Arbitrage and the Flattening Effect of Large Numbers

Louis Makowski* and Joseph M. Ostroy†

May 1995

UCLA Department of Economics Working Paper No. 737

*Department of Economics, UC Davis

†Department of Economics, UCLA

Arbitrage and the Flattening Effect of Large Numbers

Louis Makowski[*] and Joseph M. Ostroy[†]

May 1995

Abstract

In a model of an exchange economy with a continuum of agents, we show that competitive equilibrium can be regarded as resulting from the elimination of arbitrage possibilities. Arbitrage leads to a phenomenon we call the "flattening effect of large numbers," which provides a precise meaning to the statement that under perfect competition individuals cannot influence prices. There is an attractive geometry associated with arbitrage, which is highlighted in several figures.

We compare arbitrage equilibrium in a continuum economy to Walrasian equilibrium, the core, non-cooperative dynamic matching models, and to the existence of equilibrium with unbounded short-selling. We also link the demonstration of equilibrium through arbitrage with the logic of the marginalism.

KEYWORDS: Arbitrage, arbitrage cone, Walrasian equilibrium, perfectly competitive equilibrium, marginalist revolution

JEL CLASSIFICATIONS: D51 (General Equilibrium Exchange Economies).

[*]Department of Economics, UC Davis

[†]Department of Economics, UCLA

1 Introduction

The centrality of perfect competition in economics justifies the significant efforts made to understand its foundations. The point of departure is typically the Walrasian model, with its image of a perfect competitor as a price-taker. A more satisfactory model of a perfectly competitive economy would have prices emerge endogenously. Core bargaining represents one such effort (Edgeworth (1881), Shubik (1959), Debreu and Scarf (1963), Aumann (1964), Hildenbrand (1974)). Another is the non-cooperative dynamic matching and bargaining games initiated by Rubinstein and Wolinsky (1985), reformulated by Gale (1986), and further analyzed by McLennan and Sonnenschein (1991), and Osborne and Rubinstein (1990). Here we follow a different path toward the same goal; our path will share some features in common with both the core's static cooperative approach and the above, dynamic non-cooperative approach. Using a model with a continuum of agents, we show that *competitive equilibrium can be regarded as resulting from the elimination of arbitrage possibilities*, rather than from the elimination of Walrasian excess demands.

A typical companion of the elimination of arbitrage possibilities is a phenomenon we call the "flattening effect of large numbers," which provides a precise meaning to the statement that individuals cannot influence prices. The flattening effect is related to the "convexifying effect of large numbers." The convexifying effect always holds in models with a continuum of agents and a finite number of commodities, but we shall show that the flattening effect only holds generically; and we shall argue that in those exceptional cases where the flattening effect does not obtain, the economy is not perfectly competitive.

THE USE OF THE TERM "ARBITRAGE." Arbitrage means the opportunity to exploit differences in quoted prices, a basic notion of competition among individuals. Modern formalizations of the idea that arbitrage leads to a law of one price are given in the finance literature (e.g., see Ross (1976) and citations there). Here we propose a natural extension of the idea: We allow arbitragers to seek out profit opportunites based on differences in "reservation prices," i.e., differences in individuals' marginal rates of substitution (MRS's), not just differences in observed market prices. To illustrate, suppose in a pure exchange economy each individual of one type would be willing to pay 3 apples for 1 orange while each individual of another type would be willing to sell 1 orange for only 1 apple. Evidently there is an arbitrage profit to be gained—an opportunity to buy an apple at a low price and sell it at a high price—based on the differences in MRS's. Further, with arbitrarily many individuals of each type this profit could be multiplied to an arbitrarily large number. So, as with differences in market prices, an arbitrage possibility based on differences in reservation prices represents a money pump. It will be shown that if we extend the notion of arbitrage to include exploiting differences in MRS's, the elimination of arbitrage opportunities can lead to competitive equilibrium.

ARBITRAGE CONES: FLAT AND OTHERWISE. Arbitrage may occur in any economy, small or large, whether or not individuals have monopoly power. Clearly, it would be a highly questionable result if arbitrage led to competitive equilibrium regardless of the economic environment. This is not our claim. Rather, we show that arbitrage leads to competitive equilibrium provided each arbitrager's activities are at a scale that has no macroscopic signficance. Formally, to capture this we will work in model with a nonatomic continuum of individuals; but we will assume that any individual arbitrager can trade with only a finite number of other market participants.

In this continuum setting, there is a distinctive and appealing geometry associated with our arbitrage approach. This may be usefully separated into two parts. The first is that arbitrage results in the formation of an opportunity set—the arbitrager's budget set—which is a convex cone. A similar condition characterizes arbitrage in financial markets where traders are allowed unlimited short sales, and such a condition also is implicit in continuum versions of dynamic matching models. The cone condition on trading opportunities differentiates the arbitrage approach to competitive equilibrium from Walrasian price-taking and the core: prices emerge as the supporting hyperplane to a convex cone, rather than as the supporting hyperplane to a convex set.

Convex cones are "flatter" than convex sets, but they need not be flat, i.e., the cone may be "pointed." The second feature of the geometry is that, provided some agents' preferences are smooth, the boundary of each arbitrager's opportunity set will be flat (linear). Let us briefly explain the intuition. Imagine an outcome in which all arbitrage profit potentials have been exhausted; and that at this allocation the preferences of at least some individuals are smooth. Observe that each smooth trader would, to a first-order, be indifferent between his allocation and any other alllocation sufficiently close to it on the tangent line to his indifference curve. Now suppose some one individual cum arbitrager tried to trade to another position at the terms-of-trade implied by the smooth trader's MRS's. Further, to make the experiment interesting, suppose the arbitrager wanted to trade beyond an ϵ-neighborhood of the smooth individual's original allocation. Since in the continuum there are many individuals like this one smooth individual, by adding together many tiny trades the arbitrager could achieve his desired trade without hurting anyone else (to a first-order).

This illustrates the principle that, when others' indifference curves are smooth, the boundary of an arbitrager's opportunity set is enlarged by making small trades with many individuals rather than large trades with a few. In the limiting ideal the arbitrager can trade a small amount with a large number of individuals at terms-of-trade reflecting each one's marginal rate of substitution. Thus with many individuals of similar type, his opportunity set, instead of being bowed, becomes linear with slope equal to any one of his trading partners' MRS's.

FLAT CONES AS PERFECTLY ELASTIC DEMANDS AND SUPPLIES. In a perfectly competitive economy no one individual can influence prices; that is, every agent faces *perfectly elastic demands and supplies* (PEDS). Indeed, this is often taken as a defining feature of perfect competition. The arbitrage approach adds to the tradition a very appealing concrete rationale *why* agents face perfectly elastic demands and supplies: PEDS results from the flattening effect of large numbers, as explained above.

Observe how the arbitrage story yields a behavioral interpretation of *the budget line as opportunity set*: there is enough "elasticity" in a perfectly competitive economy to actually give the individual cum arbitrager *any* point on his budget line without affecting prices. Notice that this conclusion depends on the presence of some individuals with smooth preferences. If everyone had right-angled indifference curves at a Walrasian allocation, an arbitrager could move nowhere away from his allocation—except trivially, by giving away commodities—without hurting others (to a first-order); hence, his Walrasian budget set would not coincide with his arbitrage opportunities. In particular, in this case the arbitrage cone would be right-angled rather than flat; hence his Walrasian budget line would not be a true opportunity line. That is, individuals would not truly face PEDS at the Walrasian prices. (See Example 2 below.)

The arbitrage approach is formulated in the following three sections. Of particular interest is the geometry of arbitrage that is highlighted in several figures. In Section 5, we compare the results of Sections 2–4 to Walrasian equilibrium, the core, dynamic matching models, as well as to other uses of arbitrage in general equilibrium. In the concluding Section 6, we contrast the interplay between marginal utility and price in Walrasian theory and arbitrage. We argue that the arbitrage approach leads to a more complete marginalist theory of value.

2 Preliminaries

It will be convenient to avoid boundary allocations. Accordingly, following Mas-Colell (1985), we will identify each individual's consumption set X with the strictly positive orthant $\mathbf{R}^{\ell}_{++}$, and we will assume that each individual's utility function $u : X \to \mathbf{R}$ satisfies the *boundary condition*:

- for every $x \in X$, $\{y : u(y) \geq u(x)\}$ is closed relative to $\mathbf{R}^{\ell}$.

That is, all indifference curves are asymptotic to the axes. The interpretation of the boundary condition is that some amount of each commodity is required for subsistence. The utility function u is *increasing* if $u(x) > u(y)$ whenever $x > y$, where $x > y$ means $x^h \geq y^h$ $(h = 1, \ldots, \ell)$ and $x \neq y$. Let $\mathcal{U}$ denote the set of all continuous, increasing utility functions on X satisfying the boundary condition. Regard $\mathcal{U}$ as a metric space, with $u_n \to u$ meaning u_n converges to u on compacta.

