \in E\colon\ -\lambda e \le x \le \lambda e \text{ for some } \lambda > 0\}\,. \end{aligned}$$

Our representation theorem has a natural economic motivation. Fix a Riesz space E and consider an exchange economy with commodity space E and consumption sets the positive cone E^+. As Brown [13] was first to observe, all *feasible* consumptions for such an economy lie between 0 and the social endowment e. In particular, all feasible consumptions lie in the principal ideal E_e. Since Pareto optimality and the core refer only to feasible consumptions, it follows that these notions can be explored entirely within E_e. Moreover, besides its lattice structure E_e has an important additional norm structure. The formula

$$\|x\|_\infty = \inf\{\lambda > 0\colon\ -\lambda e \le x \le \lambda e\} = \min\{\lambda \ge 0\colon\ |x| \le \lambda e\}$$

defines a norm on E_e. If E is Dedekind complete, then with respect to this norm E_e is a Banach lattice[3] (in fact an abstract M-space), and hence by the classical Kakutani–Krein theorem E_e is lattice isometric to the space $C(X)$ of continuous functions on

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[2]Aliprantis and Brown [3] introduced Riesz spaces into economics, and established the existence of equilibrium for an economy specified by an aggregate excess demand function. Mas-Colell [22] exploited the lattice structure further—particularly the Riesz decomposition property—to establish the existence of equilibrium for exchange economies.

[3]Recall that a Banach lattice is a Banach space which is also a Riesz space and whose norm is compatible with the lattice structure in the sense that $|x| \le |y|$ implies $\|x\| \le \|y\|$.

some compact Hausdorff topological space X. In particular, the positive cone of E_e has non-empty interior (with respect to this norm).

As Mas-Colell [22] pointed out, non-emptiness of the interior of the positive cone is a crucial property of $L_\infty(\mu)$,[4] because it guarantees that continuous, convex preferences admit supporting prices. In spaces for which the positive cone has empty interior, supporting prices may fail to exist; the properness condition (adapted by Mas-Colell from a condition introduced by Chichilnisky and Kalman [15]) was designed precisely to guarantee the existence of supporting prices. Since the positive cone E_e^+ has non-empty interior (in E_e), supporting prices (relative to E_e) exist. With appropriate compactness assumptions this makes it possible to mimic arguments for $L_\infty(\mu)$ and obtain an equilibrium with respect to consumptions in E_e (that is, only consumptions in E_e are considered and priced). To obtain an equilibrium in the usual sense, one can exploit properness of preferences (or some variant of it) and extend the supporting prices from E_e to E. In very special cases, the idea of finding an equilibrium with respect to a restricted set of consumptions goes back to Malinvaud [21] and Peleg and Yaari [26]; the general development and the exploitation of properness is due to Aliprantis, Brown, and Burkinshaw [4, 5], Mas-Colell [22], Richard and Zame [27], Yannelis and Zame [29], and Zame [30]. For additional information, see Aliprantis and Border [2], Araujo and Monteiro [10, 11], Cherif, Deghdak, and Florenzano [14], Duffie and Zame [16], Florenzano [17], Khan and Yannelis [18], Mas-Colell [23], Mas-Colell and Richard [24], and Zame [31]. More details and further bibliography can be found in Aliprantis, Brown, and Burkinshaw [6] and Mas-Colell and Zame [25].

The point of departure for the present paper is the observation that the "appropriate compactness assumptions" actually entail a stronger conclusion about the order ideal E_e: In addition to being lattice isometric to some concrete space $C(X)$ of continuous functions, it is actually lattice isometric to some concrete space $L_\infty(\mu)$ (so that Bewley's results—and not just methods—may be applied directly). This result (indeed, a characterization of Riesz spaces which are isomorphic to some $L_\infty(\mu)$) is a consequence of the main result of this paper, which provides a simultaneous concrete representation of a Riesz space E and a principal order ideal E_e of E: we can represent E as a subspace of the subspace $L_1(\mu)$ of all integrable functions on a probability measure space (Ω, Σ, μ) in such a way that E_e is represented as $L_\infty(\mu)$.

We claim little originality for these representation results; similar results are known in the folklore of Riesz spaces, and close relatives have been established by Vulikh and Lozanovsky [28] and by Lindenstrauss and Tzafriri [19, Theorem 1.b.14, p. 25]. However, the formulations we give here seem most suitable *for economic applications*.

[4] And, in fact, this property (with a few minor extra assumptions) characterizes the $C(X)$-spaces; see [1, Section 7.5, p. 256] for details about the non-emptiness of the positive cone.