In addition to her utility function, each individual has some strictly positive endowment, $\omega \in \mathbf{R}^{\ell}_{++}$. So her exogenous characteristics are a pair (u, ω). We will restrict ourselves to pure exchange economies, which can be described simply by the distribution of individuals' exogenous characteristics. In particular, an *economy* $\mathcal{E}$ is any positive Borel measure on $\mathcal{U} \times \mathbf{R}^{\ell}_{++}$ with compact support. For any (Borel) subset $B \subset \mathcal{U} \times \mathbf{R}^{\ell}_{++}$, $\mathcal{E}(B)$ should be interpreted as the mass of individuals with characteristics B in the economy. The interpretation of the compactness assumption on the support of $\mathcal{E}$ is that individuals' characteristics are not too dispersed. Let E denote the set of all economies. Regard E as a product metric space, with $\mathcal{E}_n \to \mathcal{E}$ meaning $\mathcal{E}_n$ converges to $\mathcal{E}$ weakly and the support of $\mathcal{E}_n$ converges to the support of $\mathcal{E}$ in the Hausdorff distance.

An individual's *type* is a triple $t = (u, \omega, x) \in \mathcal{U} \times \mathbf{R}^{\ell}_{++} \times X$ consisting of the individual's preferences and endowment (her exogenous characteristics) plus her consumption decision (her endogenous characteristic). $\mathcal{T}$ denotes the set of all possible types. Regard $\mathcal{T}$ as a product metric space.

An *allocation* is any positive Borel measure μ on $\mathcal{T}$ satisfying the feasibility condition

$$\int x \, d\mu_x = \int \omega \, d\mu_\omega,$$

where μ_x denotes the marginal distribution of μ on the space X of consumption bundles (the marginal μ_ω is defined analogously). The measure μ is an allocation for the economy $\mathcal{E}$ if the marginal of μ on the space of exogenous characteristics equals $\mathcal{E}$. Let M denote the set of all allocations for all economies $\mathcal{E}$.

3 Arbitrage possibilities at μ

For any set S and any two points $x, y \in S$, y is *visible* from x if the line segment $[x, y] \subset S$. S is *star shaped (with center x)* if every point in S is visible from x. All convex sets are star shaped, but not conversely. To illustrate, the sets in panels (a) and (b) of Figure 1 are star shaped with center x. But only the darker shaded portion of the set in panel (c) is visible from x.

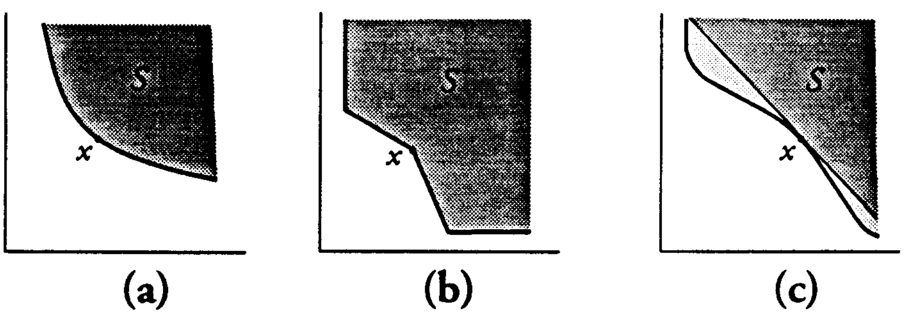

Figure 1:

Given any individual of any type $t = (u, \omega, x)$, we will assume that the individual will accept any trade offer that will leave her at least as well off as at x. But arbitragers can only "see" those trades that are visible from x. Consequently, from an arbitrager's perspective, the set of possible trades with an individual of type $t = (u, \omega, x)$ is given by

$$A(t) = \{z : u(x - z) \geq u(x) \;\&\; x - z \text{ is visible from } x\}.$$

Notice that the trades $z \in A(t)$ are described from the arbitrager's point of view: if $z^h > 0$ then the arbitrager *receives* z^h units of commodity h from an individual of type t. The restriction of $A(t)$ to trades visible from 0 makes the analysis simpler while permitting non-convexities. It can be motivated economically as being consistent with decentralized knowledge: arbitragers only have "local knowledge" about others' preferences.

A *group* is a vector of types, $(t_1, \ldots, t_i, \ldots t_n)$. Given any allocation μ, the set of possible groups that any arbitrager can form is denoted by $\mathcal{G}(\mu)$ and consists of all vectors $(t_1, \ldots, t_i, \ldots t_n)$ satisfying

1. $n < \infty$,

2. for any individual i, $t_i \in \text{supp } \mu$, and

3. for any type t, $\#\{i : t_i = t\} > 1$ only if $\mu(t) > 0$.

So $\mathcal{G}(\mu)$ includes any group consisting of only a finite number of individuals, having only individuals with types in the support of μ, and having several individuals of the same type only if that type is an atom in μ. The interpretation is that an arbitrager can make offers to only a finite number of others, and he can locate "several" individuals of the same type to trade with only if that type is an atom in μ.

An arbitrager's *arbitrage possibilites* at μ are given by

$$\begin{aligned} K(\mu) \quad = \quad \{z : \text{ for some group } (t_1, \ldots, t_i, \ldots t_n) \in \mathcal{G}(\mu), \\ z = \sum_{i=1}^{n} z_i \text{ and each } z_i \in A(t_i)\}. \end{aligned}$$

That is, z is possible for an arbitrager if and only if he can assemble a group of individuals willing to trade z with him in aggregate, where the acceptability of each individual trade z_i must be visible to the arbitrager starting from the allocation μ.

A fundamental fact is that the arbitrage possibilities at μ form a cone. For any set S, let cone S denote the smallest convex cone containing S. For any two sets S_1 and S_2, we will say that $S_1 = S_2$ *up to closure* if cl $S_1 =$ cl S_2.

Theorem 1 (characterization of arbitrage opportunities) *The closure of $K(\mu)$ is a convex cone containing the origin. In particular, up to closure*

$$K(\mu) = \cup \left(\sum_{i=1}^{n} \textit{cone } A(t_i)\right),$$

where the union is taken over all groups $(t_1, \ldots, t_i, \ldots t_n) \in \mathcal{G}(\mu)$.

(The reader may wish to skip the proof of Theorem 1 on first reading. If so, proceed to the intuition underlying the result, which is presented immediately following the proof.)

Proof: The definition of $K(\mu)$ implies $0 \in K(\mu)$, since $0 \in A(t)$ for each type $t \in \text{supp } \mu$.

To show $K(\mu)$ is a cone, suppose $z \in K(\mu)$, where $z = \sum_{i=1}^{n} z_i$, each $z_i \in A(t_i)$, and each $t_i \in \text{supp } \mu$. We will show that

(i) scaling down: $\alpha z \in K(\mu)$ for all $\alpha \in [0,1]$, and

(ii) scaling up: $2z \in \text{cl } K(\mu)$.

It readily follows that for any positive number r, $z \in K(\mu)$ implies $rz \in \text{cl } K(\mu)$. And hence it follows that $z^0 \in \text{cl } K(\mu)$ implies $rz^0 \in \text{cl } K(\mu)$; that is, cl $K(\mu)$ is a cone. (Consider a sequence $z^k \to z^0$ such that each $z^k \in K(\mu)$.)

To verify (i), observe that since each z_i is visible from 0, $\alpha z_i \in A(t_i)$ for any $\alpha \in [0,1]$. Hence $\alpha z = \sum_{i=1}^{n} \alpha z_i \in K(\mu)$.

To verify (ii), first observe that if each type t_i were an atom in μ then scaling up would be straightforward since an arbitrager could simply form a group consisting of two individuals of each type t_i. More generally, observe that for any type $t_i \in \text{supp } \mu$ having zero measure, t_i cannot by an isolated point in supp μ. (This follows from the definition of the support as the *smallest* closed set of full measure.) Consequently, there is a sequence of groups $\{(t_1, \ldots, t_i, \ldots, t_n; t_1^k, \ldots, t_i^k, \ldots, t_n^k)\} \subset \mathcal{G}(\mu)$ such that each $t_i^k \to t_i$. So there is a sequence of trades $z^k \equiv \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i^k$ in $K(\mu)$ such that each $z_i^k \to z_i$ and hence $z^k \to 2z$.

Next, to show cl $K(\mu)$ is convex, suppose $\bar{y}, \bar{z} \in K(\mu)$ and $\alpha \in [0,1]$. By (i) above, $y \equiv \alpha\bar{y}$ and $z \equiv (1-\alpha)\bar{z}$ are in $K(\mu)$. We will show

- $y + z \equiv \alpha\bar{y} + (1-\alpha)\bar{z} \in \text{cl } K(\mu)$.

It then follows, by a straightforward limiting argument, that $\bar{y}^0$, $\bar{z}^0 \in \text{cl } K(\mu)$ implies $\alpha\bar{y}^0 + (1-\alpha)\bar{z}^0 \in \text{cl } K(\mu)$ for any $\alpha \in [0,1]$.

Suppose $y = \sum_{i=1}^{m} y_i$ and $z = \sum_{i=m+1}^{n} z_i$, where each $y_i \in A(t_i)$ $(i = 1, \ldots, m)$ and each $z_i \in A(t_i)$ $(i = m+1, \ldots, n)$. If $\{t_1, \ldots, t_m\} \cap \{t_{m+1}, \ldots, t_n\} = \emptyset$ or if all these types are atoms, an arbitrager could simply form the group $(t_1, \ldots, t_n)$ and thus achieve $y + z$. Even if $\{t_1, \ldots, t_m\} \cap \{t_{m+1}, \ldots, t_n\} \neq \emptyset$ and some types are not atoms, since no zero-measure type is isolated in supp μ there is a sequence of groups $\{(t_1, \ldots, t_m; t_{m+1}^k, \ldots t_n^k)\} \subset \mathcal{G}(\mu) \to (t_1, \ldots, t_n)$. Hence there is a sequence of trades $y + z^k \to y + z$ where each $y + z^k \in K(\mu)$. So at least $y + z \in \text{cl } K(\mu)$.

Finally we show that up to closure

$$K(\mu) = \cup(\sum_{i=1}^{n} \text{cone } A(t_i)).$$

Considering first the one-member groups $(t) \in \mathcal{G}(\mu)$, from the definition of $K(\mu)$ it should be clear that $A(t) \subset K(\mu)$. Hence, from the above, cone $A(t) \subset$ cl $K(\mu)$. Similarly, considering the n-member groups $(t_1, \ldots, t_i, \ldots, t_n) \in \mathcal{G}(\mu)$, we see that $\sum_{i=1}^{n} A(t_i) \in K(\mu)$. Hence, $\sum_{i=1}^{n}$ cone $A(t_i) \subset$ cl $K(\mu)$. So, $\cup(\sum_{i=1}^{n}$ cone $A(t_i)) \subseteq$ cl $K(\mu)$. Conversely, $z \in K(\mu)$ implies $z \in \sum_{i=1}^{n}$ cone $A(t_i)$ for some group $(t_1, \ldots t_n) \in \mathcal{G}(\mu)$. So the two sets are equal up to closure. □

Convex cones are flatter than typical convex sets. Thus Theorem 1 may be interpreted as saying that, from any arbitrager's perspective, there is a "flattening effect of large numbers." To illustrate the idea, consider an allocation μ and a type $t = (u, \omega, x)$ in its support. Suppose individuals of type t have smooth, strictly convex preferences. The trades feasible with any one individual of type t are illustrated below. Notice that if the arbitrager were to

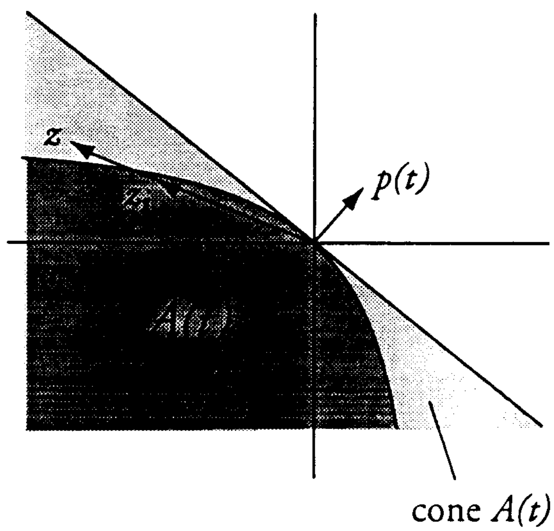

Figure 2: THE FLATTENING EFFECT OF LARGE NUMBERS

restrict his trading to only one individual of type t then z_i would be feasible, as would be all trades αz_i, $\alpha \in [0, 1]$; but the trade $z = nz_i$ $(n > 1)$ would not be feasible. Nevertheless, if t is an atom $(\mu(t) > 0)$ then an aribtrager can achieve the trade z by trading z_i with n individuals of type t. Indeed, even if t is not an atom, the arbitrager can find n types in the support of μ that are arbitrarily close to t. Thus he can achieve a trade arbitrarily close to

z, i.e., equal to z up to closure. (See the proof of Theorem 1 for details.)

More generally, let $p(t) = \nabla u(x)$. So, $p(t)$ equals type t's reservation prices at x. From the argument above, it should be clear that for any trade z in the interior of cone $A(t)$, an arbitrager can realize z by dividing it up into n smaller trades z_i, where each $z_i = \frac{1}{n} \cdot z$ and n is sufficiently large. Thus the budget line $p(t) \cdot z = 0$ is a true opportunity line for the arbitrager: he can (up to closure) achieve any trade on the line by arbitrage. Alternatively expressed, he faces perfectly elastic demands and supplies at prices $p(t)$ with individuals of type t. So the closure of $K(\mu)$ contains the cone spanned by $A(t)$.

4 Eliminating arbitrage profits

At an arbitrary allocation μ, people may have different reservation prices. Thus there may exist profits to arbitrage: buying low and selling high.

DEFINITION: The allocation μ is *arbitrage free*, denoted $\mu \in M_{AF}$, if

$$K(\mu) \cap \mathbf{R}^{\ell}{}_{++} = \emptyset.$$

So, in an arbitrage-free allocation the arbitrage possibilities set contains no "free lunches." Notice that since $K(\mu)$ is a cone, if $z \gg 0$ were in $K(\mu)$ then $n{\cdot}z \in K(\mu)$ for any $n > 0$. So an arbitrager could make unlimited profits. Evidentally, given the monotonicity of preferences, the absence of such profit potentials is a necessary condition for equilibrium (i.e., for the existence of utility-maximizing choices).

The polar cone of $K(\mu)$ is defined as

$$K^0(\mu) = \{p : pK(\mu) \leq 0,\ p \neq 0\}.$$

Arbitrage-free allocations can be characterized in terms of the polar.

Proposition 1 (characterization of arbitrage free allocations) *μ is arbitrage free iff $K^0(\mu) \neq \emptyset$. In particular,*

$$p \in K^0(\mu) \text{ iff } pA(t) \leq 0 \text{ for all } t \in \text{supp } \mu \quad (p \neq 0).$$

Hence, if at least one type in the support of μ has differentiable preferences then $\dim K^0(\mu) =$ 1 *and so*

$$cl\ K(\mu) = \{z : pz \leq 0\}$$

for any $p \in K^0(\mu)$.

Proof: μ is arbitrage free implies, by definition, that $K(\mu) \cap \mathbf{R}^\ell_{++} = \emptyset$. Hence, by the supporting hyperplane theorem, there exists a p ($p \neq 0$) such that $pK(\mu) \leq 0$. Conversely, $pK(\mu) \leq 0$ ($p \neq 0$) implies $K(\mu) \cap \mathbf{R}^\ell_{++} = \emptyset$ since $\mathbf{R}^\ell_- \subset K(\mu)$ by the monotonicity of preferences.

In particular, $p \in K^0(\mu)$ implies $pA(t) \leq 0$ for all $t \in \text{supp}\,\mu$ since $A(t) \subset K(\mu)$. Conversely, $pA(t) \leq 0$ for all $t \in \text{supp}\,\mu$ implies $p\sum_{i=1}^n \text{cone}\,A(t) \leq 0$ for all groups $(t_1, \ldots, t_n) \in \mathcal{G}(\mu)$. Hence $p\left[\cup\left(\sum_{i=1}^n \text{cone}\,A(t_i)\right)\right] \leq 0$, where the union is taken as in Theorem 1; which implies $pK(\mu) \leq 0$ by Theorem 1.

Now suppose there is at least on type $t = (u, \omega, x) \in \text{supp}\,\mu$ with differentiable preferences. For this type, $pA(t) \leq 0$ and $p'A(t) \leq 0$ implies p and p' must be colinear, since both supports must be proportional to $\nabla u(x)$. Hence, $p, p' \in K^0(\mu)$ implies p and p' must be colinear. □

$K(\mu)$ will be called *flat* (respectively, *pointed*) if $\dim K^0(\mu) = 1$ (respectively, > 1). While the elimination of arbitrage profits may result in pointed arbitrage cones, they are exceptional in a topological sense. This follows from Proposition 1 and the fact that differentiable utility functions are dense in $\mathcal{U}$. Formally, let $M_{AF}(\mathcal{E})$ denote the set of all arbitrage free allocations for the economy $\mathcal{E}$, and let

$$E_{\text{PEDS}} = \{\mathcal{E} \in E : \dim K^0(\mu) = 1 \text{ for all } \mu \in M_{AF}(\mathcal{E})\}.$$

Recall that a G_δ set is the countable intersection of open sets; hence dense G_δ subsets are large (i.e., generic) in a topological sense.

Theorem 2 E_{PEDS} *is a dense* G_δ *subset of* E.

Proof: Let $\mathcal{U}^\infty$ denote the set of all $u \in \mathcal{U}$ having, for every $n < \infty$, an nth derivative that is continuous in X.

FACT: $\mathcal{U}^\infty$ is dense in $\mathcal{U}$.

(The proof is analogous to that of Proposition 2.8.1 in Mas-Colell, 1985.)