For general information about Riesz spaces and Banach lattices, we refer the reader to Aliprantis and Burkinshaw [7, 8]. Aliprantis, Brown, and Burkinshaw [6] and Mas-Colell and Zame [25] provide surveys of the uses of Riesz spaces in general equilibrium theory, and extensive bibliographies.

2. The representation theorem

Recall that a Riesz space E is *Dedekind* (or *order*) *complete* if each non-empty subset of E which is order bounded from above has a supremum. A lattice norm $\|\cdot\|$ on a Riesz space E is said to be an *M-norm* (resp. an *L-norm*) whenever $\|y \vee z\| = \max\{\|y\|, \|z\|\}$ (resp.$\|y+z\| = \|y\| + \|z\|$) holds for all $y, z \in E^+$. An *abstract M-space*, in short *AM-space* (resp. *AL-space*), is a complete M-space (resp. L-space).

If E is a Riesz space, then as mentioned before, the principal ideal E_x generated by x is the vector space

$$E_x = \{y \in E\colon\ \exists\ \lambda \geq 0 \text{ such that } |y| \leq \lambda |x|\}.$$

The real function $\|\cdot\|_\infty\colon E_x \to \mathbb{R}$, defined by

$$\|y\|_\infty = \inf\{\lambda \geq 0\colon\ |y| \leq \lambda |x|\},$$

is a lattice seminorm on E_x whose closed unit ball in $(E_x, \|\cdot\|_\infty)$ coincides with the order interval $[-|x|, |x|]$. If E is either Dedekind complete or a Banach lattice, then $\|\cdot\|_\infty$ is a lattice norm and $(E_x, \|\cdot\|_\infty)$ is a Banach lattice and, in fact, an AM-space; see [8, Theorem 12.20, p. 187].

Now if a Banach lattice has an *order unit* e (i.e., for each $y \in E$ there exists some $\lambda > 0$ such that $|y| \leq \lambda e$), then $E_e = E$ and so the norm $\|\cdot\|_\infty$ and the original norm on E are equivalent; see [8, Corollary 12.4, p. 176]. An *AM-space with unit* is a Banach lattice with an order unit e and whose norm coincides with the $\|\cdot\|_\infty$ norm determined by e. By the above discussion, if E is either a Dedekind complete Riesz space or a Banach lattice, then every principal ideal E_x with its $\|\cdot\|_\infty$-norm is an AM-space with unit. If E is an AM-space with unit e, then there exist a (unique) compact Hausdorff topological space Q and an onto lattice isometry $T\colon E \to C(Q)$ such that $Te = \mathbf{1}$, where $\mathbf{1}$ denotes the constant function one on Q; see [8, Theorem 12.28, p. 194].

An ideal is said to be a *band* if for any net $\{x_\alpha\} \subseteq B$ that satisfies $x_\alpha \uparrow x$ in E we have $x \in B$.[5] An element $e > 0$ in a Riesz space E is said to be a *weak unit* if the band

[5]The symbol $x_\alpha \uparrow x$ means that the net $\{x_\alpha\}$ satifies $x_\alpha \geq x_\beta$ whenever $\alpha \geq \beta$ and $\sup_\alpha x_\alpha = x$.

B_e generated by e (the smallest band with respect the inclusion that contains e) is all of E. It turns out that $B_e = \{x \in E\colon\ |x| \wedge ne \uparrow |x|\}$; see [8, Theorem 3.4, p. 31].

A linear functional $f\colon E \to \mathbb{R}$ is said to be *positive* (resp. *strictly positive*) whenever $x > 0$ implies $f(x) \geq 0$ (resp. $f(x) > 0$). A positive linear functional f on a Riesz space E is said to be *order continuous* if $x_\alpha \downarrow 0$ in E implies $f(x_\alpha) \downarrow 0$ in $\mathbb{R}$.

A linear topology τ on a Riesz space E is *order continuous* if every decreasing to zero net also converges topologically to zero, i.e., $x_\alpha \downarrow 0$ implies $x_\alpha \xrightarrow{\tau} 0$. A Banach lattice E is said to have *order continuous norm* if its norm topology is order continuous, or, equivalently, whenever $x_\alpha \downarrow 0$ in E implies $\|x_\alpha\| \downarrow 0$ in $\mathbb{R}$.

An element $x > 0$ of a Banach lattice E is said to be *strictly positive* (or a *quasi-interior point*) whenever $0 < x' \in E'$ implies $x'(x) > 0$ (or, equivalently, whenever x considered as a functional on the norm dual E' of E acts as a strictly positive linear functional). Every strictly positive vector is a weak unit but a weak unit need not be a strictly positive vector; see [8, p. 259].