Using this fact, we first show that E_{PEDS} is dense in E, i.e., $\mathcal{E} \in E$ implies there is a sequence $\{\mathcal{E}^k\} \subset E_{\text{PEDS}}$ s.t. $\mathcal{E}^k \to \mathcal{E}$. Pick any $\mathcal{E}$ and any $(u, \omega) \in \text{supp}\,\mathcal{E}$. The fact implies there is a sequence $u^k \to u$ with each $u^k \in \mathcal{U}^\infty$. Let $\mathcal{E}^k$ be the economy that adds a measure

$1/k$ of individuals with characteristics (u^k, ω) to $\mathcal{E}$. That is, for every Borel subset B

$$\mathcal{E}^k(B) = \begin{cases} \mathcal{E}(B) + \frac{1}{k} & \text{if } (u^k, \omega) \in B \\ \mathcal{E}(B) & \text{otherwise.} \end{cases}$$

By construction $\mathcal{E}^k \to \mathcal{E}$; and by Proposition 1, each $\mathcal{E}^k \in E_{\text{PEDS}}$.

We next show that E_{PEDS} is a G_δ subset of E. Let

$$E_\epsilon = \{\mathcal{E} \in E : \text{ for all } \mu \in M_{AF}(\mathcal{E}),\ d(K^0(\mu), p) < \epsilon \text{ for some } p\},$$

where $d(K^0(\mu), p)$ is the Hausdorf distance between the sets $\mathcal{P}(\mu) \equiv \{p' \in K^0(\mu) : \| p' \| = 1\}$ and $\{p\}$. Notice that $\mathcal{E} \in E_{\text{PEDS}}$ iff $\mathcal{E} \in E_\epsilon$ for all $\epsilon > 0$. That is,

$$E_{\text{PEDS}} = \cap_{k=1}^{\infty} E_{\frac{1}{k}}.$$

Thus it will suffice to show E_ϵ is open for any $\epsilon > 0$.

Assume the contrary. That is, there exists an $\epsilon > 0$, $\mathcal{E} \in E_\epsilon$, and sequence $\mathcal{E}^k \to \mathcal{E}$ s.t. each $\mathcal{E}^k \notin E_\epsilon$. By assumption, for each k there exists an allocation $\mu^k \in M_{AF}(\mathcal{E}^k)$ s.t. $d(K^0(\mu^k), p) \geq \epsilon$ for all p. But since $\{\mu^k\}$ is a tight family of measures, μ^k converges weakly to a measure μ on some subsequence, say $s(k)$. [See Hildenbrand, 1974, pp. 49-50, for a discussion of tight measures. To show that the family is tight, it suffices to show that the two families of marginal distributions are tight. The tightness of the family $\{\mathcal{E}^1, \mathcal{E}^2, \ldots\}$ follows from the fact that $\mathcal{E}^k \to \mathcal{E}$, and $\mathcal{E}^k$, $\mathcal{E}$ are tight since each is a measure on a compact metric space. The tightness of the family $\{\mu_x^1, \mu_x^2, \ldots\}$ follows from the fact that $\mathcal{E}^k \to \mathcal{E}$ (in particular the aggregate endowment in $\mathcal{E}^k$ converges to the aggregate endowment in $\mathcal{E}$) and each allocation μ^k is feasible.] Further, since each allocation μ^k is feasible for $\mathcal{E}^k$ and $\mathcal{E}^k \to \mathcal{E}$, μ is a feasible allocation for $\mathcal{E}$. Now observe that the sequence $\mathcal{P}(\mu^{s(k)})$ must converge in the Hausdorf distance, at least along a subsequence, to some set $\mathcal{P}$. Further, since each $\mathcal{P}(\mu^k) \subset K^0(\mu^k)$ and $\mathcal{E}^k \to \mathcal{E}$, $\mathcal{P} \subset K^0(\mu)$. [Proof: By Proposition 1, it suffices to show $p \in \mathcal{P}(\mu)$ implies $pA(t) \leq 0$ for all $t \in \text{supp } \mu$. Suppose the contrary, that for some $t = (u, \omega, x) \in \text{supp } \mu$ and some $z' \in A(t)$, $pz' > 0$. Hence, by monotonicity of preferences, there would be a $z \in A(t)$ such that $p(x - z) < px$ and $u(x - z) > u(x)$. But supp $\mu \subset$ Li supp $\mu^{s(k)}$. Hence, for k sufficiently large, there is a type $t^k = (u^k, \omega^k, x^k) \in \text{supp } \mu^k$ and a $p^k \in K^0(\mu^k)$ such that $p^k(x^k - z) < p^k x^k$ and $u^k(x^k - z) > u^k(x^k)$. That is, $z \in A(t^k)$ and $p^k z > 0$, contradicting $p^k \in K^0(\mu^k)$.] $K^0(\mu) \neq \emptyset$ implies $\mu \in M_{AF}(\mathcal{E})$; and $\mathcal{P}(\mu^k) \to \mathcal{P}(\mu)$

along a subsequence implies $d(K^0(\mu), p) \geq \epsilon$ for all p, contradicting our assumption that $\mathcal{E} \in E_\epsilon$. $\square$

We highlight allocations with flat arbitrage cones not only because of their genericity, but also because for such allocations the supporting prices $p \in K^0(\mu)$ reflect arbitragers' true trading opportunities:

$$\text{cl } K(\mu) = \{z : pz \leq 0\}$$

implies that any arbitrager can (up to closure) achieve any trade on the budget line $p \cdot z = 0$. Alternatively expressed,

- when $K(\mu)$ is flat then all individuals truly face *perfectly elastic demands and supplies* (PEDS) at the prices $p \in K^0(\mu)$

in the sense that any individual, acting as an arbitrager, can truly buy or sell as much as he likes at these prices. Note that, by contrast, if dim $K^0(\mu) > 1$ then any prices $p \in K^0(\mu)$ still define a separating hyperplane, but they do not define true terms-of-trade in the sense of characterizing an arbitrager's trading opportunities. Thus Theorem 2 may be interpreted as saying that generically in large economies, once arbitrage profits have been eliminated, everyone faces PEDS.

Let us examine this central fact a bit more closely. For simplicity we have used the "distribution approach" to describe allocations, but there is an easy translation in terms of the "agent approach" in which the economy is regarded as consisting of a nonatomic continuum of agents. In particular, for any μ there exists a measurable mapping f from the interval $[0, b]$ into $\mathcal{T}$ such that $\mu = \lambda \circ f^{-1}$, where λ denotes Lebesgue measure on $[0, b]$ and $b = \mu(\mathcal{T})$ (see Hildenbrand [1974, p. 50]). Using this mapping, let

$$\mathcal{A}(\mu) = \int A(f(i))\, d\lambda(i).$$

$\mathcal{A}(\mu)$ represents the set of visible trades that individuals in aggregate would find at least as good as remaining at μ. For example, if the support of μ consists of only one type t with $\mu(t) = 1$, then $\mathcal{A}(\mu)$ looks identical to $A(t)$ in Figure 1, *but the trades in $\mathcal{A}(\mu)$ are of a larger order of magnitude*: each trade $z_i \in A(t)$ is infinitesimal compared to the aggregate trade $z_i \cdot \mu(t) = z_i \cdot 1 \in \mathcal{A}(\mu)$.

Notice that $\mathcal{A}(\mu)$ is convex (by Liapunov's Theorem), but it is typically not a cone. Nevertheless, there is an interesting geometrical connection between the boundaries of $\mathcal{A}(\mu)$

and $K(\mu)$. An individual arbitrager's trading possibilities are of the same order of magnitude as the trades in any one set $A(t)$. And indeed, from his perspective, $K(\mu)$ is a "blow-up" of $\mathcal{A}(\mu)$ around the origin. In particular, if $\mathcal{A}(\mu)$ is smooth at zero, $K(\mu)$ will be flat. See the figure below. To use a simile, the tiny arbitrager is like an infinitesimal ship navigating around the origin of $\mathcal{A}(\mu)$; a blow-up of $\mathcal{A}(\mu)$'s local structure defines the arbitrage possibilities he perceives. Formally:

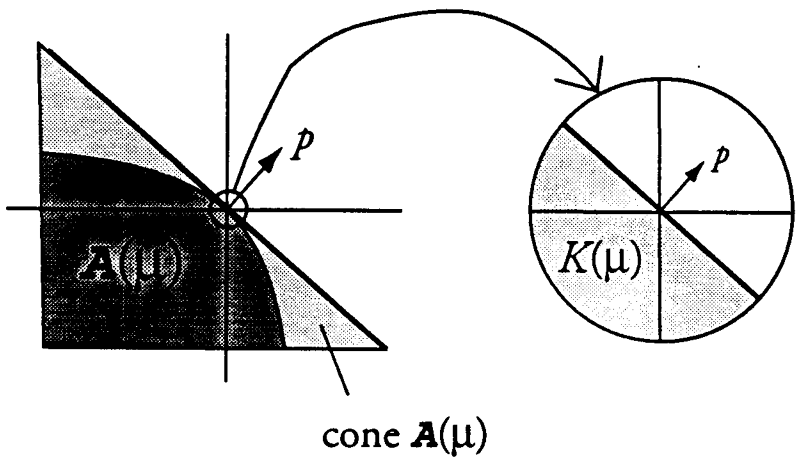

Figure 3: $K(\mu)$ IS A "BLOW-UP" OF $\mathcal{A}(\mu)$ AROUND THE ORIGIN, WITH SLOPE REFLECTING OTHERS' MRS'S

Proposition 2 (characterization) *Up to closure,* $K(\mu) = \text{cone } \mathcal{A}(\mu)$. *Thus,*

$$p \in K^0(\mu) \text{ iff } p\mathcal{A}(\mu) \leq 0 \quad (p \neq 0).$$

Proof: It suffices to show that

$$\{p : pK(\mu) \leq 0\} = \{p : p[\text{cone } \mathcal{A}(\mu)] \leq 0\},$$

since then the polars of the above cones would have to be equal, i.e., up to closure

$$K(\mu) = \text{cone } \mathcal{A}(\mu).$$

Suppose first the $p \cdot \text{cone } \mathcal{A}(\mu) \leq 0$ $(p \neq 0)$. Then $p\mathcal{A}(\mu) \leq 0$, and hence by a standard argument, $pA(t) \leq 0$ for all $t \in \text{supp } \mu$. So, by Proposition 1, $p \in K^0(\mu)$. Conversely, $p \in K^0(\mu)$ implies $pA(t) \leq 0$ for all $t \in \text{supp } \mu$, hence $\int pA(f(i))d\lambda(i) = p\mathcal{A}(\mu) \leq 0$, which implies $p[\text{cone } \mathcal{A}(\mu)] \leq 0$. $\square$

Underlying Figure 3 is the fact that in continuum economies, ***there are two infinitesimal margins of analysis***:

- the traditional commodity margin, which is infinitesimal compared to any one individual's trades, and
- each arbitrager, whose entire trading is infinitesimal compared to the size of the economy as a whole, as illustrated in Figure 3.

It is this second infinitesimal margin of analysis that leads to the flattening effect of large numbers. By trading a little bit more, z_i, with many others at (almost) their reservation prices $p \in K^0(\mu)$ the arbitrager can effectively "blow up" others' commodity margins into his trading opportunity set. To introduce a suggestive terminology, when $\mathcal{A}(\mu)$ is smooth at the origin let us refer to the price vector p that supports $\mathcal{A}(\mu)$ as reflecting *aggregate marginal rates of substitution.* To see that the terminology is justifed, observe that the boundary of $\mathcal{A}(\mu)$ may be viewed as the economy's aggregate indifference curve relative to remaining at μ, i.e., relative to trading zero more. Hence the slope of $\mathcal{A}(\mu)$ at 0 may be viewed as the economy's *aggregate MRS.* More specifically, Propositions 1 and 2 together imply that $p\mathcal{A}(\mu) \leq 0$ iff p is a supporting price for each type in the support of μ. So if everyone's preferences are smooth, p also measures each type's MRS's; but in $\mathcal{A}(\mu)$ these MRS's have been scaled up from a size that is infinitesimal relative to any one arbitrager (i.e., the commodity margin) to a size that is of the same order of magnitude as any arbitrager's entire trading (i.e., the individual margin). Using this terminology we have found that, generically in large economies, once arbitrage profits have been eliminated each individual's trading opportunities reflect aggregate marginal rates of substitution.

Some analogue of the set $\mathcal{A}(\mu)$ plays a prominent role in traditional general equilibrium theory (e.g., see the set G in Debreu [1959, p. 95, figure 2]). In particular, price vectors that support Pareto optima are intimately related to price vectors that support $\mathcal{A}(\mu)$. In traditional theory it is of little significance whether or not $\mathcal{A}(\mu)$ is smooth: convexity suffices to obtain supporting prices. The interesting fact is that in large economies, if (and only if) $\mathcal{A}(\mu)$ is smooth then the budget line that supports $\mathcal{A}(\mu)$ defines a true opportunity line in the sense that an individual can attain *any* point on the line via arbitrage.

Remark: The flattening effect of large numbers depends on the presence of a large number of competing economic agents; however, this effect does not point to a discontinuity at infinity.

To illustrate, consider an pure-exchange economy with a finite number of individuals, $\hat{\mathcal{E}} = \{(u_i, \omega_i)\}_{i=1,\ldots,n}$. For simplicity, suppose each u_i is strictly quasi-concave. Let $\hat{\mu} = \{(u_i, \omega_i, x_i)\}_{i=1,\ldots,n}$ be an allocation for $\hat{\mathcal{E}}$. To facilitate comparisons with the continuum, define the arbitrage possibilities at $\hat{\mu}$ for any individual i by

$$K_i(\hat{\mu}) = \sum_{j \neq i} A(t_j),$$

where $t_j = (u_j, \omega_j, x_j)$.

By contrast, let $\mathcal{E}$ be a continuum economy containing a unit measure of individuals with characteristics (u_i, ω_i), where $i = 1, \ldots, n$. And let μ be the allocation for $\mathcal{E}$ with $\mu(t_i) = 1$, $i = 1, \ldots, n$. Observe that Theorem 1 implies

$$K(\mu) = \sum_i \text{cone } A(t_i).$$

The key distinctions between the finite and continuum versions of arbitrage are that

- $K_i(\hat{\mu})$ depends on i; $K(\mu)$ does not.
- $K_i(\hat{\mu})$ is a convex set, but not a cone; $K(\mu)$ is a cone.

Observe what happens as the finite economy gets larger. Let $\hat{\mathcal{E}}^r$ be the r-fold replica of $\hat{\mathcal{E}}$; similarly, let $\hat{\mu}^r$ be the r-fold replica of the allocation $\hat{\mu}$. Define the arbitrage possibilities at $\hat{\mu}^r$ for any individual of any type i by

$$K_i(\hat{\mu}^r) = r \sum_{j \neq i} A(t_j) + (r-1) A(t_i).$$

Since each set $A(t_j)$ contains 0, evidently

$$K_i(\hat{\mu}) \subset K_i(\hat{\mu}^2) \subset K_i(\hat{\mu}^3) \subset \cdots \subset K(\mu).$$

Further, $\lim K_i(\hat{\mu}^r) = K(\mu)$. So the flattening effect of large numbers becomes more pronounced as the economy gets larger and larger. See Figure 4 below.

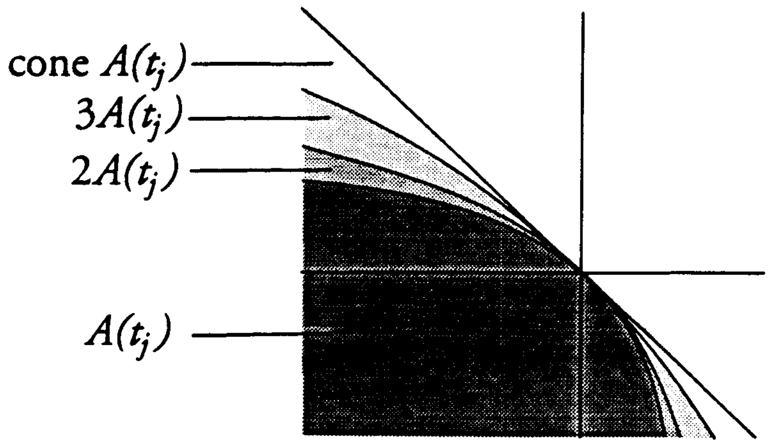

Figure 4: THE FLATTENING EFFECT OF LARGE NUMBERS AS A FINITE ECONOMY IS REPLICATED

5 Arbitrage Equilibrium

DEFINITION: An allocation μ is an *arbitrage equilibrium* (denoted $\mu \in M_{AE}$) if for every type $t = (u, \omega, x) \in \operatorname{supp} \mu$,

$$u(x) \geq u(\omega + z) \text{ for all } z \in K(\mu).$$

An allocation μ is a ***perfectly competitive arbitrage equilibrium*** (denoted $\mu \in M_{PC}$) if it is an arbitrage equilibrium and $K(\mu)$ is flat.

That is, an arbitrage equilibrium is an allocation μ that cannot be improved upon by any individual given his endowment and arbitrage possibilities at μ. The equilibrium is perfectly competitive if each individual's arbitrage possibilities are flat, so he can sell or buy as much as he wants at the terms of trade established by the elimination of arbitrage profits; i.e., he truly faces perfectly elastic demands and supplies (PEDS) at the prices $p \in K^0(\mu)$. See Figure 5 below.