And now we are ready to state and prove a representation theorem for a certain class of Riësz spaces.

Theorem 2.1 *Let E be a Dedekind complete Riesz space E with a weak unit $e > 0$ admitting a strictly positive order continuous linear functional φ. Then there exist a probability measure space (Ω, Σ, μ), a norm dense order ideal F of $L_1(\mu)$ and an onto lattice isomorphism $T\colon E \to F$ such that:*

1. $Te = \mathbf{1}$.
2. $T\colon (E_e, \|\cdot\|_\infty) \to (L_\infty(\mu), \|\cdot\|_\infty)$ *is an onto lattice isometry.*

Moreover, if τ is any Hausdorff locally convex-solid topology on E and $\varphi \in E'$, where $E' = (E,\tau)'$ is the topological dual, then the mappings

3. $T\colon (E,\tau) \to (L_1(\mu), \|\cdot\|_1)$ *and* $T\colon (E, \sigma(E,E')) \to (L_1(\mu), \sigma(L_1(\mu), L_\infty(\mu))$

are both continuous.

If, in addition, τ is order continuous, then the restrictions

4. $T\colon ([0,e],\tau) \to ([0,\mathbf{1}], \|\cdot\|_1)$ *and* $T\colon ([0,e], \sigma(E,E')) \to ([0,\mathbf{1}], \sigma(L_1(\mu), L_\infty(\mu))$

are both homeomorphisms.

Proof: (1) & (2) Let $\varphi\colon E \to \mathbb{R}$ be an order continuous strictly positive functional. Replacing φ by $\frac{\varphi}{\varphi(e)}$, we can suppose that $\varphi(e) = 1$. Define a function $\|\cdot\|\colon E \to \mathbb{R}$ by the formula

$$\|y\| = \varphi(|y|),\ \ y \in E.$$

It is easily checked, that $\|\cdot\|$ is an L-norm on E which is order continuous (because φ is order continuous). Let L be the completion of E in this norm. Clearly, L is an AL-space.

We claim that E is an ideal in L. Clearly, E is a Riesz subspace of L. We must verify that whenever $0 \le z \le y \in E$ and $z \in L$, then $z \in E$. Since E is norm dense in its completion L, there exists a sequence $\{z_n\} \subseteq E$ converging to z; that is, $\|z_n - z\| \to 0$. Since $z \le y$ and the mapping $u \mapsto u \wedge y$ is norm continuous, we can assume (replacing $\{z_n\}$ by $\{z_n \wedge y\}$) that $0 \le z_n \le y$ holds for each n. Also, by choosing an appropriate subsequence, we can suppose that $\|z_n - z\| < \frac{1}{n2^n}$ for each n. Now let

$$w_{k,m} = \bigvee_{i=k}^{k+m} z_i,$$

and note that $0 \le w_{k,m} \le y$ for all k and m. Since E is Dedekind complete, there exists some $w_k \in E$ satisfying $w_{k,m} \uparrow_m w_k$ in E, and since $\|\cdot\|$ is order continuous we have $\|w_{k,m} - w_k\| \xrightarrow[m\to\infty]{} 0$. From

$$|w_{k,m} - z| = \Big|\bigvee_{i=k}^{k+m} z_i - z\Big| = \Big|\bigvee_{i=k}^{k+m} (z_i - z)\Big| \le \bigvee_{i=k}^{k+m} |z_i - z| \le \sum_{i=k}^{k+m} |z_i - z|,$$

we see that $\|w_{k,m} - z\| \le \sum_{i=k}^{k+m} \|z_i - z\| \le \frac{1}{k}$, and so $\|w_k - z\| = \lim_{m\to\infty} \|w_{k,m} - z\| \le \frac{1}{k}$ for each k. In particular, $\lim_{k\to\infty} \|w_k - z\| = 0$. Clearly, the sequence $\{w_k\}$ is decreasing and bounded from below by zero. Using once more the Dedekind completeness of E, it follows that $w_k \downarrow w$ holds for some $w \in E$. Now from the order continuity of the norm, we infer that $\|w_k - w\| \to 0$. But then $\|z - w\| = \lim_{k\to\infty} \|z - w_k\| = 0$, proving that $z = w \in E$.[6]

Next, by the Kakutani–Bohnenblust–Nakano representation theorem (see [8, Theorem 12.26, p. 192]), we can find a probability measure space (Ω, Σ, μ) and an onto lattice isometry $\pi\colon L \to L_1(\mu)$ such that $\pi(e) = \mathbf{1}$. Write $F = \pi(E)$ for the image of E and $T\colon E \to F$ for the restriction of π to E. It is evident that T is a lattice isomorphism and $T(e) = \mathbf{1}$.