An "arbitrage equilibrium" receives its name from the fact that any such equilibrium must be arbitrage free (otherwise, since preferences are monotonic, no one would have a utility maximizing choice). The second part of an arbitrage equilibrium—after the elimination of free lunches—is that each arbitrager continues to trade on his "own account" as long as he can improve upon his lot at the terms-of-trade implicit in the slope of (the boundary

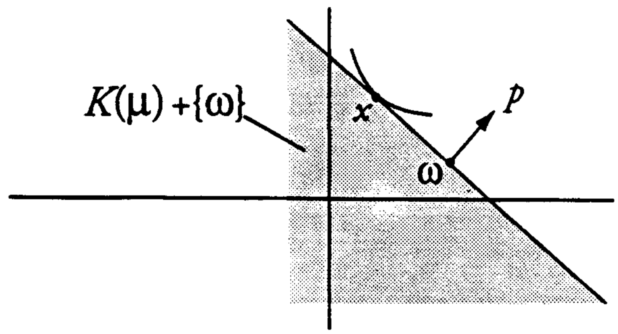

Figure 5: A PERFECTLY COMPETITIVE ARBITRAGE EQUILIBRIUM

of) $K(\mu)$. Notice the individual is assumed to launch his arbitrage efforts starting from his endowment point ω rather than from x. This is similar to the hypothesis of no false trading in the *tâtonnement* approach to Walrasian equilibrium, to recontracting in the core, or to the hypothesis in Nash equilibrium that every individual can revise his strategy in reaction to the choices by others.

Figures 3 and 5 can be combined to provide an alternative picture of a perfectly competitive arbitrage equilibrium. See Figure 6. It may be regarded as a variation on the Edgeworth box picture of a competitive equilbrium. The variation has some interesting ad-

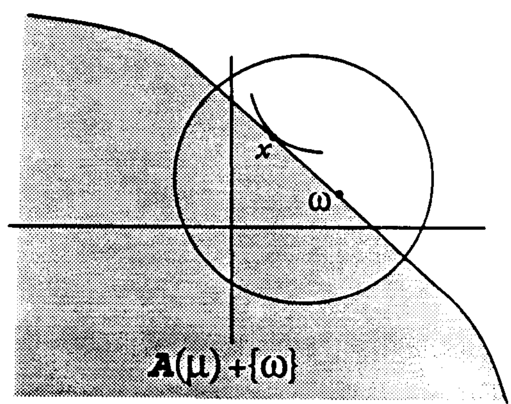

Figure 6: A VARIATION ON THE EDGEWORTH BOX. Others' MRS's determine each trader's opportunity set.

vantages. First, it does not require just two traders, where the price-taking assumption is dubious. Second it emphasizes the importance of *others'* MRS's: they determine the slope of each trader's opportunity set. Relatedly, and perhaps most importantly, it explicitly brings out the flattening effect of large numbers: each agent truly faces PEDS at the market clearing prices, he does not simply "act" as a price-taker.

6 Comparisons

6.1 Walrasian equilibrium

In a Walrasian equilibrium prices are given exogenously, and each individual is assumed to act as a price-taker.

DEFINITION: A pair (μ, p) is a *Walrasian equilibrium* if $p \neq 0$ and for every type (u, ω, x) in $\operatorname{supp} \mu$

- x maximizes u on $\{y : py \leq p\omega\}$.

Let M_{WE} denote the set of Walrasian allocations.

Any Walrasian allocation is arbitrage free, indeed it is an arbitrage equilibrium.

Proposition 3 $M_{WE} \subset M_{AE}$.

Proof: Let (μ, p) be any Walrasian equilibrium. In view of Proposition 1, it will suffice to show that $p \in K^0(\mu)$. Consider an arbitrary $z \in K(\mu)$, where $z = \sum_{i=1}^{n} z_i$, each $z_i \in A(t_i)$, and each $t_i = (u_i, \omega_i, x_i) \in \operatorname{supp} \mu$. Since μ is Walrasian, for each i

$$p(x_i - z_i) \geq p\omega_i \;\&\; px_i = p\omega_i.$$

Hence, $pz = \sum_i pz_i \leq 0$. That is, $p \in K^0(\mu)$. □

The inclusion may be strict, as illustrated by the following example.

Example 1. Consider a pure-exchange economy with only two goods. Everyone has preferences $u(x_1, x_2) = \min\{x_1, x_2\}$. Half the people have an endowment of $\omega = (0, 1)$ each; the other half have $\omega = (1, 0)$ each. Consider the allocation μ with 4 equally-sized atoms, illustrated in the Edgeworth box in Figure 7. Since the two goods are perfect complements, $K(\mu) = \mathbf{R}^{\ell}_{-}$, the negative orthant. So for each type, $u(x) > u(\omega + z)$ for all $z \in K(\mu)$. That is, the allocation is an arbitrage equilibrium. But since half the people trade at prices p^L (a relatively low price for good 2) and the other at p^H, the allocation does not satisfy the Law of One Price. It is not Walrasian.[1]

[1] In the example, indifference curves are taken as right-angled only for convenience. It should be clear that kinked but not right-angled indifference curves would suffice. The latter are consistent with our assumption that any $u \in \mathcal{U}$ is increasing.

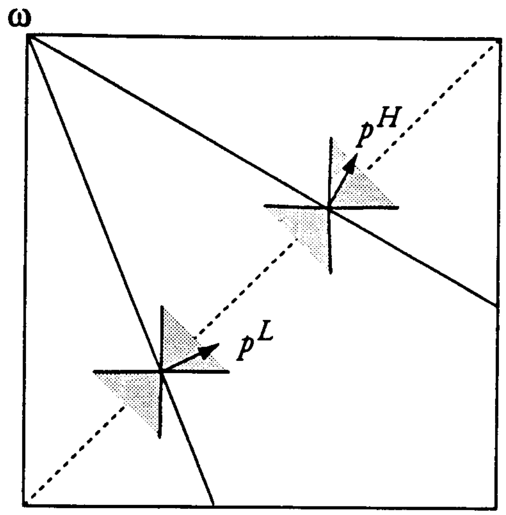

Figure 7: AN ARBITRAGE EQUILIBRIUM MAY NOT BE WALRASIAN

But perfectly competitive arbitrage equilibria do satisfy the Law of One Price and, indeed, are Walrasian.

Proposition 4 $M_{PC} \subset M_{WE}$.

Proof: Let $\mu \in M_{PC}$ and $p \in K^0(\mu)$. We will show the (μ, p) form a Walrasian equilibrium. Since μ is an arbitrage equilibrium, for each $t = (u, \omega, x) \in \text{supp } \mu$,

$$u(x) \geq u(\omega + z) \text{ for all } z \in K(\mu).$$

Hence, since $K(\mu)$ is flat,

$$u(x) \geq u(\omega + z) \text{ for all } z \text{ satisfying } pz \leq 0.$$

That is, $u(x) \geq u(y)$ for all y satisfying $py \leq p\omega$. □

In particular, perfectly competitive arbitrage equilibria include all Walrasian allocations in which individuals truly face perfectly elastic demands and supplies. Again the inclusion may be strict. Even in large economies there may be Walrasian equilibria in which some individuals do *not* face PEDS at the market clearing prices. This is illustrated by the following variant of the first example.

Example 2. Again consider the economy of Example 1. But now consider the allocation μ in which all trading occurs at prices p. It is still the case that $K(\mu) = \mathbf{R}^{\ell}_{-}$. So dim $K^0(\mu) > 1$, and the allocation is not in M_{PC}. Observe that while $p \in K^0(\mu)$, the budget line defined by p does not reflect a true opportunity line. For example, traders cannot reach x from ω via arbitrage. See Figure 8 below.

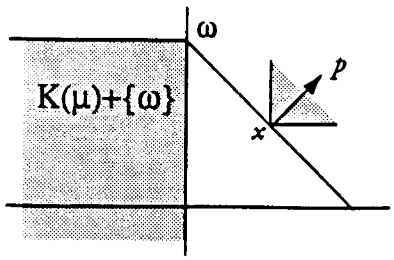

Figure 8: WALRASIAN ALLOCATIONS MAY NOT BE PERFECTLY COMPETITIVE

But such examples are exceptional. In most economies, all arbitrage equilibria will be perfectly competitive. Let $M_{PC}(\mathcal{E})$ denote the set of allocations for the economy $\mathcal{E}$ that are perfectly competitive; define $M_{AE}(\mathcal{E})$ and $M_{WE}(\mathcal{E})$ analogously. Call $\mathcal{E}$ a *perfectly competitive economy* if all its arbitrage equilibria are perfectly competitive, i.e., if

$$M_{AE}(\mathcal{E}) = M_{PC}(\mathcal{E});$$

and let E_{PC} denote the set of all perfectly competitive economies. Since any arbitrage equilibrium is arbitrage free, $E_{\text{PEDS}} \subset E_{PC}$. Hence it immediately follows for Theorem 2 that:

Corollary 1 *The set of perfectly competitive economies, E_{PC}, contains a dense G_δ subset of E.*

Thus, while for some economies $\mathcal{E}$

$$M_{PC}(\mathcal{E}) \subset M_{WE}(\mathcal{E}) \subset M_{AE}(\mathcal{E}),$$

generically the three coincide.

Nevertheless there are other bases for comparison. Compared to Walrasian equilibrium, in a perfectly competitive arbitrage equilibrium prices emerge *endogenously*: they reflect aggregate marginal rates of substitution after arbitrage has equalized all individuals' MRS's (recall Figure 3). Further, in a perfectly competitive arbitrage equilibrium, price-taking behavior can be endogenously justified: PEDS implies that "price taking" is the best that any self-interested arbitrager/trader can do in equilibrium (recall Figures 5 or 6). The next subsection amplifies on these themes.

6.2 The core

Since the core and Walrasian allocations are known to coincide in the continuum, the above discussion implies

$$M_{PC} \subset M_{\text{core}} \subset M_{AE},$$

where M_{core} denotes the set of allocations in the core. Examples 1 and 2 illustrate that both inclusions may be strict. In particular, Example 1 shows that arbitrage equilibria need not satisfy equal-treatment, hence need not be in the core. Example 2 shows that the core may be huge in economies that possess no perfectly competitive arbitrage equilibria. But Corollary 1 implies that such examples are exceptional: generically the core will coincide with the set of perfectly competitive arbitrage equilibria.

In spirit, perfectly competitive arbitrage equilibrium and the core share a common goal. They both represent efforts to open the "black box" called price-taking behavior, to give a bargaining story which leads to prices. Indeed, core bargaining may be viewed as a form of arbitrage: individuals form improving coalitions as long as they perceive "arbitrage profits." (See Mas-Colell (1982).) Compared to core bargaining, the arbitrager of our model acts more individualistically and myopically: Given any status quo allocation μ, he only considers the possibility of suggesting local changes to others around their status quo. He does not consider forming a group in which all members drop their existing contracts to form a new self-sufficient subeconomy. For example, the allocation in Example 1 above is not in the core: individuals having endowments (0,1) and trading at p^L could improve their lot by forming a coalition with individuals having endowments (1,0) and trading at p^H—if both the former *and latter* recontracted simultaneously. We do not deny that such more global group recontracting may occur and may be viewed as a form of arbitrage. The interesting fact is that even without it, our simpler form of arbitrage is generically sufficient to reach a perfectly competitive equilibrium.[2]

It may be argued that our less global view of arbitrage accords better with the idea that knowledge of preferences is private knowledge, dispersed over the countless individuals in the economy. It also may be argued that our view accords better with the idea of arbitrage in the finance literature: differences in prices (in our case, reservation prices) figure centrally

[2] An arbitrage equilibrium may be interpreted as allowing for individual recontracting, but not group recontracting. The reader may want to look at the definition of arbitrage equilibrium again, to view it in this light.

in the process of arbitrage as modelled above.

We emphasize a third advantage to our model of arbitrage equilibrium. At least in informal discussions, a central feature of perfectly competitive equilibrium is that agents face perfectly elastic demands and supplies at the market-clearing prices. For example, PEDS is used to informally justify the price-taking assumption in Walrasian theory (it also figures prominently in Marshallian partial equilibrium analysis). But PEDS plays no essential role in the logic of core bargaining. Indeed, core equivalence does *not* ensure PEDS, as Example 2 illustrates. While generically such examples are exceptional, they do show that core equivalence does not give a complete picture of perfectly competitive equilibrium, as informally understood. By contrast, in perfectly competitive arbitrage equilibrium, PEDS appears explicitly in the *formal* definition of equilibrium. Indeed, the image of a flat arbitrage cone may be viewed as the central one around which the current theory is constructed. If this image is considered useful, then arbitrage equilibrium can be viewed as a complementary theory to core equivalence, one that helps to flesh out the image of a perfect competitor.

There are some interesting similarities between our concept of arbitrage equilibrium and the f-core introduced by Kaneko and Wooders (e.g., see their 1989 article with Peter Hammond). They argue that a satisfactory picture of a perfectly competitive economy should include two features: (1) individuals should be effective in the pursuit of their own interests, but (2) individuals should be ineffective in influencing broad economic aggregates. They argue that the continuum is the natural setting for (2). But the traditional core concept, when applied to the continuum, requires improving coalitions to have positive measure; so any one individual has no voice, i.e., (1) is not captured. On this basis they argue that for continuum economies their f-core (which restricts improving coalitions to only a finite number of participants), provides a more satisfactory picture of a perfectly competitive economy than does the traditional core. Observe that in our model of arbitrage equilibrium, both the above desiderata are satisifed. In particular, recall that we have restricted any arbitrager to only forming groups with a *finite* number of participants. To Kaneko and Wooder's list we would add a third, and we feel equally important, desideratum: (3) individuals should face PEDS. It is this third feature of perfect competition that arbitrage equilibrium pictures in a more satisfactory way, we believe, than does the core.

6.3 Dynamic Matching

After completing an earlier version, it was pointed out to us that there is an intimate connection between arbitrage equilibria and the non-cooperative equilibria in dynamic matching and bargaining games (Rubinstein and Wolinsky (1985), Gale (1986), McLennan and Sonnenschein (1991); see Rubinstein and Osborne (1990) for an introductory exposition and further references). The central finding in this literature, due to Gale, is that these games' subgame perfect equilibria are Walrasian. The main idea leading to this result is that any player can effectively ensure himself a linear opportunity line since he can always adopt the strategy of making a sequence of small take-it-or-leave-it offers to individuals about to leave the market, at terms of trade reflecting their MRS's. The equilibrium notion of subgame perfection implies that these individuals will accept such offers since they are about to leave the market anyway. In our language, the above strategy allows each player to form a ***flat*** arbitrage cone as his trading opportunity set. Note that Gale assumes individuals have differentiable utility functions, so the arbitrage cone will indeed be flat, not pointed (recall Proposition 1).

In the course of extending Gale's result, McLennan and Sonnenschein (1991) offer a characterization of Walrasian equilibrium to help explain its connection with dynamic matching games. (Their characterization is, in turn, related to work of Schmeidler and Vind (1972) and Vind (1978) on fair allocations.) They show that given any allocation, if individuals' preferences are continuously differentiable and if there is a trading possibilities set Z satisfying:

(i) individuals can choose not to trade at all or can trade as many times as they like (i.e., $0 \in Z = Z + Z$),

(ii) others will accept any utility-increasing trade relative to the given allocation (i.e., Z includes all such trades), and

(iii) each individual's component in the given allocation is at least as good as what she could achieve via trading in Z starting from her endowment

then the given allocation must be Walrasian.

From our perspective, the McLennan and Sonnenschein characterization encapsulates much of our reasoning leading to Proposition 5, identifying their set Z with our $K(\mu)$.

However, in their characterization the pivotal set Z is hypothesized rather than derived; indeed, McLennan and Sonnenschein first prove their result for an economy with only a finite number of individuals. They then go on to show how the set Z arises in the equilibrium of a dynamic matching-and-bargaining model with a continuum of agents and differentiable preferences.

As a foundation for perfectly competitive equilibrium, our arbitrage approach lies between dynamic matching models and the static characterization of McLennan-Sonnenschein. It is considerably more condensed than Gale's result but, on the other hand, it is considerably less condensed than conditions (i)–(iii) above. Any fully articulated extensive form game must, of necessity, be very specific about institutions. While some real-world markets may fruitfully be viewed as involving random bilateral matching, there also are many markets that do not fit this mold. For example, retailers often set posted prices, while buyers decide which retail shops to patronize; in such markets there is an asymmetry between sellers and buyers, and matching is not random. Nevertheless, such markets are often very competitive. So the principles of arbitrage and the flattening effect of large numbers will still apply. Further condensation to make $K(\mu)$ into the abstract set Z would leave open the question of where Z comes from.

Because our approach is more abstract than the extensive game-form treatment, it complements that literature by highlighting principles that might otherwise be lost amidst the detailed game theoretic analysis. In particular, it highlights the importance of "arbitrage" and the "flattening effect of large numbers" in the above dynamic games. Relatedly, it gives economic content to the differentiability assumption about preferences that is essential to obtain the literature's central result. Differentiability implies each player effectively faces perfectly elastic demands and supplies (PEDS) at the economy's Walrasian prices, e.g., as in Figure 5. Arbitrage and PEDS have strong links to the meaning of perfect competition; and, as far as we know, neither has been previously used to give interpretive content to Gale's game theoretic equivalence result.

The complementary relationship is two-way. From our perspective, it is nice to be able to point to this literature as providing an interesting extensive form game in which our *assumption* in forming the arbitrage cone $K(\mu)$—that individuals will accept any utility-increasing offers from an arbitrager—can be strategically justified.

6.4 Finance

In the Introduction we discussed the relation between our usage of "arbitrage" and the common usage. Here we focus on a narrower matter. We compare the role played by the elimination of arbitrage profits in our model with the "no market arbitrage" condition in the literature on the existence of Walrasian equilibrium in models with unbounded short selling and/or non-monotonic preferences (e.g., models with asset markets).

Hart (1974) observed that unlimited short selling could lead to existence problems because some investors may want to sell short indefinitely large amounts of some assets while other (more optimistic) investors may want to to take substantial long positions in the same assets. He introduced a no market arbitrage condition to rule out the possibility. Hammond (1983), Page (1987), Werner (1987), Nielsen (1989), Chilchilnisky (1992) and Page and Wooders (1994) (among others) elaborated on Hart's work, leading to a much more general theorem on the existence of Walrasian equilibrium, one that is more useful in applications to finance.