[6] Another way of proving that E is an ideal in L which requires some knowledge of Riesz space theory goes as follows. Since $\|\cdot\|$ is order continuous, the norm dual A of $(E, \|\cdot\|)$ is an ideal in the order continuous dual $E_n^\sim$ of E (and, of course, A is an AM-space). Now by a well known theorem of H. Nakano (see [8, Theorem 5.5, p. 59]) it follows that the natural embedding of E into its double dual $(E, \|\cdot\|)'' = A'$ is an order dense Riesz subspace. Since E is Dedekind complete, it follows that E is an ideal of the AL-space A'; see [8, Theorem 7.19, p. 100]. But then $L = \overline{E}$, the norm closure of E in A'. This shows that L is an AL-space containing E as an ideal and having e as a weak unit satisfying $\|e\| = \varphi(e) = 1$.

Since E is an order ideal in L and $e \in E$, it follows that $E_e = L_e$, i.e., the order ideal generated by e in E and the order ideal generated by e in L coincide. Since π is a lattice isomorphism of L onto $L_1(\mu)$, it sends any norm dense ideal in L to a norm dense ideal in $L_1(\mu)$, and hence F must be a norm dense ideal in $L_1(\mu)$. Furthermore, $T(E_e) = L_\infty(\mu)$ since $T(e) = \mathbf{1}$. The formulas defining the norms in E_e and $L_\infty(\mu)$ guarantee that $T: (E_e, \|\cdot\|_\infty) \to (L_\infty(\mu), \|\cdot\|_\infty)$ is also a lattice isometry.

(3) Now let τ be a locally convex-solid topology on E in which φ is continuous, and let $y_\alpha \xrightarrow{\tau} 0$ in E. Since τ is locally solid, we also have $|y_\alpha| \xrightarrow{\tau} 0$ in E. The τ-continuity of φ implies $\varphi(|y_\alpha|) \to 0$ in $\mathbb{R}$. But then:

$$\|T(y_\alpha)\|_1 = \|y_\alpha\| = \varphi(|y_\alpha|) \to 0 .$$

This shows that $T: (E, \tau) \to (L_1(\mu), \|\cdot\|_1)$ is continuous, as asserted. To verify the weak continuity claim, assume $y_\alpha \xrightarrow{\sigma(E,E')} 0$ and let $f \in L_\infty(\mu)$. The τ-continuity of T just obtained implies that the composition $f \circ T$ is a τ-continuous linear functional on E, i.e., $f \circ T$ belongs to E'. Therefore,

$$f(T(y_\alpha)) = (f \circ T)(y_\alpha) \to 0,$$

for each $f \in L_\infty(\mu)$, proving that $T(y_\alpha) \xrightarrow{\sigma(L_1, L_\infty)} 0$.

(4) Now assume that τ is also order continuous. To see that $T: ([0, e], \tau) \to ([0, \mathbf{1}], \|\cdot\|_1)$ is a homeomorphism, it suffices to show that for every sequence $\{f_n\} \subset [0, \mathbf{1}]$ with $\|f_n\|_1 \to 0$ there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $T^{-1}(g_n) \xrightarrow{\tau} 0$. To this end, fix such a sequence $\{f_n\}$. By passing to a subsequence, we can assume that $\|f_n\|_1 < 4^{-n}$ for each n. Let

$$A_n = \{\omega \in \Omega: \ f_n(\omega) \geq 2^{-n}\} \ \text{ and } B_n = \Omega \setminus A_n = \{\omega \in \Omega: \ f_n(\omega) < 2^{-n}\},$$

and note that $\mu(A_n) < 2^{-n}$ for each n. If $C_n = \bigcup_{i=n}^\infty A_i$ then $\mu(C_n) \leq 2^{1-n}$ for each n and so the set $C = \bigcap_{i=1}^\infty C_i$ satisfies $\mu(C) = 0$. As T is a lattice isomorphism and $C_n \downarrow C$, it follows that $T^{-1}(\chi_{C_n}) \downarrow 0$ in E. Consequently, by the order continuity of τ we have so $T^{-1}(\chi_{C_n}) \xrightarrow{\tau} 0$. From $f\chi_{A_n} \leq \chi_{C_n}$ and $f_n\chi_{B_n} \leq 2^{-n}\mathbf{1}$, we see that

$$\begin{aligned} 0 \leq T^{-1}(f_n) &= T^{-1}(f_n\chi_{A_n} + f_n\chi_{B_n}) = T^{-1}(f_n\chi_{A_n}) + T^{-1}(f_n\chi_{B_n}) \\ &\leq T^{-1}(\chi_{C_n}) + 2^{-n}e \xrightarrow{\tau} 0 , \end{aligned}$$

from which it follows that $T^{-1}(f_n) \xrightarrow{\tau} 0$, as claimed.[7]

[7]This conclusion can be also derived from the following (quite deep) theorem of I. Amemiya: *All Hausdorff locally solid order continuous topologies on a Riesz space induce the same topology on the order intervals*; see [7, Theorem 12.9, p. 87].