The central "no market arbitrage condition" that emerged from this literature requires that there is no feasible set of net trades which can be repeated indefinitely without eventually making somebody worse off. Notice this is a restriction on preferences which rules out groups having diametrically opposed tastes. Formally, the recession cones of individuals' preferred-to sets and their polars play a key role in the analysis. Werner and Nielsen's approaches to proving existence are somewhat different, although related: Nielsen shows that no market arbitrage is sufficient for existence, while Werner obtains existence using a price characterization of no market arbitrage.

In the above existence theory the recession cones of individuals' preferred-to sets and their polars play a central role. Analogously, in our theory the arbitrage cone and its polar play a central role. But the similarity is superficial: The relevant cones in the two theories involve very different constructs; they are not the same cones. For example, the recession cones highlighted in the existence literature are typically not flat; their slopes do not summarize individuals' trading opportunities.

That the similarities are only superficial should come as no surprise, given that the two theories have quite different goals. The above literature seeks to to find sufficient conditions to prove the existence of Walrasian equilibrium without restricited short selling. Since the question posed pertains to the existence of *Walrasian equilibrium*, price-taking behavior is assumed. Indeed, since the models analyzed involve only a finite number of participants, individuals could influence market-clearing prices if they tried, i.e., PEDS is not satisifed. Within this context, the remarkable finding is that a no market arbitrage condition does the trick: it allows one to prove the existence of prices that will clear all markets when individuals act as price-takers but are not restricted in their short selling. By contrast, our goal is very different: it is to give a story of how arbitrage may lead to market-clearing prices without assuming price-taking behavior; and further, to justify price-taking behavior *in equilibrium* via the flatness of the arbitrage cone (hence PEDS) that emerges from active arbitraging rather than passive price-taking.

7 Conclusion: The place of marginal utility in the theory of value

Arbitrage provides another, and we believe stronger, statement of the "marginalist logic" behind the competitive theory of value. Jevons (1879) is credited with one of the earliest appeals to arbitrage. Further, his application was to a model of pure exchange. Jevons' reasoning was that, first, independent of marginal utility considerations there would be a single exchange rate in the market between any pair of commodities; and that therefore, second, individuals would regulate their purchases and sales to equate their individual MRS's to the market exchange rate. In our version of arbitrage, marginal utility enters at the first step to show how individuals' MRS's lead to the law of one price.

Marginal utility was, and is still regarded as, the central ingredient of the marginal revolution. In the Walrasian description of equilibrium marginal utility is the key to the formation of (price-taking) individuals' demand and supply schedules, which naturally leads to (a) the description of equilibrium as the equality of aggregate demand and supply and to (b) the *tâtonnement* view of the equilibration process. Parts (a) and (b) represent a unified construction driven by marginal utility. Notice how the identification of marginalism with price-taking behavior reinforces *tâtonnement* as apparently *the* logical path to competitive equilibrium.

But the connection that Walras established between marginal utility and competitive equilibrium is only one way to proceed, it is not the only way. Walras exploited only the marginal utility underpinnings of the consequences of perfect competition (i.e., price-taking behavior) rather than the marginal utility underpinnings of perfect competition itself. Although the two margins are not the same, "marginal utility" underlies both the maximizing behavior of a price-taker and also the maximizing behavior of an arbitrager. From the individual's point of view, when she acts as a passive price-taker, her own MRS's are central; but when she acts as an active arbitrager, *others' MRS's* are central: they determine the slope of her arbitrage cone and hence her trading possiblities. Notice how arbitrage provides an alternative logical link between marginal utility and competitive equilibrium.

Our proposal is to replace *tâtonnement* by arbitrage as the equilibration story behind competitive equilibrium. It has the following implication: while demand and supply functions (individual and aggregate) may be essentials of partial equilibrium theory, they are dispensable elements of competitive general equilibrium theory. It is interesting to observe that Walras' ideas on general equilibrium followed a more or less contrary path. Even before he saw how to make marginal utility the engine of his general equilibrium system, Walras had already formulated his conception of general equilibrium in terms of the equality of demand and supply schedules. For this development of Walras's ideas see Jaffé (1976), where he concludes: "Instead of climbing up from marginal utility to the level of his general equilibrium system, Walras actually climbed down from that level to marginal utility." In this respect, therefore, we may say that the arbitrage approach represents a "non-Walrasian" formulation of competitive equilibrium in which marginal utility figures even more prominently.[3]

Finally, the fact remains that an arbitrage equilibrium is (modulo the flat cone condition) a Walrasian equilibrium that is explicitly situated in a thick markets environment, where it has historically been understood to belong. Does this coincidence mean that there are no practical distinctions to be drawn between the arbitrage and Walrasian versions of competitive equilibrium? In our view, there is an important difference. The Walrasian tradition of price-taking reinforces the view that the perfect competitor responds passively to his environment whereas in the arbitrage approach the perfect competitor is actively

[3]Comparisons between arbitrage and Walrasian equilibria ignore the issue of existence. To demonstrate the existence of equilibrium through arbitrage, it might appear that one would first have to demonstrate the existence of Walrasian equilibrium. To emphasize the integrity of the arbitrage approach to perfectly competitive equilibrium, in an earlier version of this paper we have given a self-contained demonstration of the existence of perfectly competitive arbitrage equilibrium.

opportunistic. The difference in "psychology" between the competitor-as-price-taker versus the competitor-as-arbitrager are alternative perspectives which can significantly influence the way one interprets market behavior.

References

[1] R. J. Aumann. Markets with a continuum of traders. *Econometrica*, 32:39–50, 1964.

[2] G. Chichilnisky. Markets, arbitrage and social choices. Technical report, Department of Economics, Columbia University, 1992.

[3] G. Debreu. *Theory of Value.* Wiley, New York, 1959.

[4] G. Debreu and H. Scarf. A limit theorem on the core of an economy. *International Economic Review*, 4:236–246, 1963.

[5] P. Dubey, A. Mas-Colell, and M. Shubik. Efficiency properties of strategic market games: An axiomatic approach. *Journal of Economic Theory*, 22(2):339–362, 1980.

[6] F. Y. Edgeworth. *Mathematical Psychics.* Kegan Paul, London, 1881.

[7] D. Gale. Bargaining and competition Part i: Characterization. *Econometrica*, 54:785–806, 1986.

[8] P. Hammond. Overlapping expectations and Hart's condition for equilibrium in a securities model. *Journal of Economic Theory*, 31:170–75, 1983.

[9] P. Hammond, M. Kaneko, and M. Wooders. Continuum economies with finite coalitions: Core, equilibria and widespread externalities. *Journal of Economic Theory*, 49:113–134, 1989.

[10] O. Hart. On the existence of equilibrium in a securities model. *Journal of Economic Theory*, 9:293–311, 1974.

[11] W. Hildenbrand. *Core and Equilibria of a Large Economy.* Number 5 in Princeton studies in mathematical economics. Princeton University Press, Princeton, NJ, 1974.

[12] W. Jaffè. Menger, jevons and walras de-homogenized. *Economic Inquiry*, 14:511–524, 1976.

[13] W. S. Jevons. *The Theory of Political Economy.* Macmillan, London, second edition, 1879.

[14] F. Page Jr. On equilibrium in Hart's securities exchange model. *Journal of Economic Theory*, 41:392–404, 1987.

[15] F. Page Jr. and M. Wooders. Arbitrage in markets with unbounded short sales: Necessary and sufficient conditions for nonemptiness of the core and existence of equilibrium. Technical report, Department of Economics, University of Toronto, 1994.

[16] A. Mas-Colell. Perfect competition and the core. *Review of Economic Studies*, XLIX:15–30, 1982.

[17] A. Mas-Colell. *The Theory of General Economic Equilibrium: A Differentiable Approach.* Cambridge University Press, London, 1985.

[18] A. McLennan and H. Sonnenschein. Sequential bargaining as a noncooperative foundation for walrasian equilibrium. *Econometrica*, 59:1395–1424, 1991.

[19] L. Nielsen. Asset market equilibrium with short selling. *Review of Economic Studies*, 56:467–74, 1989.

[20] M. Osborne and A. Rubenstein. *Bargaining and Markets.* Academic Press, San Diego, 1990.

[21] S. Ross. The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13:341–360, 1976.

[22] A. Rubenstein and A. Wolinsky. Equilibrium in a market with sequential bargaining. *Econometrica*, 53:1133–50, 1985.

[23] D. Schmeidler and K. Vind. Fair net trades. *Econometrica*, 40:637–42, 1972.

[24] M. Shubik. Edgeworth market games. In R. Luce and A. Tucker, editors, *Contributions to the Theory of Games, Vol. IV*, pages 267–78. Princeton University Press, Princeton, 1959.

[25] K. Vind. Equilibrium with respect to a simple market. In G. Schwodiauer, editor, *Equilibrium and Disequilibrium in Economic Theory.* D. Reidel Publishing Co., Dordrecht, 1978.

[26] D. A. Walker, editor. *William Jaffeé's Essays on Walras.* Cambridge University Press, 1983.

[27] J. Werner. Arbitrage and the existence of competitive equilibrium. *Econometrica*, 55(6):1403–1418, 1987.