Finally, to see that $T: ([0,e], \sigma(E,E')) \to ([0,\mathbf{1}], \sigma(L_1(\mu), L_\infty(\mu))$ is a homeomorphism, note first that its continuity follows from (3). Since τ is order continuous and E is Dedekind complete, it follows that the order interval $[0,e]$ is $\sigma(E,E')$-compact [8, Theorem 11.13, p. 170]. Since the weak topology $\sigma(L_1(\mu), L_\infty(\mu))$ is Hausdorff, and a one-to-one continuous mapping from a compact space onto a Hausdorff space is a homeomorphism, we conclude that

$$T: ([0,e], \sigma(E,E')) \to ([0,\mathbf{1}], \sigma(L_1(\mu), L_\infty(\mu))$$

is indeed a homeomorphism, and the proof of the theorem is finished. ∎

For future references, we now isolate a special case of the preceding theorem which is an important characterization of Dedekind complete AM-spaces with units.

Corollary 2.2 *For a Dedekind complete AM-space E with unit e the following two statements are equivalent.*

1. *There exists a probability measure space (Ω, Σ, μ) and an onto lattice isometry $T: E \to L_\infty(\mu)$ with $Te = \mathbf{1}$.*
2. *E admits a strictly positive order continuous linear functional.*

Proof: If (Ω, Σ, μ) is a probability measure space, then $L_\infty(\mu)$ is Dedekind complete with order unit $\mathbf{1}$, and integration against μ yields a strictly positive order continuous linear functional on $L_\infty(\mu)$. Furthermore, if $T: E \to L_\infty(\mu)$ is a lattice isometry, then the mapping $\varphi: E \to \mathbb{R}$, defined by $\varphi(x) = \int_\Omega T(x)\, d\mu$, is a strictly positive order continuous linear functional on E. Hence (1) implies (2).

Conversely, assume that φ is a strictly positive linear functional on E. Since e is an order unit, the ideal it generates is all of E; that is, $E_e = B_e = E$. Now Theorem 2.1 applies and yields the desired lattice isometry. ∎

In light of these results, the reader might well ask for simple sufficient conditions that guarantee Dedekind completeness, or the existence of a strictly positive element, or the existence of a strictly positive functional. For Banach lattices, simple conditions are well known:

- Every separable Banach lattice has a strictly positive element. (For, if $\{x_1, x_2, \ldots\}$ is a countable dense set of non-zero elements, then $x = \sum_{n=1}^\infty \frac{1}{2^n} \cdot \frac{|x_n|}{\|x_n\|}$ is a strictly positive element.)
- Every Banach lattice with order continuous norm and every dual Banach lattice is Dedekind complete; see [8, Theorem 12.9, p. 179]. (A dual Banach lattice is a Banach lattice which is the norm dual of another Banach lattice.)

- If E is a Banach lattice with order continuous norm and $0 < x \in E$, then there exists a continuous (and hence order continuous) positive linear functional φ on E which is strictly positive on the order ideal E_x. If x is also strictly positive, then φ is strictly positive on all of E; see [8, Theorem 12.14, p. 183].

The representation theorem tells us that an abstract Riesz space (satsifying appropriate conditions) can be represented as a space of equivalence classes of integrable functions. The following result of G. Ya. Lozanovsky [20, Theorem 7] tells us that a Banach lattice which is already presented as a space of equivalence classes of measurable functions can be realized after a "change of variable" as a space of equivalence classes of integrable functions between L_∞ and L_1.

Theorem 2.3 (Lozanovsky) *Let (Ω, Σ, μ) be a finite measure space and let $L^0(\mu)$ denote the Riesz space of all (equivalence classes) of measurable real functions on Ω. If E is an order dense ideal of $L^0(\mu)$ which is also a Banach lattice (with respect to some lattice norm), then there exists a non-negative function $h \in L^0(\mu)$ such that*

$$L_\infty(\mu) \subseteq hE = \{hf\colon\ f \in E\} \subseteq L_1(\mu)\,.$$

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